MAE-598

Design Optimization

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Homework - 2

① given,

$$f(x_1,x_2) = 2x_1^2 - 4x_1x_2 + 1.5\,x_2^2 + x_2$$

now,

$$\text{gradt. } g = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

now,

$$g = 0 \Rightarrow \begin{bmatrix} 4x_1 - 4x_2 \quad ① \\ -4x_1 + 3x_2 + 1 \quad ② \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_1 - 4x_2 = 0$

$x_1 = \frac{4x_2}{4} \Rightarrow x_1 = x_2$

so,

$-4x_1 + 3x_2 + 1 = 0$

$-4x_1 + 3x_1 = -1$

$x_1(-1) = -1$

$\Rightarrow \boxed{x_1 = 1}$ so, $\boxed{x_2 = 1}$

now,

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \text{ now, } |H| = \begin{vmatrix} 4 & -4 \\ -4 & 3 \end{vmatrix} = 4(3) - (-4)(-4)$$

$= 12 - 16 = -4 < 0$

Indefinite eigen value so, saddle point exist.

at $f(\bar{x}_1, \bar{x}_2) = f(1,1) = 2(1)^2 - 4(1)(1) + 1.5(1)^2 + 1 = 0.5$

on taylor expansion,

$$f(x) = f_0 + g^T(x - x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0)$$

$$= 0.5 + [0 \;\; 0](x - x_0) + \frac{1}{2}(x - x_0)^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}(x - x_0)$$

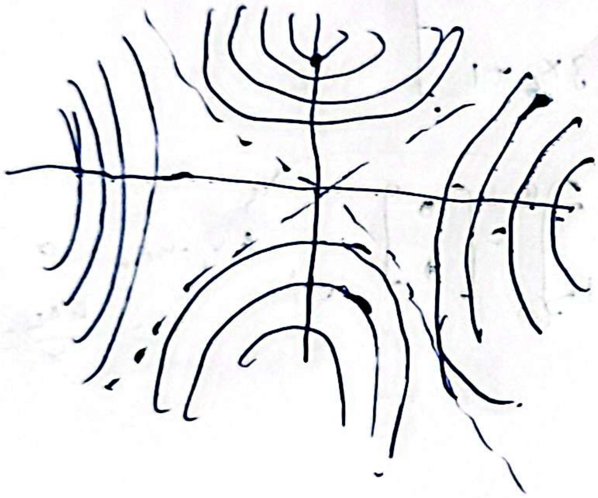
$$= 0.5 + (0)(x_1 - 1) + (0)(x_2 - 1) + \frac{1}{2}(x_1 - 1)^2(4) + (x_1 - 1)(x_2 - 1)(-4)$$

$$+ \frac{1}{2}(x_2 - 1)^2(3)$$

$\Rightarrow f(x) - 0.5 = 2(x_1^2 - 2x_1 + 1) - 4(x_1 x_2 - x_1 - x_2 + 1) + \frac{3}{2}(x_2^2 - 2x_2 + 1) \leq 0$

$0 \geq 2x_1^2 + \frac{3}{2}x_2^2 + x_2 - 4x_1x_2 + 3$

so it gives



It can be both sides

② (a) given

$$x_1 + 2x_2 + 3x_3 = 1 \qquad (-1, 0, 1)^T$$

$$\min_{\bar{x}} d^2 = (x_1+1)^2 + x_2^2 + (x_3-1)^2$$

$$\text{s.t } x_1 + 2x_2 + 3x_3 = 1$$

$$f_1 = x_1^2 + 2x_1 + x_2^2 + x_3^2 - 2x_3 + 2$$

$$\nabla f_1 = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2 \\ 2x_2 \\ 2x_3 - 2 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \lambda = 2, 2, 2 \Rightarrow \text{p.d}$$

$\Downarrow$

objective is convex.

constraint are $x_1 + 2x_2 + 3x_3 = 1 \Rightarrow$ Hyperplane which is convex

so, the problem is convex.

to convert to unconstrained prob.

$$x_1 = 1 - 2x_2 - 3x_3$$

$$f_2 = (1 - 2x_2 - 3x_3)^2 + 2(1 - 2x_2 - 3x_3) + x_2^2 + x_3^2 - 2x_3 + 2$$

$$= 5x_2^2 + 12x_2x_3 - 8x_2 + 10x_3^2 - 14x_3 + 5$$

$$\min_{x_2, x_3} f_2$$

$$\nabla f_2 = \begin{bmatrix} 10x_2 + 12x_3 - 8 \\ 12x_2 + 20x_3 - 14 \end{bmatrix}$$

$$H = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix} \Rightarrow \lambda_i = 2, 28 \Rightarrow p.d$$

convex prob

So, sol. will be min.

$\nabla f_2 = 0$ when

$10x_2 + 12x_3 - 8 = 0 \Rightarrow x_2 = -1/7$

$12x_2 + 20x_3 - 14 = 0 \Rightarrow x_3 = 11/14$

$x_1 = 1 - 2(-1/7) - 3(11/14)$

$= \frac{-15}{14}$

$\bar{x} = \left(\frac{-15}{14}, -1/7, 11/14\right)$

(3) Hyperplane as a convex set -

$a^T x = c$ & $a^T y = c$

now,

consider line segment join x & y by t

$f(t) = t x + (1-t) y$

now,

$a^T f(t) = a^T (t x + (1-t) y)$

$= t (a^T x) + (1-t) (a^T y)$

now,

$a^T f(t) = t c + (1-t) c$ [$\because a^T x = c$, $a^T y = c$]

$a^T f(t) = t c + c - \cancel{tc}$

$a^T f(t) = c \rightarrow$ Hyperplane.

$a^T x = c$ is a convex set because it contains all the line segment joining any 2 points within it.

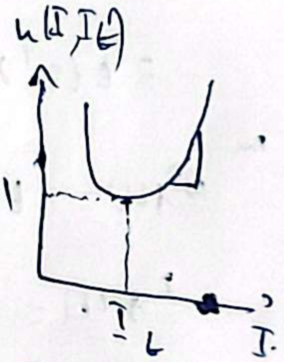
(4) Given,

$$\min_{P} \ \max_{I_L} \{ h(a_k^T P, I_t) \}$$

s.t. $0 \le P_i \le P_{max}$

$$h(I, I_t) = \begin{cases} I_t / I & I \le I_t \\ I / I_t & I \ge I_t \end{cases}$$

(a) Now,



$$\frac{\partial h}{\partial p} = \frac{dh}{dI} \cdot \frac{\partial (a^T p)}{\partial p} = h' a$$

$$\frac{\partial^2 h}{\partial p^2} = \frac{\partial^2 h}{\partial I^2} \cdot \frac{\partial (a^T p)}{\partial p} a^T = h'' a a^T$$

$h'' \ge 0 \rightarrow h$ is a convex set w.r.t I

$$\left.\begin{array}{l} I < I_t \rightarrow h'' > 0 \\ I > I_t \rightarrow h'' = 0 \end{array}\right\} h'' \text{ is p.s.d}$$

Since $h(I, I_t)$ is a convex set $h(a^T p, I_t)$ is also a convex set

(b) C_n^{10} $\begin{cases} P_1 + P_2 + \dots + P_{10} \le P^* \\ P_2 + P_3 + \dots + P_{11} \le P^* \\ \vdots \end{cases}$

$$[1, \dots, 1, \ 0, \dots, 0] \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

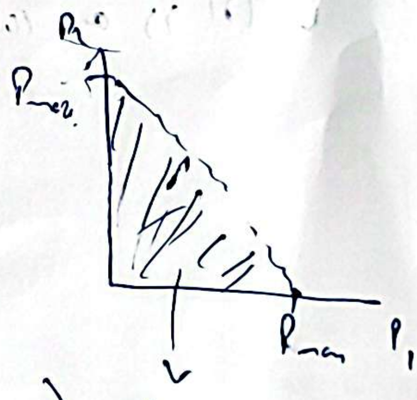
linear constraint so, strictly convex so, unique sol.

(c) No more than 10 lamps to be switched on.

2 lamps P_1, P_2 but no more than 1 can be on.

The region is not a convex set

→ If expanded to 10 lamps still it will not be a convex set



so may (or) may not have unique sol.) not a set region.

(5) given,

$$C^{*}(y) = \max_{x} \{xy - c(x)\}$$

$$\frac{d\,C^{*}(y)}{dy} = x$$

$$\frac{d^2 C^{*}(y)}{dy} = 0$$

so,

$C^{*}(y)$ is a linear function wrt to y.

so, $C^{*}(y)$ is a convex set wrt to y