Topological Sort on a DAG

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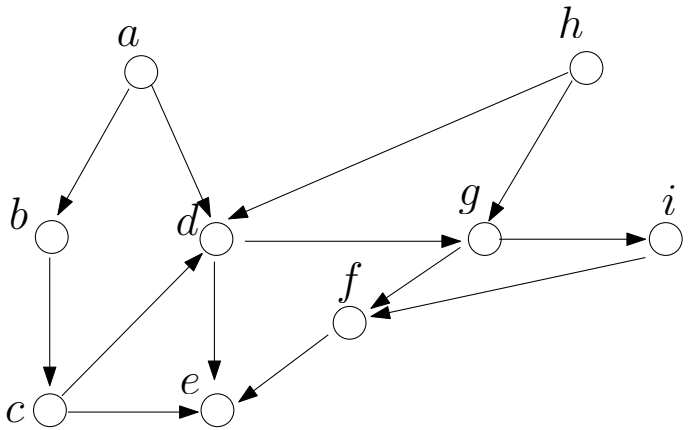
As mentioned earlier, depth first search (DFS) algorithm is surprisingly powerful. Indeed, we have already used it to detect efficiently whether a directed graph contains any cycle. In this lecture, we will use it to settle another classic problem—called topological sort—in linear time. The algorithm is very elegant, and simple enough to make this lecture a very short one.

Topological Order

Let $G = (V, E)$ be a directed acyclic graph (DAG).

A topological order of G is an ordering of the vertices in V such that, for any edge (u, v), it must hold that u precedes v in the ordering.

Example



The following are two possible topological orders:

- $h, a, b, c, d, g, i, f, e$.
- $a, h, b, c, d, g, i, f, e$.

An ordering that is not a topological order:

- $a, h, d, b, c, g, i, f, e$ (because of edge (c, d)).

Remarks:

- A directed cyclic graph has no topological orders (think: why?).
- Every DAG has a topological order.
  - This will be a corollary from our subsequent discussion.

The Topological Sort Problem

Let $G = (V, E)$ be a directed acyclic graph (DAG). The goal of topological sort is to produce a topological order of G.

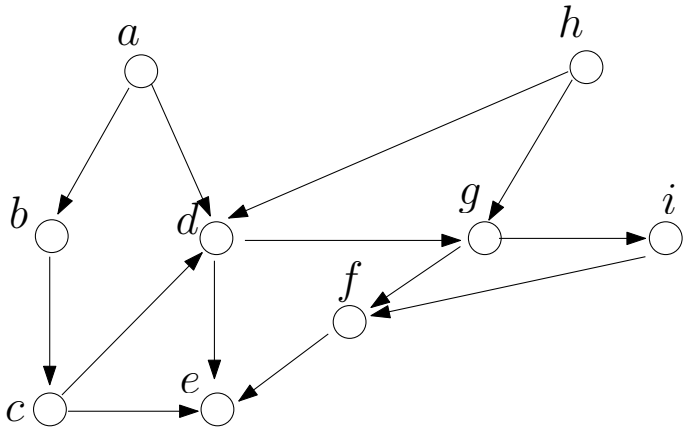
Algorithm

Very simple:

1. Create an empty list L.
2. Run DFS on G. Whenever a vertex v turns black (i.e., it is popped from the stack), append it to L.
3. Output the reverse order of L.

The total running time is clearly $O(|V| + |E|)$.

Example 1

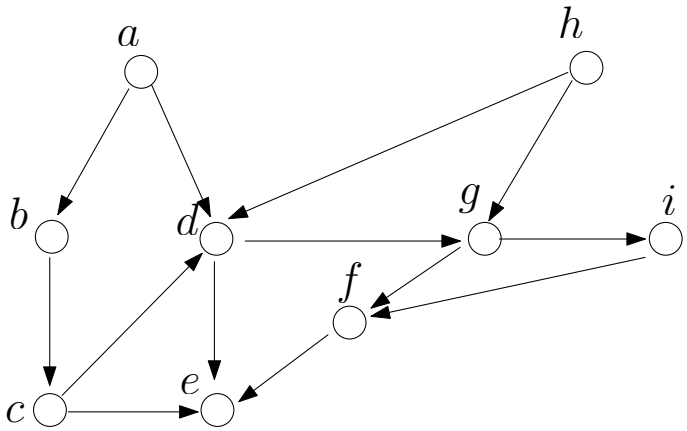


Suppose that we run DFS starting from a. The following is one possible order by which the vertices turn black:

- $e, f, i, g, d, c, b, a, h$.

Therefore, we output $h, a, b, c, d, g, i, f, e$ as a topological order.

Example 2



Suppose that we run DFS starting from d, then restarting from h, and then from a. The following is one possible order by which the vertices turn black:

- $e, f, i, g, d, h, c, b, a$.

Therefore, we output $a, b, c, h, d, g, i, f, e$ as a topological order.

We will now prove that the algorithm is correct.

Proof: Take any edge (u, v). We will show that u turns black after v, which will complete the proof.

Consider the moment when u enters the stack. We argue that currently v cannot be in the stack. Suppose that v was in the stack. As there must be a path chaining up all the vertices in the stack bottom up, we know that there is a path from v to u. Then, adding the edge (u, v) forms a cycle, contradicting the fact that G is a DAG.

Now it remains to consider:

- v is black at this moment: Then obviously u will turn black after v.
- v is white: Then by the white path theorem of DFS, we know that v will become a proper descendant of u in the DFS-forest. Therefore, u will turn back after v.

The correctness of our algorithm also proves:

> Every DAG has a topological order.