Algorithms: Greedy Algorithms

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Greedy Algorithms

- **Greedy algorithms** make decisions that “seem” to be the best following some **greedy** criteria.
- In **Off-Line** problems:
  - The whole input is known in advance.
  - Possible to do some preprocessing of the input.
  - Decisions are irrevocable.
- In **Real-Time** and **On-Line** problems:
  - The present cannot change the past.
  - The present cannot rely on the un-known future.

How and When to use Greedy Algorithms?

- **Initial solution:** Establish trivial solutions for a problem of a small size. Usually $n = 0$ or $n = 1$.
- **Top bottom procedure:** For a problem of size n, look for a greedy decision that reduces the size of the problem to some $k < n$ and then, apply **recursion**.
- **Bottom up procedure:** Construct the solution for a problem of size n based on some greedy criteria applied on the solutions to the problems of size $k = 1, \ldots, n-1$.

The Coin Changing Problem

- **Input:**
  - Integer coin denominations $d_n > \cdots > d_2 > d_1 = 1$.
  - An integer amount to pay: $\mathcal{A}$.
- **Output:** Number of coins n_i for each denomination d_i to get the exact amount.
  - $\mathcal{A} = n_n d_n + n_{n-1} d_{n-1} + n_2 d_2 + n_1 d_1$.
- **Goal:** Minimize total number of coins.
  - $\mathcal{N} = n_n + \cdots + n_2 + n_1$.
- **Remark:** There is always a solution with $\mathcal{N} = \mathcal{A}$ since $d_1 = 1$.

Examples

- **USA:** $d_6 = 100$, $d_5 = 50$, $d_4 = 25$, $d_3 = 10$, $d_2 = 5$, $d_1 = 1$.
  - $\mathcal{A} = 73 = 2 \cdot 25 + 2 \cdot 10 + 3 \cdot 1$.
  - $\mathcal{N} = 2 + 2 + 3 = 7$.

- **Old British:** $d_3 = 240$, $d_2 = 20$, $d_1 = 1$.
  - $\mathcal{A} = 307 = 1 \cdot 240 + 3 \cdot 20 + 7 \cdot 1$.
  - $\mathcal{N} = 1 + 3 + 7 = 11$.

Greedy Solution

- **Idea:** Use the largest possible denomination and update $\mathcal{A}$.
- **Implementation:**

  Coin-Changing($d_n > \cdots > d_2 > d_1 = 1$)
      for $i = n$ **downto** 1
          $n_i = \lfloor \mathcal{A}/d_i \rfloor$
          $\mathcal{A} = \mathcal{A} \bmod d_i = \mathcal{A} - n_i d_i$
      Return($\mathcal{N} = n_n + \cdots + n_2 + n_1$)

- **Correctness:** $\mathcal{A} = n_n d_n + n_{n-1} d_{n-1} + n_2 d_2 + n_1 d_1$.
- **Complexity:** $\Theta(n)$ division and mod integer operations.

Optimality

- **Greedy** is optimal for the USA system.

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- A coin system for which **Greedy** is not optimal:
  - $d_3 = 4$, $d_2 = 3$, $d_1 = 1$ and $\mathcal{A} = 6$:
  - **Greedy:** $6 = 1 \cdot 4 + 2 \cdot 1 \quad \Rightarrow \quad \mathcal{N} = 3$.
  - **Optimal:** $6 = 2 \cdot 3 \quad \Rightarrow \quad \mathcal{N} = 2$.

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  - **Optimal:** $6 = 2 \cdot 3 \quad \Rightarrow \quad \mathcal{N} = 2$.

- A coin system for which **Greedy** is very “bad”:
  - $d_3 = x + 1$, $d_2 = x$, $d_1 = 1$ and $\mathcal{A} = 2x$:
  - **Greedy:** $2x = 1 \cdot (x + 1) + (x - 1) \cdot 1 \quad \Rightarrow \quad \mathcal{N} = x$.
  - **Optimal:** $2x = 2 \cdot x \quad \Rightarrow \quad \mathcal{N} = 2$.

Efficiency

- **Optimal solution:** Check all possible combinations.
  - Not a polynomial time algorithm.
- **Another optimal solution:** Polynomial in both n and $\mathcal{A}$.
  - Not a strongly polynomial time algorithm.
- **Objective:**
  - Find a solution that is polynomial only in n.
  - Probably impossible!?

The Knapsack Problem

- **Input:**
  - A thief enters a store and finds n items $I_1, \ldots, I_n$.
  - The value of item I_i is $v(I_i)$ and its weight is $w(I_i)$.
    - Both are positive integers.
  - The thief can carry at most weight W.
  - The thief either takes all of item I_i or doesn't take item I_i.

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  - The thief either takes all of item I_i or doesn't take item I_i.

- **Goal:** Carry items with maximum total value.
  - Which are these items?
  - What is their total value?

A General Greedy Scheme

- **Order** the items according to some **greedy criterion**.
  - Assume this order is $J_1, J_2, \ldots, J_n$.
  - Assume J_1 is the most desired item and J_n is the least desired item.
- If J_1 is not too heavy ($w(J_1) \leq W$):
  - **Take** item J_1.
  - **Continue recursively** with $J_2, J_3, \ldots, J_n$ and updated maximum weight $W - w(J_1)$.
- If J_1 is too heavy ($w(J_1) > W$):
  - **Ignore** item J_1.
  - **Continue recursively** with $J_2, J_3, \ldots, J_n$ and the same maximum weight W.

A General Greedy Scheme – Implementation

Non-Recursive Knapsack$(I_1, \ldots, I_n, w(\cdot), v(\cdot), W)$

  Let $J_1, \ldots, J_n$ be the new order on the items.
  $S = \emptyset$ **(* the set of items the thief takes *)**
  $V = 0$ **(* the value of these items *)**
  for $i = 1$ **to** n
    if $w(J_i) \leq W$ **then**
      $S = S \cup \{J_i\}$
      $V = V + v(J_i)$
      $W = W - w(J_i)$
  Return(S, V)

Greedy Criteria

- **Greedy criterion I:** Order the items by their **value** from the **most expensive** to the **cheapest**.
- **Greedy criterion II:** Order the items by their **weight** from the **lightest** to the **heaviest**.
- **Greedy criterion III:** Order the items by their **ratio** of value over weight from the **largest** ratio to the **smallest** ratio.

The three criteria are not optimal

- **Counter example for Greedy-by-Value and Greedy-by-Ratio:**
  - 3 items and maximum weight is $W = 10$. Weights and values are: $I_1 = \langle 6, 10 \rangle$, $I_2 = \langle 5, 6 \rangle$, and $I_3 = \langle 5, 6 \rangle$.
  - **Optimal** takes items I_2 and I_3 for a profit of 12.
  - **Greedy-by-Value** or **Greedy-by-Ratio** take only item I_1 for a profit of 10.

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  - **Optimal** takes items I_2 and I_3 for a profit of 12.
  - **Greedy-by-Value** or **Greedy-by-Ratio** take only item I_1 for a profit of 10.

- **Counter example for Greedy-by-Weight:**
  - 3 items and maximum weight is $W = 10$. Weights and values are: $I_1 = \langle 6, 13 \rangle$, $I_2 = \langle 5, 6 \rangle$, and $I_3 = \langle 5, 6 \rangle$.
  - **Optimal** takes only item I_1 for a profit of 13.
  - **Greedy-by-Weight** takes items I_2 and I_3 for a profit of 12.

Very Bad Counter Examples for Criteria I and II

- **Counter example for Greedy-by-Value:**
  - n items and maximum weight is W. Weights and values are: $I_1 = \langle W, 2 \rangle, I_2 = \langle 1, 1 \rangle, \ldots, I_3 = \langle 1, 1 \rangle$.
  - **Optimal** takes items $I_2, \ldots, I_n$ for a profit of $n - 1$.
  - **Greedy-by-Value** takes only item I_1 for a profit of 2.
  - The ratio is $(n - 1)/2$.

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  - **Greedy-by-Value** takes only item I_1 for a profit of 2.
  - The ratio is $(n-1)/2$.

- **Counter example for Greedy-by-Weight:**
  - 2 items and maximum weight is 2. Weights and values are: $I_1 = \langle 1, 1 \rangle$ and $I_2 = \langle 2, x \rangle$ for a very large x.
  - **Optimal** takes item I_2 for a profit of x.
  - **Greedy-by-Weight** takes item I_1 for a profit of 1.
  - The ratio is x.

A Very Bad Counter Example for Criterion III

- **Counter example for Greedy-by-Ratio:**
  - 2 items and maximum weight is W. Weights and values are: $I_1 = \langle 1, 2 \rangle$ and $I_2 = \langle W, W \rangle$.
  - **Optimal** takes items I_2 for a profit of W.
  - **Greedy-by-Ratio** takes item I_1 for a profit of 2.
  - The ratio is almost $\frac{W}{2}$.

A Very Bad Counter Example for Criterion III

- **Counter example for Greedy-by-Ratio:**
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  - **Optimal** takes items I_2 for a profit of W.
  - **Greedy-by-Ratio** takes item I_1 for a profit of 2.
  - The ratio is almost $\frac{W}{2}$.

- **A $1/2$ guaranteed approximation algorithm:**
  - **Greedy-by-Ratio** guarantees half of the profit of **Optimal** with a **tweak**.
  - Select either the output of greedy or the one item with the maximum value whose weight is at most W.

The Fractional Knapsack Problem

- The thief can take portions of items.
- If the thief takes a fraction $0 \leq p_i \leq 1$ of item I_i:
  - Its value is $p_i v(I_i)$.
  - Its weight is $p_i w(I_i)$.

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  - Its weight is $p_i w(I_i)$.

- **Theorem: Greedy-by-Ratio** is **optimal**.

Proof

- Assume that **Greedy-by-Ratio** fails on the input $I_1, \ldots, I_n$ and the weight W.
- Let the portions taken by **Optimal** be $p_1, \ldots, p_n$.
  - $p_i = 1$: all of item I_i is taken.
  - $p_i = 0$: none of item I_i is taken.
  - $0 < p_i < 1$: some but not all of item I_i is taken.
- Since **Greedy-by-Ratio** fails, there exist I_i and I_j such that:
  - $\frac{v(I_i)}{w(I_i)} > \frac{v(I_j)}{w(I_j)}$ and $p_i < 1$ and $p_j > 0$.
- Because each unit of weight of item I_i has more value than each unit of weight of item I_j, it is more profitable to take more of item I_i and less of item I_j.
- A **contradiction** to the optimality of **Optimal**.

The $0-1$ Knapsack Problem

- **Optimal solution:** Check all possible sets of items.
  - Not a polynomial time algorithm.
- **Another optimal solution:** Polynomial in both n and W.
  - Not a strongly polynomial time algorithm.
- **Objective:**
  - Find a solution that is polynomial only in n.
  - Probably impossible!?
  - However, **Greedy-by-Ratio** produces "good" solutions.

The Activity-Selection Problem

- **Input:**
  - Activities $A_1, \ldots, A_n$ that need the service of a common resource.
  - Activity A_i is associated with a time interval $[s_i, f_i)$ for $s_i < f_i$.
    - A_i needs the service from time s_i until just before time f_i.
- **Mutual Exclusion:** The resource serves at most one activity at any time.
- **Definition:** A_i and A_j are compatible if either $f_i \leq s_j$ or $f_j \leq s_i$.
- **Goal:** Find a maximum size set of compatible activities.

Example

- **Input:** 3 activities $A_1 = [1, 4)$, $A_2 = [3, 6)$, $A_3 = [5, 8)$.

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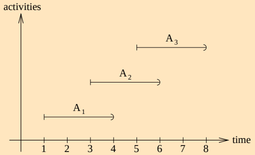
- **A graphical representation:**



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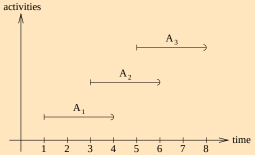
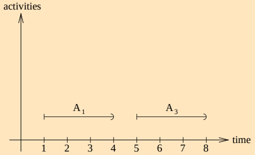


- **The best solution:**



Static vs. Dynamic Greedy

- **Static:** The **greedy** criterion is determined in advance and cannot be changed during the execution of the algorithm.
- **Dynamic:** The **greedy** criterion may be modified during the execution of the algorithm based on prior decisions.
- **Remark:** A static criterion is also a dynamic criterion.

A General Static Greedy Scheme

- **Maintain** a set $\mathcal{S}$ of the activities that have been selected so far.
- Initially, $\mathcal{S} = \emptyset$ and at the end, *S* is an optimal solution.
- **Order** the activities following some greedy criterion and **consider** the activities according to this order.
- Let *A* be the current considered activity. **If** *A* is compatible with all the activities in *S*:
  - **Then add** *A* to $\mathcal{S}$.
  - **Else ignore** *A*.
- **Continue** until there are no activities to consider

A General Dynamic Greedy Scheme

- **Maintain** two sets of activities:
  - $\mathcal{S}$ those that have been selected so far.
  - $\mathcal{R}$ those that can still be selected.
  - Initially, $\mathcal{S} = \emptyset$ and $\mathcal{R} = \{A_1, \ldots, A_n\}$.
  - At the end, S is an **optimal** solution and $R = \emptyset$.
- **Select** a "good" activity A from $\mathcal{R}$, following some greedy criterion.
- **Add** A to $\mathcal{S}$.
- **Delete** from $\mathcal{R}$ the activities that are not compatible with activity A.
- **Continue** until $\mathcal{R}$ is empty.

Greedy Criteria

- **Four criteria:**
  - **Prefer short** activities.
  - **Prefer** activities intersecting **few** other activities.
  - **Prefer** activities that start **earlier**.
  - **Prefer** activities that terminate **earlier**.
- **Optimality:** Only the fourth criterion is **optimal**.
- **Remarks:**
  - All four criteria are static in their nature.
  - The second criterion has a dynamic version.

An Optimal Greedy Solution

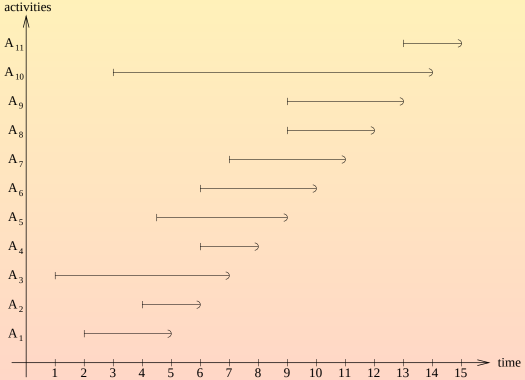
Preprocessing$(A_1, \ldots, A_n)$
Sort the activities according to their finish time
Let this order be $A_1, \ldots, A_n$ (*$i < j \Rightarrow f_i \leq f_j$ *)

Greedy-Activity-Selector$(A_1, \ldots, A_n)$
$\mathcal{S} = \{A_1\}$ (* A_1 terminates the earliest *)
$j = 1$ (* A_j is the current selected activity *)
for $i = 2$ **to** n (* scan all the activities *)
if $s_i \geq f_j$ (* check compatibility *)
then (* select A_i that is compatible with $\mathcal{S}$ *)
$\mathcal{S} = \mathcal{S} \cup \{A_i\}$
$j = i$
else (* A_i is not compatible *)
Return$(\mathcal{S})$

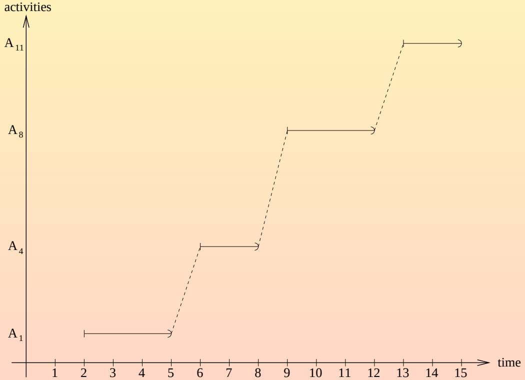
Correctness and Complexity

- **Correctness:** By definition.
- **Complexity:**
  - The sorting can be done in $O(n \log n)$ time.
  - There are $O(1)$ operations per each activity.
  - All together: $O(n \log n) + n \cdot O(1) = O(n \log n)$ time.

Example - Input



Example - Output



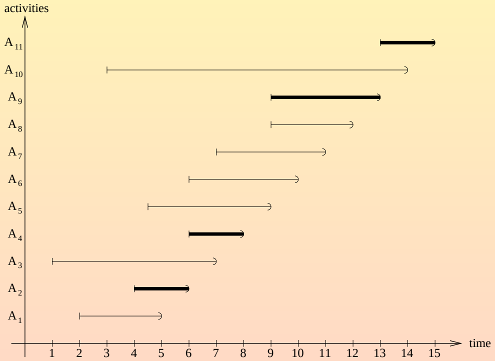
Optimality

- Let $\mathcal{T}$ be an optimal set of activities.
- Transform $\mathcal{T}$ to $\mathcal{S}$ preserving the size of $\mathcal{T}$.
- Let $A_1, \ldots, A_n$ be ordered by their *finish* time.
- Let A_i be the first activity that is in $\mathcal{T}$ and not in $\mathcal{S}$.
- All the activities in $\mathcal{T}$ that finish before A_i are also in $\mathcal{S}$.

Optimality

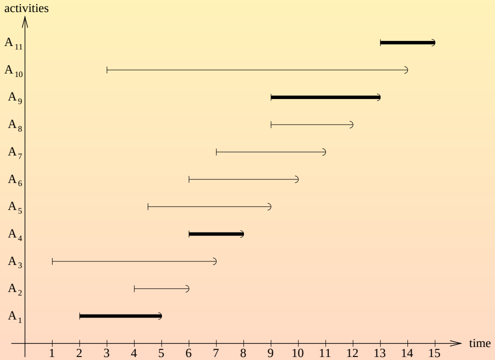
- $A_i \notin \mathcal{S} \Rightarrow \exists A_j \in \mathcal{S}$ that is not in $\mathcal{T}$ in which $j < i$.
- A_j is compatible with all the activities in $\mathcal{T}$ that finish before it since they are all in $\mathcal{S}$.
- A_j is compatible with all the activities in $\mathcal{T}$ that finish after A_i since it finishes before A_i.
- Therefore, $\mathcal{T} \cup \{A_j\} \setminus \{A_i\}$ is a solution with the same size as $\mathcal{T}$ and hence optimal.
- Continue this way until $\mathcal{T}$ becomes $\mathcal{S}$.

Example



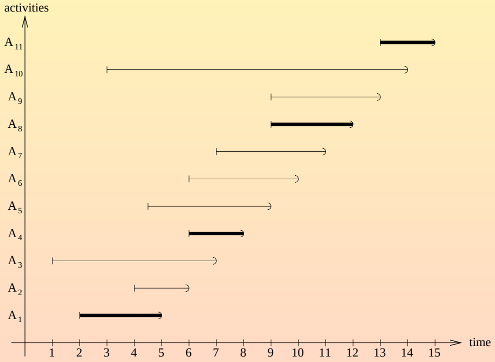
Another optimal solution with 4 activities.

Example



A third optimal solution: after the first transformation.

Example



The greedy solution: after the second transformation.

Huffman Codes

- **Input:**
  - An alphabet of n symbols $a_1, \ldots, a_n$.
  - A frequency f_i for each symbol a_i:
    - $\sum_{i=1}^{n} f_i = 1$.
  - A File $\mathcal{F}$ containing L symbols from the alphabet.
    - a_i appears exactly $n_i = f_i \cdot L$ times in $\mathcal{F}$.

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- **Output:**
  - For symbol a_i, $1 \leq i \leq n$: A binary codeword w_i of length ℓ_i.
  - A compressed (encoded) binary file $\mathcal{F}'$ of $\mathcal{F}$.

Huffman Codes – Goals

- L' the length of $\mathcal{F}'$ should be minimal.
- An efficient algorithm to find the n codewords.
  - Good polynomial running time: ($O(n \log n)$).
- Efficient encoding and decoding of the file
  - Should be done in $O(B)$-time.
  - B is the size of the original file in bits.

Example

- A file with the alphabet a, b, c, d, e, f containing 100 symbols.
  - $n_a = 45, n_b = 13, n_c = 12, n_d = 16, n_e = 9, n_f = 5.$
- **Code I:**
  - $w_a = 000, w_b = 001, w_c = 010, w_d = 011, w_e = 100, w_f = 101.$
  - Length of encoded file is 300.
- **Code II:**
  - $w_a = 0, w_b = 101, w_c = 100, w_d = 111, w_e = 1101, w_f = 1100.$
  - Length of encoded file is 224
    - $1 \cdot 45 + 3 \cdot 13 + 3 \cdot 12 + 3 \cdot 16 + 4 \cdot 9 + 4 \cdot 5 = 224.$
- **Remark:** Code II is optimal, $\approx 25\%$ better than code I.

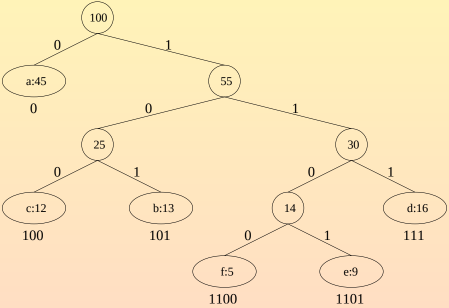
Prefix Free Codes

- **Definition:** A prefix free code is a code in which no codeword is a prefix of another codeword.
- **Examples:** Both code I and code II are prefix free.
- **Proposition:** A code in which the lengths of all the codewords is the same is a prefix free code.
- **Theorem:** Always exists an optimal prefix free code.
- **Encoding:** “Easy” using tables.
- **Decoding:** By scanning the coded text once.

Binary Tree Representation for Prefix Free Codes

- A code can be represented by a rooted and ordered binary tree with n leaves.
- Each leaf stores a codeword.
- The codeword corresponding to a leaf is defined by the unique path from the root to the leaf:
  - 0 for going left.
  - 1 for going right.

Example: Code II



- A leaf is represented by the symbol and its frequency.
- An internal node is labelled by the sum of the frequencies of all the leaves in its subtree.

Binary Tree Representation

- **Proposition:** The binary tree represents a prefix free code since a path to a leaf cannot be a prefix of any other path.
- **Complexity Parameters:**
  - $f(x)$ the frequency of a leaf x.
  - $\ell(x)$ the length of the path from the root to x.
  - The cost of the tree is: $B(T) = \sum_{\text{a leaf } x}(f(x) \cdot \ell(x))$.
    - $B(T)$ is the average length of a codeword.
  - The length of the encoded file: $\sum_{\text{a leaf } x}(n(x) \cdot \ell(x))$.

A Structural Claim

- **Lemma:** Let T be a tree that represents an optimal code. Then each internal node in the tree has two children.

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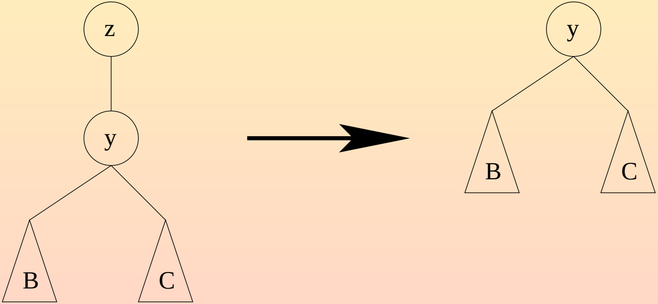
- **Lemma:** Let T be a tree that represents an optimal code. Then each internal node in the tree has two children.

- **Proof:**
  - Let z be an internal node with only one child y.
  - There are 2 cases:
    - Case I: z is the root.
    - Case II: z is not the root.

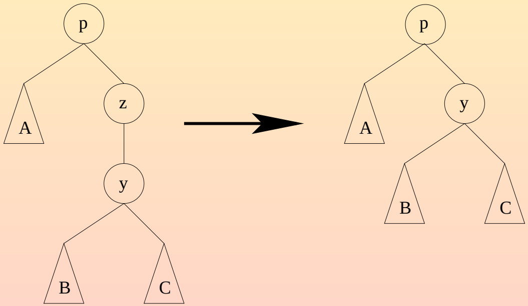
Case I

- z is the root: Make y the new root.



Case II

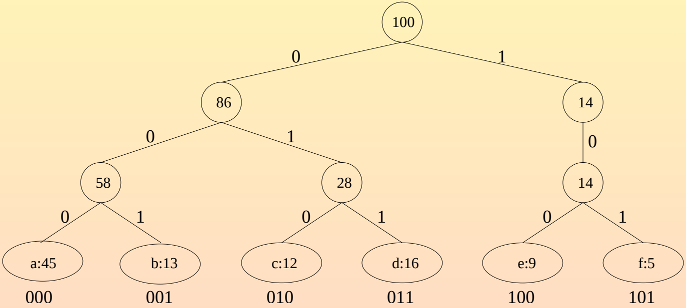
- *z* is not a root and *p* is its parent: Bypass *z* by making *y* the child of *p*.



Proof

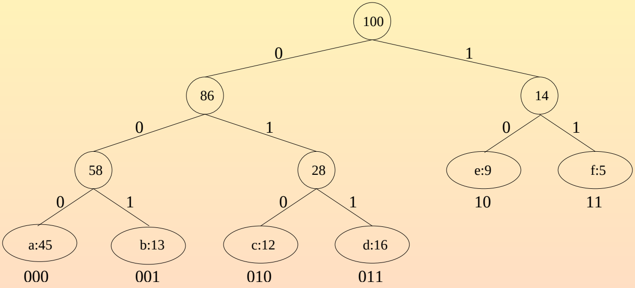
- In both cases:
  - $\ell(x)$ of all the leaves in the sub-tree rooted at z is reduced by 1.
  - These are the only changes.
  - As a result the cost of the tree is improved.
  - A contradiction to the optimality of the code.

Example: Code I



$$B(T) = 300$$

Example: Improving Code I

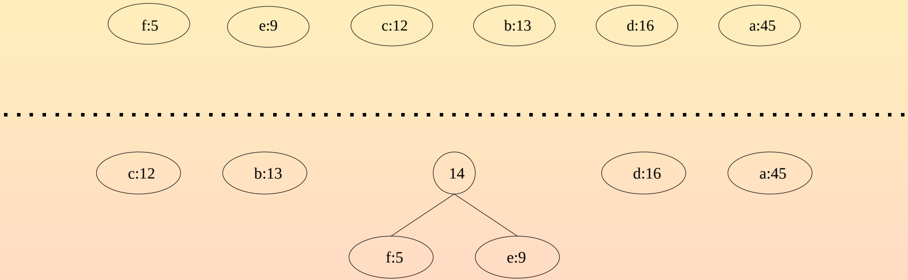


$$B(T) = 3 \cdot 86 + 2 \cdot 14 = 286$$

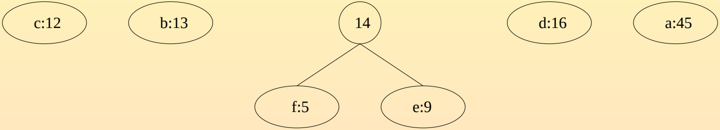
Huffman Algorithm

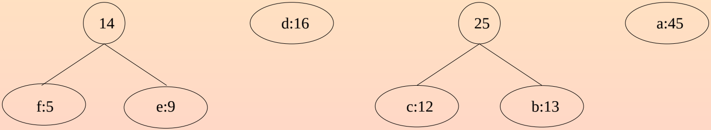
- **Construct** a coding tree bottom-up.
- **Maintain** a forest with n leaves in all of its trees. Each tree is optimal for its leaves.
- Initially, there are n singleton trees in the forest. Each tree is a leaf.
- The frequency of a tree is the sum of the frequencies of all of its leaves.
- **Greedy** step:
  - **Find** the two trees with the minimum frequencies.
  - **Combine** them together into one tree.
  - The frequency of the new tree is the sum of the frequencies of the two combined trees.
- **Terminate** when there is only one tree in the forest.

Example

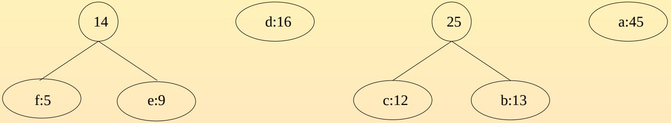


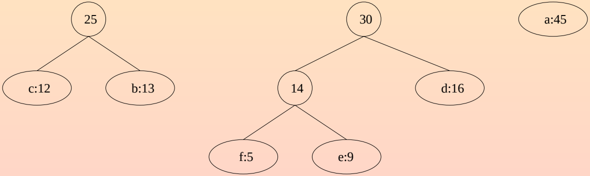
Example



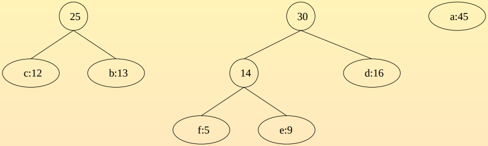


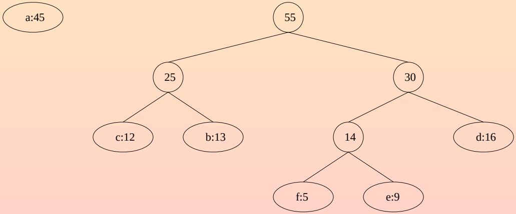
Example



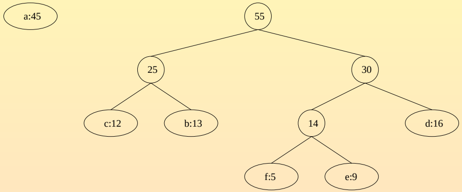


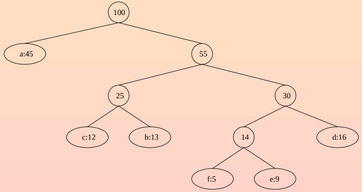
Example





Example


a:45
55
25
30
c:12
b:13
14
d:16
f:5
e:9


100
a:45
55
25
30
c:12
b:13
14
d:16
f:5
e:9

Huffman Code Animation

http://www.cs.auckland.ac.nz/~jmor159/PLDS210/huffman.html

Correctness

- Huffman algorithm generates a binary tree with n leaves.
- A binary tree represents a prefix free code.

Implementation – Data Structure

- A forest of binary trees.
  - Initially, the forest contains n singleton trees.
  - At the end, the forest contains one tree.
- The frequencies of the trees in the forest are maintained in a priority queue Q.
  - Initially, the queue contains the n original frequencies.
  - At the end, the queue contains one frequency which is the sum of all original frequencies.

Implementation – Procedure

Huffman$(\langle a_1, f_1\rangle, \ldots, \langle a_n, f_n\rangle)$
  Build-Queue$(\{f_1, \ldots, f_n\}, Q)$
  for $i = 1$ to $n - 1$ **(* the combination loop *)**
    $z =$**Allocate-Node**()   **(* creating a new root *)**
    $x = left(z) =$**Extract-Min**(Q)
      (* lightest tree is the left sub-tree *)
    $y = right(z) =$**Extract-Min**(Q)
      (* second lightest tree is the right sub-tree *)
    $f(z) = f(x) + f(y)$ **(* frequency of new root *)**
    Insert$(Q, f(z))$ **(* inserting the new root to the queue *)**
  return **Extract-Min**(Q) **(* last tree is the Huffman code *)**

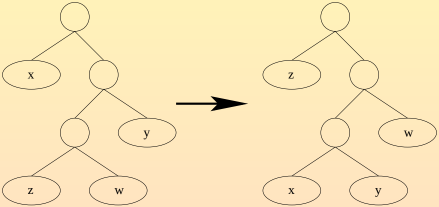
Complexity

- Implement the priority queue with a **Binary Heap**.
- The complexity of **Build-Queue** is $O(n)$.
- The complexity of **Extract-Min** and **Insert** is $O(\log n)$.
- The loop is executed $O(n)$ times.
- The complexity of all the **Extract-Min** and the **Insert** operations is $O(n \log n)$.
- The total complexity is: $O(n \log n)$.

Optimality - First Lemma

- Let $\mathcal{A}$ be an alphabet.
- Let x and y be the two symbols in $\mathcal{A}$ with the smallest frequencies.
- Then, there exists an optimal tree in which:
  - x and y are adjacent leaves (differ only in their last bit).
  - x and y are the farthest leaves from the root.

Proof

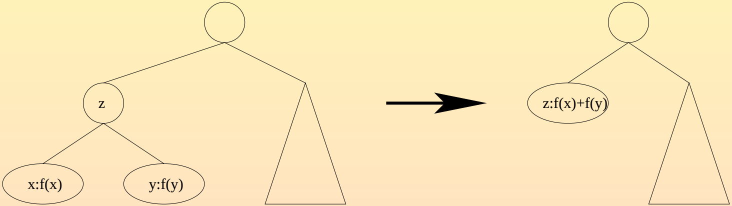


- Let z and w be adjacent leaves in an optimal tree that are the farthest from the root.
- Exchanging z and w with x and y yields a tree with a smaller or equal cost.

Optimality - Second Lemma

- Let T be an optimal tree for the alphabet $\mathcal{A}$.
- Let x, y be adjacent leaves in T and let z be their parent.
- Let $\mathcal{A}'$ be $\mathcal{A}$ with a new symbol z replacing x and y with frequency: $f(z) = f(x) + f(y)$.
- Let T' be the tree T without the leaves x and y and with z as a new leaf.
- Then T' is an optimal tree for the alphabet $\mathcal{A}'$.

Proof



- Let T'' be an optimal tree with smaller cost than T'.
- Replacing z in T'' with the two leaves x and y creates a tree with a smaller cost than T.
- A contradiction to the optimality of T.

Optimality

- **Theorem:** Huffman code is optimal.
- **Proof by Induction:**
  - The first lemma implies that the first greedy step is a first step towards an optimal solution.
  - The second lemma justifies the inductive steps, applying again and again the first lemma.