# Kalman Filtering Demo

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# What is it?

- Linear Quadratic Estimator
  - Filters out noise from data
- Typically used for dynamic systems

# Why Kalman filter?

- Optimal in minimum mean squared error sense
  - For special cases that almost never occur in reality:
    - Linear system
    - Known system parameters
    - Additive white Gaussian noise
    - Markov process
- Tells how accurate its estimates are
- Easily combines measurements from different sensor types
- It works

# Types of Kalman Filters

A few of the many:

- Extended Kalman Filter
  - Non-linear models
  - Approximates the system models
- Iterated Extended Kalman Filter
  - Improves on EKF at a computational cost
- Information Filter
  - Information matrix tends to be sparse, simplifying computations
- Unscented Kalman Filter
  - Non-linear models
  - Approximates estimate distribution
- Square-root Kalman Filter
  - Handles numerical instability when variances have a large order of magnitude difference

# System

$$\underline{x}_k = \mathbf{F}_{k-1}\,\underline{x}_{k-1} + \mathbf{B}_{k-1}\,\underline{u}_{k-1} + \mathbf{G}_{k-1}\,\underline{w}_{k-1}$$

$$\underline{z}_k = \mathbf{H}_k\,\underline{x}_k + \underline{v}_k$$

- $\underline{x}$ system state vector
- $\mathbf{F}$ system transition matrix
- $\underline{u}$ control input vector
- $\mathbf{B}$ input coupling matrix
- $\mathbf{G}$ noise to state coupling matrix
- $\underline{w}$ process noise

- $k$ time index
- $\underline{z}$ measurement
- $\mathbf{H}$ measurement sensitivity matrix
- $\underline{v}$ observation noise

# Kalman Filter Equations

Propagation

$$\underline{\hat{x}}_{k|k-1} = \mathbf{F}_k \; \underline{\hat{x}}_{k|k}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \; \mathbf{P}_{k|k} \; \mathbf{F}_k^T + \mathbf{Q}_{k-1}$$

$$\mathbf{S}_{k|k} = \mathbf{H}_k \; \mathbf{P}_{k|k-1} \; \mathbf{H}_k + \mathbf{R}_{k-1}$$

$$\mathbf{K}_{k|k} = \mathbf{P}_{k|k-1} \; \mathbf{H}_k^T \; \mathbf{S}_{k|k}^{-1}$$

Update

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_{k|k} \; \mathbf{H}_k) \; \mathbf{P}_{k|k-1}$$

$$\underline{\hat{x}}_{k|k} = \underline{\hat{x}}_{k|k-1} + \mathbf{K}_{k|k}(\underline{z}_k - \mathbf{H}_k \; \underline{\hat{x}}_{k|k-1})$$

# Extended Kalman Filter Equations

Propagation

$$\underline{\hat{x}}_{k|k-1} = \mathbf{F}_k \ \underline{\hat{x}}_{k|k}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \ \mathbf{P}_{k|k} \ \mathbf{F}_k^T + \mathbf{Q}_{k-1}$$

$$\mathbf{S}_{k|k} = \mathbf{H}_k \ \mathbf{P}_{k|k-1} \ \mathbf{H}_k + \mathbf{R}_{k-1}$$

$$\mathbf{K}_{k|k} = \mathbf{P}_{k|k-1} \ \mathbf{H}_k^T \ \mathbf{S}_{k|k}^{-1}$$

Update

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_{k|k} \ \mathbf{H}_k) \ \mathbf{P}_{k|k-1}$$

$$\underline{\hat{x}}_{k|k} = \underline{\hat{x}}_{k|k-1} + \mathbf{K}_{k|k}(\underline{z}_k - \mathbf{H}_k \ \underline{\hat{x}}_{k|k-1})$$

Approximate non-linear functions by using Jacobian (matrix of derivatives) evaluated at current state

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} f_x(x_{k-1}, y_{k-1}) \\ f_y(x_{k-1}, y_{k-1}) \end{bmatrix}$$

$$\mathbf{F}_k = \begin{bmatrix} \frac{df_x(x_{k-1}, y_{k-1})}{dx_{k-1}} & \frac{df_x(x_{k-1}, y_{k-1})}{dy_{k-1}} \\ \frac{df_y(x_{k-1}, y_{k-1})}{dx_{k-1}} & \frac{df_y(x_{k-1}, y_{k-1})}{dy_{k-1}} \end{bmatrix}$$

# State Transition Model

1-D Constant Angular Velocity Model:

$$\theta_k = \theta_{k-1} + T\omega_{k-1} + \frac{T^2}{2} w_{k-1}$$

$$\omega_k = \omega_{k-1} + T w_{k-1}$$

- $\theta$ orientation angle
- $\omega$ angular velocity
- $T$ time interval between measurements
- $w$ angular velocity noise

$$\underline{x}_k = \begin{bmatrix} \theta_k \\ \omega_k \end{bmatrix} \qquad \underline{u}_k = 0 \qquad \underline{w}_k = w_k$$

$$\mathbf{F}_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B}_k = 0 \qquad \mathbf{G}_k = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

# 3D State Transition Model

$$\underline{x}_k = \begin{bmatrix} \theta_{xk} & \omega_{xk} & \theta_{yk} & \omega_{yk} & \theta_{zk} & \omega_{zk} \end{bmatrix}^T$$

$$\mathbf{F}_k = \begin{bmatrix} 1 & T & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{G}_k = \begin{bmatrix} \frac{T^2}{2} & 0 & 0 \\ T & 0 & 0 \\ 0 & \frac{T^2}{2} & 0 \\ 0 & T & 0 \\ 0 & 0 & \frac{T^2}{2} \\ 0 & 0 & T \end{bmatrix}$$

# Gyroscope Measurement Model

1-D Measurement Model:

$$z_k = \omega_k + v_k$$

$$\mathbf{H}_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \mathbf{R}_k = v_k$$

3-D Measurement Model:

$$\mathbf{H}_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_k = \begin{bmatrix} v_{xk} & 0 & 0 \\ 0 & v_{yk} & 0 \\ 0 & 0 & v_{zk} \end{bmatrix}$$

# Note on Gyroscope Measurement Model

Orientation is unobservable with only gyroscope measurements

- Orientation is not uniquely determinable
- Error grows, unbounded
    - Trace of covariance matrix grows over time

# Accelerometer Measurement Model

1-D Measurement Model

$$z_k = 1250 \sin\left(\frac{\pi}{180}\theta_k\right) + v_k$$

$$\frac{dz_k}{d\hat{\theta}_{k-1|k-1}} = 1250\frac{\pi}{180}\cos\left(\frac{\pi}{180}\hat{\theta}_{k-1|k-1}\right)$$

$$\frac{dz_k}{d\hat{\omega}_{k-1|k-1}} = 0$$

$$\mathbf{H}_k = \begin{bmatrix} 1250\frac{\pi}{180}\cos\left(\frac{\pi}{180}\hat{\theta}_{k-1|k-1}\right) & 0 \end{bmatrix}$$

$$\mathbf{R}_k = \sigma_k$$

# Combined Measurement Model

$$
\mathbf{H}_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1250\frac{\pi}{180}\cos\left(\frac{\pi}{180}\hat{\theta}_{xk-1|k-1}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1250\frac{\pi}{180}\cos\left(\frac{\pi}{180}\hat{\theta}_{yk-1|k-1}\right) & 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{R}_k = \begin{bmatrix} v_{xk} & 0 & 0 & 0 & 0 \\ 0 & v_{yk} & 0 & 0 & 0 \\ 0 & 0 & v_{zk} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{xk} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{yk} \end{bmatrix}
$$

# Sensors and Connections to Arduino

As used in the code at:

https://github.com/RenasonceGent/GyroAcceKF