② given,

$b \in \mathbb{R}^n$ & is a vector

$A \in \mathbb{R}^{n \times n}$ & is a matrix

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = b^T x + x^T A x$

(a) The gradient is

$$\nabla f(x) = \frac{\partial f(x)}{\partial x}$$

$$= \frac{\partial (b^T x + x^T A x)}{\partial x}$$

$$\Rightarrow \nabla f(x) = b + \cancel{2A}(A + A^T)x$$

assuming A is symmetric

$$\boxed{\nabla f(x) = b + 2Ax}$$

The Hessian is

$$H(f(x)) = \frac{\partial^2 f(x)}{\partial x^2}$$

$$= \frac{\partial^2 (b^T x + x^T A x)}{\partial x}$$

$$= \frac{\partial (b + 2Ax)}{\partial x}$$

$$\boxed{H(f(x)) = 2A}$$ (assuming A is symmetric)

(b) The first order Taylor expansion series is

$$f_1(\lambda) = f(\lambda_0) + \nabla f(\lambda_0)^T(\lambda - \lambda_0)$$

$$= 0 + (b^T + 0)^T(\lambda - 0)$$

$$= 0 + b^T\lambda$$

$$\boxed{f_1(\lambda) = b^T}$$

For second order Taylor series expansion,

$$f_2(\lambda) = f(\lambda_0) + \nabla f(\lambda_0)^T(\lambda - \lambda_0) + \frac{1}{2}(\lambda - \lambda_0)^T H(\lambda_0)(\lambda - \lambda_0)$$

$$= 0 + b^T + \frac{1}{2}(\lambda)^T 2A(\lambda)$$

$$= b^T + \frac{1}{2}2\lambda^T A\lambda$$

$$\boxed{f_2(\lambda) = b^T + \lambda^T A\lambda}$$

→ All Taylor approximations are approximate and not accurate completely because all Hessian, gradient and Taylor expansions are done locally and they are accurate at that local point but not everywhere. But, they are accurate in this problem.

③ given,

$A \in R^{n \times n}$ is a square matrix.

(a) The necessary and sufficient conditions for A to be Positive definite if all eigen values are Positive

i.e $\lambda_i > 0$

(b) The necessary and sufficient conditions for A to have a full rank is if all eigen values are non zero.

i.e $\lambda_i \neq 0$

(c) If $y \in R^n$; $y \neq 0$ $\Rightarrow$ $A^T y = 0$

then $Ax = b$ will have a solution only if $y^T b = 0$

Same as ② Question ②

(4) For Stingler diet.

let,

define x_i = quantity of food type 'i' to be purchased

c_i = unit price of food type

so, the problem would be

$$\text{minimize: } \sum_{i=1}^{N} c_i x_i$$

The constraints considered ~~are~~ to ensure nutrition requirements is

$$\sum_{i=1}^{N} a_{i,j} x_i \geq b_j \quad \text{where, } j = 1, 2, 3, \ldots, M$$

where,

$a_{i,j}$ = Quantity of nutrition type j in food type 'i'

b_j = required quantity of nutrition type 'j' for a month