REG No. SST S04 | SBN S04

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Question (1)

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Simple Linear Regression Model

The model is

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i = 1\ 2\ 3\ 4 \dots n$$

Simple since there is only one x_i

It is linear in β_0 and β_1

$E(y_i) = \beta_0 + \beta x_i$ is linear

$y = \beta_0 + \beta_1 x_1 + e^{\beta_2 x_2} + \varepsilon_i$ is not linear

$y = \beta_0 + e^{\beta_1 x_1} + \varepsilon_i$ is not linear also.

- A simple linear regression model assumes that the following specification is true in the population

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

- Where other unobserved factors determining y are captured by the error term ε.
- To estimate the parameters β_0 & β_1 in the model Assumptions 1-5, Linearity, identification, exogeneity, spherical error terms and data generation are required for Ordinary least squares. (OLS)
- Assumptions 1-6, ie including normality are required to use Maximum likelihood to estimate β_0 & β_1 (MLE)

Assumptions summarized

i $E(\varepsilon_i) = 0$ for all $i = 1 2 \dots n \Rightarrow E(y_i) = \beta_0 + \beta_1 x_i$

ii $Var(\varepsilon_i) = \sigma^2$ for all $i = 1 2 \dots n \Rightarrow Var(y_i) = \sigma^2$ Constant

iii Covariance$(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j \Rightarrow Cov(y_i\, y_j) = 0$ uncorrelatedness

iv Normality $\varepsilon_i \sim N(0, \sigma^2)$, ε_i and y_i are independent and uncorrelated.

Ordinary least squares estimator

- Least squares Method does not require any distributional assumptions. (It does not require normality)
- MLE estimation requires the normality assumption (6)
- ordinary least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ Minimizes the sum of squared residuals (SSR)

$$Q = \sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \hat{\varepsilon}'\hat{\varepsilon} = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

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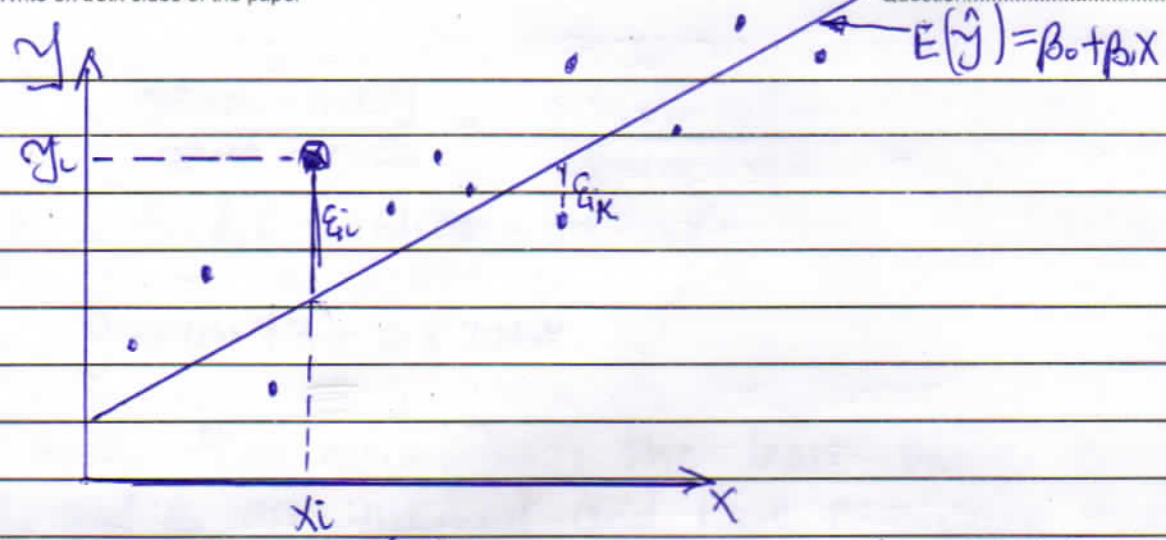

$\hat{y}_i$ estimates $E(y_i) = \beta_0 + \beta_1 x_i$ not $\beta_0 + \beta_1 x + \varepsilon_i$.

The minimum is acquired by.

$$\frac{\partial Q}{\partial \hat{\beta}_0} = -2 \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\partial Q / \partial \hat{\beta}_1 = -2 \sum x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

Simplify

$$n\hat{\beta}_0 + \hat{\beta}_1 \sum x_i = \sum y_i$$

$$\hat{\beta}_0 \sum x_i + \hat{\beta}_1 \sum x_i^2 = \sum x_i y_i$$

Solving for

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} \quad \text{OR} \quad \hat{\beta}_1 = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

Note

$$S_{xx} = \sum_i (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$S_{xy} = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

$$S_{yy} = \sum y_i^2 - \frac{(\sum y_i)^2}{n}$$

Example: Is a student performance in final exam (y) determined by continuous assessment (x)

y	95	80	0	0	79	77	72	66	98	90	0	95	35	50	72	55	75	66	$\sum y =$
x	96	77	0	0	78	64	89	47	90	93	18	86	0	30	59	77	74	67	$\sum x =$
x^2																			$\sum x^2 =$
xy																			$\sum xy =$
y^2																			$\sum y^2 =$

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$$\hat{\beta}_1 = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2} = \frac{81{,}195 - 18(58.056)(61.389)}{80199 - 18(58.056)^2} = 0.8726$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = 61.389 - 0.8726(58.056) = 10.73$$

$$\hat{y} = 10.73 + 0.8726x.$$

With the three assumptions the least squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased and have minimum variance among all linear unbiased estimators "MVUE"

$$E(\hat{\beta}_1) = \beta_1$$

$$E(\hat{\beta}_0) = \beta_0$$

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i-\bar{x})^2} = \frac{\sigma^2}{S_{xx}}$$

$$Var(\hat{\beta}_0) = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}}{\sum(x_i-\bar{x})^2}\right) = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)$$

If $\sigma^2 = E(y_i - E(y_i))^2$ using $\hat{y}$ as estimate for $E(y_i)$ weestimate $\hat{\sigma}^2$ by

$$S^2 = \frac{\sum(y_i - \hat{y})^2}{n-2}.$$

Note

$$Var(y_i) = E(y_i - E(y_i))^2 = E(y_i - \beta_0 - \beta_1 x_i)^2 = E(\hat{\varepsilon}_i^2) = \sigma^2$$

$$S^2 = \frac{\sum(y_i - \hat{y}_i)^2}{n-2} = \frac{\sum_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)}{n-2} = \frac{SSE}{n-2}$$

$\hat{\varepsilon} = (y_i - \hat{y}_i)$ is refered to residuals

$$E(S^2) = E\frac{SSE}{n-2} = \frac{(n-2)\sigma^2}{n-2} = \sigma^2 \quad \text{unbiased}$$

S^2 is an unbiased estimators of σ^2

$$SSE = \sum(y_i - \hat{y}_i)^2 = S_{yy} - \frac{S_{xy}^2}{S_{xx}} \qquad S^2 = \frac{SSE}{n-2}$$

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Question ④

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Hypothesis Testing and Confidence Intervals for $\hat{\beta}_1$

$H_0: \beta_1 = 0$ Vs $H_1\ \beta_1 \neq 0$:

$H_0: \beta_1 = 0$ There is no linear relationship between y and x

Assuming $\varepsilon \sim N(0, \sigma^2)$ or $y_i \sim N(\beta_0 + \beta_1 x_1, \sigma^2)$

(i) $\hat{\beta}_1$ is $N\left[\beta_1, \sigma^2/S_{xx}\right]$

(ii) $\frac{(n-2)S^2}{\sigma^2}$ is $\chi^2(n-2)$

(iii) $\hat{\beta}_1$ and S^2 are independent.

Test statistics for $H_0: \beta_1 = 0$ Vs $\beta_1 \neq 0$

$$t = \frac{\hat{\beta}_1}{S/\sqrt{S_{xx}}} \sim t(n-2) \quad \text{If } \beta = 0$$

Note

$$t = \frac{\text{Estimate}}{\text{Standard error of estimate}} = \frac{\hat{\beta}_1}{S_e(\hat{\beta}_1)}$$

$$\text{Reject } H_0 \text{ If } |t| \geq t_{\alpha/2}(n-2)$$

Detailed Hypothesis

Case 1: $H_0: \beta_1 \leq 0$ Vs $H_1: \beta_1 > 0$

Case 2: $H_0: \beta_1 \geq 0$ Vs $H_1: \beta_1 < 0$

Case 3: $H_0: \beta_1 = 0$ Vs $H_1: \beta_1 \neq 0$

Test statistic (T.S) $t = \frac{\hat{\beta}_1 - 0}{S_e/\sqrt{S_{xx}}}$

Rejection Region For d.f. $n-2$ and type 1 error α

Case 1. Reject H_0 If $t > t_\alpha$

2. Reject H_0 if $t < -t_\alpha$

3. Reject H_0 if $|t| > t_{\alpha/2}$

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Question. ⑤

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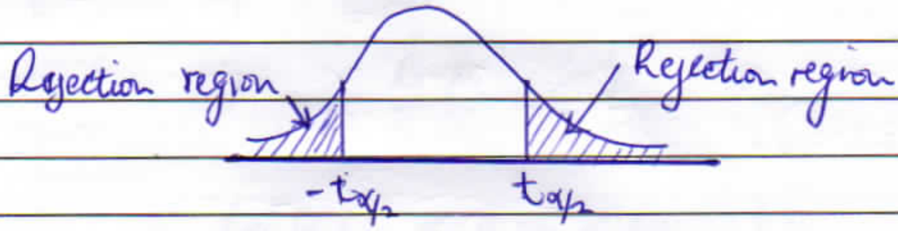
For case 3: $H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$



Example on Marks above: Test $H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$

$$t = \frac{\hat{\beta}_1}{S/\sqrt{S_{xx}}} = \frac{0.8726}{13.8547/139.753} = 8.8025$$

$t = 8.8025 > t_{0.025}(16) = 2.120$ Reject H_0.

Confidence interval for Slope $\hat{\beta}_1$

~~95%~~ A $100(1-\alpha)\%$ Confidence interval for β_1 is

$$\hat{\beta}_1 \pm t_{\alpha/2}(n-1)\frac{S}{\sqrt{S_{xx}}} \quad \text{Where } S = \frac{S_{YY} - \frac{S_{xy}^2}{S_{xx}}}{n-2}$$

$$\beta_1 \pm t_{\alpha/2}(n-1)\frac{S}{\sqrt{S_{xx}}}$$

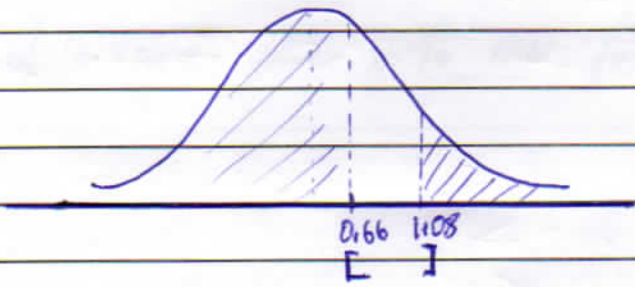
A 95% Confidence interval for β_1 for the Marks example

$$\hat{\beta}_1 \pm t_{0.025}^{(16)} \frac{S}{\sqrt{(x_i - \bar{x})^2}}$$

$0.8726 \pm 2.120(0.09914)$

0.8726 ± 0.2102

$(0.6624, 1.0828)$



Confidence interval for Slope β_1

$$\left| \frac{\hat{\beta}_1 - \beta_1}{S_e \sqrt{1/S_{xx}}} \right| < t_{\alpha/2} \quad \text{is a } (1-\alpha)100\% \text{ Confidence I.}$$

$$-t_{\alpha/2} < \frac{\hat{\beta}_1 - \beta_1}{S_e\sqrt{1/S_{xx}}} < t_{\alpha/2}$$

$$-t_{\alpha/2} S_e \sqrt{\frac{1}{S_{xx}}} < \hat{\beta}_1 - \beta_1 < t_{\alpha/2} S_e \sqrt{1/S_{xx}}$$

$$\hat{\beta}_1 - t_{\alpha/2} S_e \sqrt{\frac{1}{S_{xx}}} < \beta < \hat{\beta}_1 + t_{\alpha/2} S_e \sqrt{1/S_{xx}} \quad \text{is the}$$

$(1-\alpha)100\%$ Confidence interval for β_1 where t has $n-2$ df.

F test for H_0 $\beta_1 = 0$ (Recall ANOVA)

$H_0: \beta_1 = 0$ vs H_1 $\beta_1 \neq 0$

Test Statistics $F = \dfrac{SS(\text{Regression})/n-1}{SS\,\text{Error}/n-2} = \dfrac{MS\,Reg}{MSE}$

Rejection Region: With $df_1 = 1$ $df_2 = n-2$, reject H_0 if $F > F_\alpha$

$$SS\,Reg = \sum(\hat{y} - \bar{y})^2$$
$$SSE = \sum(y_i - \bar{y})^2$$

Confidence Interval for intercept β_0

$$\sigma_{\hat{\beta}_0} = S_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}}$$

The $(1-\alpha)100\%$ Confidence interval for β_0 is given by

$$\beta_0 \pm t_{\alpha/2} S_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}} \qquad \text{for } t \text{ with } n-2 \text{ df.}$$

Q Determine the 95% CI for $\hat{\beta}_1$ for example 1.1.

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Write on both sides of the paper Question 7

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Inference about the Correlation coefficient ρ_{yx}

The t statistic for β_1 can be expressed in terms of r as follows

$$t = \frac{\hat{\beta}_1}{\delta / \sqrt{\sum (x_i - \bar{x})^2}}$$

$$t = r_{yx} \frac{\sqrt{n-2}}{\sqrt{1-r_{yx}^2}}$$

The sample correlation r_{yx} is the basis for estimation and significance testing of the population correlation ρ_{yx}

Hypothesis

Case 1: $H_0\ \rho_{yx} \leq 0$ vs. $H_1\ \rho_{xy} > 0$

Case 2: $H_0\ \rho_{yx} \geq 0$ vs. $H_a\ \rho_{yx} < 0$

Case 3: $H_0\ \rho_{yx} = 0$ vs. $H_1\ \rho_{yx} \neq 0$

Time Series T.S. $t = r_{yx} \frac{\sqrt{n-2}}{\sqrt{1-r^2_{yx}}}$

R.R With n-2 df and type I error probability α,

1. $t > t_\alpha$
2. $t < -t_\alpha$
3. $|t| > t_{\alpha/2}$

Check assumptions and draw Conclusions

Exercise

Soil PH :	3.3	3.4	3.4	3.5	3.6	3.6	3.7	3.7	3.8	3.8
Growth Red :	17.78	21.59	23.84	15.13	23.45	20.87	17.78	20.09	17.78	12.46

QI Examine the scatter plot and decide whether a straight line is reasonable model. Is regression significant?

QII Identify the Least squares model estimates for $Y = \beta_0 + \beta_1 X + \epsilon$

QIII Predict the growth retardation for a soil PH of 4.0

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Simple Linear Regression.

The Model: The simple linear regression mode for n observation can be written as

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i = 1,2,3,\ldots,n$$

Linear in β_0 and β_1

$y = \beta_0 + \beta_1 x_i^2 + \varepsilon_i$ is Linear in β_0 & β_1

$y = \beta_0 + e^{\beta_1 x_i} + \varepsilon_i$ is not Linear ~~in~~

Model

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i = 1\,2\ldots n$$

Assumption

(i) $E(\varepsilon_i) = 0$ for all $i = 1\,2\ldots n$ or $E(y_i) = \beta_0 + \beta_1 x_i$

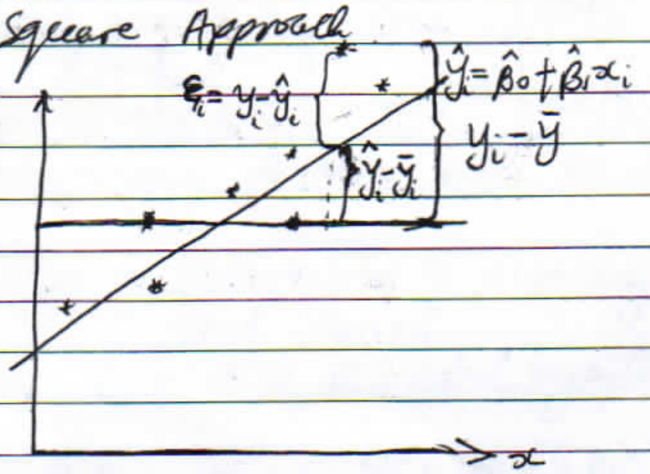
(ii) $Var(\varepsilon_i) = \sigma^2$ for all $i = 1\,2\ldots n$ or $Var(y_i) = \sigma^2$; Homoscedestic

(iii) $Cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$ or $Cov(x_i, y_i) = 0$

$$Var(y_i) = E[y_i - Ey_i]^2 = E(y_i - \beta_0 - \beta_1 x_i)^2 = E(\varepsilon_i^2) = \sigma^2$$

Estimation of β_0, β_1 and σ^2

Least square Approach



Total variation $= y_i - \bar{y}$

error (Residual) $= y_i - \hat{y}$

explained by regression $= \hat{y}_i - \bar{y}_i$

$y_i - \hat{y}_i = (y_i - \bar{y}) - (\hat{y}_i - \bar{y})$ from the sum

$$(y_i - \bar{y}) = (y_i - \hat{y}) + (\hat{y}_i - \bar{y}_i)$$

Sum of squares

$$\sum (y_i-\bar{y})^2 = \sum(\hat{y}_i-\bar{y})^2 + \sum(y_i-\hat{y})^2 + \underbrace{2\sum(\hat{y}_i-\bar{y})(y_i-\hat{y}_i)}_{0}$$

(Sum of squares about mean SST corrected) = (Sum of squares due to regression $SSR(b_1/b_0)$) + (Sum of square about regression Residual SS)

$$SST = SSR + SSE.$$

Therefore to test if regression is significant ie
$H_0: \beta_1 = 0$ Vs H_1 $\beta_1 \neq 0$
We have the anova.

ANOVA Table.

Source of variation	Degrees of freedom	Sum of Squares SS	Mean Square	F
Due to regression	1	$\sum(\hat{y}_i-\bar{y})^2$	MS_{reg}	$F_c = \frac{MS_{Reg}}{S^2}$
About regression Residual	$n-2$	$\sum(y_i-\hat{y}_i)^2$	$S^2 = \frac{SSE}{n-2}$	
Total corrected for $\bar{y}$	$n-1$	$\sum(y_i-\bar{y})^2$		

An accompanying p value will be available (given)

Pvalue = probability of observing a value Larger than that observed.

In this case it is a 2 sided hypothesis

$$p\text{-value} = 2 * P(F \geq F_c(1, n-2))$$

- If ~~#~~ F_c is Large reject H_0 $\beta_1 = 0$: $\Rightarrow$ implying that most variation is caused by (explained by) regression.
- Reject H_0 if pvalue is too small

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Question (10)

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Coefficient of Determination

Coefficient of Determination is defined as

$$r^2 = \frac{SSR}{SST} = \frac{\sum(\hat{y}_i - \bar{y})^2}{\sum(y_i - \bar{y})^2}$$

SSR - Regression sum of squares $= \sum(\hat{y}_i - \bar{y})^2$
SST - Total sum of squares $= \sum(y_i - \bar{y})^2$

$$SST = SSR + SSE$$

$$\sum(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$\sum(\hat{y} - \bar{y}) = \sum(y_i - \bar{y})^2 - \sum(y_i - \hat{y}_i)^2$$

$$SSR = SST - SSE$$

$$r^2 = \frac{SST - SSE}{SST} = 1 - \frac{SSE}{SST}.$$

proportion of variation in y explained by the model or accounted by regression.

$$r = \frac{S_{xy}}{\sqrt{S_x^2 S_y^2}} = \frac{\sum(x_i - \bar{x})\sum(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}$$

$$r^2 = \frac{SSR}{SST} = \frac{14873.0}{17944} = 0.8288$$

$$t = \frac{\hat{\beta}_1}{s/\sqrt{\sum_i (x_i - \bar{x})^2}} = \frac{\sqrt{n-2}\, r}{\sqrt{1-r^2}}$$

The fit of the regression is "good" if the sum $\sum \varepsilon_i^2$ ($= SSE$) is small ie the unexplained part of the variance of y is small. If SSE is small then R^2 is close to 1

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Question F11

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Coefficient of Determination (Multiple Correlation Coefficient)

$$r^2 = \frac{SSR}{SST} = \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum (y_i - \bar{y})^2} \quad \text{Where } SST = SSR + SSE$$

$$SST \sum (y_i - \bar{y})^2 = \sum_i (\hat{y}_i - \bar{y})^2 + \sum_i (y_i - \hat{y})^2$$

r^2 gives the proportion of variation in y that is explained by the model or accounted for by regression on x.

Sample Correlation Coefficient r between x and y

$$r = \frac{S_{xy}}{\sqrt{S_{xx}^2 S_{yy}^2}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{[\sum (x_i - \bar{x})^2][\sum_{i=1}^{n} (y_i - \bar{y})^2]}}$$

For Example 1.1 $\quad r^2 = \frac{SSR}{SST} = \frac{45\,873.0}{17\,944.3} = 0.8288$

$$r = \sqrt{0.8288} = 0.91$$

Exercise

y	x	x^2	y^2	xy
25	10			
55	18			
50	25			
75	40			
110	50			
138	63			
90	42			
60	30			
10	5			
100	55			

(i) Determine the least square estimators $\hat{\beta}_0$ and $\hat{\beta}_1$

(ii) Determine r^2 and explain how well your model fits the data

(iii) Use ANOVA to test whether regression is significant.