Combinatorial Analysis

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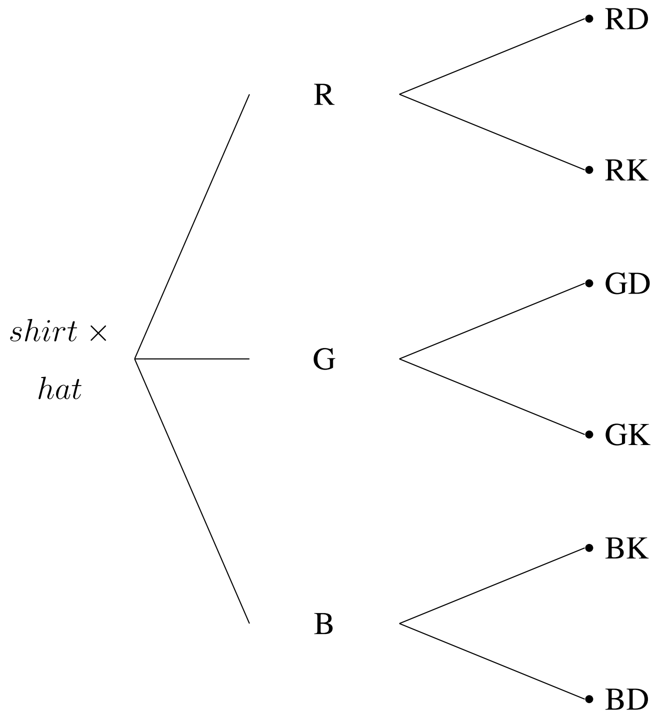
1 Combinatorial analysis

This chapter presents an introduction to the basic mathematical theory of counting known as *Combinatorial Analysis*. We will walk through the basic principles of combinatorics and introducte the *fundamental principle of counting* and the notions of *permutation* and *combinations*. This chapter has also another goal to familiarize the readers with the mathematical notation.

1.1 The fundamental principle of counting

Definition of the fundamental principle of counting. A basic counting principle is the *rule of product* or *multiplication principle*. Suppose that two tests are to be performed. Then if there are m outcomes from the first test and n outcomes from the second, then there are mn results from the two tests.

Let us explain this in an informal way first. Imagine that we have three shirts a blue shirt, a gray shirt, and a red shirt lets us indicate these as $shirt = \{B, G, R\}$ and two pairs of hats a khaki and a dark blue lets denote these as $hats = \{K, D\}$ and we want to see which combination is better for us. How many possible combinations are there? The answer is 6, namely $shirt \times hat = \{BK, BD, GK, GD, RK, RD\}$



Proof of the fundamental principle of counting. To proof the fundamental principle of counting one should enumerate all the possible outcomes of the two tests; i.e.,

$$\begin{pmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (m,1) & (m,2) & \cdots & (m,n) \end{pmatrix} \tag{1}$$

so, the possible outcomes from both tests is (i, j) if the first test results in its i_{th} outcomes and the second test in its j_{th} outcomes. Hence, the set of possible outcomes consists of m rows, each containing n elements.

If there is k number of tests, then there is a total of $n_1 \times n_2 \ldots n_k$ possible outcomes of the k tests.

Example. How many different English words of 9 letters each are potentially possible if 4 of these letters are vowels and 5 are consonants?

Solution. Following the fundamental principle of counting the answer is $6 \times 6 \times 6 \times 6 \times 20 \times 20 \times 20 \times 20 \times 20 = 4147200000$. Note, however than not all 4147200000 are legal combinations due to the phonotactic constraints of the English language.

1.2 Permutations

To find how many possible arrangements are possible with the three letters a, b, c then there are 6 possible permutations: $1 \times 2 \times 3 = 6$.

In general to determine the number of permutations in set of n things that does not include repetitions, then this number is $n!$, i.e.,

$$n(n{-}1)(n{-}2)\cdots 3 \times 2 \times 1 = n! \tag{2}$$

The $n!$ is also know as factorial and it is equal to the number of rearrangements of n things.

Example. How many different arrangements are possible for the letters of the word *letters*?

Solution. there are $7!$ possible combinations for the word *letters*. However, note that in this word there are 2E and 2T $(2! \times 2!)$. So, the result is $7!/2! \times 2! =$

$1260.$

Overall, when there are cases when we have repetitions then generic form is this

$$\frac{n!}{n_1!n_2!\dots n_k!} \tag{3}$$

R Function for Factorials. The R function for calculating the factorial is factorial(); for example:

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1.3 Combinations

There can be n ways to select an item from S; to select a second item there are $n-1$ ways, to selected a third item there are $n-2$ ways etc. So, to specify the number of different sets of k items that could be formed from a total of n items, we apply the following:

$$\frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \tag{4}$$

when $0 \leq k \leq n$, and let it equal 0 otherwise. For instance, we might want to know how many 3 letter words could be formed from the 5 letters: i, n, a, m, e, and s. To this purpose, we need to work as follows. There are 5 ways to choose the first letter; 4 ways to chose the second letter, and 3 ways to select the final letter. So there are $5 \times 4 \times 3$ ways of selecting the group of 3. Note that in this case there no constraints for the order of these letters. So, to get all the possible combinations of 3 letters in a group, then each group of three letters should be counted 6 times (i.e., $1 \times 2 \times 3$). So the total number of 3 letters that could be created is:

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10 \tag{5}$$

The result provided by 4 is $\binom{n}{k}$, that is the $\binom{n}{k}$ is the number of possible combinations of n items selected k at a time.

Example. 4 students should be selected from a class of 15 students. How may different groups of students are possible?

Solution There are 1365 combinations calculated as follows:

$$\binom{n}{k} = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365 \tag{6}$$

Example. There are 20 consonants and 6 vowels in the English alphabet, how many different letter combinations are possible consisting of 3 vowels and 4 consonants.

Solution There are $\binom{20}{4}$ combinations of 4 consonants and $\binom{6}{3}$ combinations of 3 vowels. Therefore, by applying the fundamental principle there are $\binom{20}{4} \times \binom{6}{3} = \binom{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} \times \binom{6 \times 5 \times 4}{3 \times 2 \times 1} = 96900$.

R Function for Combinations. The R function for calculating combinations is choose(). The function combn() enumerates all possible combinations.

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2 Applications

2.1 Binomial Theorem

The $\binom{n}{k}$ are known as binomial coefficients, because of the crucial role in the *Binomial Theorem*. The binomial theorem is the following:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{7}$$

The binomial theory is employed in elementary algebra to describe the algebraic expansion of powers of a binomial. We know that for very simple binomials the expansion is easy but becomes extremely complex as the power of the binomial increases **??**:

$$
\begin{aligned}
&(x+y)^0 = 1, \\
&(x+y)^1 = x+y, \\
&(x+y)^2 = x^2 + 2xy + y^2, \\
&(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3, \\
&(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4, \\
&(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5, \\
&(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6, \\
&(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7, \\
&\qquad \ldots
\end{aligned}
$$

The binomial theorem is a way to calculate the n^{th} power of any binomial. So, suppose that the expansion of the $(x+y)^3$ binomial is unknown and we want to calculated it using the binomial theorem, then we proceed as follows:

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + \binom{4}{4} y^4 \tag{8}$$

Now the binomial coefficients need to be calculated from the type, we discussed in Section 1.3 and repeated here as 9:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{9}$$

So, we calculate the coefficients as follows: for the $\binom{4}{0}x^4 = \frac{4!}{0!4!} = 1$, for the $\binom{4}{1}x^4 = \frac{4!}{1!3!} = 4$, for the $\binom{4}{2}x^4 = \frac{4!}{2!2!} = 6$, for the $\binom{4}{3}x^4 = \frac{4!}{3!1!} = 4$, and for the $\binom{4}{4}x^4 = \frac{4!}{4!0!} = 1$.

Then, we simply substitute the values as follows:

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \tag{10}$$

2.2 The multinomial theorem

A generalization of the binomial theorem describe in Section 2.1 to polynomials, is the multinomial theorem, which describes how to expand a power of a sum based on the powers of the terms in the designated sum. The general form follows the basic counting principle and it is the following:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_k-1}{n_k}$$
$$=\frac{n!}{(n-n_1)!n_1!}\frac{(n-n1)!}{(n-n1-n2)!n2!}\cdots\frac{(n-n_1-n_2-\ldots-n_k-1)!}{0!n_k!}$$
$$=\frac{n!}{n_1!n_2!\ldots n_k!}$$

To find how many divisions of size $n_1, n_2, \ldots, n_k$, where $\sum_{i=1}^{k} n_i = n$, are possible for a set of n distinct items, we can proceed as follows: first we find the divisions for the first group, which are $\binom{n}{n_1}$, the identify the divisions for the second group, which are $\binom{n-n_1}{n_2}$, then for the third group, which are $\binom{n-n_1-n_2}{n_3}$, then for the fourth group etc. So that the $\frac{n!}{n_1!n_2!\ldots n_k!}$ expresses the number of divisions of n items into k distinct groups of sizes, that is $n_1, n_2, \ldots n_k$.

Example. Computing the polynomial $(x + y + c)^3$:

$$(x+y+c)^3 = x^3+y^3+c^3+3x^2y+3x^2c+3y^2x+3y^2c+3c^2x+3c^2b+6xyc \tag{11}$$

Example. There are 20 students, and there are three activities that can attend: a theater class can accept 10 students, a music class can accept 5 students, and the dancing class can accept 5 students. How many different divisions of the 8 students are possible into the 3 classes?

Solution.

$$\frac{20!}{10!5!8!} = 138567 \tag{12}$$