Lecture # 2 Continue :

Review of Dynamic Systems and Feedback control

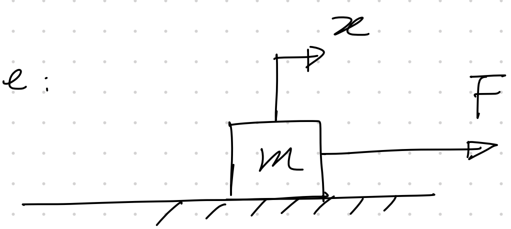
Topics :

- poles and zeros , stability
- Frequency response and Bode plots
- Phase margin and gain margin
- Loop - shaping example

Review of Lecture # 1:

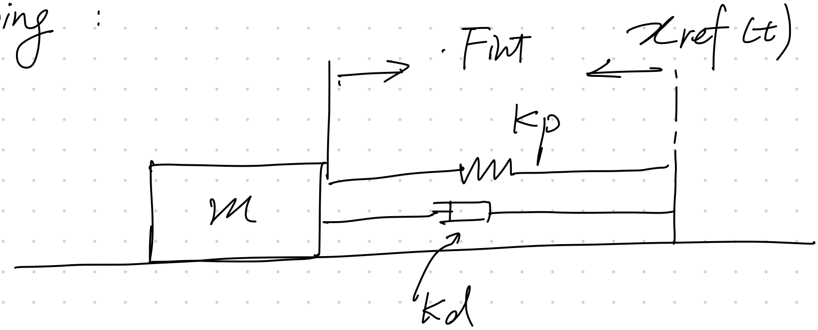
- Physical Meaning of PID Control

Example:



Goal: let x (measured position) to track a desired trajectory $x_{ref}(t)$ as good as possible

physically, PID Controller Implements the following:



where $F_{int} = K_I \int_0^t (x_{ref} - x)\, dt$

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- In practice, we never implement PID controller as it is.

- Practical Implementation : Lead-Lag format:

$$C(s) = K_p \left(1 + \frac{1}{T_i s}\right)\left(\frac{\alpha \tau s + 1}{\tau s + 1}\right)$$

s : Laplace variable

K_p : proportional gain

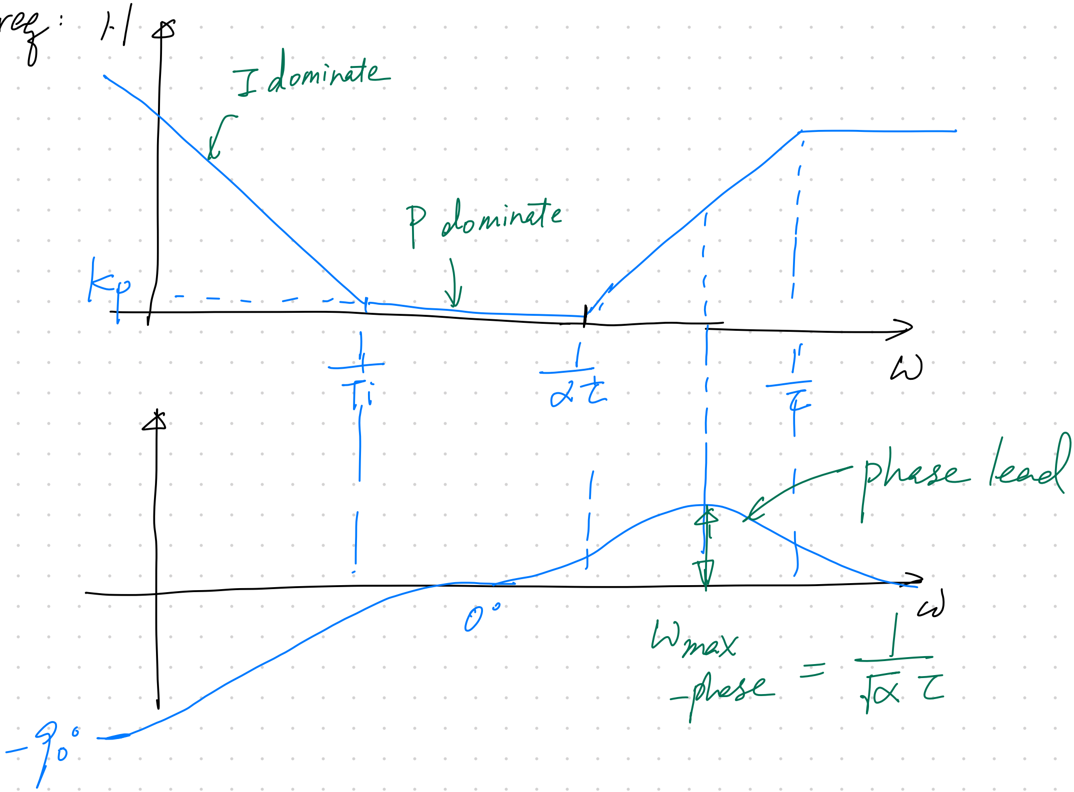
T_i : Integral time const

τ : Lead time constant

α : Lead separation constant

This form implements PID Controller with a Low-pass filter to bound high-freq amplification.

Freq: 1.1

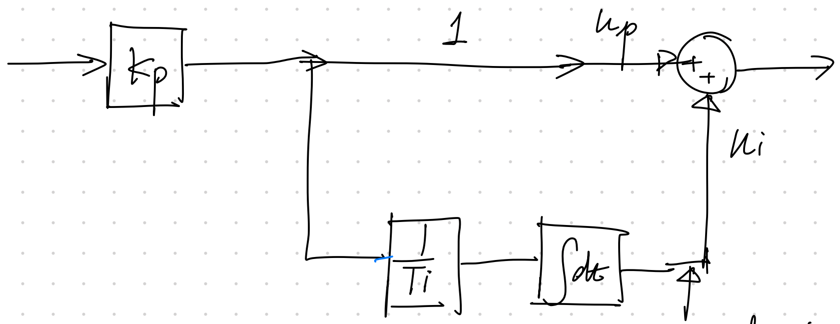


We will discuss parameter selection for these values today.

Another "Must" in implementation:

Anti-wind up

Implemetation of PI controller:



need to keep track of its state.

Integrator implementation:

if $|u_{ik}|$ > Threshold

$$u_{ik} = u_{i\,k-1}$$

else

$$u_{i_k} = u_{i_{k-1}} + \underbrace{\frac{k_p}{T_i} e_k \cdot dt}_{\text{term being accumulated}}$$

dt ← sample time

end

Today: ① Poles and zeros, stability

② Bode plots, freq response

③ Phase margin, gain margin

④ Loop-shaping example

① Poles and zeros:

Consider a transfer function:

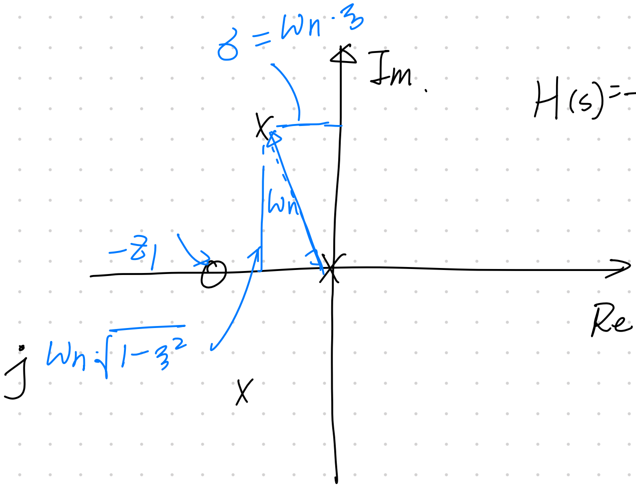
$$H(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Poles: $p_1 \cdots p_n$ (roots of denominator)

Zeros: $z_1 \cdots z_m$ (roots of numeritor)

Note they are complex numbers

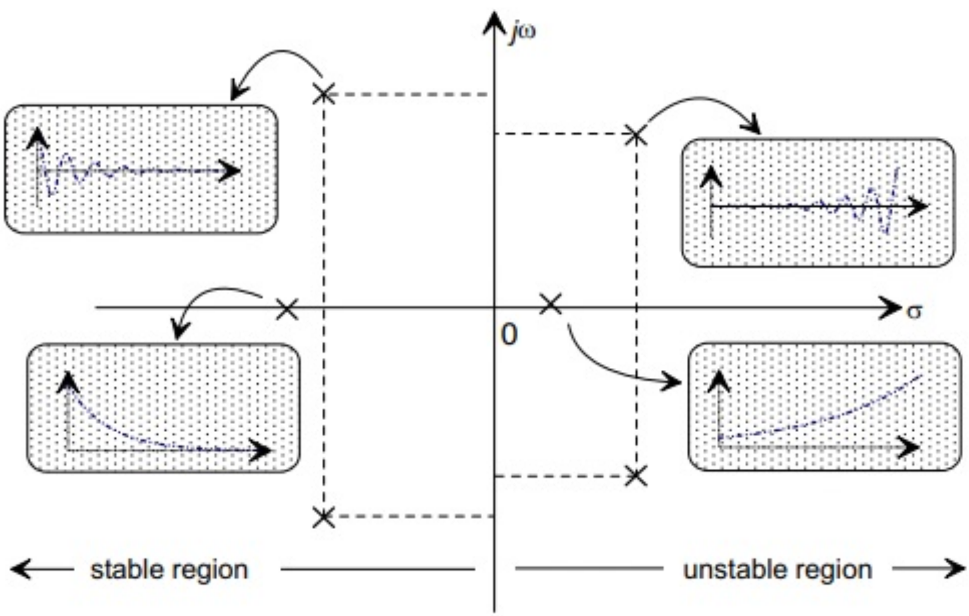
Ploting them on complex plane as:



$$H(s) = \frac{s + z_1}{s\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1\right)}$$

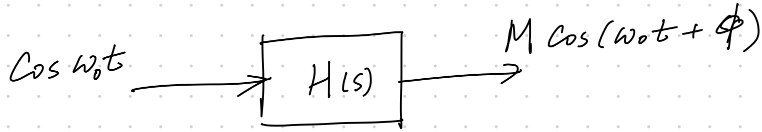
Poles determine system stability:

If all poles on left plane: Stable

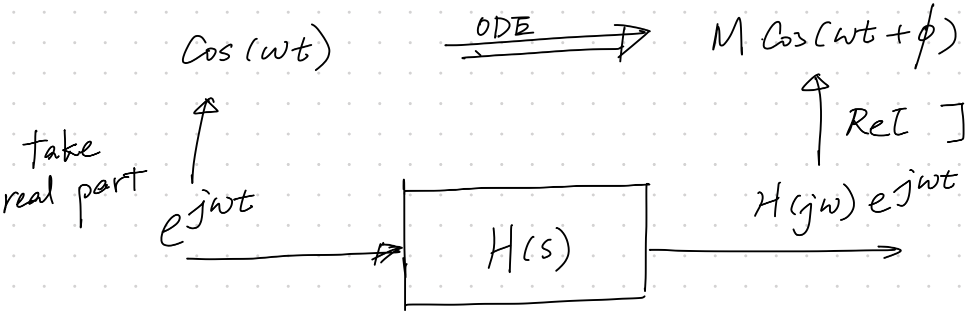


2 Frequency Responses and Bode Plots

For a linear dynamic system; frequency response is its graphical representation to a sinusoidal steady-state input.



where $M = |H(j\omega_0)|$ and $\phi = \angle H(j\omega_0)$



For a transfer function $H(s)$, when evaluating frequency response, lets substitute in $s = j\omega$.,

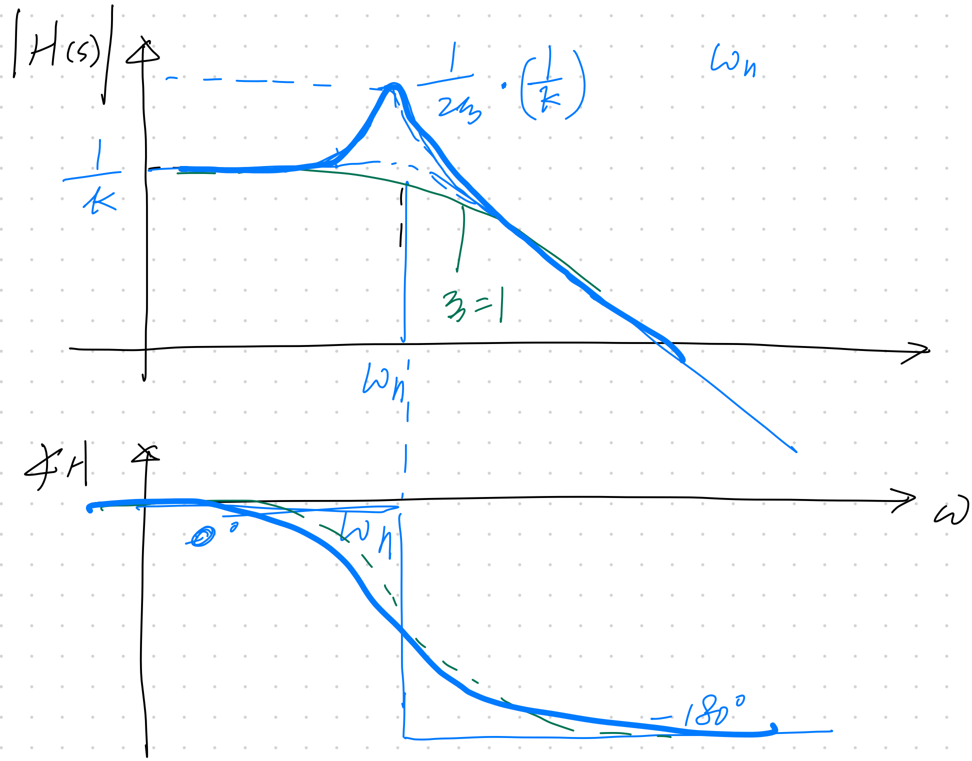
Then $H(j\omega)$ is a complex amplitude, which has a magnitude M, and a phase angle ϕ

Bode plot : Represent the freq - response in graphics.

example: $m\ddot{x} + b\dot{x} + k = f$,

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$= \frac{1}{k\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1\right)}$$



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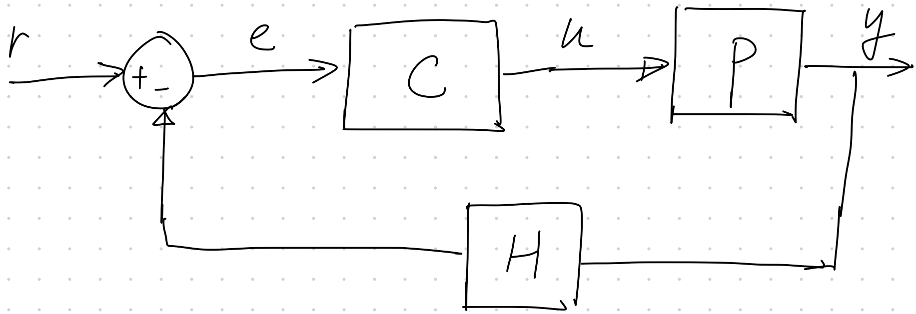
E) Loop-shaping Design for feedback systems.

- Loop-shaping is the most important perspective for practical controller design

- While computational tools like Matlab make it easier, we MUST have an ability to think through the design by hand

- Works for both analytical dynamic models and measured freq-response data.

For feedback control loop:



Define: $C \cdot P \triangleq$ Forward path

$H \triangleq$ Feedback path

$L \triangleq C \cdot P \cdot H$:

Loop transfer function

or

Loop gain

or

Loop return ratio

$$CL(s) \triangleq \frac{Y(s)}{R(s)} = \frac{C(s)\,P(s)}{1 + L(s)}$$

Stability depends on poles of the closed-loop system,

i.e. roots of $\boxed{1 + L(s) = 0}$

BUT: we design in $L(s)$, specifically, $C(s)$

Controller

$1 + L(s) = 0$ depend on $L(s)$ in complicated way

- Nyquist test solves this question:

How to design for acceptable $1 + L(s) = 0$ roots through designing $L(s)$

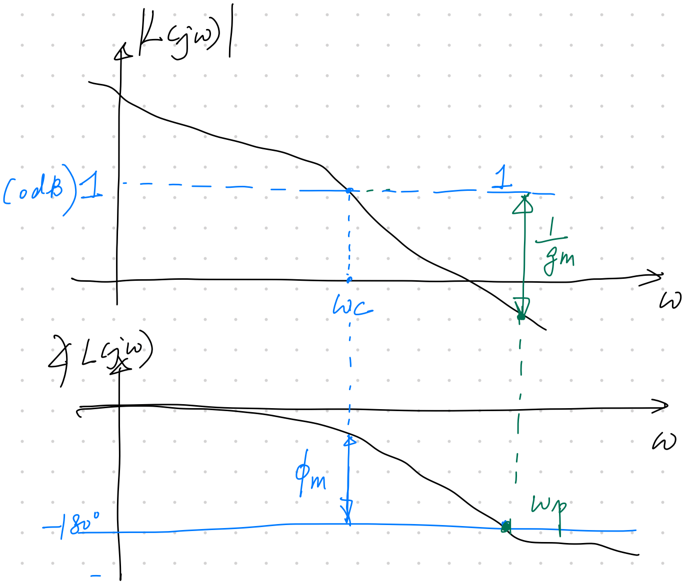
Intuition: find roots for $L(s) = -1$, i.e.

$$|L| = 1 \text{ and } \measuredangle L = -180^\circ$$

- For most systems, full Nyquist approach is not needed

They can be designed via Bode Plots, ie loop-shaping

Phase margin and Gain margin are definitions for a Bode plot for $L(j\omega)$



☆ $\omega_c \triangleq$ crossover frequency

$(|L(j\omega)| = 1)$

☆ $\phi_m \triangleq$ phase margin:

$= 180° + \angle L(j\omega)$

☆ $\omega_p \triangleq$ phase cross over freq

$\angle L(j\omega) = 1$

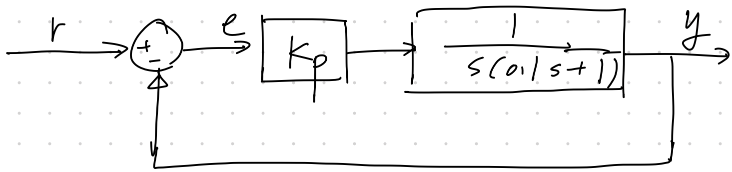
Note: only defined for $L(j\omega)$.

☆ $g_m \triangleq$ gain margin

$|L(j\omega_p)| = \frac{1}{g_m}$

These value are strongly correlate to closed-loop performance:

example:



Matlab example:

K	ω_c	ϕ_m	ω_n	ζ
1	1	84°	3.16	1.6
10	10	45°	10	0.5
100	32	19.5°	31.6	0.16
1000	100	6.3°	100	0.05

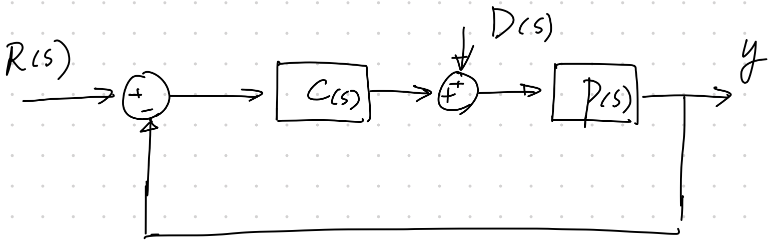
Rule of Thumb:

$\omega_n \approx \omega_c$ (Bandwidth)

$\zeta \approx \phi_m / 100$ (relative stability)

Important equations for design.

Also note: Higher loop gain is typically better as long as we can achieve an acceptable relative stability



$$Y(s) = R(s)\underbrace{\frac{C \cdot P}{1 + CP}} + D(s)\underbrace{\frac{P(s)}{1 + C(s)P(s)}}$$

$\Rightarrow 1$ as $C(s) \to \infty$ $\qquad$ $\Rightarrow 0$ as $C(s) \to \infty$

But watch out for noise in sensors at high freq.

Loop-shape captures this trade off between:

1) Large loop gain for a wide range of frequency

2) Relative stability

Loop-shaping design :

1) Model or measure $P(s)$, select $C(s)$
2) Plot Bode plot for $L(s)$
3) Find ω_c , ϕ_m , ω_p , g_m
4) System bandwidth defined as ω_c
   - ω_c is an accurate measure of loop performance
   - Some literature uses $\omega_{BW} \triangleq -3dB$ in CL
5) System's damping ratio $\zeta \approx \frac{\phi_m}{100}$
   - Typically we design for ϕ_m in $30^\circ \sim 60^\circ$
   - In special cases , lower is ok.

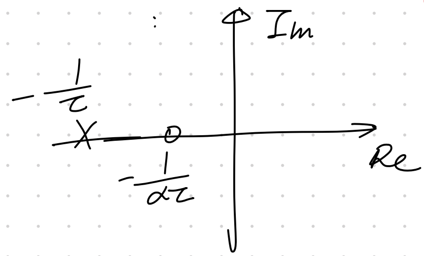
So far we have not talked about controller selection yet.

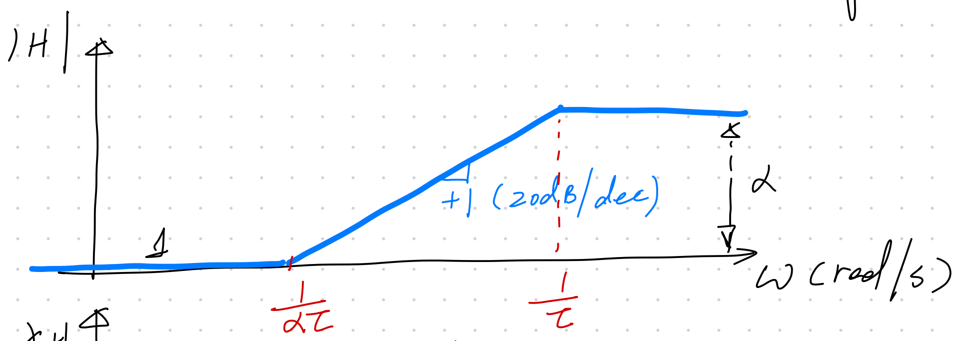
- How to select ? $\begin{cases} \text{Proportional} : k_p \\ \text{P-I} : \quad k_p\left(1+\frac{1}{T_i s}\right) \\ \text{PID / lead lag ?} \end{cases}$

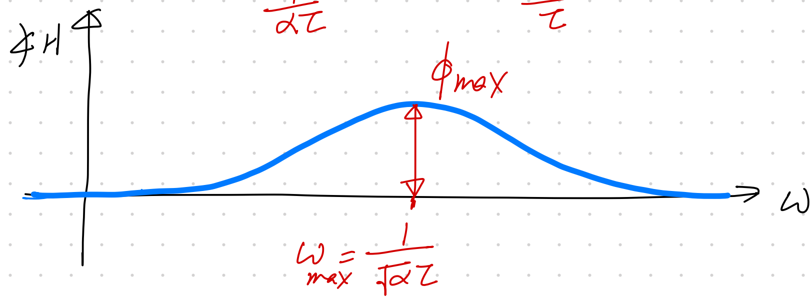
Going back to it now.

Revist lead controller :

$$H(s) = \frac{\alpha \tau s + 1}{\tau s + 1}$$







How much phase lead it can provide ?

Consider $\quad Lead(s) = \dfrac{\alpha \tau s + 1}{\tau s + 1}$,

Let $s = j\omega$, evaluate phase at $\omega = \dfrac{1}{\sqrt{\alpha}\,\tau}$

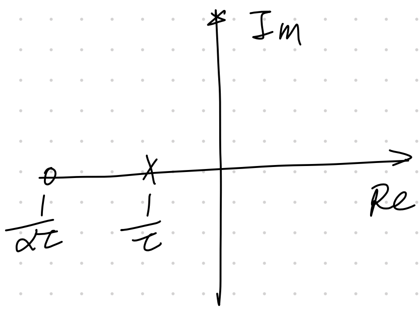
$$\Rightarrow Lead(j\omega) = \frac{j\sqrt{\alpha} + 1}{j/\sqrt{\alpha} + 1}$$

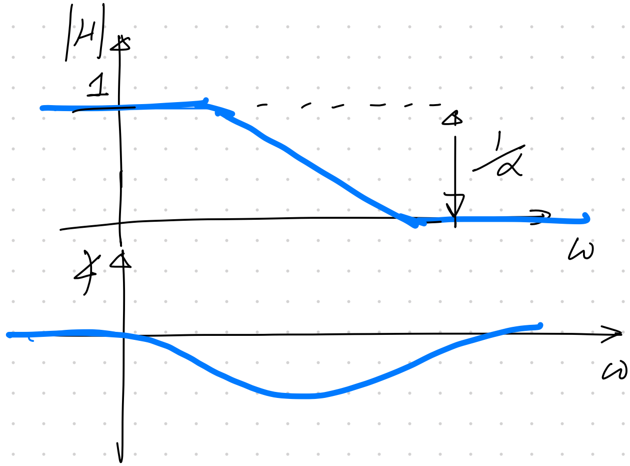
$$\phi_{peak} = \operatorname{atan}(\sqrt{\alpha}) - \operatorname{atan}\left(\frac{1}{\sqrt{\alpha}}\right)$$

often used

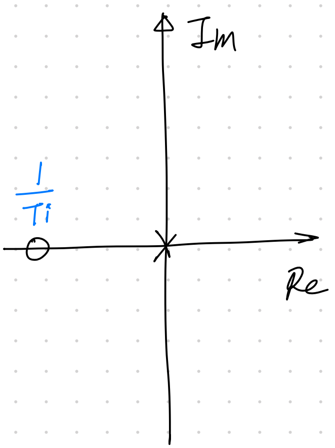
$$\alpha = 10 \rightarrow \phi_{peak} = 55^\circ$$

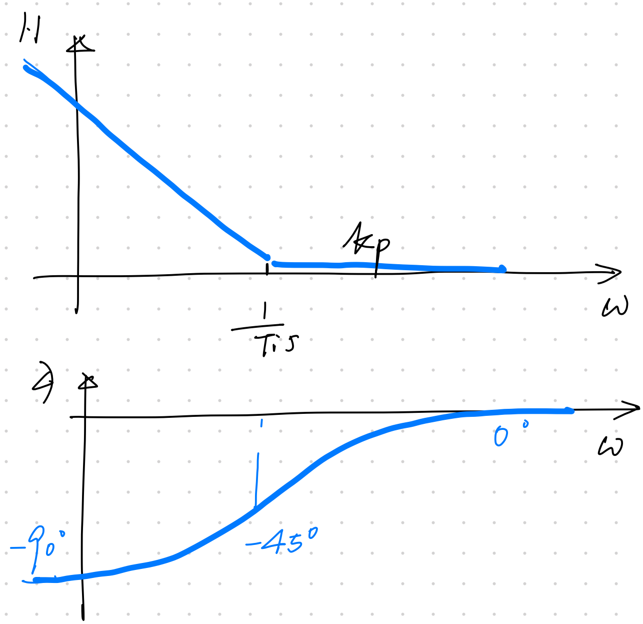
Lag compensators : $H(s) = \dfrac{\alpha\tau s + 1}{\tau s + 1}, \quad \alpha < 1$





PI controllers can be viewed as lag controller with the pole located at zero frequency.

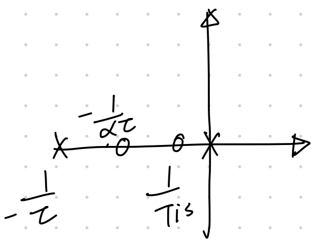


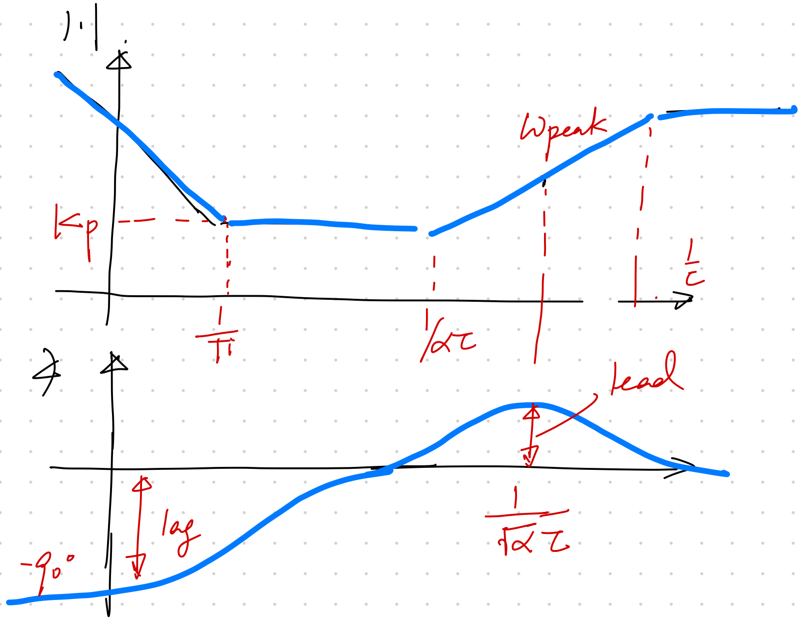


Going back to our PID Controller :

$$e \rightarrow \boxed{K_p} \rightarrow \boxed{1+\frac{1}{T_i s}} \rightarrow \boxed{\frac{\alpha\tau s+1}{\tau s+1}} \rightarrow u$$

$\alpha > 1$

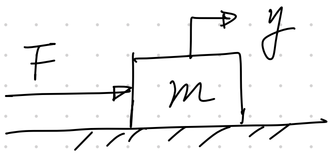




Lets design parameters with loop shaping,

i.e. manipulate the shape of $L(j\omega)$.

Back to our design example:



$$P(s) = \frac{1}{ms^2}$$

$m = 10\,kg$

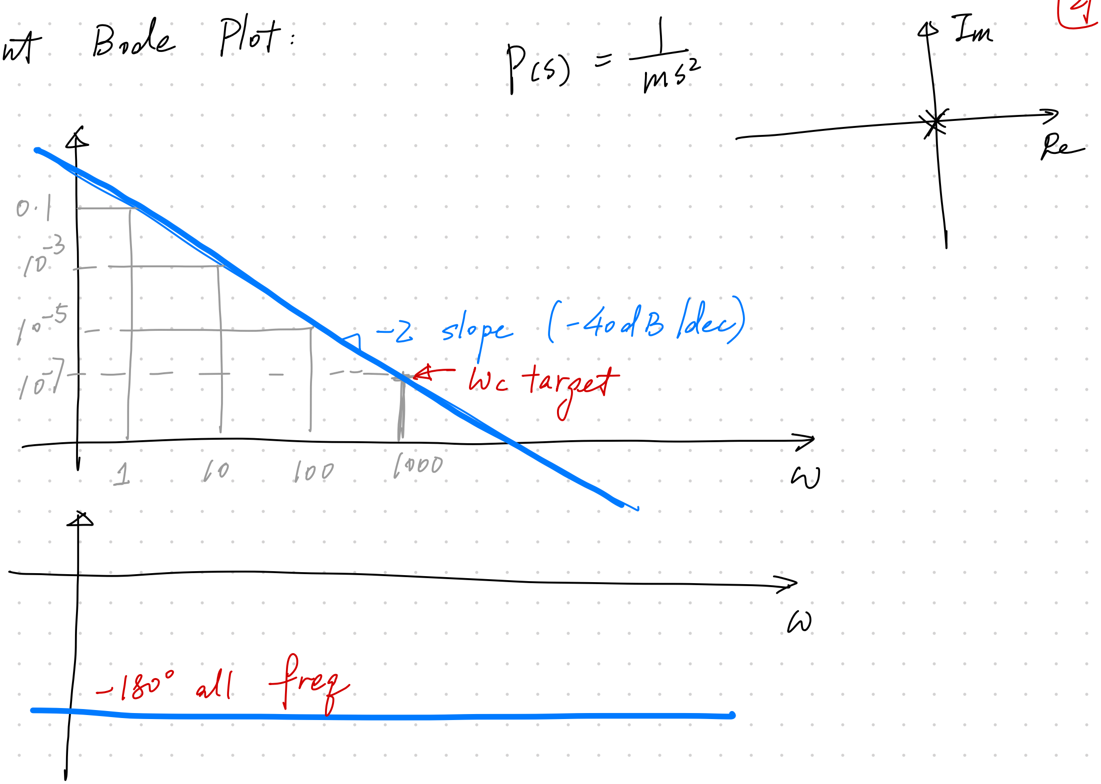
Design Goal:
- $\omega_c = 1000$ rad/s (31.6 Hz),
- $\phi_m > 45°$

Step ①: Design lead compensator to stabilize the system

Step ②: Add Integral term for high loop gain at low freq, while still satisfying the specs.

Plant Bode Plot:

$$P(s) = \frac{1}{ms^2}$$



at target $w_c = 1000$ rad/s:

$$|P(j\omega)| = 10^{-7}$$

$$\measuredangle P(j\omega) = -180^\circ$$ ← need phase lead of at least 45° from the controller

Step ① Design lead controller:

$$C(s) = k_p \cdot \frac{\alpha \tau s + 1}{\tau s + 1}$$

recall ϕ_{max} is at frequency $\omega_{max} = \frac{1}{\sqrt{\alpha}\,\tau}$

- select $\alpha = 10$ (typical value) $\Rightarrow \phi_{max} = 55°$

- Place the peak phase at the target crossover freq to satisfy phase margin requirement:

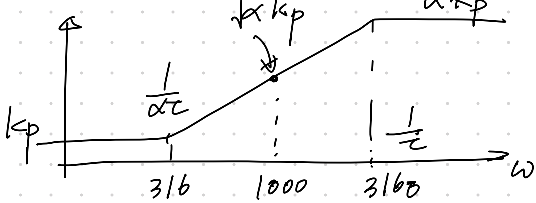
$$\omega_{max} = \frac{1}{\sqrt{\alpha}\,\tau} = 1000 \text{ rad/s}$$

$$\Rightarrow \quad \tau = \frac{1}{\sqrt{\alpha}\,\omega_{c\,target}} = \frac{1}{\sqrt{10} \cdot 1000} \simeq 0.316 \text{ ms}$$

Now α and τ fixed, lets find k_p to make

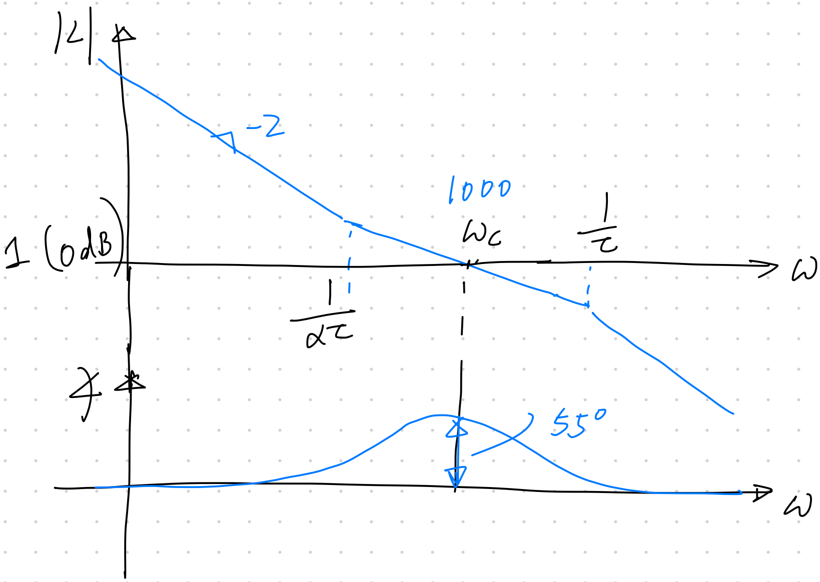
$$|L| = 1 \quad \text{at} \quad \omega_c = 1000 \text{ rad/s}$$

lead filter



$$\Rightarrow \quad |P(j1000)| \cdot |C(j1000)| = 1 \quad \Rightarrow \quad \boxed{k = 3.16 \times 10^6}$$

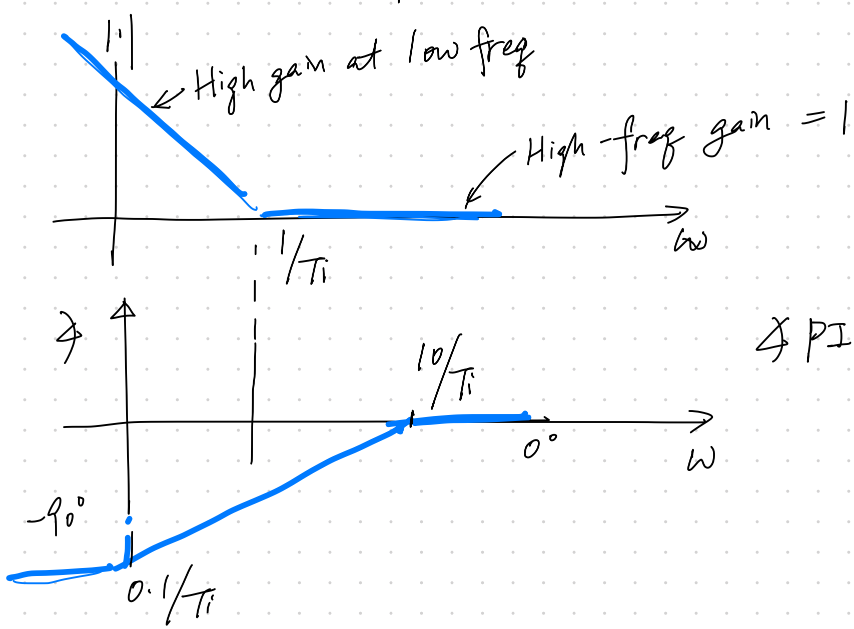
Now our loop looks like:



Step 1 Done

Now, consider PI Controller:

$$PI(s) = 1 + \frac{1}{T_i s}$$



$∡ PI = -90° + \tan^{-1} T_i \omega$

We can only allow 10° phase lag at ω_c,

while we want to maximize loop gain at low freq

$$\Rightarrow \quad \max \frac{1}{T_i} \quad \text{s.t.} \quad \angle PI(j\omega_c) > -10^\circ$$

$$\Rightarrow \quad \tan^{-1} T_i \omega_c = 80^\circ$$

$$\Rightarrow T_i = \frac{1}{\omega_c} \tan 80^\circ = \frac{5.7}{\omega_c} = \underline{5.7 \text{ msec}}$$

Now out design is complete.

Matlab Simulation $\Rightarrow$

In Lab 1, we will do this design for an analog circuit with the measured data.

Demo.