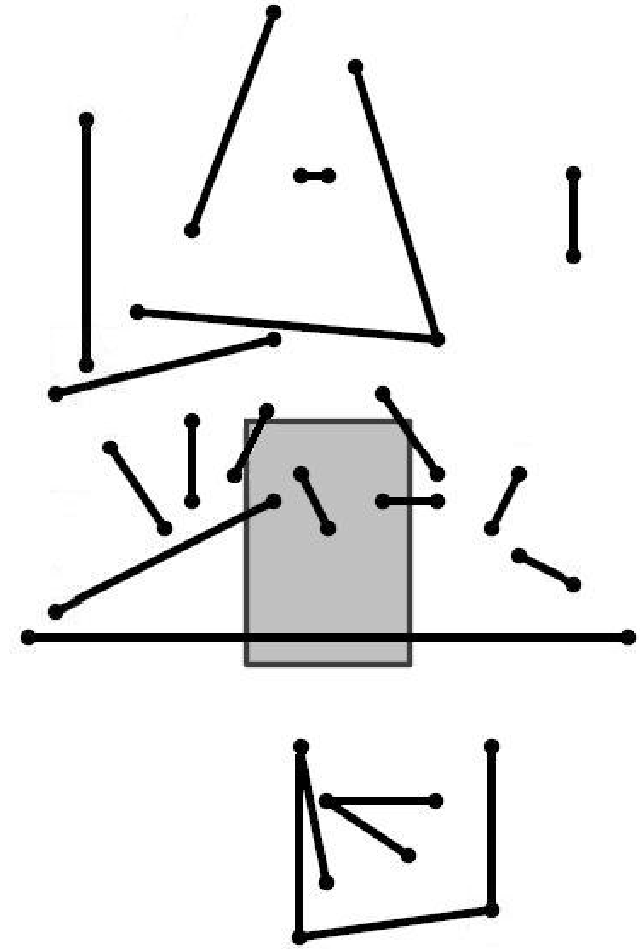
segment tree

By Zohre Akbari
January2014

Arbitrarily oriented segments

Two cases of intersection:

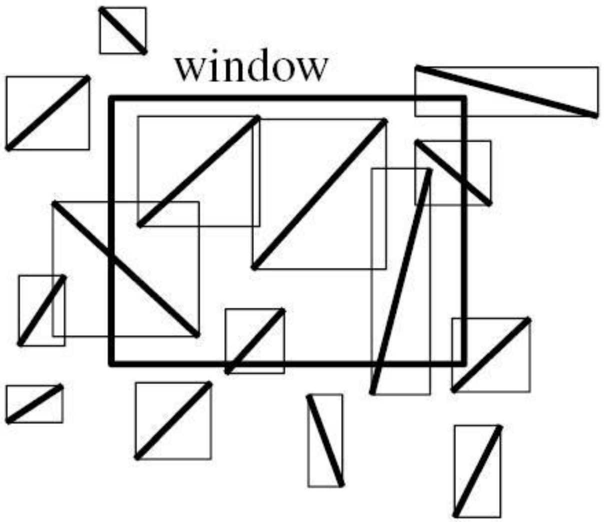
- An endpoint lies inside the query window; solve with range trees
- The segment intersects the window boundary; solve how?



Arbitrarily oriented segments

A simple solution:

- Replace each line segment by

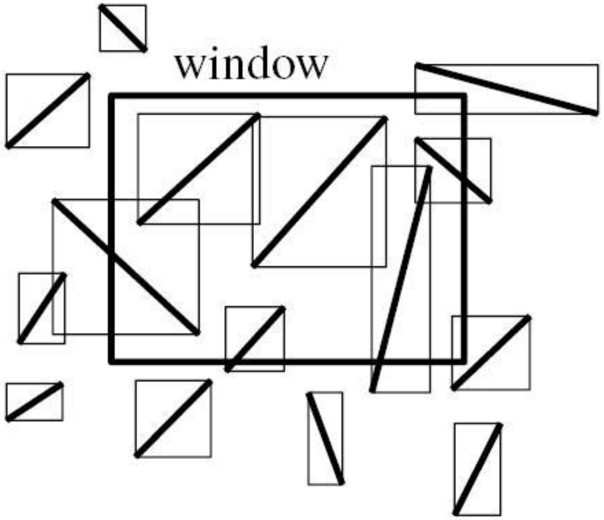


- So we could search in the 4n bounding box sides.

Arbitrarily oriented segments

A simple solution:

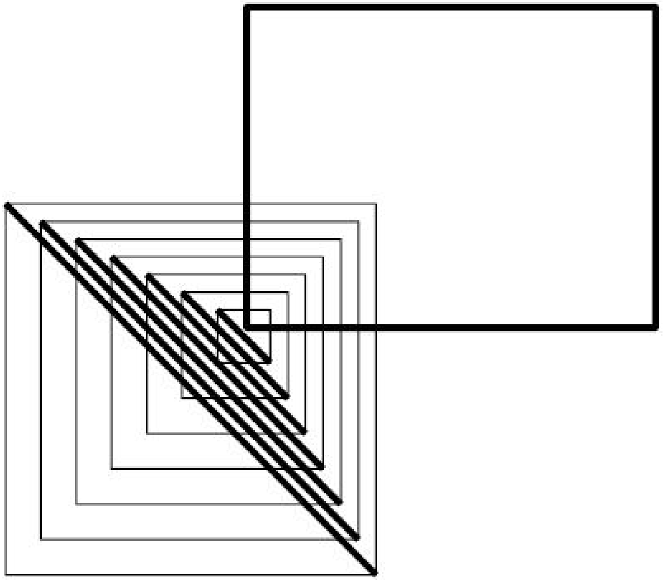
- Replace each line segment by its bounding box.



- So we could search in the $4n$ bounding box sides.

In the worst case:

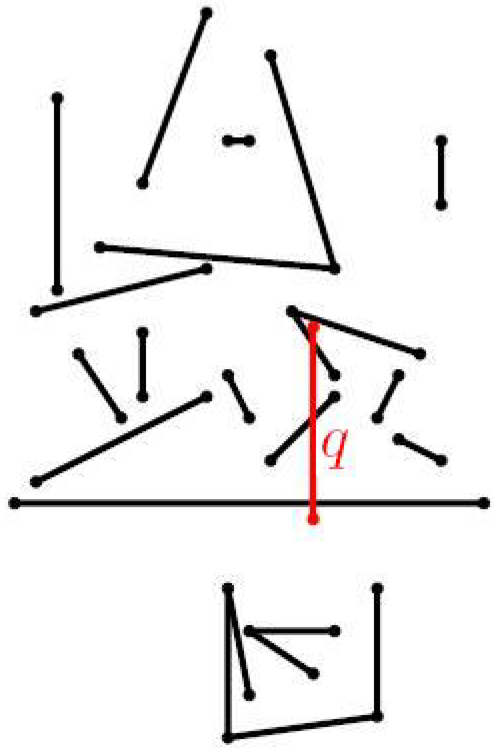
- The solution is quite bad:



- All bounding boxes may intersect W whereas none of the segments do.

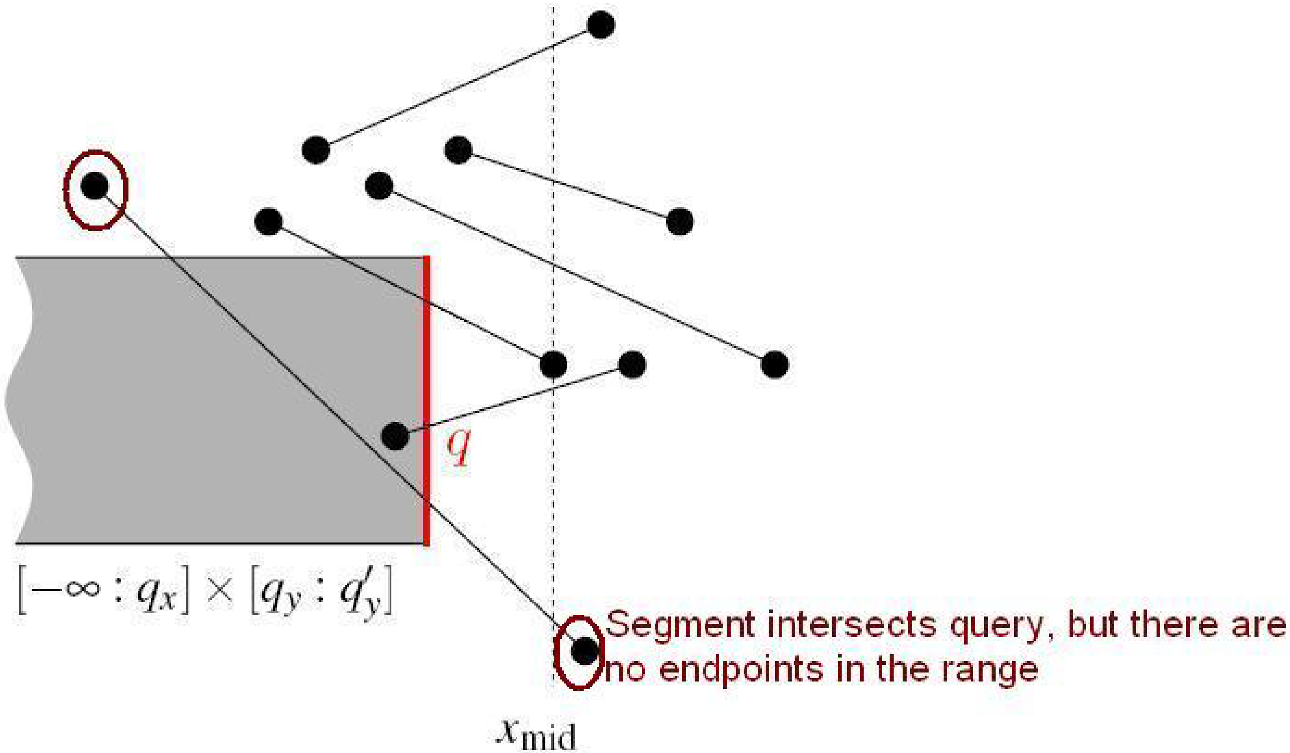
Current problem of our intesect:

Given a set S of line segments with arbitrary orientations in the plane, and we want to find those segments in S that intersect a vertical query segment $q:=q_x \times [q_y : q'_y]$.

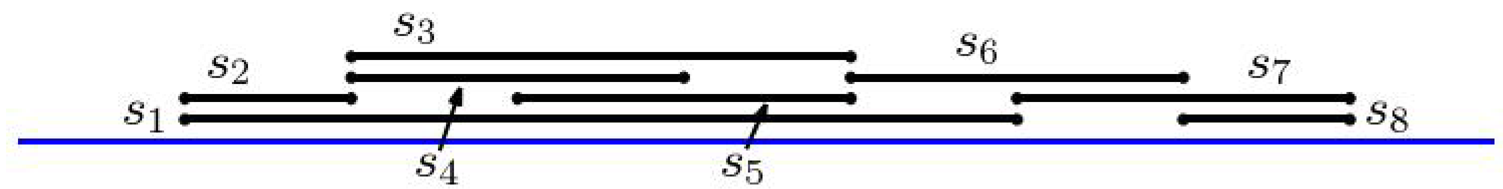


Why don't interval trees work ?

If the segments have arbitrary orientation, knowing that the right endpoint of a segment is to the right of *q* doesn't help us much.



- Given a set $S = \{s_1, s_2, . . . , s_n\}$ of n segments(Intervals) on the real line, preprocess them into a data structure so that the ones containing a query point (value) can be reported efficiently



- The new structure is called the **segment tree**.

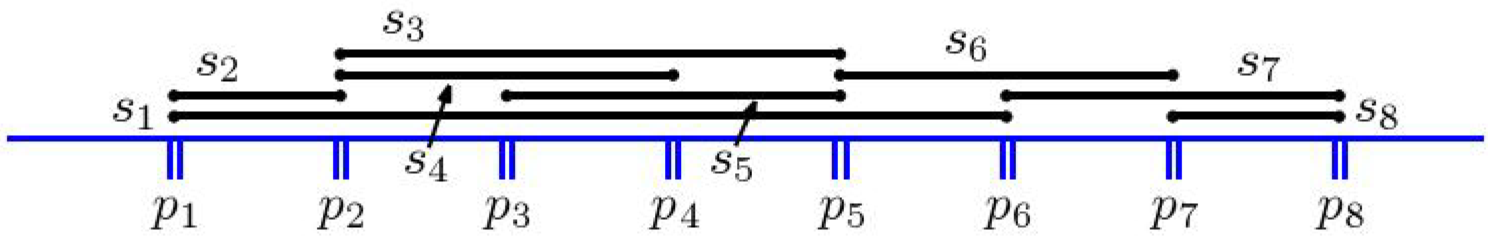
Locus approach

- The locus approach is the idea to partition the parameter space into regions where the answer to a query is the same.

Locus approach

- The locus approach is the idea to partition the parameter space into regions where the answer to a query is the same.

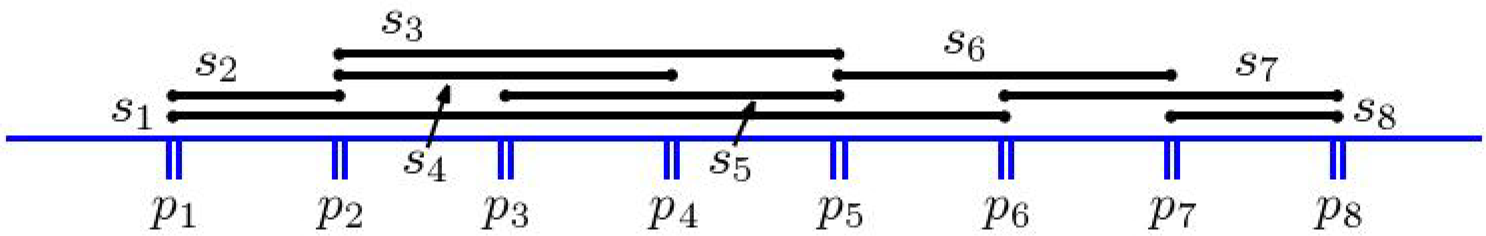
- Our query has only one parameter, q_x, so the parameter space is the real line. Let $p_1, p_2, . . . , p_n$ be the list of distinct interval endpoints, sorted from left to right; $m \leq 2n$



Locus approach

- The locus approach is the idea to partition the parameter space into regions where the answer to a query is the same.

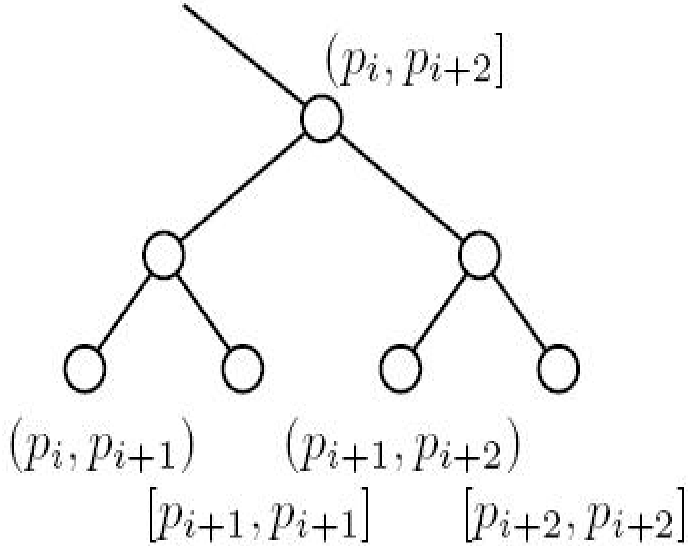
- Our query has only one parameter, q_x, so the parameter space is the real line. . Let $p_1, p_2, \ldots, p_n$ be the list of distinct interval endpoints, sorted from left to right; $m \leq 2n$

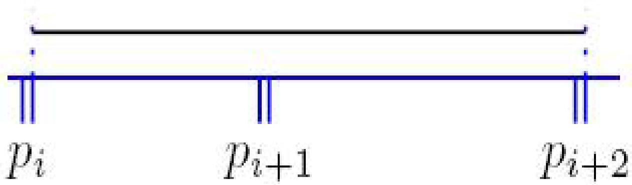


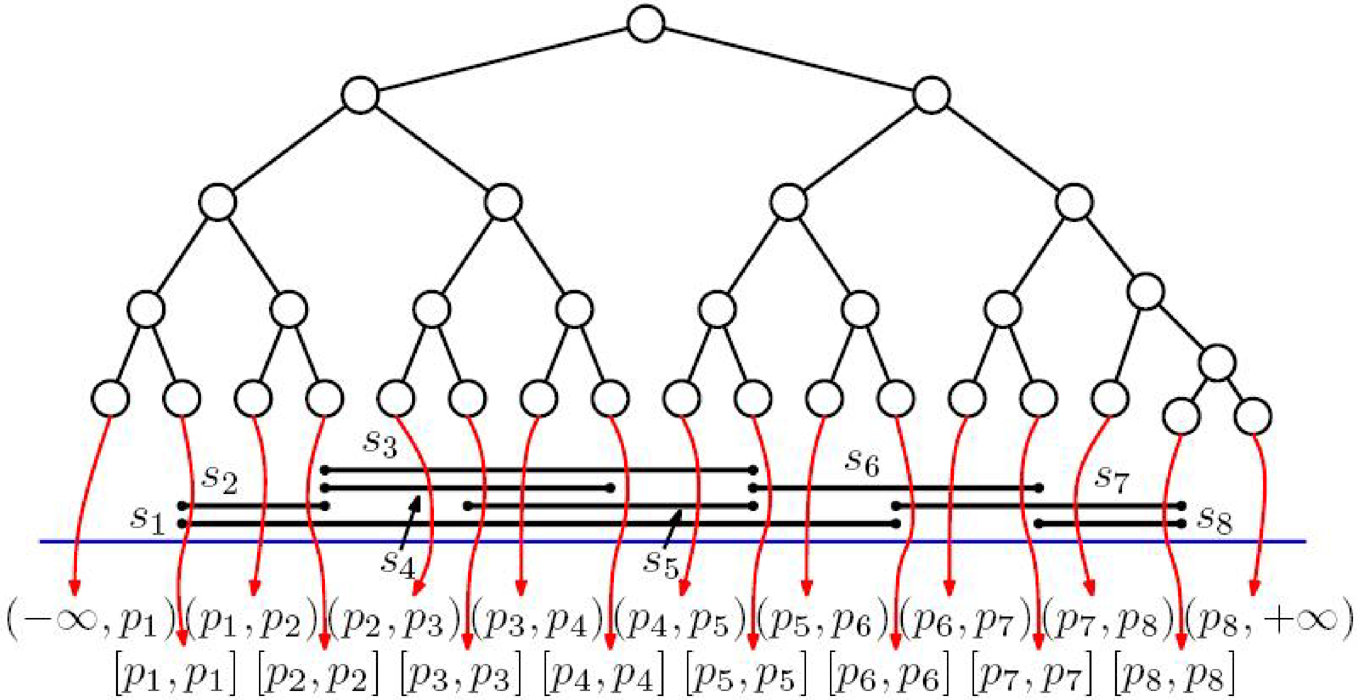
- The real line is partitioned into
- $(-\infty, p_1)$, $[p_1, p_1]$, (p_1, p_2), $[p_2, p_2]$, (p_2, p_3), . . . , $(p_m, +\infty)$, these are called the elementary intervals.

Locus approach

- We could make a binary search tree that has a leaf for every elementary interval.
- We denote the elementary interval corresponding to a leaf μ by $Int(\mu)$.
- all the segments (intervals)in S containing $Int(\mu)$ are stored at the leaf μ
- each internal node corresponds to an interval that is the union of the elementary intervals of all leaves below it





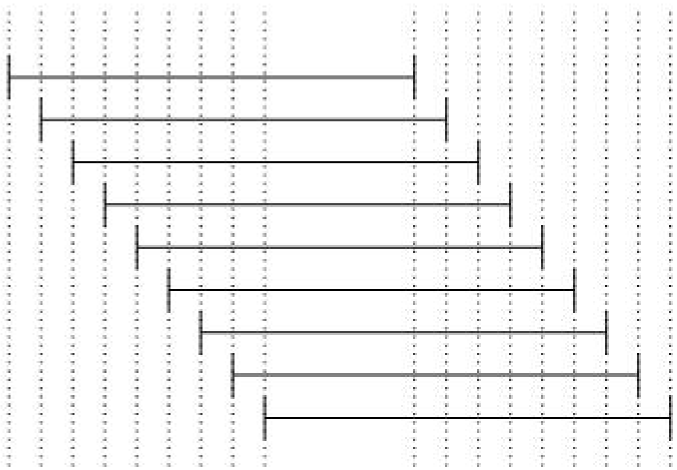

s_1
s_2
s_3
s_4
s_5
s_6
s_7
s_8
$(-\infty, p_1)$ (p_1, p_2) (p_2, p_3) (p_3, p_4) (p_4, p_5) (p_5, p_6) (p_6, p_7) (p_7, p_8) $(p_8, +\infty)$
$[p_1, p_1]$ $[p_2, p_2]$ $[p_3, p_3]$ $[p_4, p_4]$ $[p_5, p_5]$ $[p_6, p_6]$ $[p_7, p_7]$ $[p_8, p_8]$

Query time

We can report the k intervals containing q_x in $O(\log n + k)$ time.

storage

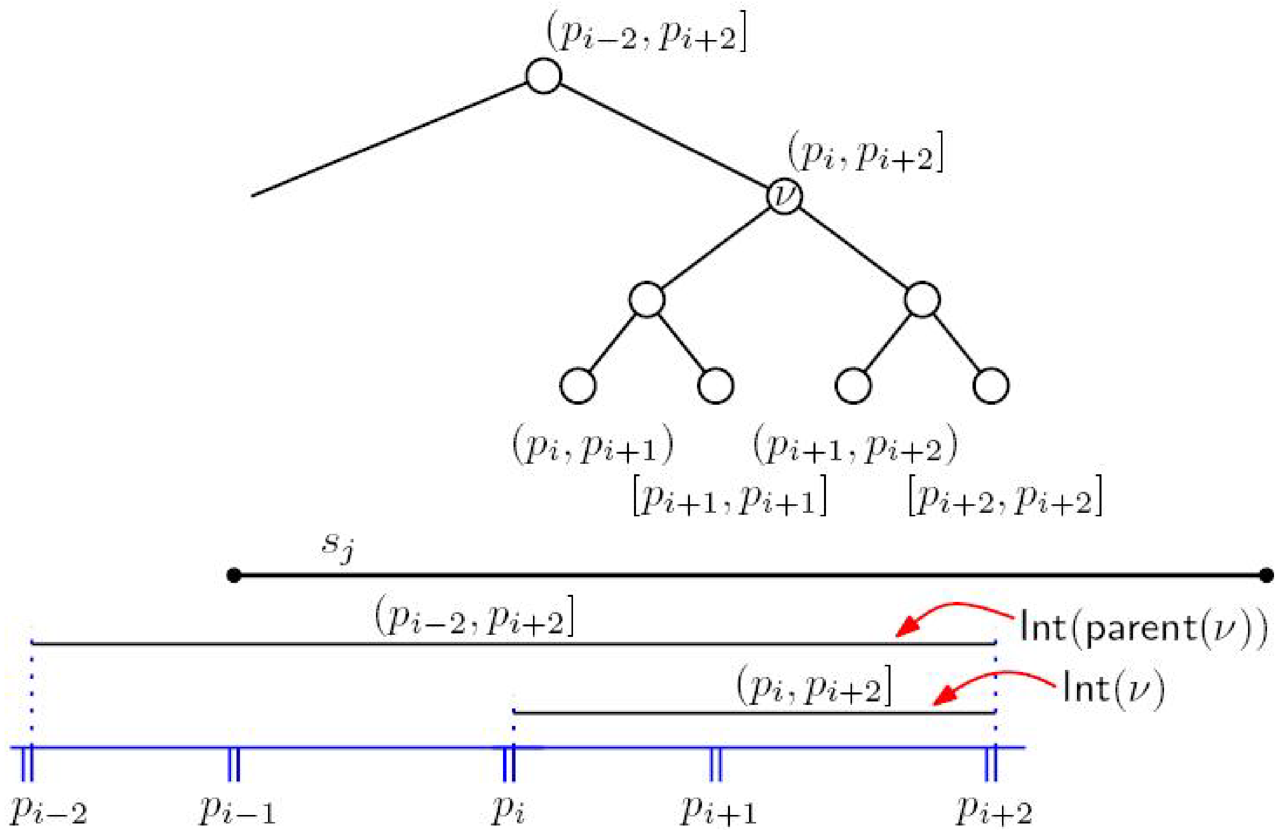
$O(n^2)$ storage in the worst case:



Reduce the amount of storage

- To avoid quadratic storage, we store any segment s_j with v iff

$\boldsymbol{Int(v) \subseteq s_j}$ but $\boldsymbol{Int(parent(v)) \nsubseteq s_j}$.

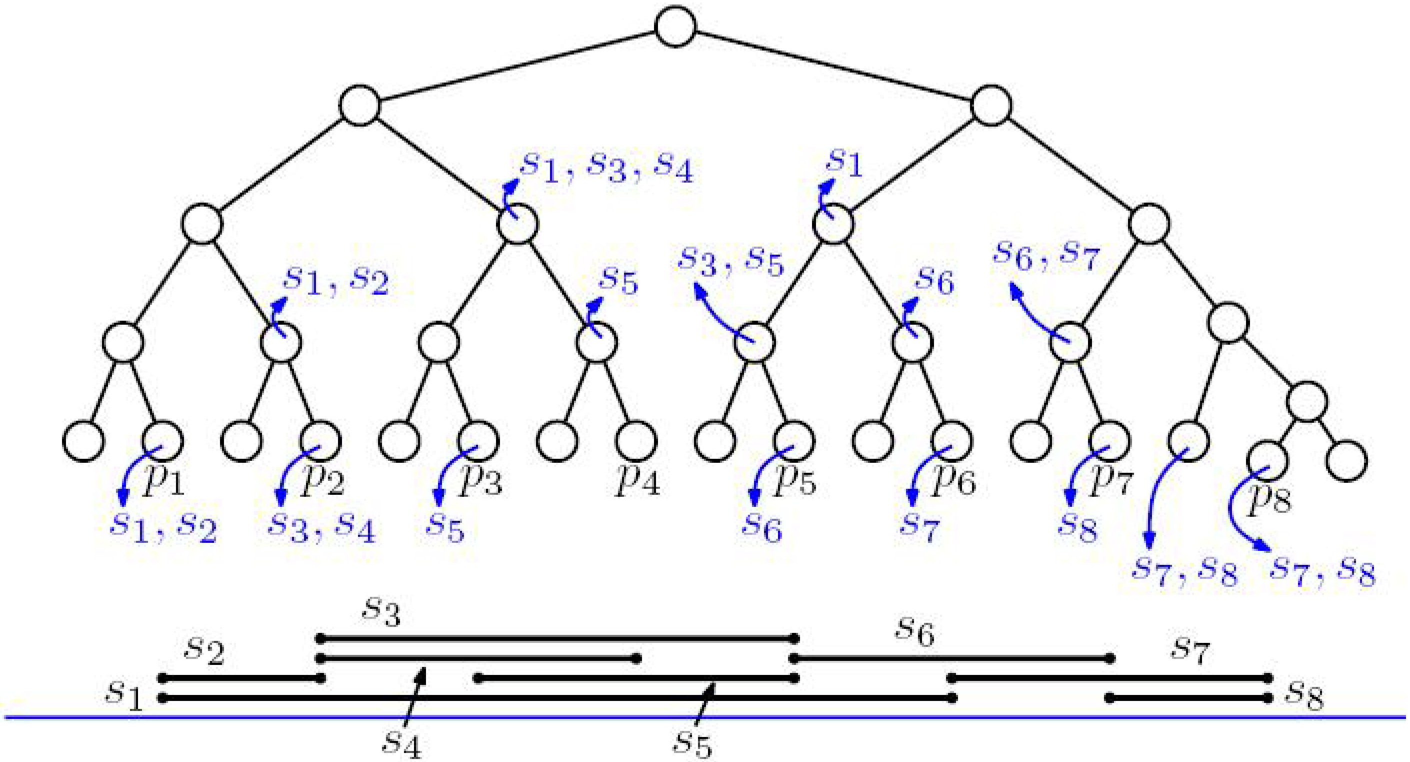


- The data structure based on this principle is called a *segment tree*.

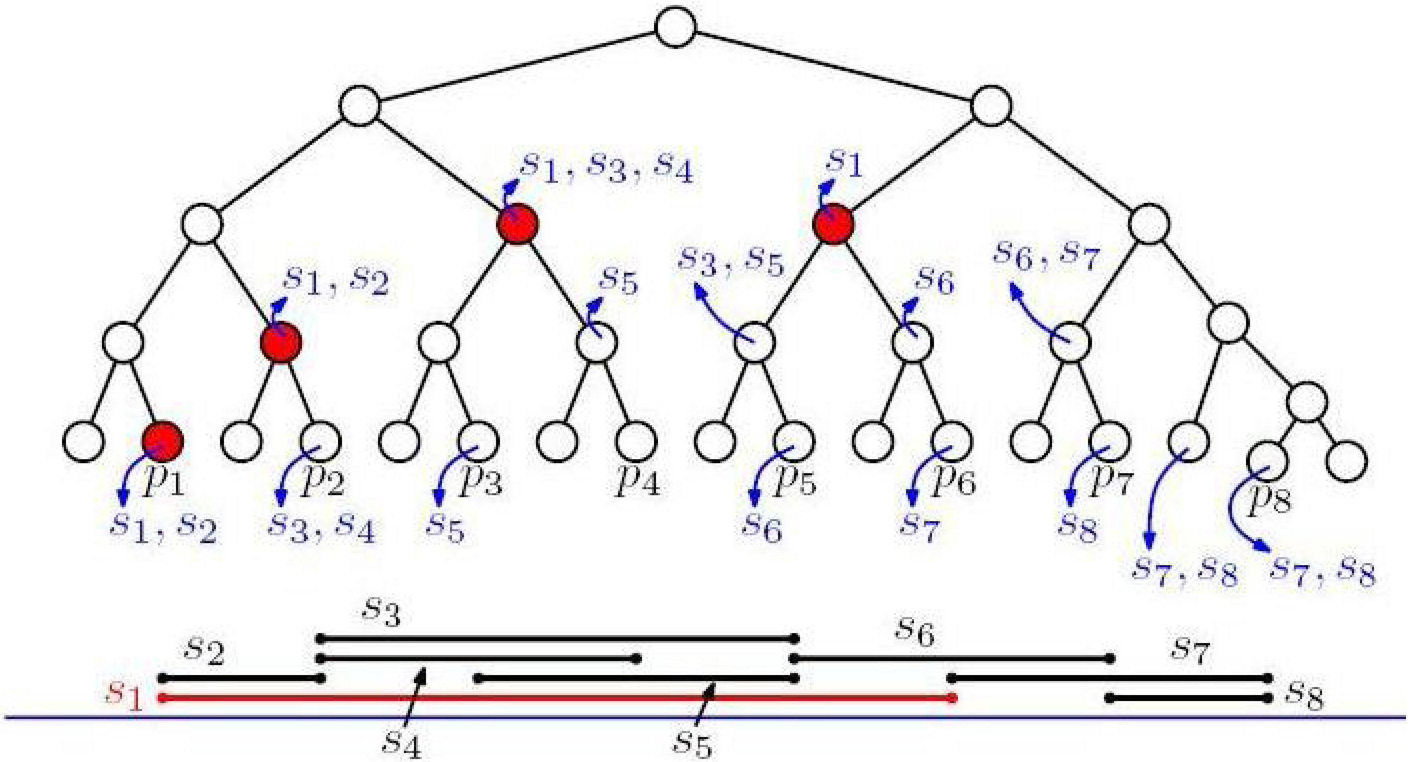
Segment tree

- A segment tree on a set S of segments is a balanced binary search tree on the elementary intervals defined by S, and each node stores its interval, and its canonical subset of S in a list.

- The canonical subset of a node v contains segments s_j such that $\boldsymbol{Int(v)} \subseteq \boldsymbol{s_j}$ but $\boldsymbol{Int(parent(v))} \nsubseteq \boldsymbol{s_j}$.



s_1, s_3, s_4
s_1
s_1, s_2
s_5
s_3, s_5
s_6
s_6, s_7
p_1
p_2
p_3
p_4
p_5
p_6
p_7
p_8
s_1, s_2
s_3, s_4
s_5
s_6
s_7
s_8
s_7, s_8
s_7, s_8
s_3
s_6
s_2
s_7
s_1
s_8
s_4
s_5



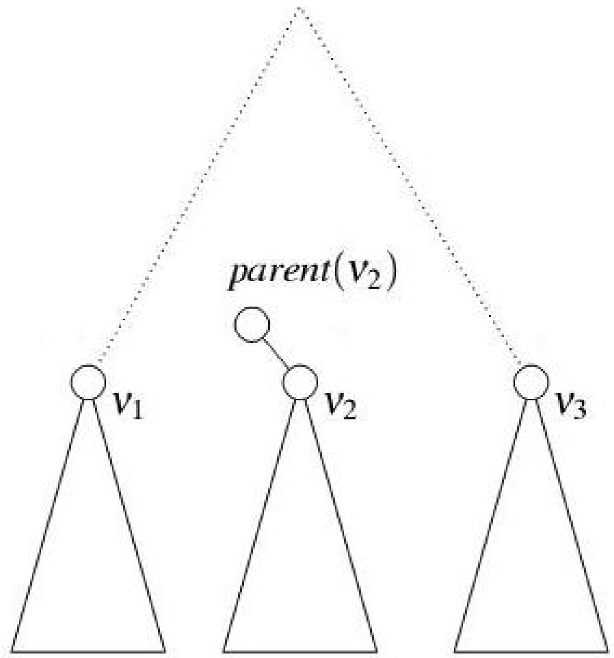
s_1, s_3, s_4
s_1
s_1, s_2
s_5
s_3, s_5
s_6
s_6, s_7
p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8
s_1, s_2
s_3, s_4
s_5
s_6
s_7
s_8
s_7, s_8
s_7, s_8
s_3
s_2
s_6
s_7
s_1
s_8
s_4
s_5

Lemma 10.10

A segment tree on a set of n intervals uses $O(n \log n)$storage.

Proof.

We claim that any segment is stored for at most two nodes at the same depth of the tree.



Query algorithm

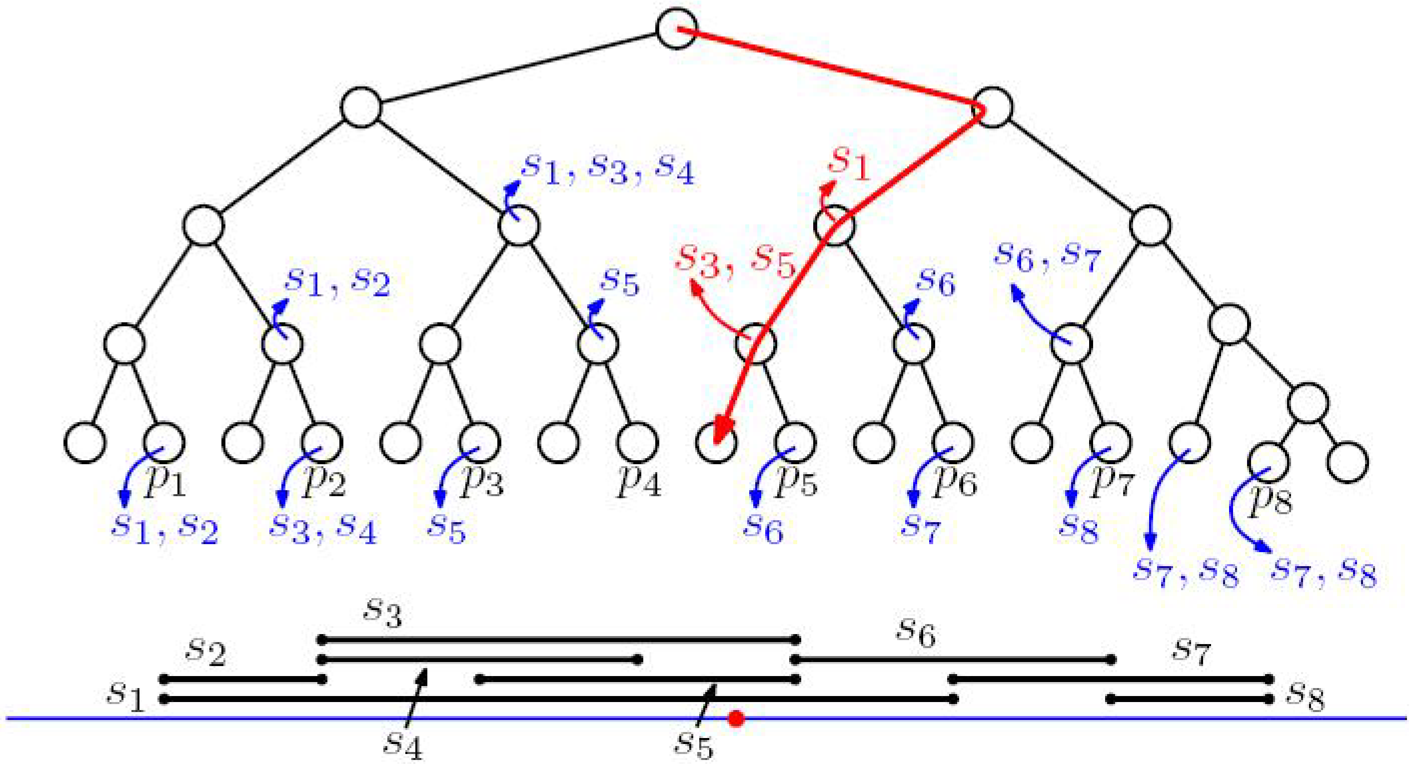
Algorithm QUERYSEGMENTTREE(ν, q_x)

Input. The root of a (subtree of a) segment tree and a query point q_x.

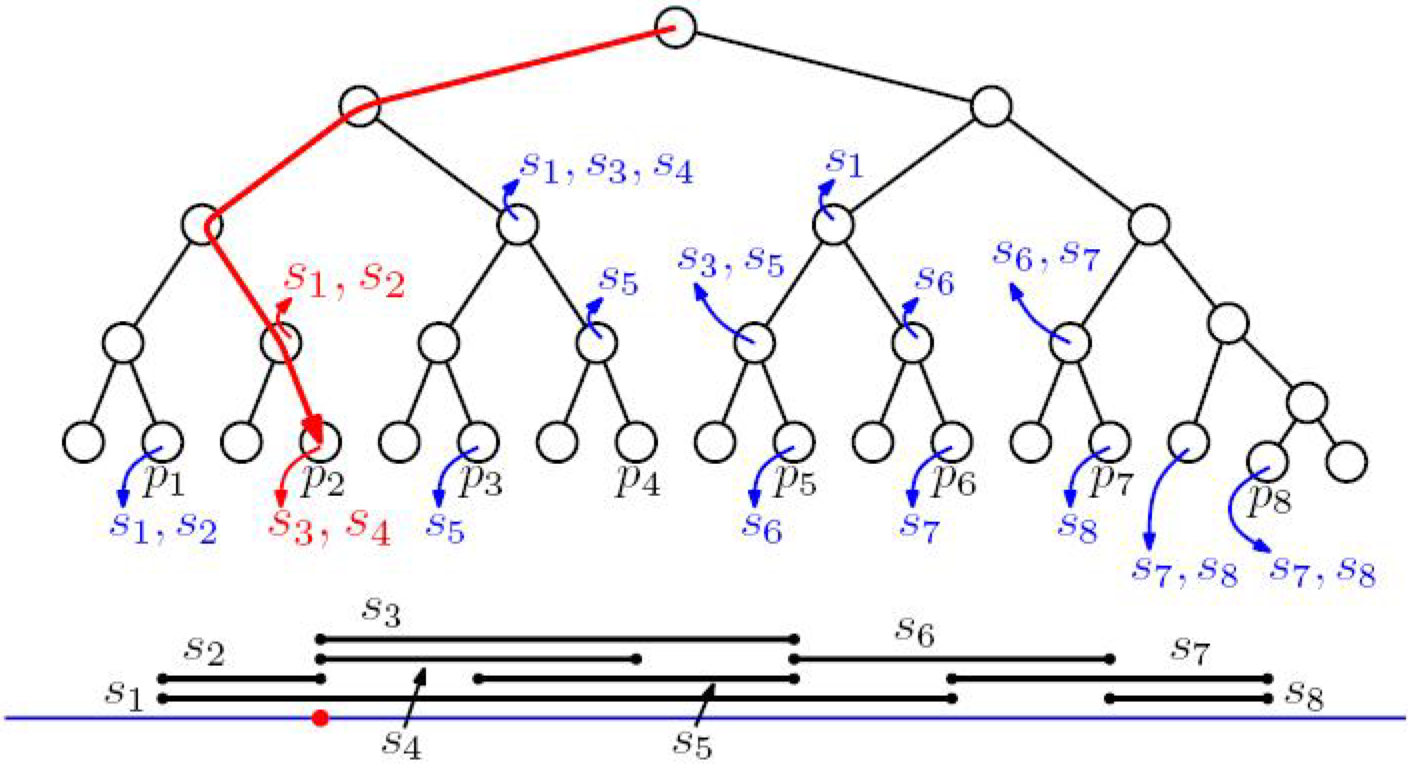
Output. All intervals in the tree containing q_x.

1. Report all the intervals in $I(\nu)$.
2. **if** ν is not a leaf
3. **then if** $q_x \in \mathrm{Int}(lc(\nu))$
4. **then** QUERYSEGMENTTREE($lc(\nu), q_x$)
5. **else** QUERYSEGMENTTREE($rc(\nu), q_x$)

Example query



Example query



Lemma 10.11

- Using a segment tree, the intervals containing a query point q_x can be reported in $O(\log n + k)$ time, where k is the number of reported intervals.

Segment Tree Construction

- Build tree :
- - Sort the endpoints of the segments take $O(n \log n)$ time.This give us the elementary intervals.
- - Construct a balanced binary tree on the elementary intervals,this can be done bottom-up in $O(n)$ time.

Segment Tree Construction

- Build tree :
- - Sort the endpoints of the segments take $O(n \log n)$ time.This give us the elementary intervals.
- - Construct a balanced binary tree on the elementary intervals,this can be done bottom-up in $O(n)$ time.
- Compute the canonical subset for the nodes.To this end we insert the intervals one by one into the segment tree by calling :

Segment Tree Construction

- Build tree :
- \- Sort the endpoints of the segments take $O(n \log n)$ time.This give us the elementary intervals.
- \- Construct a balanced binary tree on the elementary intervals,this can be done bottom-up in $O(n)$ time.
- Compute the canonical subset for the nodes.To this end we insert the intervals one by one into the segment tree by calling :

Algorithm INSERTSEGMENTTREE($\nu, [x : x']$)
Input. The root of a (subtree of a) segment tree and an interval.
Output. The interval will be stored in the subtree.

if $\mathrm{Int}(\nu) \subseteq [x : x']$
    then store $[x : x']$ at ν
    else **if** $\mathrm{Int}(lc(\nu)) \cap [x : x'] \neq \emptyset$
            then INSERTSEGMENTTREE($lc(\nu), [x : x']$)
        if $\mathrm{Int}(rc(\nu)) \cap [x : x'] \neq \emptyset$
            then INSERTSEGMENTTREE($rc(\nu), [x : x']$)

How much time does it take to insert an interval $[x : x']$ into the segment tree?

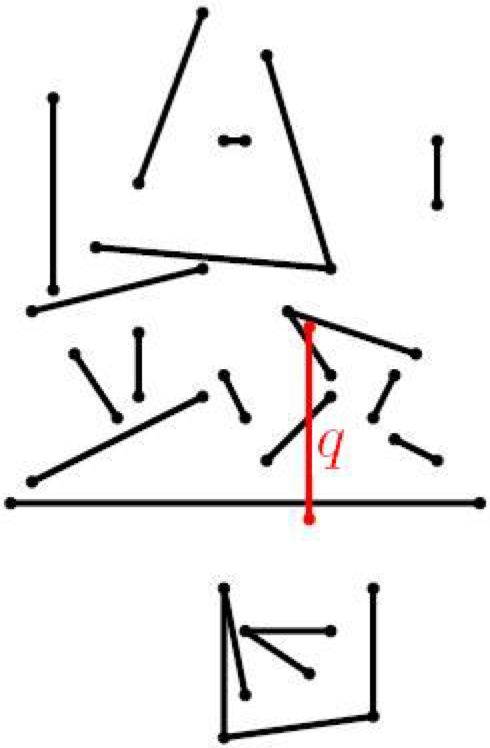
- an interval is stored at most twice at each level of τ
- There is also at most one node at every level whose corresponding interval contains x and one node whose interval contains x'.
- So we visit at most 4 nodes per level.
- Hence, the time to insert a single interval is $O(\log n)$, and the total time to construct the segment tree is $O(n \log n)$.

Theorem 10.12

- A segment tree for a set I of n intervals uses $O(n \log n)$ storage and can be built in $O(n \log n)$ time. Using the segment tree we can report all intervals that contain a query point in $O(\log n + k)$ time, where k is the number of reported intervals.

Back to windowing problem

Let S be a set of arbitrarily oriented, disjoint segments in the plane. We want to report the segments intersecting a vertical query segment $q := q_x \times [q_y : q'_y]$.

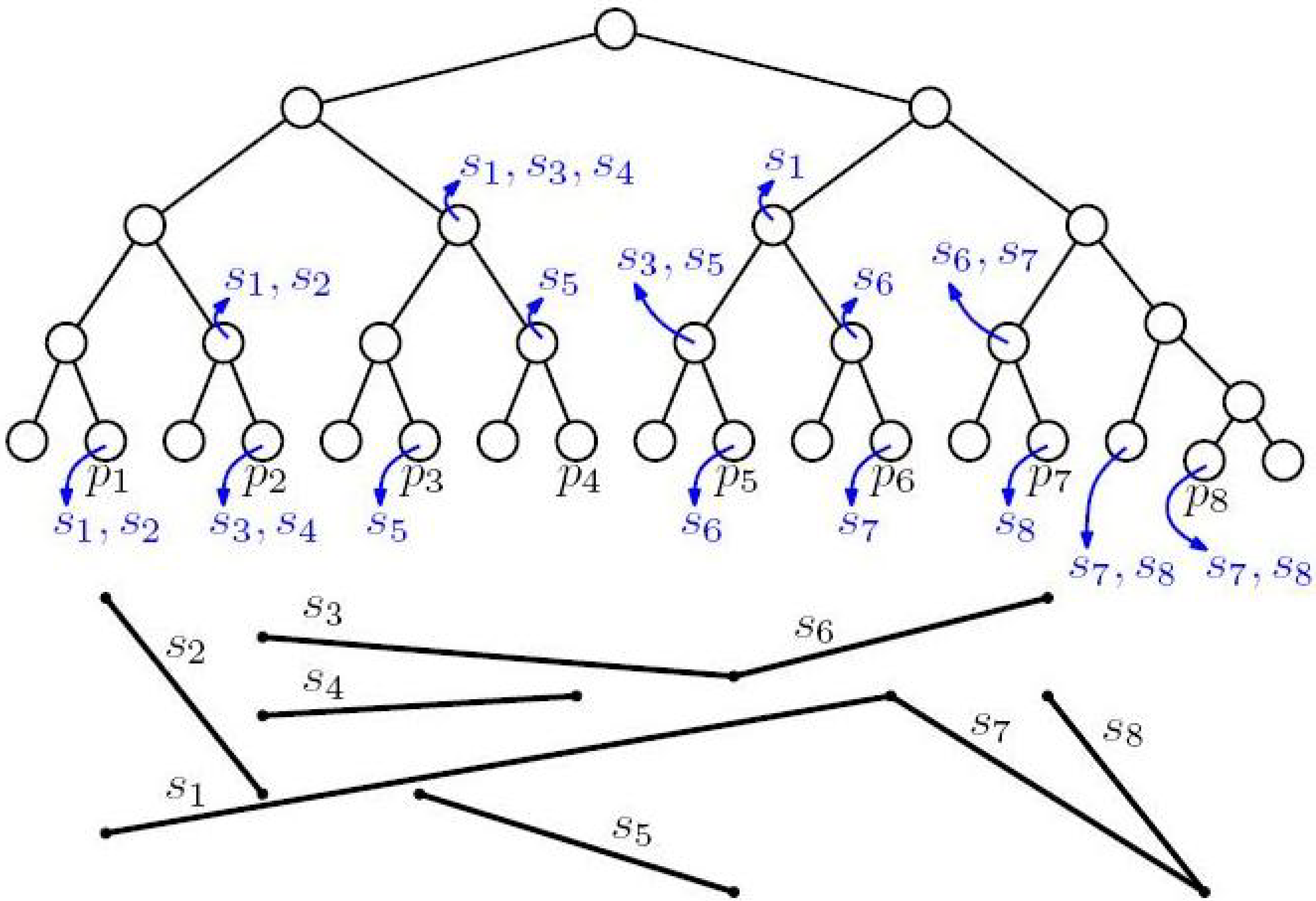


- Build a segment tree $\mathcal{T}$ on the x-intervals of the segments in S.

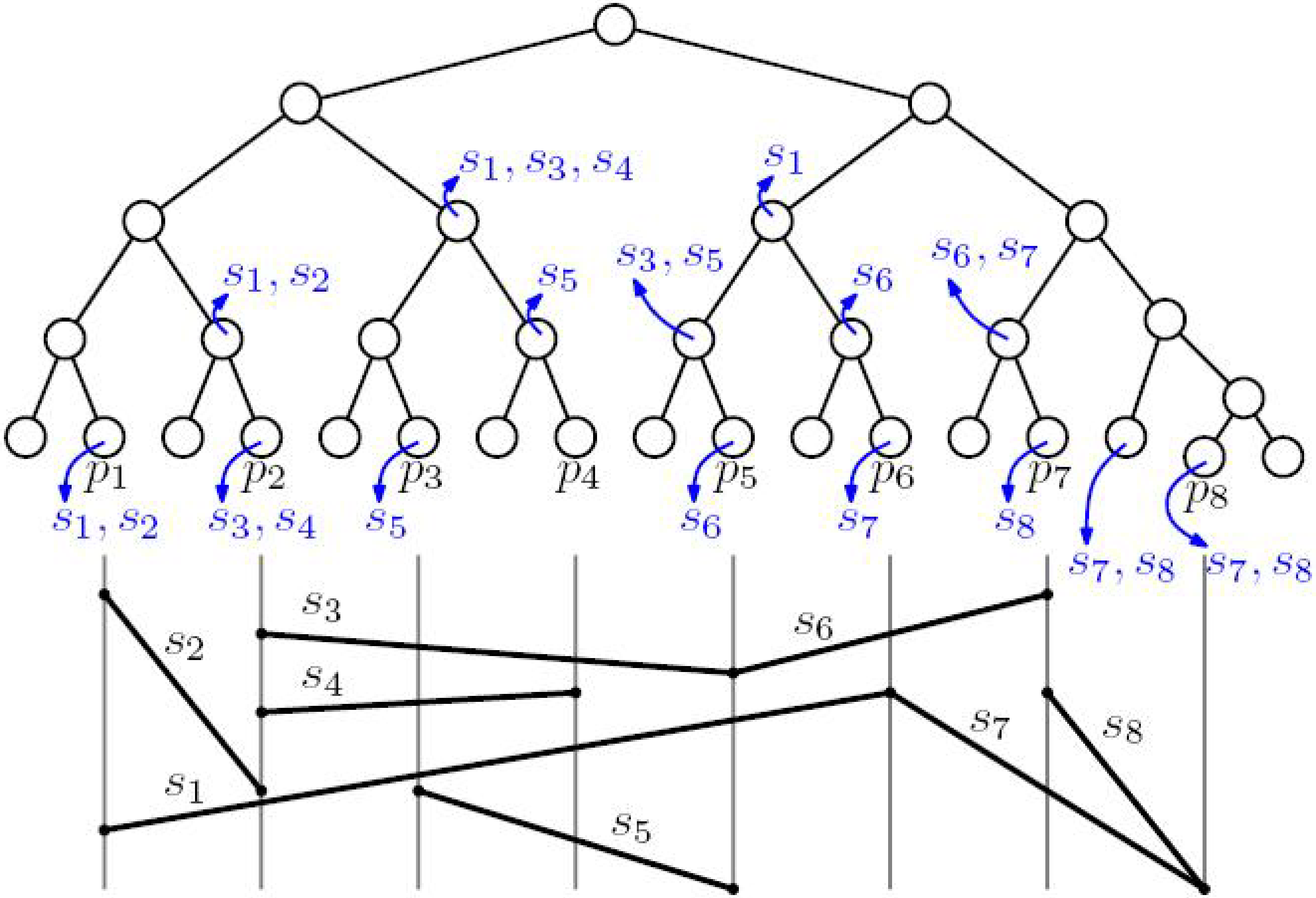
- Build a segment tree $\mathcal{T}$ on the x-intervals of the segments in S.
- A node v in $\mathcal{T}$ can now be considered to correspond to the vertical slab $Int(v) \times (-\infty : +\infty)$.

- Build a segment tree $\mathcal{T}$ on the *x*-intervals of the segments in *S*.
- A node *v* in $\mathcal{T}$ can now be considered to correspond to the vertical slab $Int(v) \times (-\infty : +\infty)$
- A segment s_i is in the canonical subset of *v*, if it crosses the slab of *v* completely, but not the slab of the parent of *v*.

- Build a segment tree T on the x-intervals of the segments in S.
- A node v in T can now be considered to correspond to the vertical slab $Int(v) \times (-\infty : +\infty)$
- A segment s_i is in the canonical subset of v, if it crosses the slab of v completely, but not the slab of the parent of v.
- We denote canonical subset of v with $S(v)$.



s_1, s_3, s_4
s_1
s_1, s_2
s_5
s_3, s_5
s_6
s_6, s_7
p_1
p_2
p_3
p_4
p_5
p_6
p_7
p_8
s_1, s_2
s_3, s_4
s_5
s_6
s_7
s_8
s_7, s_8
s_7, s_8
s_3
s_6
s_2
s_4
s_7
s_8
s_1
s_5

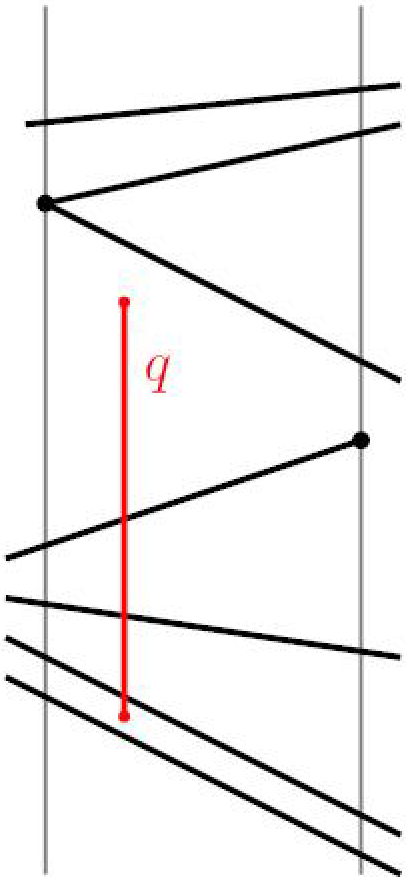


s_1, s_3, s_4
s_1
s_1, s_2
s_5
s_3, s_5
s_6
s_6, s_7
p_1
p_2
p_3
p_4
p_5
p_6
p_7
p_8
s_1, s_2
s_3, s_4
s_5
s_6
s_7
s_8
s_7, s_8
s_7, s_8
s_3
s_6
s_2
s_4
s_7
s_8
s_1
s_5

Querying

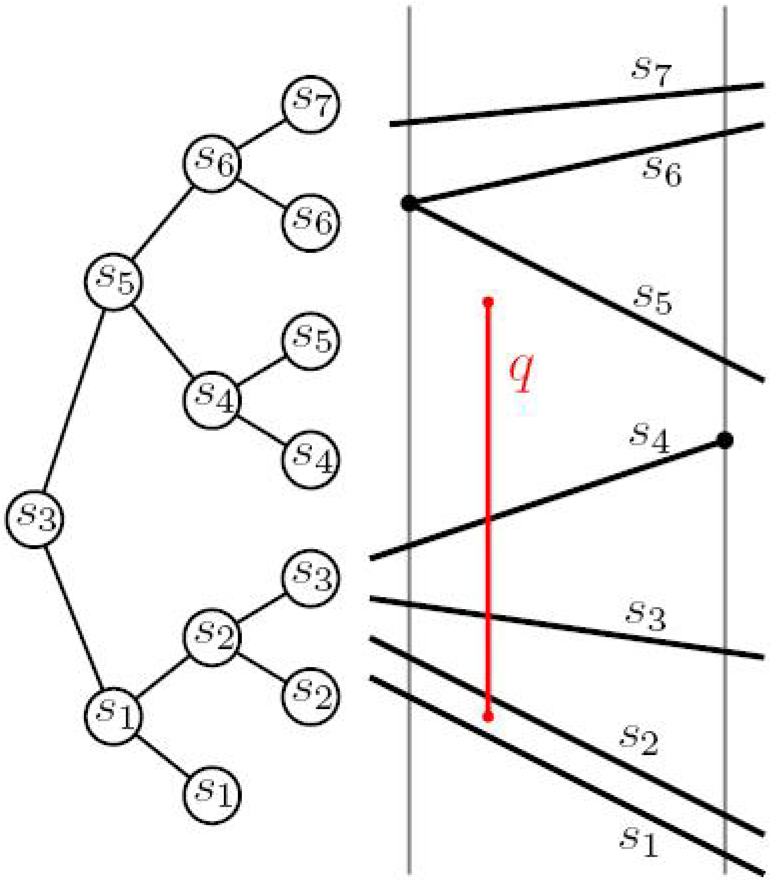
- When we search with q_x in $\mathcal{T}$ we find $O(\log n)$ canonical subsets that collectively contain all the segments whose x-interval contains q_x.

- A segment s in such a canonical subset is intersected by q if and only if the lower endpoint of q is below s and the upper endpointof q is above s



Querying

- segments in the canonical subset $S(v)$ do not intersect each other. This implies that the segments can be ordered vertically.
- we can store $S(v)$ in a search tree $\mathcal{T}(v)$ according to the vertical order.



Query time

- A query with q_x follows one path down the main tree(segment tree)

- And at every node v on the search path we search with endpoints of in $\mathcal{T}(v)$ to report the segments in $S(v)$ intersected by q (a 1-dimensional range query).
- The search in $\mathcal{T}$ (v) takes $O(\log n + k_v)$ time,where k_v is the number of reported segments at (v).
- Hence, the total query time is $O(\log^2 n + k)$.

Storage

- Because the associated structure of any node v uses storage linear in the size of $S(v)$, the total amount of storage remains $O(n \log n)$.

- Data structure can be build in $O(n \log n)$ time.

Theorem 10.13

Let S be a set of n disjoint segments in the plane. The segments intersecting a vertical query segment can be reported in $O(\log^2 n + k)$ time with a data structure that uses $O(n \log n)$ storage, where k is the number of reported segments. The structure can be built in $O(n \log n)$ time.

Corollary 10.14

- Let S be a set of n segments in the plane with disjoint interiors. The segments intersecting an axis-parallel rectangular query window can be reported in $O(\log^2 n + k)$ time with a data structure that uses $O(n \log n)$ storage, where k is the number of reported segments. The structure can be built in $O(n \log n)$ time