

An introduction to the general linear model

Application to biomarker discovery with omics data.

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2023-02-20

Outline

- 1) A recap on linear models
- 2) Linear models with R
- 3) Linear models for omics data
- 4) Exercises

A recap on linear models

What is a linear model (in statistics)

- Linear models appear when we assume that the relation between:
 - one variable,
 - and another (set of) variable(s)

can be represented through a linear relation, that is, one of the form

$$y \simeq a + b \times x \quad \text{or:} \quad y \simeq a + b \times x + c \times z + \dots$$

- Linear models have been thoroughly used because of their flexibility and many of the important techniques in statistics are "just" linear models.

Regression, ANOVA are linear models

- Linear models provide a flexible way to modelling relations and building explanatory or predictive models, as seen in **linear regression**.

$$Y_i = \beta_0 + \beta_1 \times X_{1i} + \beta_2 \times X_{2i} + \cdots + \epsilon_i$$

- But they also provide a convenient setting to
 - describe experimental designs and
 - to analyze data that has been obtained from experiments performed according to the design they describe, as seen in **ANOVA**.

$$Y_{ij} = \mu_i + e_{ij} = \mu + \tau_i + e_{ij}, \quad i = 1 \dots k, \quad j = 1 \dots r.$$

Linear models. An example

- Consider a study comparing **two diets** in mice, a "standard" vs a "high-fat" one.
- If we consider that the weight of the animals is a linear function of the diets, the following linear model can be written:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, i = 1, \dots, N$$

- where:
 - Y_i represents the weights of the i -th experimental unit-
 - $x_i = 1$ when mouse i receives the high fat diet and
 $x_i = 0$ when it receives the standard diet.
- The main goal of the study are:
 - To *estimate* the mean weight of mice after having been nurrished with each diet
 - To *compare* the difference in weight between the standard and "high fat" diets.

The general linear model

The diagram shows the general linear model equation: $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i, \quad i \in 1, \dots$. Annotations include: "model parameters" with arrows pointing to β_0 , β_1 , and β_2 ; "observed data" with an arrow pointing to Y_i ; and "0s or 1s, for now" with arrows pointing to x_{i1} and x_{i2} .

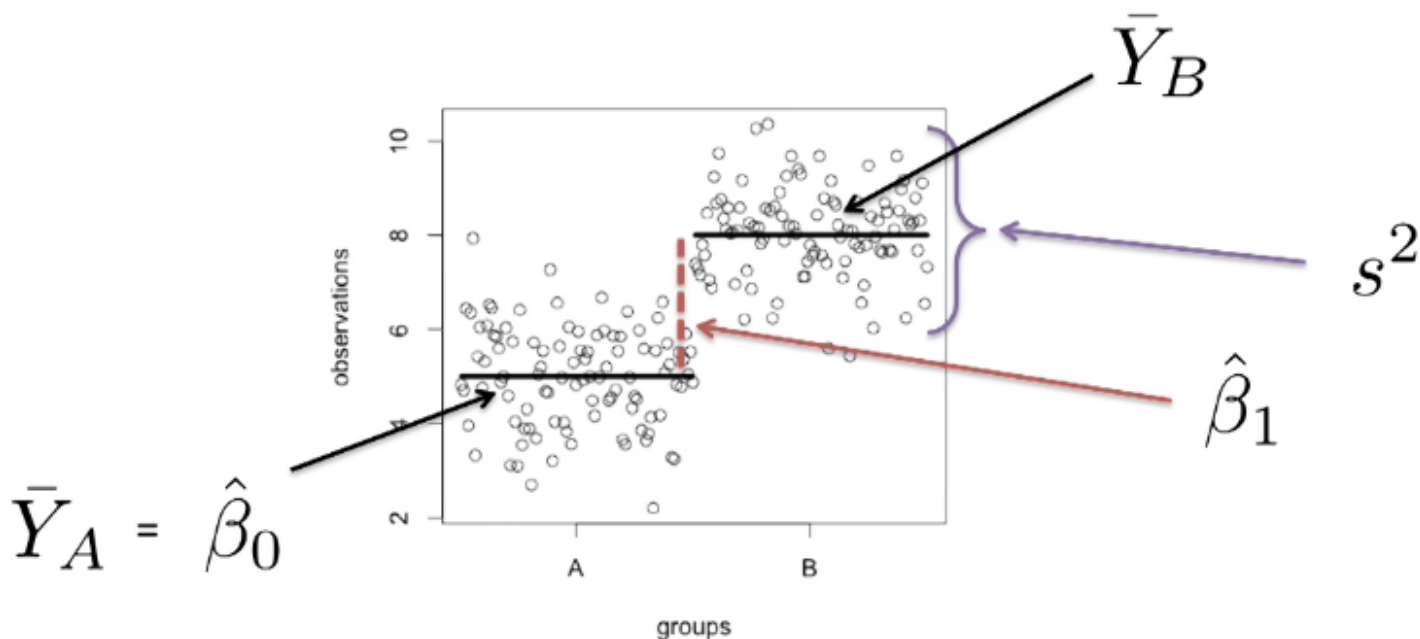
$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i, \quad i \in 1, \dots$$

- Linear models assume a linear relation between a *response* ("dependent") variable, and one or more *explanatory* (or "independent") variables(s).
- Many common problems can be re-written as linear models.

Common statistical tests are linear models

Comparison of 2 groups (t-test)

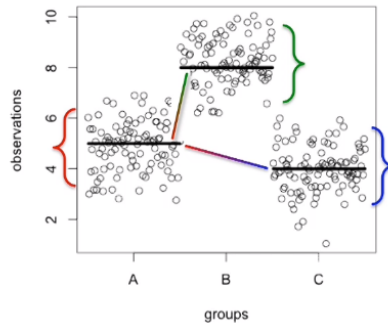
- The coefficients represent the (differential) effect of diet
- Comparison between diets can be based on the means or on the coefficients.



$$\bar{Y}_A = 1 \times \hat{\beta}_0 + 0 \times \hat{\beta}_1$$

$$\bar{Y}_B = 1 \times \hat{\beta}_0 + 1 \times \hat{\beta}_1$$

Comparison of three groups (ANOVA)



$$\bar{Y}_A = \hat{\beta}_0$$

$$\bar{Y}_B = \hat{\beta}_0 + \hat{\beta}_1$$

$$\bar{Y}_C = \hat{\beta}_0 + \hat{\beta}_2$$

- With more than two diets the scenario is similar
 - β_0 represents the "control" or standard
 - β_1, β_2 the effect of each diet added to the standard
- Comparisons can be based on
 - The means
 - The coefficients

- Notice that, although the linear model postulates a relation between *parameters* we will work with their *estimates* so, there is an error that we need to control in order for our conclusions (inferences) to be reliable.

Estimating a linear model

- For linear models to be useful, we have to *estimate* the unknown values β_i .
- The standard approach in science is to find the values that minimize the distance of the fitted model to the data.
- The following is called the least squares (LS) equation:

$$RSS = \sum_{i=1}^n \{y_i - (\beta_0 + \beta_1 x_i)\}^2$$

- This quantity is called the residual sum of squares (RSS).
- Once we find the values that minimize the RSS, we will call the values the *least squares estimates (LSE)* and denote them with $\hat{\beta}_i$.

Matrix notation for linear models

- Linear models can be written as:
 - explicit expressions (above) or
 - using matrix notation (below)

```
> beta <- c(1,1)
> sum((Y - beta[0] - beta[1]*x[,1] - beta[2]*x[,2] - beta[3]*x[,3] - beta[4]*x[,4])^2)
```

$$\sum_{i=1}^N (Y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \beta_3 x_{i3} - \beta_4 x_{i4})^2$$

$$(Y - \beta X)^T (Y - \beta X)$$

- Matrix notation is generally preferred.
 - More compact notation
 - More efficient computations.

Two groups in matrix notation

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

This *linear model* can be written in more compact notations as:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

The matrix \mathbf{X} is called the *design matrix*

Three groups in matrix notation

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}$$

Fitting linear models

- A linear model can be *fitted* by solving the *normal equations*

simple, fast matrix multiplication:

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

- Error estimates for the model coefficients can also be obtained:

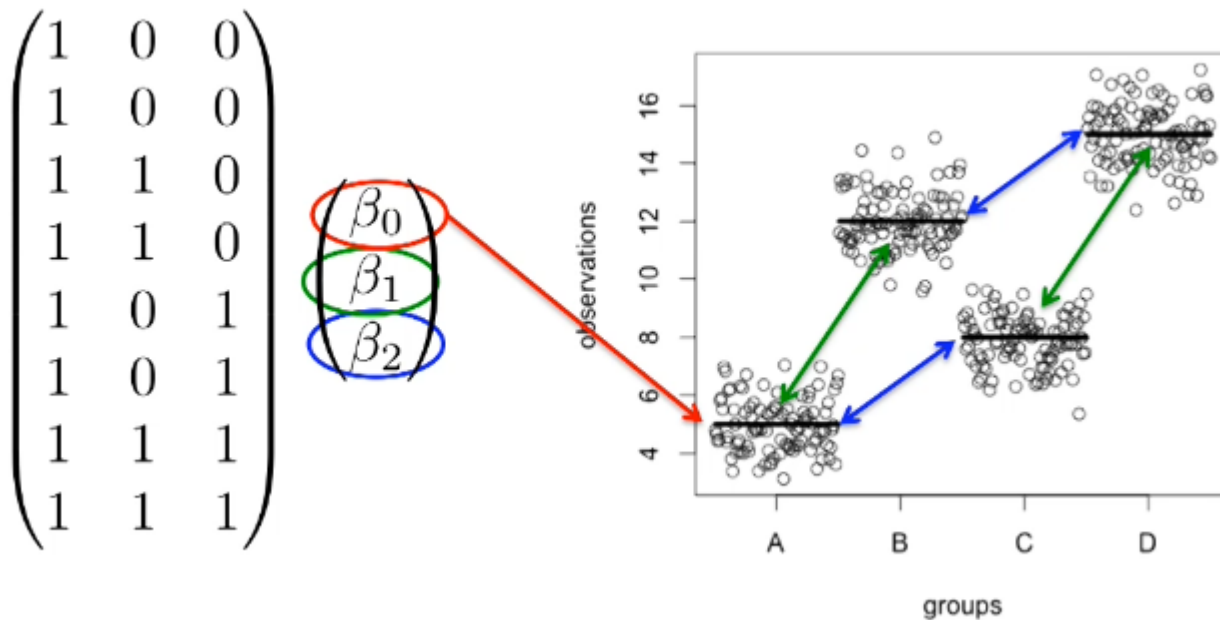
$$\text{se}(\hat{\beta}_i) = \sqrt{s^2 (X^T X)^{-1}_{ii}}$$

Significance testing with linear models.

- Assuming a series of assumptions hold
 - Variance homogeneity
 - Linearity of relations
 - Independence and *normality* of error terms
- A test can be built to test the significance of the model coefficients

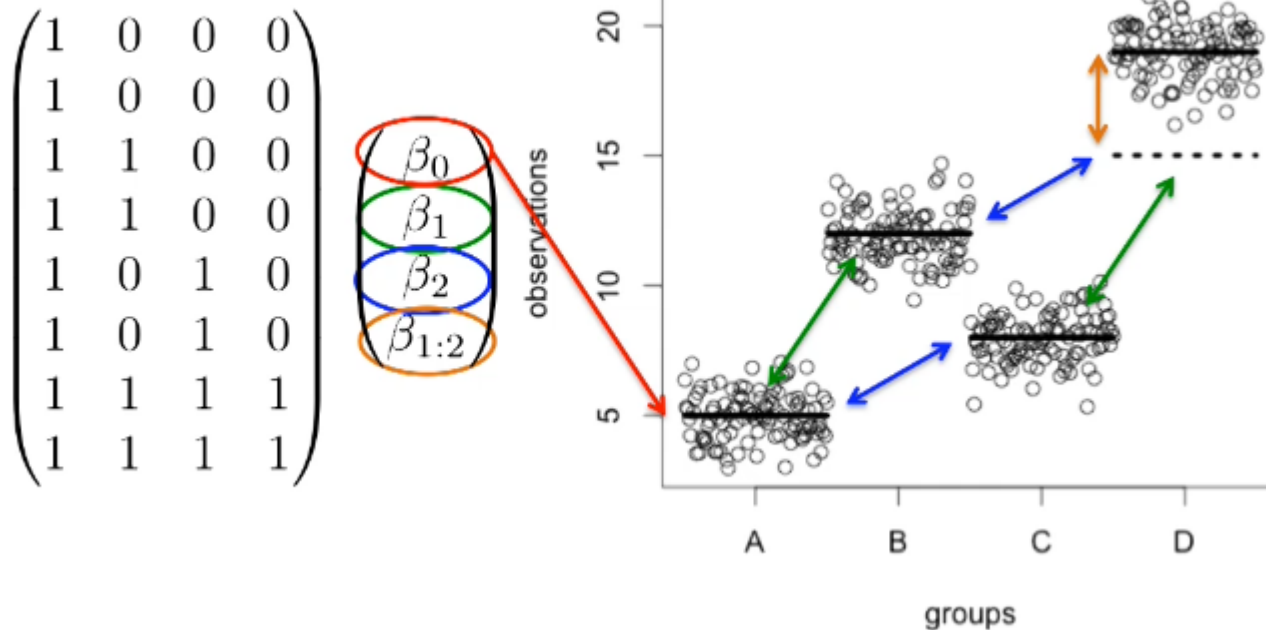
$$\frac{\text{signal}}{\text{noise}} = \frac{\hat{\beta}_i}{\text{se}(\hat{\beta}_i)}$$

More linear models: crossed designs



- We assume effects are additive!

Crossed designs with interaction



- Interaction shows additional to additive effects

How is a linear models usually applied?

- Estimating the parameters values to know mean effect of one factor or of each level of this factor.
- Comparing the parameters' values to decide if different treatments have the same effect, or have any effect at all.
- Adjusting the effect of distinct covariables, which may not be of direct interest on the study but whose effect may interfere in the relation between the response variable and the explanatory ones.
 - A common quote in papers: *"The study of ... controlled for age, race, and sex ..."*,
 - Usually, what it means is that we have included these variables in the model expecting the variability that they explain is separated from that explained by the other components of the model.

$$Y_{\{32 \text{ yo}, \text{Caucasian}, \text{female}\}} = \beta_0 + \beta_{25-35 \text{ yo}} + \beta_{\text{Caucasian}} + \beta_{\text{female}} + \varepsilon$$

Linear models in R

Fitting linear models in R

- In practice, when using R, we *rarely fit a model by solving the normal equations*
- The usual, and most practical way to do it, is to use `lm` function.
- The `lm` function requires
 - Either a `formula` relating the variables to be included in a linear model
 - Or a design matrix, that we can create using the `model.matrix` function, and which, implicitly, defines that model.

The importance of the design matrix

- The choice of design matrix is a *critical step* in linear modeling as
 - it encodes which coefficients will be fit in the model,
 - and the inter-relationship between the samples.
- Defining which design matrix we use is equivalent to defining the parameters of the model (or the model's *parametrization*).
 - Same data can be modelled differently, if parameters receive different meanings.
 - In practice this represents using distinct design matrices
- A typical example: How does the meaning of a one-way factor ANOVA change if we consider a model with or without an intercept?
 - How is it reflected in the design matrix?

Model matrix for two groups

- Suppose we have two groups, 1 and 2, with two samples each.
- We might start to encode this experimental design like so:

```
x ← c(1,1,2,2)
f ← formula(~ x)
model.matrix(f)
```

```
##      (Intercept) x
## 1              1 1
## 2              1 1
## 3              1 2
## 4              1 2
## attr(,"assign")
## [1] 0 1
```

Model matrix for two groups (2)

- Note that an intercept will be included by default, so the formula could equivalently be written: `~ x + 1`.
- We can then inspect the design matrix which is formed by this:

```
model.matrix(f)
```

```
##      (Intercept) x  
## 1              1 1  
## 2              1 1  
## 3              1 2  
## 4              1 2  
## attr("assign")  
## [1] 0 1
```

model.matrix requires factors

- Note, this is not the design matrix we wanted.
- We should instead first tell R that these values should not be interpreted numerically, but as different levels of a factor variable:

```
x ← factor(c(1,1,2,2))  
model.matrix(~ x)
```

```
##      (Intercept) x2  
## 1              1  0  
## 2              1  0  
## 3              1  1  
## 4              1  1  
## attr(,"assign")  
## [1] 0 1  
## attr(,"contrasts")  
## attr(,"contrasts")$x  
## [1] "contr.treatment"
```

- Now we have achieved the correct design matrix.
- Or have we?

The role of the intercept term

- Note that the previous matrix has one intercept column and one group column although there are two groups indeed.
 - The first group's values are represented by the basal or "overall mean".
 - The second group's are represented by one column.
- An alternative representation is possible setting the intercept to zero.
- Both representations are equivalent and for one-factor designs it's up to you which one to choose

```
x ← factor(c(1,1,2,2))  
model.matrix(~ x + 0)
```

```
##      x1 x2  
## 1   1  0  
## 2   1  0  
## 3   0  1  
## 4   0  1  
## attr(,"assign")  
## [1] 1 1  
## attr(,"contrasts")  
## attr(,"contrasts")$x  
## [1] "contr.treatment"
```

Design matrix for more than 2 groups

- How is the design matrix for an experiment with 3 groups?
- We proceed like in the previous case

```
x ← factor(c(1,1,2,2,3,3))  
model.matrix(~ x)
```

```
##   (Intercept) x2 x3  
## 1           1  0  0  
## 2           1  0  0  
## 3           1  1  0  
## 4           1  1  0  
## 5           1  0  1  
## 6           1  0  1  
## attr(,"assign")  
## [1] 0 1 1  
## attr(,"contrasts")  
## attr(,"contrasts")$x  
## [1] "contr.treatment"
```

- Again the first group is implicit in the intercept but it can be set explicitly by setting the intercept to zero.

An alternative parametrization

- An alternate formulation of design matrix is possible by specifying `+0` in the formula:

```
x ← factor(c(1,1,2,2,3,3))  
model.matrix(~ x + 0)
```

```
##      x1 x2 x3  
## 1   1  0  0  
## 2   1  0  0  
## 3   0  1  0  
## 4   0  1  0  
## 5   0  0  1  
## 6   0  0  1  
## attr(,"assign")  
## [1] 1 1 1  
## attr(,"contrasts")  
## attr(,"contrasts")$x  
## [1] "contr.treatment"
```

- This representation allows fitting a separate coefficient for each group.

Linear Models for Omics Data

Linear models for omics data

- For the analysis of omics data a very popular option is the `limma` package.
- `limma` extends some R functionalities to make them easy to use in the analysis of omics data using linear models.
- Besides this it includes extensions to the standard linear model to improve analysis capabilities.
- In the following we show
 - How to create a design matrix from a "targets" file containing information on groups.
 - How to create a contrasts matrix to define the comparisons to be done.
 - How to do the comparisons and how to interpret the resulting analysis tables.

Example: Comparing 3 types of tumors

- This example study is based on a paper published in <http://www.ncbi.nlm.nih.gov/pubmed/15897907> whose data are available in GEO as series GSE1561 series on the following link <http://www.ncbi.nlm.nih.gov/geo/query/acc.cgi?acc=GSE1561>
- The researchers investigated three types of breast cancer tumors: apocrine (APO), basal (BAS) and luminal (LUMI).
- The classification is based on the resistance of tumors to estrogen and androgen receptors.
 - Tumors classified as "APO" are negative for estrogen receptor (ER-) and positive for the androgen receptor (AR+).
 - Those classified as "LUMI" are ER + and AR + and
 - Those classified as "BAS" are ER- and AR.

Identifying groups and comparisons

- The assignment of each sample to an experimental group can be obtained from this link: <http://www.ncbi.nlm.nih.gov/geo/gds/profileGraph.cgi?gds=1329>
- Obviously this is an observational study but its analysis can be done using a linear model approach as well.
- We will usually proceed in three steps:
 1. Identify the experimental factors and their levels.
 2. Write the design matrix associated with this study design.
 3. Build the contrast matrix that defines the comparisons we are interested in.
- In this example we have identified three groups and we wish to compare each tumor type with the other two
 1. "APO" vs "LUMI"
 2. "APO" vs "BAS"
 3. "LUMI" vs "BAS"

Defining the groups

```
library (limma)
targetsLocation← "https://raw.githubusercontent.com/alexsanchezpla/Ejemplo_de_MDA_cor
targets← read.delim(targetsLocation, row.names=1)
head(targets)
```

##	Sample	Ids	SampleIDs	Group	Apocrine.grade	AR.repeat.length
## GSM26878.CEL	GSM26878	PF14 EnPnT2N1G2	PF14	A	3	18
## GSM26883.CEL	GSM26883	PF19 EnPuT4N0Gu	PF19	A	2	18
## GSM26887.CEL	GSM26887	PF23 EnPnT2N0G2	PF23	A	3	18
## GSM26903.CEL	GSM26903	PF39 EnPuT4N0Gu	PF39	A	3	17
## GSM26910.CEL	GSM26910	PF46 EnPnT4N1G3	PF46	A	3	19
## GSM26888.CEL	GSM26888	PF24 EnPnTiN0G3	PF24	B	2	NA

Creating the design matrix

```
design<-matrix(  
  c(1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,  
    0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,  
    0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1),  
  nrow=15,byrow=F)  
colnames(design) <-c("A", "B", "L")  
rownames(design)<- targets$Sample
```

```
print(design)
```

```
##           A B L  
## GSM26878 1 0 0  
## GSM26883 1 0 0  
## GSM26887 1 0 0  
## GSM26903 1 0 0  
## GSM26910 1 0 0  
## GSM26888 0 1 0  
## GSM26889 0 1 0  
## GSM26892 0 1 0  
## GSM26898 0 1 0  
## GSM26906 0 1 0  
## GSM26879 0 0 1  
## GSM26896 0 0 1  
## GSM26897 0 0 1  
## GSM26907 0 0 1  
## GSM26911 0 0 1
```

Another way to create the design matrix

```
design2 ← model.matrix(~ 0+targets$Group  
colnames(design2)←c("A", "B", "L")  
rownames(design2)← targets$Sample
```

```
print(design2)
```

```
##           A B L  
## GSM26878 1 0 0  
## GSM26883 1 0 0  
## GSM26887 1 0 0  
## GSM26903 1 0 0  
## GSM26910 1 0 0  
## GSM26888 0 1 0  
## GSM26889 0 1 0  
## GSM26892 0 1 0  
## GSM26898 0 1 0  
## GSM26906 0 1 0  
## GSM26879 0 0 1  
## GSM26896 0 0 1  
## GSM26897 0 0 1  
## GSM26907 0 0 1  
## GSM26911 0 0 1  
## attr(,"assign")  
## [1] 1 1 1  
## attr(,"contrasts")  
## attr(,"contrasts")$`targets$Group`34 / 40  
## [1] "contr.treatment"
```

Defining the questions: The contrast

matrix

```
cont.matrix ← makeContrasts (  
  AvsB = B-A,  
  AvsL = L-A,  
  BvsL = L-B,  
  levels=design)  
cont.matrix
```

```
##           Contrasts  
## Levels AvsB AvsL BvsL  
##      A   -1   -1    0  
##      B    1    0   -1  
##      L    0    1    1
```

Fitting the model and estimating the contrasts

- Once we have the design and contrast matrices we can proceed to estimate the model fit the contrasts and check the results
- Here we first need the data.

```
dataLocation← "https://raw.githubusercontent.com/alexsanchezpla/Ejemplo_de_MDA_con_B:  
dataMatrix ← read.csv2(dataLocation, row.names = 1)  
colnames(dataMatrix)=rownames(targets)
```

```
## [1] TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE
```

```
dim(dataMatrix)
```

```
## [1] 6201 15
```

Fit the model and the contrasts

```
require(limma)
fit<-lmFit(dataMatrix, design)
fit.main<-contrasts.fit(fit, contrast.matrix)
fit.main<-eBayes(fit.main)
```

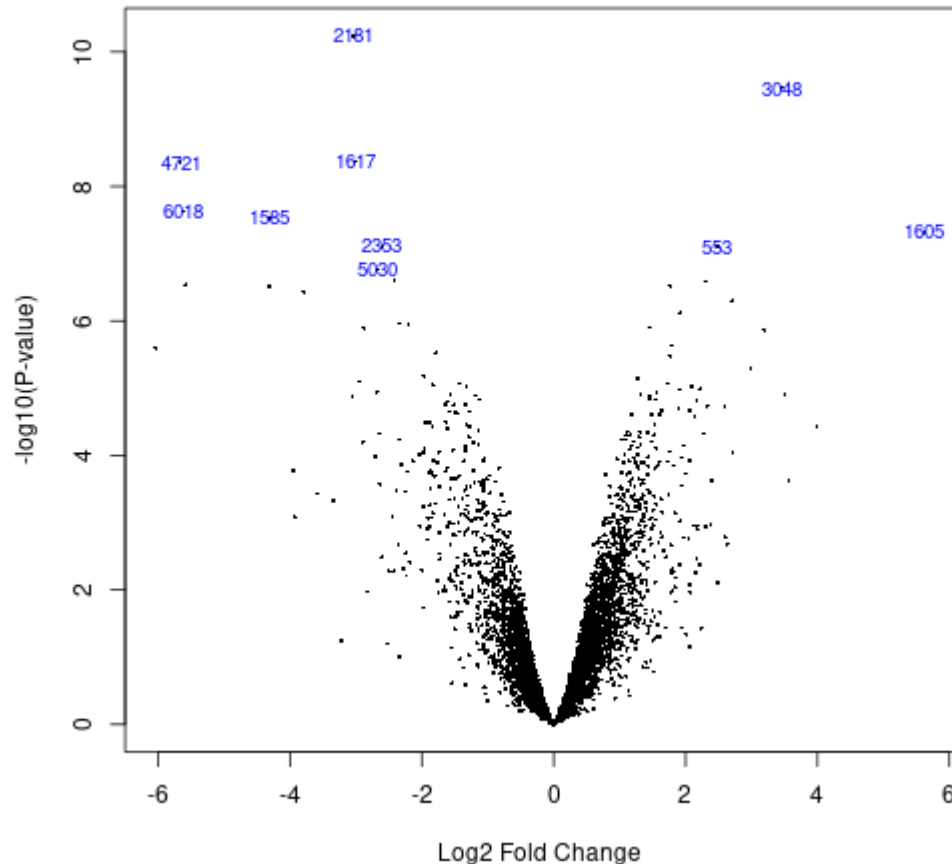
```
topTab_AvsB <- topTable (fit.main, number=nrow(fit.main), coef="AvsB", adjust="fdr")
head(topTab_AvsB)
```

##		logFC	AveExpr	t	P.Value	adj.P.Val	B
##	204667_at	-3.038344	8.651157	-14.338491	5.758201e-11	3.570660e-07	14.654208
##	215729_s_at	3.452290	6.137595	12.805722	3.406870e-10	1.056300e-06	13.162737
##	220192_x_at	-3.016315	9.521883	-10.852451	4.304210e-09	6.846708e-06	10.938163
##	214451_at	-5.665059	7.432823	-10.833890	4.416519e-09	6.846708e-06	10.915058
##	217528_at	-5.622086	6.763101	-9.670746	2.379541e-08	2.951106e-05	9.384907
##	217284_x_at	-4.313116	9.133307	-9.530910	2.942044e-08	3.040603e-05	9.189568

```
# topTab_AvsL <- topTable (fit.main, number=nrow(fit.main), coef="AvsL", adjust="fdr",
# topTab_BvsL <- topTable (fit.main, number=nrow(fit.main) , coef="BvsL", adjust="fdr",
```

Volcano plots provide visualization

```
volcanoplot(fit.main, coef="AvsB", highlight=10)
```



Exercises

Exercises on linear models for omics

- You can go through the exercises in the document "Exercises on linear models for microarrays"
 - Start by reading the experiment description
 - Postulate the type of experimental design
 - Write the linear model
 - Create the design and the contrast matrix
 - Get the data
 - Fit the model
 - Examine the results