

Absolute calibration of gravitational wave detector using gravity field and photon pressure

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Absolute calibration of the gravitational wave detectors is an essential to evaluate the various parameters of the gravitational wave sources. The photon calibrator is primary calibrator for calibrating the absolute displacement of the mirror by using the photon pressure. Current technological limit of the absolute calibration uncertainty is 3.5 % corresponding to the uncertainty of the laser power standard of the standard metrology institutes from nine country. In order to reduce the uncertainty of the photon calibrator, we propose a new method using the combination of photon calibrator and gravity field calibrator. The gravity field calibrator gives the modulation to mirror using the gravity gradient. In previous study, uncertainty of the distance between the test mass and gravity field calibrator is one of the serious systematic error of the absolute calibration. To cancel this uncertainty, we newly propose the method of quadrupole and hexapole mass distribution. We also estimated the uncertainty of this method. The estimated precision of absolute calibration is 0.17%, which is 10 times less than that of previous method.

I. INTRODUCTION

The discovery of the gravitational wave (GW) gave us the new probe for observing our universe [1]. The typical strain sensitivity, h , of 2nd generation interferometric detectors (IFO), such as Advanced LIGO [2], Advanced Virgo [3], and KAGRA [4, 5], are around $10^{-23}/\sqrt{\text{Hz}}$ at 100 Hz. By using the GW signals from compact binary coalescences, we can derive the parameters such as masses, spins, luminosity distance, orbital inclination and the sky location of the binary system from the detected waveforms. The precision of the derived parameters are potentially limited by the calibration accuracy. As the number of detected sources increases and we detect higher signal-to-noise ratio (SNR) events, the calibration uncertainty will become the dominant source of the errors to extract physics information. In particular, the uncertainty on the absolute GW signal amplitude directly propagates to the error on the estimation of the distance to the sources. The detection of GW signal from a Binary Neutron Star (BNS) system, GW170817 [6] in both GW and Electro-Magnetic (EM) waves opened a new era of multi-messenger astronomy. These observations allow us to use GW170817 as a standard siren [7–10] to determine the absolute luminosity distance to the source directly from the GW signal measurements. Assuming the event rate of $3000 \text{ Gpc}^{-3}\text{yr}^{-1}$ which is consistent with the bounds from GW170817 at 90% confidence [6], we expect to detect GW signals from about 50 BNS standard sirens in the next few observing runs. They can constrain the Hubble constant (H_0) measure-

ment to 2% or less [11], and eventually resolve the $3\text{-}\sigma$ tension of the H_0 measurement between Cepheid-SN distance ladder [12] and CMB data assuming a ΛCDM model. [13] The systematic errors on the calibration of the absolute GW signal amplitude must be suppressed at the sub-% level to achieve the high precision H_0 measurement with the GW standard sirens.

An interferometer measures the change of distance difference along the two arms of interferometer. Then, the fluctuations in the degree of freedom of differential arm length (DARM) is suppressed by the DARM control loop. The reconstruction signal of DARM fluctuation at the observation frequency is excited by the gravitational waves. We can reconstruct the gravitational waveform with the calibrated error and control signals of this DARM loop. To calibrate the signals, the accurate modelings of the actuator and sensing function are essential. To understand the model, we need to measure the transfer function and monitor the time dependency of the transfer function using continuous sine curve (calibration lines). The residual of the time-dependent models corresponds to uncertainty of the detection.

To reduce the calibration systematic uncertainty, we need to inject well parameterized calibration line by photon calibrator (Pcal) or other calibration source for monitoring the time variation of the response of IFO. The first generation of the photon calibrator was developed by the Glasgow and GEO600 [14, 15]. They proposed the modulation method with photon pressure for understanding the response of interferometer. The second generation photon calibrator is developed by LIGO [3, 16–18]. LIGO

and Virgo employ the second generation photon calibrator for the calibration of time-dependent response of IFO [3]. The third generation photon calibrator is developed by KAGRA [19]. KAGRA employ the 20 W laser and independent modulation system for injected beams. LIGO and KAGRA use a power sensor called to as the Gold standard, which is calibrated by the laser power standard of NIST in Boulder, CO [20]. However, it has a challenging issue of the absolute calibration due to the accuracy of the absolute laser power of laser standard between national metrology institute from nine countries. The systematic discrepancies between nine countries are as large as 3.5 % [21].

The dynamic gravity field calibrator (Gcal) is one of the candidates to be able to solve the uncertainty problem of absolute calibration. The technologies of the system are established and tested in Forward and Miller [22], Weber [23, 24], University of Tokyo [25–29] and Rome university group [30]. Related techniques using gravity field calibrator for the calibration are discussed in Matone et al [31]. It can modulate the test mass using gravity gradient with rotor. The amplitude of displacement of the mirror is determined by masses, distance, frequency, radius, and gravity constant.

In this paper, we propose how we can achieve sub-percent uncertainty of the calibration. We focus on the combination method of the photon calibrator and gravity field calibrator. In section II, we explain how to calibrate with photon calibrator. In chapter III, we show the principle of multipole moment gravity and modulation method. In section IV, we show how to calibrate the absolute displacement with Pcal and Gcal. In section V, we discuss the systematic error by changing parameters.

II. PHOTON CALIBRATOR

Pcal relies on the photon radiation pressure from the power modulated laser beams reflecting on the test mass to apply periodic force via the recoil of photons [16]. advanced LIGO, advanced Virgo and KAGRA employ the photon calibrator for the calibration of the interferometer response [3, 19, 32]. Each gravitational detectors placed the 1047 nm laser around the end test mass. The displacement of the test mass can be described as

$$x = \frac{P \cos \theta}{2c} s(\omega) \left(1 + \frac{M}{I} \vec{a} \cdot \vec{b} \right), \quad (1)$$

where P is absolute laser power, θ is incident angle of the Pcal laser, M is mass of test mass, ω is angular frequency of the laser power modulation, \vec{a} and \vec{b} are position vector of Pcal laser beams. The schematic view is shown in Fig. 1. $I = Mh^2/12 + Mr^2/4$ is the moment of inertia, where h and r are thickness and radius of test mass. $s(\omega)$ is transfer function between force and displacements. We can regard the $s(\omega)$ as $1/(M\omega^2)$ above 20 Hz.

The laser power is stabilized less than design sensitivity. The schematic view of the photon calibrator is shown

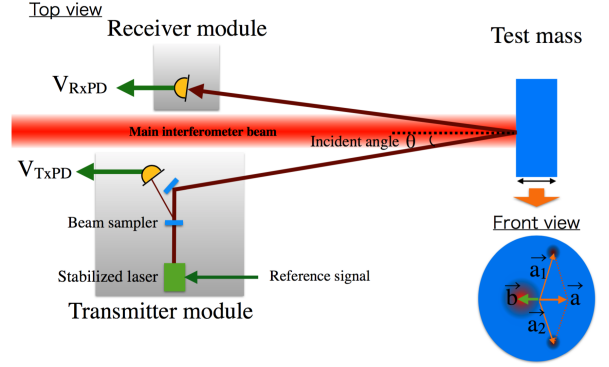


FIG. 1. Schematic view of photon calibrator. We place the stabilized laser on the transmitter module. The injected signal at the test masses is monitored by using the response of photo detector power between the transmitter module, V_{TxPD} and receiver module, V_{RxPD} . The geometrical factor is characterized by the position vectors of photon calibrator beams, $\vec{a} = \vec{a}_1 + \vec{a}_2$, and the main beam, \vec{b} .

in Fig. 1. The stabilized laser is mounted on the transmitter module. The power is monitored by the response of the photo detectors at the transmitter module, V_{TxPD} , and receiver module, V_{RxPD} . The largest relative uncertainty of photon calibrator is that of laser power. LIGO and KAGRA use the working standard to cross-calibrate the relative response of each interferometer. The relative uncertainty of each calibrator is 0.51 % [16]. The second largest relative uncertainty is an optical efficiency of the optical path. We calibrate the injected power from the outside of the vacuum chamber. Therefore, we need to consider the optical efficiency due to the transmittance of the vacuum window and reflectance of the mirrors. The measured uncertainty of optical efficiency in LIGO is 0.37 %. For the absolute calibration, the detector, so called “Gold standard”, is calibrated with the NIST laser power standard. After that, the responses of “Working standard” of Hanford, Livingston and KAGRA are calibrated by the “Gold standard” in LIGO Hanford observatory. However, the result of comparison between accuracies of the absolute laser powers of each institutes has 3.5 % uncertainty. It implies that the serious systematic error for distance estimation because the uncertainty of the absolute calibration propagate to the distance of the GW source.

III. GRAVITY FIELD CALIBRATOR

To solve the uncertainty problem of the absolute calibration, we propose the method of gravity field calibrator. The gravity field calibrator generate the dynamic gravity field at the end of test mass by rotating the multipole masses. The rotor placed in the vacuum chamber for isolating the acoustic noise. To monitor the frequency,

TABLE I. Specification summary of LIGO, Virgo and KAGRA photon calibrator

	KAGRA	advanced LIGO	advanced Virgo
Mirror material	Sapphire	Silica	Silica
Mirror mass	23 kg	40 kg	40 kg
Mirror diameter	220 mm	340 mm	350 mm
Mirror thickness	150 mm	200 mm	200 mm
Distance of Pcal from ETM	36 m	8 m	1.5 m
Pcal laser power	20 W	2 W	3 W
Pcal laser frequency	1047 nm	1047 nm	1047 nm
Incident angle	0.72 deg	8.75 deg	30 deg

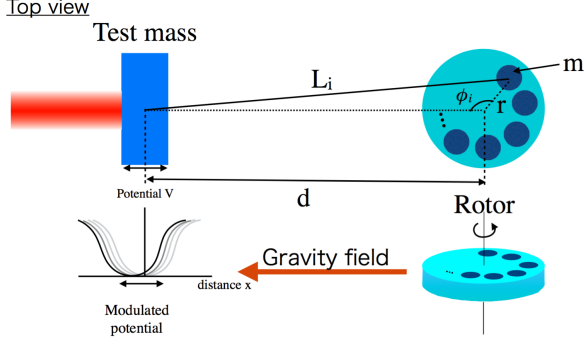


FIG. 2. Schematic view of gravity field calibrator. We placed the rotor at the same height and the distance of d away from test masses. Multipole mass generate the gravitational potential at the test mass position.

we mount the 10-bit encoder. We read the response of encoder using 16 bit ADC system. We calculated the displacement by changing dynamic gravity field of multipole moment with N peaces of masses. We assumed the suspended test mass for the interferometer and disk with multipole masses as shown in Fig 2. We put the masses, m , at the positions of the radius, r . The distance between the center of mass of mirror and disk is assumed d . We rotate the disk at the angular frequency of $\omega_{\text{rot}} = 2\pi f_{\text{rot}}$.

First we calculate the distance with N pieces of masses which are separated by radius of r from the center of the rotor mass and arranged at equal intervals, respectively. Distance between i -th mass and center of test mass is written as

$$L_i = d \sqrt{1 + \left(\frac{r}{d}\right)^2 - 2 \left(\frac{r}{d}\right) \cos \phi_i}, \quad (2)$$

where the angle of i -th mass is assumed as $\phi_i = \omega_{\text{rot}} t + 2\pi i/N$. The gravity potential at the center of test mass can be described as

$$V = \sum_{i=0}^N V_i \quad (3)$$

$$= -GMm \sum_{i=0}^N L_i^{-1} \quad (4)$$

$$= -\frac{GMm}{d} \sum_{i=0}^N \sum_{n=0}^{\infty} \left(\frac{r}{d}\right)^n P_n \left(\cos \left(\omega_{\text{rot}} t + \frac{2\pi i}{N} \right) \right), \quad (5)$$

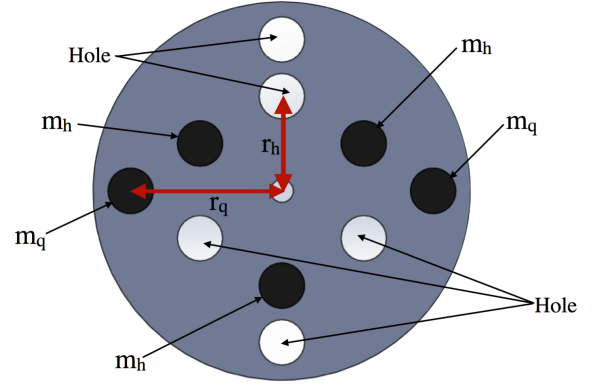


FIG. 3. Configuration of the rotor with quadrupole and hexapole mass distributions. m_q and m_h are masses of quadrupole and hexapole. r_q and r_h are radiuses of quadrupole and hexapole.

where P_n is Legendre polynomial, and V_i is potential of a mass. The equation of motion of test mass is

$$Ma = \left| \frac{\partial V}{\partial d} \right| = \frac{GMm}{d^2} \sum_{i=0}^N \sum_{n=0}^{\infty} (n+1) \left(\frac{r}{d}\right)^n \times P_n \left(\cos \left(\omega_{\text{rot}} t + \frac{2\pi i}{N} \right) \right), \quad (6)$$

where a is acceleration of test mass.

We place the quadrupole and hexapole masses in the same rotor as shown in Fig. 3. We put the hole between each mass. The hole can increase the gravity gradient twice effectively. Therefore, we can describe the equation of motion as

$$Ma = \left| \frac{\partial V}{\partial d} \right| = \frac{2GMm}{d^2} \sum_{i=0}^N \sum_{n=0}^{\infty} (n+1) \left(\frac{r}{d}\right)^n \times P_n \left(\cos \left(\omega_{\text{rot}} t + \frac{2\pi i}{N} \right) \right). \quad (7)$$

We will calculate the displacement of quadrupole and hexapole in the section III A and III B.

A. Displacement of test mass (Quadrupole)

We calculate the displacement of the quadrupole masses distribution corresponding to $N = 2$. The masses and radiuses of quadrupole are assumed as m_q and r_q . The equation of motion of test mass is described as

$$Ma = \frac{2GMm_q}{d^2} \sum_{n=0}^{\infty} (n+1) \left(\frac{r_q}{d}\right)^n \times \sum_{i=0}^1 P_n \left(\cos (\omega_{\text{rot}} t + \pi i) \right). \quad (8)$$

If we assume $r \ll d$, the displacement of the time-dependent lower harmonics can be written by

$$x = \sum_{k=1}^{\infty} x_{kf} \cos(k\omega_{\text{rot}} t) \sim x_{2f} \cos(2\omega_{\text{rot}} t) = x_{2f} \cos \omega t, \quad (9)$$

where k is the number of the harmonics. The amplitude of 2-f rotation is described as

$$x_{2f} = 9 \frac{GMm_q r_q^2}{d^4} s(\omega). \quad (10)$$

B. Displacement of test mass (Hexapole)

We also calculate the displacement of the hexapole masses distribution, which corresponds to $N = 3$. The masses and radiuses of hexapole are assumed as m_h and r_h . The equation of motion of test mass is described as

$$Ma = \frac{2GMm_h}{d^2} \sum_{n=0}^{\infty} (n+1) \left(\frac{r_h}{d} \right)^n \quad (11)$$

$$\times \sum_{i=0}^2 P_n \left(\cos \left(\omega_{\text{rot}} t + \frac{2\pi i}{3} \right) \right). \quad (12)$$

If we assume $r \ll d$, the displacement of the time-dependent lower harmonics can be written by

$$x = \sum_{k=1}^{\infty} x_{kf} \cos(k\omega_{\text{rot}} t) \sim x_{3f} \cos(3\omega_{\text{rot}} t) = x_{3f} \cos \omega t, \quad (13)$$

where amplitude of 3-f is described as

$$x_{3f} = 15 \frac{GMm_h r_h^3}{d^5} s(\omega). \quad (14)$$

IV. ABSOLUTE POWER CALIBRATION BY USING GCal AND PCal

In this section, we discuss about absolute laser power calibration using interferometer. Figure 4 shows the configuration of the calibration by using the combination of photon calibrator and gravity field calibrator. First, we modulate the mirror position using gravity field calibrator. We can measure the signal of x_{2f} and x_{3f} in the response of interferometer. Second, we send the interferometer signal to the excitation port of photon calibrator as a reference signal port of feedback control as shown in Fig. 4. The photon calibrator cancel the displacement by gravity field calibrator. Third, we measure the response of the detector of transmitter module and receiver module, whose units are volt. The output signal of transmitter module, V_{TxPD} and receiver module, V_{RxPD} should be corresponding to displacement from gravity field. By using Eq (I),(10), and (14), the modulated powers are

$$P_{2f} = 18 \frac{Gcm_q M r_q^2}{d^4 \cos \theta} \frac{1}{1 + \frac{M}{I} \vec{a} \cdot \vec{b}} \quad (15)$$

$$P_{3f} = 30 \frac{Gcm_h M r_h^3}{d^5 \cos \theta} \frac{1}{1 + \frac{M}{I} \vec{a} \cdot \vec{b}} \quad (16)$$

Fourth, we demodulate the signal of the transmitter and receiver modules using the measured encoder signal of

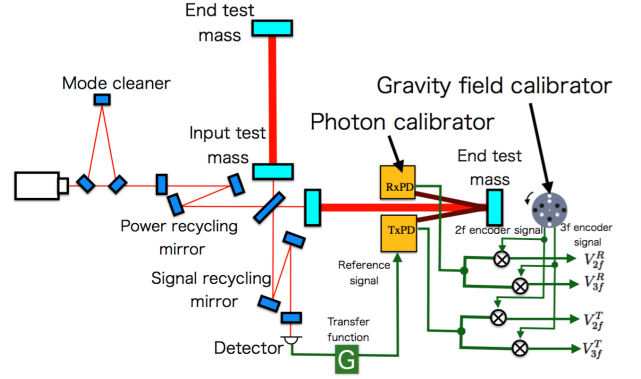


FIG. 4. Test setup of the absolute calibration. We place the gravity field calibrator behind of the test mass. The frequency of the gravity field calibrator is monitored by the encoder output. The error signal of differential arm length of the interferometer send to the reference port of the photon calibrator for canceling of the modulation of dynamic gravity field with feedback technique, where G is transfer function. Output signals from the photon calibrator are synchronized with the force from the gravity field calibrator. We demodulate the output signals by using 2-f and 3-f signal monitored by encoder.

gravity field calibrator. The demodulated signals are

$$V_{2f}^T = \rho_T P_{2f}, \quad (17)$$

$$V_{2f}^R = \rho_R P_{2f}, \quad (18)$$

$$V_{3f}^T = \rho_T P_{3f}, \quad (19)$$

$$V_{3f}^R = \rho_R P_{3f}, \quad (20)$$

where ρ_T and ρ_R are transfer function from power to voltage of the photo detector output at the transmitter and receiver modules. Therefore, we can measure the distance with the ratio of response between 2-f and 3-f components:

$$d = \frac{5}{3} \frac{V_{2f}^T}{V_{3f}^T} \frac{m_h}{m_q} \frac{r_h^3}{r_q^2} = \frac{5}{3} \frac{V_{2f}^R}{V_{3f}^R} \frac{m_h}{m_q} \frac{r_h^3}{r_q^2}. \quad (21)$$

Finally, we calculate the displacement formula of Pcal calibrated by Gcal. we insert the equation (10) to the Eq. (I):

$$x = \frac{P \cos \theta}{2c} s(\omega) \left(1 + \frac{M}{I} \vec{a} \cdot \vec{b} \right) \quad (22)$$

$$= 9 \frac{Gm_q M r_q^2}{d^4} \frac{P}{P_{2f}} s(\omega), \quad (23)$$

$$= \frac{729}{625} \frac{Gm_q^5 r_q^{10}}{m_h^4 r_h^{12}} \frac{V_{3f}^R}{V_{2f}^R} V_{\text{in}} s(\omega), \quad (24)$$

where we assume $P(\omega) = \rho_R V_{\text{in}}$, and V_{in} is amplitude of the input voltage. The factor of $(Gm_q^5 r_q^{10})/(m_h^4 r_h^{12})$ is measurable values before the operation. V_{3f}^R/V_{2f}^R is measured during the calibration between Gcal and Pcal. The

TABLE II. The assumed parameters. G is gravity constant [33]. θ is incident angle of the Pcal beams. M is mass of test mass. $1 + \frac{I}{M} \vec{a} \cdot \vec{b}$ is geometrical factor.

	Value	Relative uncertainty
G	$6.67408 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ sec}^{-2}$	0.0047 %
$\cos \theta$	1.000	0.07 %
M	22.89 kg	0.02 %
m_q	4.485 kg	0.004 %
m_h	4.485 kg	0.004 %
r_q	0.200 m	0.010 %
r_h	0.125 m	0.016 %
$1 + \frac{I}{M} \vec{a} \cdot \vec{b}$	1	0.3 %

interval of the calibration between Pcal and Gcal depend on the response stability of the photon calibrator. In the case of LIGO, they calibrate the absolute laser power using Working standard every month. Therefore, we should operate the gravity field calibrator with same interval or less. We do not need to operate the Gcal during the observation. In this method, we reconstruct the signal with photon calibrator calibrated by the gravity field calibrator.

During the operation, Gcal may contaminate the noise floor by the acoustic noise and/or vibration noise. However, we can avoid this noise effect by changing the rotation frequency. We do not pay attention to the noise in the observation because we only calibrate the absolute displacement before the observation. We consider the advantage of the demodulation. When we cancel of the modulation of Gcal with Pcal, the transfer functions of the Gcal and Pcal are also canceled. Therefore, the estimated displacement is not depend on the frequency. So, we can reduce the rotation systematic error with the demodulation technique.

V. ESTIMATION OF UNCERTAINTY

In evaluating the accuracy of the estimated displacement, we discuss the systematic error by changing the operation frequency and distance. After that, we discuss the uncertainty of the displacement of the mirror. The following discussions are assumed with KAGRA basic parameters as listed in Table I and parameters of Gcal as listed in Table II. The assumed parameters of the calibrators are listed in Table II. We assumed these number in the following section.

A. Systematic error of the higher order term

In order to achieve the precision less than 1 %, we need to consider the operation position due to the higher order of Legendre polynomial. This is because that higher order also include the 2-f and 3-f components. The n -th order of the Legendre polynomial is calculated in Eq.(7).

TABLE III. The calculated quadrupole($N = 2$) displacement. n is order of Legendre polynomial, where $\omega = n\omega_{\text{rot}}$.

	n=1	n=2	n=3	n=4	n=5	n=6	n=7
1f	0	0	0	0	0	0	0
2f	0	$9 \frac{Gmr^2}{d^4 \omega^2}$	0	$\frac{25}{4} \frac{Gmr^4}{d^6 \omega^2}$	0	$\frac{735}{128} \frac{Gmr^6}{d^8 \omega^2}$	0
3f	0	0	0	0	0	0	0
4f	0	0	0	$\frac{175}{16} \frac{Gmr^4}{d^6 \omega^2}$	0	$\frac{441}{64} \frac{Gmr^6}{d^8 \omega^2}$	0
5f	0	0	0	0	0	0	0
6f	0	0	0	0	0	$\frac{1617}{128} \frac{Gmr^6}{d^8 \omega^2}$	0

TABLE IV. The calculated hexapole($N = 3$) displacement. n is order of Legendre polynomial, where $\omega = n\omega_{\text{rot}}$.

	n=1	n=2	n=3	n=4	n=5	n=6	n=7
1f	0	0	0	0	0	0	0
2f	0	0	0	0	0	0	0
3f	0	0	$15 \frac{Gmr^3}{d^5 \omega^2}$	0	$\frac{315}{32} \frac{Gmr^5}{d^7 \omega^2}$	0	$\frac{189}{64} \frac{Gmr^7}{d^9 \omega^2}$
4f	0	0	0	0	0	0	0
5f	0	0	0	0	0	0	0
6f	0	0	0	0	0	$\frac{4851}{256} \frac{Gmr^6}{d^8 \omega^2}$	0

The effect of higher order factor is mitigated by the factor of $(r/d)^n$. Table III and IV shows the calculated displacement of the higher order term. To compare the higher order effect, we calculate the ratio between the Finite Element Method (FEM) and calculation by changing the r/d is shown in Fig 5. The results shows that higher the order of polynomials are, the less the systematic errors are. We need to place the mirror at least 2 m away from the mirror and use the sum of first and second order equation to reduce the systematic error. In the following calculation, we assume the distance as 2 m.

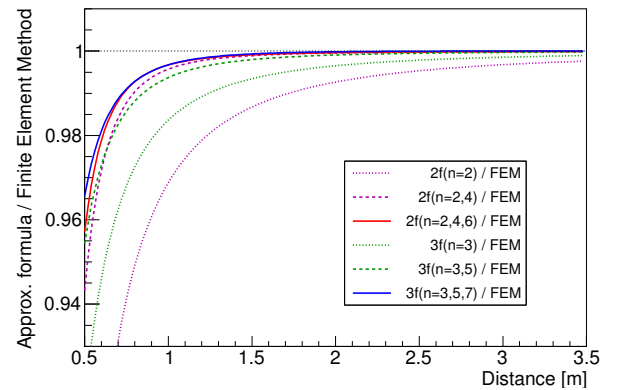


FIG. 5. The displacement ratio of the higher order effect by changing r/d . Dotted curves are included with the first order term. Dashed curves are included with sum of first order and second order. Solid Curves are assumed with sum of first, second and third order terms. The analytical result is listed in Table III and IV. To achieve the precision less than 1 %, we need to include the higher order terms.

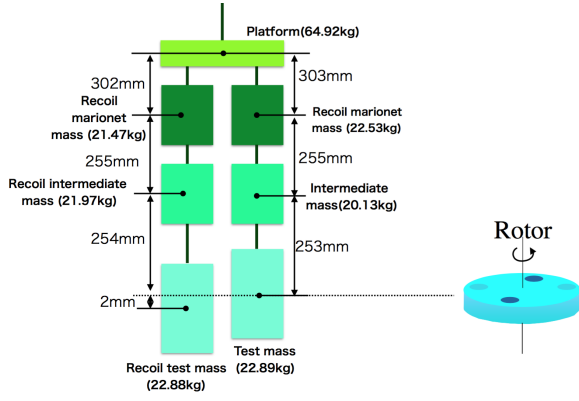


FIG. 6. Schematic view of the suspension system. The parameter of the height and mass is the assumed value.

B. Systematic error of the transfer function

The gravity field calibrator can modulate the mirrors with gradient of gravity potential. However, its gravity gradient act the masses of suspension system as shown in Fig. 6. We simulated the transfer function by assuming the cryogenic suspension system in KAGRA [34]. The transfer function is calculated by the suspension rigid-body simulation code, called SUMCON [35]. We estimated the total displacement including all the masses. Figure 7 shows the displacement ratio between the sensed motion and free mass motion as a function of frequency. The simulation result is in good agreement with free mass motion at the frequency larger than 10 Hz. The structures of low frequency are corresponding to the resonant peak of the suspension system. Therefore, we can neglect the intermediate mass effect and regard as free mass motion larger than 10 Hz. Therefore, we need to operate the rotor larger than 10 Hz for reducing the error less than 0.1 %. We assumed the rotation frequency as 16 Hz, which is corresponding to 32 Hz and 48 Hz at the operation frequency of 2-f and 3-f components. We used this assumption in the following section.

C. Uncertainty of displacement and laser power

In this section, we estimate the typical displacement based on the Table. II. We neglect above the second order Legendre polynomial in the following discussion to simplify the discussion. The estimated displacements of 2-f and 3-f are described as

$$x_{2f}^{\text{rms}} = 1.18 \times 10^{-16} [\text{m}] \times \left(\frac{G}{6.67408 \times 10^{-11} [\text{m}^3 \text{kg}^{-1} \text{sec}^{-2}]} \right) \times \left(\frac{m_q}{4.485 [\text{kg}]} \right) \times \left(\frac{r_q}{0.200 [\text{m}]} \right)^2 \times \left(\frac{2 [\text{m}]}{d} \right)^4 \times \left(\frac{2\pi \times 32 [\text{Hz}]}{\omega} \right)^2 \quad (25)$$

$$x_{3f}^{\text{rms}} = 2.13 \times 10^{-18} [\text{m}] \times \left(\frac{G}{6.6742 \times 10^{-11} [\text{m}^3 \text{kg}^{-1} \text{sec}^{-2}]} \right) \times \left(\frac{m_h}{4.485 [\text{kg}]} \right) \times \left(\frac{r_h}{0.125 [\text{m}]} \right)^3 \times \left(\frac{2 [\text{m}]}{d} \right)^5 \times \left(\frac{2\pi \times 48 [\text{Hz}]}{\omega} \right)^2 \quad (26)$$

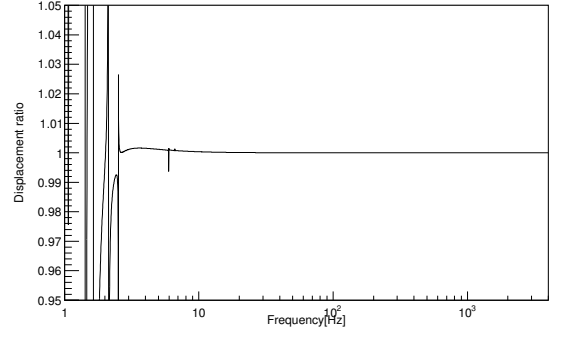


FIG. 7. The displacement ratio of the transfer function of multi pendulum by changing modulation frequency, where relations of the modulation frequency, f , modulation angular frequency, ω , and rotation angular frequency, ω_{rot} are described as. $n\omega_{\text{rot}} = \omega = 2\pi f$. If we put the gravity field calibrator, it act the upper masses and it makes systematic error of the transfer function.

We define the signal-to-noise ratio (SNR) with the ratio of RMS displacement to the design noise spectrum density of IFO of KAGRA at 32 Hz for 2-f and 48 Hz for 3-f. By using this result, we estimate the SNR of the peaks.

$$\text{SNR}_{2f} = 392 \times \left(\frac{3.0 \times 10^{-19} [\text{m}/\sqrt{\text{Hz}}]}{n_{32\text{Hz}}} \right) \times \left(\frac{T}{1 [\text{sec}]} \right)^{\frac{1}{2}} \times \left(\frac{x_{2f}^{\text{rms}}}{1.178 \times 10^{-16} [\text{m}]} \right), \quad (27)$$

$$\text{SNR}_{3f} = 73 \times \left(\frac{2.9 \times 10^{-20} [\text{m}/\sqrt{\text{Hz}}]}{n_{48\text{Hz}}} \right) \times \left(\frac{T}{1 [\text{sec}]} \right)^{\frac{1}{2}} \times \left(\frac{x_{2f}^{\text{rms}}}{2.130 \times 10^{-18} [\text{m}]} \right) \quad (28)$$

where T is integration time. When we integrate the signal larger than 10 min, we can measure the V_{2f}^{R} and V_{3f}^{R} with enough of SNR. This method are applicable to the measurement of the absolute laser power. The estimated powers are

$$P_{2f} = 0.093 [\text{W}] \times \left(\frac{G}{6.6742 \times 10^{-11} [\text{m}^3 \text{kg}^{-1} \text{sec}^{-2}]} \right) \times \left(\frac{m_q}{4.485 [\text{kg}]} \right) \times \left(\frac{r_q}{0.200 [\text{m}]} \right)^2 \times \left(\frac{2 [\text{m}]}{d} \right)^4 \times \left(\frac{1}{\cos \theta} \right) \times \left(\frac{1}{1 + \frac{M}{I} \vec{a} \cdot \vec{b}} \right)^2 \quad (29)$$

$$P_{3f} = 0.0038 [\text{W}] \times \left(\frac{G}{6.6742 \times 10^{-11} [\text{m}^3 \text{kg}^{-1} \text{sec}^{-2}]} \right) \times \left(\frac{m_h}{4.485 [\text{kg}]} \right) \times \left(\frac{r_h}{0.125 [\text{m}]} \right)^3 \times \left(\frac{2 [\text{m}]}{d} \right)^5 \times \left(\frac{1}{\cos \theta} \right) \times \left(\frac{1}{1 + \frac{M}{I} \vec{a} \cdot \vec{b}} \right)^2. \quad (30)$$

The estimated $V_{2f}^{\text{T}}/V_{3f}^{\text{T}}$ by changing distance are shown in Fig. 8. We estimate the laser power by using the following

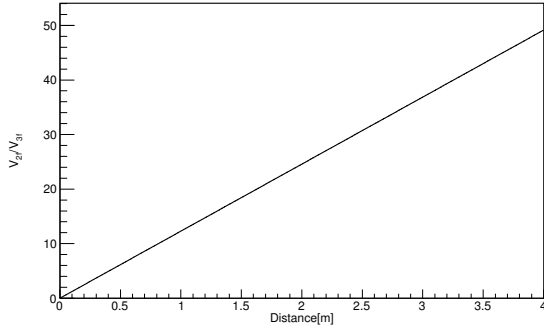


FIG. 8. The response of V_{2f}/V_{3f} by changing distance between test mass and gravity field calibrator.

equations:

$$\left(\frac{\delta P_{2f}}{P_{2f}}\right)^2 \sim 16 \left(\frac{\delta V_{2f}^R}{V_{2f}^R}\right)^2 + 16 \left(\frac{\delta V_{3f}^R}{V_{3f}^R}\right)^2 + \left(\frac{\delta P_{sys}}{P_{sys}}\right)^2 \quad (31)$$

$$\left(\frac{\delta P_{3f}}{P_{3f}}\right)^2 \sim 16 \left(\frac{\delta V_{2f}^R}{V_{2f}^R}\right)^2 + 16 \left(\frac{\delta V_{3f}^R}{V_{3f}^R}\right)^2 + \left(\frac{\delta P_{sys}}{P_{sys}}\right)^2 \quad (32)$$

where $\delta P_{sys}/P_{sys}$ is the relative systematic error of the power due to the machining tolerance of the rotor masses and radiuses as disproved by

$$\begin{aligned} \frac{\delta P_{sys}}{P_{sys}} \sim & \frac{\delta G}{G} + \frac{\delta M}{M} + \frac{\delta \cos \theta}{\cos \theta} + \frac{\delta \left(1 + \frac{M}{I} \vec{a} \cdot \vec{b}\right)}{\left(1 + \frac{M}{I} \vec{a} \cdot \vec{b}\right)} \\ & + \frac{12}{\sqrt{6}} \frac{\delta r_h}{r_h} + \frac{10}{2} \frac{\delta r_q}{r_q} + \frac{5}{2} \frac{\delta m_q}{m_q} + \frac{4}{\sqrt{6}} \frac{\delta m_h}{m_h} \end{aligned} \quad (33)$$

We consider the mitigation effect of the masses and radiuses due to the tolerance and uncertainty of the measurement instruments. The values of masses and radiuses have a variance due to the fabrication tolerance. The errors of m_q , r_q , m_h , and r_h are mitigated by the factor of $1/\sqrt{6}$ and $1/\sqrt{4}$. The uncertainty of the quadrupole and hexapole masses are limited by the accuracy of an electronic balance. In this case, we use the masses made of Tungsten. The density of Tungsten is 19.25 g/cm^3 . The diameter and thickness of mass are 0.06 m and 0.08 m , respectively. Therefore, the mass of the rotor mass is 4.485 kg . To measure this mass, we assumed that we use an electronic balance whose catalog number and accuracy are CG-6000 and 0.2 g , respectively. Therefore, the relative uncertainty of the mass of rotor mass is 0.04% .

To make the rotor disk, we use the NC milling machine. The typical accuracy is less than 0.02 mm . For the measuring of the shape, we employ the three-dimension coordinate measuring machine (CMM) [36]. The precision of CMM is $2 \text{ } \mu\text{m}$. We can measure the shape of the rotor and masses with enough of uncertainty using CMM.

The estimated relative uncertainties of the powers are 0.52% . One of the largest uncertainties is the geometri-

cal factor of the Pcal laser. The geometrical factor uncertainty is assumed 0.3% , which is same number of LIGO.

Finally, the estimated relative uncertainty of displacement is written as

$$\left(\frac{\delta x}{x}\right)^2 \sim \left(\frac{\delta V_{in}}{V_{in}}\right)^2 + \left(\frac{\delta s(\omega)}{s(\omega)}\right)^2 + 25 \left(\frac{\delta V_{2f}^R}{V_{2f}^R}\right)^2 + 16 \left(\frac{\delta V_{3f}^R}{V_{3f}^R}\right)^2 + \left(\frac{\delta x_{sys}}{x_{sys}}\right)^2, \quad (34)$$

where $\delta x_{sys}/x_{sys}$ is the relative systematic error of the displacement. This factor is written by

$$\frac{\delta x_{sys}}{x_{sys}} = \frac{\delta G}{G} + \frac{\delta M}{M} + \frac{12}{\sqrt{6}} \frac{\delta r_h}{r_h} + \frac{10}{2} \frac{\delta r_q}{r_q} + \frac{5}{2} \frac{\delta m_q}{m_q} + \frac{4}{\sqrt{6}} \frac{\delta m_h}{m_h} \quad (35)$$

We assumed the mitigation factors of radiuses and masses. The contribution of the radius uncertainty is amplified by the $O(100)$ factor. To reduce the noise of the displacement, we need to reduce the uncertainty of the shape of the rotor and masses. The uncertainty of the $V_{2f}^R, V_{3f}^R, V_0^R$ are much less than that of other contributions. We can reduce the uncertainty of these values with long time integration time due to the statistics. Each the uncertainty is listed in Table. II. The estimated total uncertainty of the displacement is 0.17% .

VI. CONCLUSION

Photon calibrator is one of the powerful calibrators in advanced LIGO, advanced Virgo and KAGRA. It can calibrate the response of IFO and its uncertainty is essential for estimation of parameters of the gravitational wave source. In particular, the distance of the source is strongly depend on the absolute laser power of the photon calibrator. In previous study, the Gold standard, which response is calibrated by the laser power standard of NIST, is used for the absolute laser power calibration of the photon calibrator. However, current limit of the absolute laser power between each country is about 3.5% . It is directly propagate to the uncertainty of absolute displacement of the gravitational wave detector.

To solve the problem, we proposed the combination method of the photon calibrator and gravity field calibrator. Gravity field calibrator can modulate the mirror using dynamic gravity field. By canceling the displacement of the test mass displacement using the photon calibrator, we can calibrate the absolute laser power using gravity field calibrator.

This method has an advantage of a direct comparison of the amplitude of the injected power and gravity field. In previous study, we need to consider the uncertainty of the optical efficiency through the window and mirrors and geometrical factor of the laser position. This is because we put the Working standard at the outside of the chamber. However, the method of gravity field can compare the displacement directly. By using this method, we can calibrate the uncertainty of optical efficiency and absolute power of the laser. The estimated uncertainty of the power of this method is 0.52% . It imply that we can make a new power standard using interferometer with 10 times improvement.

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