Advanced Statistical Inference, 2024-2025
Problems to discuss in class

Unit I. Part 2. Testing Statistical Hypothesis

1. The following figures (Cushny and Peebles' data), are quoted quote by R.A. Fisher[1] from a "Student's" paper, and show the result of an experiment with ten patients on the effect of two supposedly soporific drugs, A and B, in producing sleep.

Patient.	A.	B.	Difference (B $\sim$ A).
I	+0.7	+1.9	+1.2
2	- 0.6	+0. 8	+2.4
3	-0.2	+1.1	+1 .3
4	- 1.2	+0.1	$+1 \cdot 3$
5	-0.1	-0.1	0.0
6	+3.4	+4.4	+1.0
7	+3.7	+5.5	+1.8
8	+0.8	+1.6	+0.8
9	0.0	+4.6	$+4 \cdot 6$
Io	+2.0	+3.4	+1.4
Mean ($\bar{x}$)	+0.75	+2.33	$+1 \cdot 58$

Table 1: Additional Hours of Sleep galned by the Use OF TWO TESTED DRUGS

The last column gives a controlled comparison of the efficacy of the two drugs as soporifics,

(a) Propose and apply a test of significance to help decide if both drugs can be considered to have the same soporific effect.

(b) Answer the previous question assuming that the researchers had decide to record only the sign of the difference, but not the numerical value.

(c) Answer the first question assuming that soporific A and B were not tested on the same subjects but, instead on two independent (not matched) groups of subjects.

2. Assume a certain process that leads to a TRUE/FALSE decision is expected to be *fair*, wihch means that either TRUE or FALSE are expected to habppen with probability 0.5

   In order to check this fairness, from time to time, the process is repeated 100 times and the number of TRUE results is recorded. If this number is above 60 or below 40 the process is declared out-of-control or unfair and it is re-adjusted.

(a) Show how to turn this decision rule into a test with critical region $W = \{\tilde{x}\, s.t. \, |X - 50| > 10\}$

(b) Calculate the probability of the type I error. Use the Normal approximation to a Binomial with the continuity correction.

(c) Plot the power function depending on the values of p.

[1] Statistical Methods for Research Workers, p 121

3. Let $X_1, \ldots, X_n$ be a simple random sample from a Uniform distribution on$(0, \theta)$. We would like to test $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$. Let $Y_n = max(X_1, \ldots, X_n)$ and consider the procedure that has as critical region *all the results such that* $Y_n \leq 1.5$.

   (a) Find the power function of such procedure.

   (b) Calculate the size of the procedure.

4. A team of researchers plans a study to see if a certain drug can increase the speed at which mice move through a maze. An average decrease of 2 seconds through the maze would be considered effective, so the researchers would like to have a good chance of detecting a change this large or larger. Would 20 mice be a large enough sample? Assume the standard deviation is $\sigma = 3$ sec. and that the researchers will use a significance level of $\alpha = 0.05$.

5. Suppose the researchers in the previous example want a 95% chance of rejecting $H_0 : \mu = 0$ at $\alpha = 0.01$ if the true change is a 1.5 sec. decrease in time. What is the smallest number of mice that should be included in the study?

6. Let $X_1, \ldots, X_n$ be a simple random sample from a Gamma distribution with parameters $(3, \theta)$, with probability density function

$$f(x; \theta) = \frac{1}{2\theta^3} x^2 e^{-x/\theta} I_{(0,\infty)}(x).$$

   (a) Find the most powerful test of size $\alpha \leq 0.05$ of the hypotheses $H_0 : \theta = \theta_0$ or $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

   (b) Use the results obtained in order to give the uniformly most powerful test to contrast H_0 against $H_1' : \theta > \theta_0$.

7. Let $X_1, \ldots, X_n$ be a simple random sample from a Poisson distribution with parameter λ, unknown ($\lambda > 0$).

   (a) Prove that the joint probability function of $(X_1, \ldots, X_n)$ has a monotonic likelihood ratio in the statistic $\sum_{i=1}^{n} X_i$.

   (b) Prove that, for $n = 10$, there exists a UMP procedure to test the hypothesis: $H_0 : \lambda \leq 1$ versus $H_A : \lambda > 1$ with significance level $\alpha_0 = 0.0143$.

   (c) Prove that, for $n = 10$, there exists a UMP procedure to test the hypothesis: $H_0 : \lambda \geq 1$ versus $H_A : \lambda < 1$ with significance level α_0 for some α_0 such that $0 < \alpha_0 < 0.03$.

8. Let Y_1 , ..., Y_n be independent identically distributed random variables with pdf:

$$f(y) = \frac{y}{\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right]$$

   where $\sigma > 0$ and $y \geq 0$.

   (a) Show $W = Y^2 \sim \sigma^2 \chi_2^2$. Equivalently, show $Y^2/\sigma^2 \sim \chi_2^2$.

   (b) What is the UMP (uniformly most powerful) level α test for $H_0 : \sigma = 1$ versus $H_1 : \sigma > 1$?

   (c) If $n = 20$ and $\alpha = 0.05$, then find the power $\beta(\sqrt{1.9193})$ of the above UMP test if $\sigma = \sqrt{1.9193}$.

9. We wish to compare the mean survival time (in weeks) of two groups of mice which are used used to test a treatment for a liver disease. All mice are affected by the disease and half one group has been treated by a placebo (y) while the other has been given the drug being tested (z). Placebo and treatmen have been assigned at random to an otherwise homogeneous sample of mice. The resulting survival times are:

$$z = \quad 94, 197, 16, 38, 99, 141, 23$$
$$y = \quad 52, 104, 146, 10, 51, 30, 40, 27, 46.$$

   (a) Assuming the data are exponentially distributed build a likelihood ratio test and use it to test the null hypothesis of mean equality versus the alternative hypothesios that the mean survival times are different.
   (b) Implement a permutation test in R to compare the two group means. Use it with the data and a number of 1000 permutations to obtain a permutation p-value.
   (c) Compare the results of both tests and comment about the pros and cons of each method.

Examen exercises

10. Let X be a random variable from the location logistic family with density function given by

$$f(x|\theta) = \frac{e^{(x-\theta)}}{\left(1 + e^{(x-\theta)}\right)^2}$$

and distribution function given by

$$F(x|\theta) = \frac{e^{(x-\theta)}}{1 + e^{(x-\theta)}}$$

for $-\infty < x < \infty$ and $-\infty < \theta < \infty$.

   (a) Find the most powerful test of size α to test $H_0 : \theta = 0$ *versus* $H_1 : \theta = 1$ with a sample of size $n = 1$.
   (b) Compute the power of this test for size $\alpha = 0.2$.
   (c) Prove that the test in part 1 is as well UMP of size α for the one-sided test of $H_0 : \theta = 0$ *versus* $H_1 : \theta > 0$.