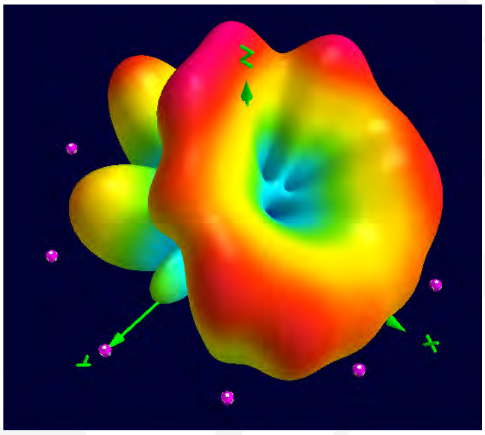
*Bo Thidé*

# ELECTROMAGNETIC FIELD THEORY




SECOND EDITION

# ELECTROMAGNETIC FIELD THEORY

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## Second Edition

Bo Thidé

Swedish Institute of Space Physics
Uppsala, Sweden

and

Department of Physics and Astronomy
Uppsala University, Sweden

and

Galilean School of Higher Education
University of Padua
Padua, Italy

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The cover graphics illustrates the linear momentum radiation pattern of a radio beam endowed with orbital angular momentum, generated by an array of tri-axial antennas. This graphics illustration was prepared by JOHAN SJÖHOLM and KRISTOFFER PALMER as part of their undergraduate Diploma Thesis work in Engineering Physics at Uppsala University 2006–2007.

To the memory of professor
LEV MIKHAILOVICH ERUKHIMOV (1936–1997)
dear friend, great physicist, poet
and a truly remarkable man.

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# PREFACE TO THE SECOND EDITION

This second edition of the book ELECTROMAGNETIC FIELD THEORY is a major revision and expansion of the first edition that was published on the Internet (www.plasma.uu.se/CED/Book) in an organic growth process over the years 1997–2008. The main changes are an expansion of the material treated, an addition of a new chapter and several illustrative examples, and a slight reordering of the chapters.

The main reason for attempting to improve the presentation and to add more material is that this new edition is now being made available in printed form by Dover Publications and is used in an extended Classical Electrodynamics course at Uppsala University, at the last-year undergraduate, Master, and beginning post-graduate/doctoral level. It has also been used by the author in a similar course at the Galilean School of Higher Education (Scuola Galileiana di Studi Superiori) at University of Padova. It is the author's hope that the second edition of his book will find a wid use in Academia and elsewhere.

The subject matter starts with a description of the properties of electromagnetism when the charges and currents are located in otherwise free space, *i.e.*, a space that is free of matter and external fields (*e.g.*, gravitation). A rigorous analysis of the fundamental properties of the electromagnetic fields and radiation phenomena follows. Only then the influence of matter on the fields and the pertinent interaction processes is taken into account. In the author's opinion, this approach is preferable since it avoids the formal logical inconsistency of introducing, very early in the derivations, the effect on the electric and mangetic fields when conductors and dielectrics are present (and *vice versa*) in an *ad hoc* manner, before constitutive relations and physical models for the electromagnetic properties of matter, including conductors and dielectrics, have been derived from first principles. Curved-space effects on electromagnetism are not treated at all.

In addition to the Maxwell-Lorentz equations, which postulate the beaviour of electromagnetic fields due to electric charges and currents on a microscopic classical scale, chapter chapter 1 also introduces Dirac's symmetrised equations that incorporate the effects of magnetic charges and currents. In chapter chapter 2, a stronger emphasis than before is put on the axiomatic foundation of electrodynamics as provided by the Maxwell-Lorentz equations that are taken as the postulates of the theory. Chapter chapter 3 on potentials and gauges now provides a more comprehensive picture and discusses gauge invariance in a more satisfactory manner than the first edition did. Chapter chapter 4 is new and deals with symmetries and conserved quantities in a more rigourous, profound and detailed way than in the first edition. For instance, the presentation of the theory of electromagnetic angular momentum and other observables (constants of motion) has been substantially expanded and put on a firm basis. Chapter chapter 9 is a complete rewrite and combines material that was scattered more or less all over the first edition. It also contains new material on wave propagation in plasma and other media. When, in chapter chapter 9, the macroscopic Maxwell equations are introduced, the inherent approximations in the derived field quantities are clearly pointed out. The collection of formulæ in appendix F on page 213 has been augmented quite substantially. In appendix M on page 231, the treatment of dyadic products and tensors has been expanded significantly and numerous examples have been added throughout.

I want to express my warm gratitude to professor CESARE BARBIERI and his entire group, particularly FABRIZIO TAMBURINI, at the Department of Astronomy, University of Padova, for stimulating discussions and the generous hospitality bestowed upon me during several shorter and longer visits in 2008, 2009, and 2010 that made it possible to prepare the current major revision of the book. In this breathtakingly beautiful northern Italy where the cradle of renaissance once stood, intellectual titan GALILEO GALILEI worked for eighteen years and gave birth to modern physics, astronomy and science as we know it today, by sweeping away Aristotelian dogmas, misconceptions and mere superstition, thus most profoundly changing our conception of the world and our place in it. In the process, Galileo's new ideas transformed society and mankind irreversibly and changed our view of the Universe, including our own planet, forever. It is hoped that this book may contribute in some small, humble way to further these, once upon a time, mind-boggling—and dangerous—ideas of intellectual freedom and enlightment.

Thanks are also due to JOHAN SJÖHOLM, KRISTOFFER PALMER, MARCUS ERIKSSON, and JOHAN LINDBERG who during their work on their Diploma theses suggested improvements and additions and to HOLGER THEN and STAFFAN YNGVE for carefully checking some lengthy calculations and to the numerous undergraduate students, who have been exposed to various draft versions of this second edtion. In particular, I would like to mention BRUNO STRANDBERG.

This book is dedicated to my son MATTIAS, my daughter KAROLINA, my four grandsons MAX, ALBIN, FILIP and OSKAR, my high-school physics teacher, STAFFAN RÖSBY, and my fellow members of the CAPELLA PEDAGOGICA UPSALIENSIS.

*Padova, Italy* BO THIDÉ
*February,* 2011 `www.physics.irfu.se/∼bt`

# PREFACE TO THE FIRST EDITION

Of the four known fundamental interactions in nature—gravitational, strong, weak, and electromagnetic—the latter has a special standing in the physical sciences. Not only does it, together with gravitation, permanently make itself known to all of us in our everyday lives. Electrodynamics is also by far the most accurate physical theory known, tested on scales running from sub-nuclear to galactic, and electromagnetic field theory is the prototype of all other field theories.

This book, ELECTROMAGNETIC FIELD THEORY, which tries to give a modern view of classical electrodynamics, is the result of a more than thirty-five year long love affair. In the autumn of 1972, I took my first advanced course in electrodynamics at the Department of Theoretical Physics, Uppsala University. Soon I joined the research group there and took on the task of helping the late professor PER OLOF FRÖMAN, who was to become my Ph.D. thesis adviser, with the preparation of a new version of his lecture notes on the Theory of Electricity. This opened my eyes to the beauty and intricacy of electrodynamics and I simply became intrigued by it. The teaching of a course in Classical Electrodynamics at Uppsala University, some twenty odd years after I experienced the first encounter with the subject, provided the incentive and impetus to write this book.

Intended primarily as a textbook for physics and engineering students at the advanced undergraduate or beginning graduate level, it is hoped that the present book will be useful for research workers too. It aims at providing a thorough treatment of the theory of electrodynamics, mainly from a classical field-theoretical point of view. The first chapter is, by and large, a description of how Classical Electrodynamics was established by JAMES CLERK MAXWELL as a fundamental theory of nature. It does so by introducing electrostatics and magnetostatics and demonstrating how they can be unified into one theory, classical electrodynamics, summarised in Lorentz's microscopic formulation of the Maxwell equations. These equations are used as an axiomatic foundation for the treatment in the remainder of the book, which includes modern formulation of the theory; electromagnetic waves and their propagation; electromagnetic potentials and gauge transformations; analysis of symmetries and conservation laws describing the electromagnetic counterparts of the classical concepts of force, momentum and energy, plus other fundamental properties of the electromagnetic field; radiation phenomena; and covariant Lagrangian/Hamiltonian field-theoretical methods for electromagnetic fields, particles and interactions. Emphasis has been put on modern electrodynamics concepts while the mathematical tools used, some of them presented in an Appendix, are essentially the same kind of vector and tensor analysis methods that are used in intermediate level textbooks on electromagnetics but perhaps a bit more advanced and far-reaching.

The aim has been to write a book that can serve both as an advanced text in Classical Electrodynamics and as a preparation for studies in Quantum Electrodynamics and Field Theory, as well as more applied subjects such as Plasma Physics, Astrophysics, Condensed Matter Physics, Optics, Antenna Engineering, and Wireless Communications.

The current version of the book is a major revision of an earlier version, which in turn was an outgrowth of the lecture notes that the author prepared for the four-credit course Electrodynamics that was introduced in the Uppsala University curriculum in 1992, to become the five-credit course Classical Electrodynamics in 1997. To some extent, parts of those notes were based on lecture notes prepared, in Swedish, by my friend and Theoretical Physics colleague BENGT LUNDBORG, who created, developed and taught an earlier, two-credit course called Electromagnetic Radiation at our faculty. Thanks are due not only to Bengt Lundborg for providing the inspiration to write this book, but also to professor CHRISTER WAHLBERG, and professor GÖRAN FÄLDT, both at the Department of Physics and Astronomy, Uppsala University, for insightful suggestions, to professor JOHN LEARNED, Department of Physics and Astronomy, University of Hawaii, for decisive encouragement at the early stage of this book project, to professor GERARDUS T'HOOFT, for recommending this book on his web page 'How to become a *good* theoretical physicist', and professor CECILIA JARLSKOG, Lund Unversity, for pointing out a couple of errors and ambiguities.

I am particularly indebted to the late professor VITALIY LAZAREVICH GINZBURG, for his many fascinating and very elucidating lectures, comments and historical notes on plasma physics, electromagnetic radiation and cosmic electrodynamics while cruising up and down the Volga and Oka rivers in Russia at the ship-borne Russian-Swedish summer schools that were organised jointly by late professor LEV MIKAHILOVICH ERUKHIMOV and the author during the 1990's, and for numerous deep discussions over the years.

Helpful comments and suggestions for improvement from former PhD students TOBIA CAROZZI, ROGER KARLSSON, and MATTIAS WALDENVIK, as well as ANDERS ERIKSSON at the Swedish Institute of Space Physics in Uppsala and who have all taught Uppsala students on the material covered in this book, are gratefully acknowledged. Thanks are also due to the late HELMUT KOPKA, for more than twenty-five years a close friend and space physics colleague working at the Max-Planck-Institut für Aeronomie, Lindau, Germany, who not only taught me the practical aspects of the use of high-power electromagnetic radiation for studying space, but also some of the delicate aspects of typesetting in TeX and LaTeX.

In an attempt to encourage the involvement of other scientists and students in the making of this book, thereby trying to ensure its quality and scope to make it useful in higher university education anywhere in the world, it was produced as a World-Wide Web (WWW) project. This turned out to be a rather successful move. By making an electronic version of the book freely downloadable on the Internet, comments have been received from fellow physicists around the world. To judge from WWW 'hit' statistics, it seems that the book serves as a frequently used Internet resource. This way it is hoped that it will be particularly useful for students and researchers working under financial or other circumstances that make it difficult to procure a printed copy of the book. I would like to thank all students and Internet users who have downloaded and commented on the book during its life on the World-Wide Web.

*Uppsala, Sweden* BO THIDÉ
*December,* 2008 www.physics.irfu.se/∼bt

# 1

# FOUNDATIONS OF CLASSICAL ELECTRODYNAMICS

The classical theory of electromagnetism deals with electric and magnetic fields and their interaction with each other and with charges and currents. This theory is classical in the sense that it assumes the validity of certain mathematical limiting processes in which it is considered possible for the charge and current distributions to be localised in infinitesimally small volumes of space.[1] Clearly, this is in contradistinction to electromagnetism on an atomistic scale, where charges and currents have to be described in a nonlocal quantum formalism. However, the limiting processes used in the classical domain, which, crudely speaking, assume that an elementary charge has a continuous distribution of charge density, will yield results that agree perfectly with experiments on non-atomistic scales, however small or large these scales may be.[2]

[1] Accepting the mere existence of an electrically charged particle requires some careful thinking. In his excellent book *Classical Charged Particles*, FRITZ ROHRLICH writes

> 'To what extent does it make sense to talk about an electron, say, in classical terms? These and similar questions clearly indicate that ignoring philosophy in physics means not understanding physics. For there is no theoretical physics without some philosophy; not admitting this fact would be self-deception.'

[2] Electrodynamics has been tested experimentally over a larger range of spatial scales than any other existing physical theory.

It took the genius of JAMES CLERK MAXWELL to consistently unify, in the mid-1800's, the theory of *Electricity* and the then distinctively different theory *Magnetism* into a single super-theory, *Electromagnetism* or *Classical Electrodynamics* (CED), and also to realise that optics is a sub-field of this super-theory. Early in the 20th century, HENDRIK ANTOON LORENTZ took the electrodynamics theory further to the microscopic scale and also paved the way for the Special Theory of Relativity, formulated in its full extent by ALBERT EINSTEIN in 1905. In the 1930's PAUL ADRIEN MAURICE DIRAC expanded electrodynamics to a more symmetric form, including magnetic as well as electric charges. With his relativistic quantum mechanics and field quantisation concepts, Dirac had already in the 1920's laid the foundation for *Quantum Electrodynamics* (*QED*), the relativistic quantum theory for electromagnetic fields and their interaction with matter for which RICHARD PHILLIPS FEYNMAN, JULIAN SEYMOUR SCHWINGER, and SIN-ITIRO TOMONAGA were awarded the Nobel Prize in Physics in 1965. Around the same time, physicists such as SHELDON GLASHOW, ABDUS SALAM, and STEVEN WEINBERG were able to unify electrodynamics with the weak interaction theory, thus creating yet another successful super-theory, *Electroweak Theory*, an achievement which rendered them the Nobel Prize in Physics 1979. The modern theory of strong interactions, *Quantum Chromodynamics* (*QCD*), is heavily influenced by CED and QED.

In this introductory chapter we start with the force interactions in classical electrostatics and classical magnetostatics, introduce the corresponding static electric and magnetic fields and postulate two uncoupled systems of differential equations for them. We continue by showing that the conservation of electric charge and its relation to electric current lead to a dynamic connection between electricity and magnetism and how the two can be unified into Classical Electrodynamics. This theory is described by a system of coupled dynamic field equations — the microscopic versions of Maxwell's differential equations introduced by Lorentz — which, in chapter 2, we take as the axiomatic foundation of the theory of electromagnetic fields and the basis for the treatment in the rest of the book.

At the end of this chapter 1 we present Dirac's symmetrised form of the Maxwell-Lorentz equations that incorporate magnetic charges and magnetic currents into the theory in a symmetric way. In practical work, such as in antenna engineering, magnetic currents have proved to be a very useful concept. We shall make some use of this symmetrised theory of electricity and magnetism.

## 1.1 Electrostatics

The theory that describes physical phenomena related to the interaction between stationary electric charges or charge distributions in a finite space with stationary boundaries is called *electrostatics*. For a long time, electrostatics, under the name *electricity*, was considered an independent physical theory of its own, alongside other physical theories such as Magnetism, Mechanics, Optics, and Thermodynamics.[3]

[3] The physicist, mathematician and philosopher PIERRE MAURICE MARIE DUHEM (1861–1916) once wrote:

> 'The whole theory of electrostatics constitutes a group of abstract ideas and general propositions, formulated in the clear and concise language of geometry and algebra, and connected with one another by the rules of strict logic. This whole fully satisfies the reason of a French physicist and his taste for clarity, simplicity and order....'

### 1.1.1 Coulomb's law

It has been found experimentally that the force interaction between stationary, electrically charged bodies can be described in terms of two-body mechanical forces. Based on these experimental observations, Coulomb[4] postulated, in 1775, that in the simple case depicted in figure 1.1 on the facing page, the mechanical force on a static electric charge $q$ located at the *field point* (*observation point*) $\mathbf{x}$, due to the presence of another stationary electric charge $q'$ at the *source point* $\mathbf{x}'$, is directed along the line connecting these two points, repulsive for charges of equal signs and attractive for charges of opposite signs. This postu-

[4] CHARLES-AUGUSTIN DE COULOMB (1736–1806) was a French physicist who in 1775 published three reports on the forces between electrically charged bodies.

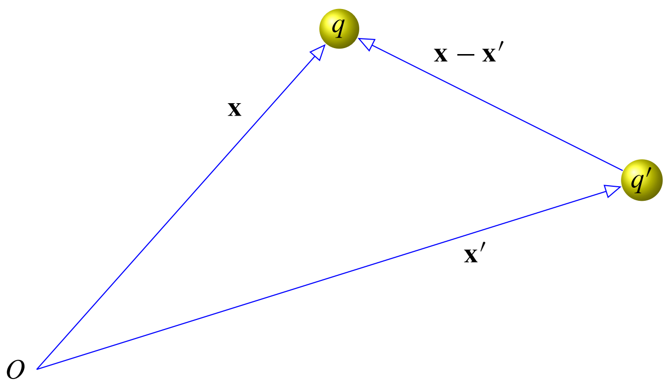


Figure 1.1: Coulomb's law postulates that a static electric charge $q$, located at a point $\mathbf{x}$ relative to the origin $O$, will experience an electrostatic force $\mathbf{F}^{\text{es}}(\mathbf{x})$ from a static electric charge $q'$ located at $\mathbf{x}'$. Note that this definition is independent of any particular choice of coordinate system since the mechanical force $\mathbf{F}^{\text{es}}$ is a true (polar) vector.

late is called *Coulomb's law* and can be formulated mathematically as

$$\mathbf{F}^{\text{es}}(\mathbf{x}) = \frac{qq'}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{qq'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{qq'}{4\pi\varepsilon_0}\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.1}$$

where, in the last step, formula (F.114) on page 220 was used. In *SI units*, which we shall use throughout, the electrostatic force[5] $\mathbf{F}^{\text{es}}$ is measured in Newton (N), the electric charges $q$ and $q'$ in Coulomb (C), *i.e.* Ampere-seconds (As), and the length $|\mathbf{x}-\mathbf{x}'|$ in metres (m). The constant $\varepsilon_0 = 10^7/(4\pi c^2)$ Farad per metre ($\text{Fm}^{-1}$) is the *permittivity of free space* and $c$ $\text{ms}^{-1}$ is the speed of light in vacuum.[6] In *CGS units*, $\varepsilon_0 = 1/(4\pi)$ and the force is measured in dyne, electric charge in statcoulomb, and length in centimetres (cm).

[5] Massive particles also interact gravitationally but with a force that is typically $10^{-36}$ times weaker.

[6] The notation $c$ for speed stems from the Latin word 'celeritas' which means 'swiftness'. This notation seems to have been introduced by WILHELM EDUARD WEBER (1804–1891), and RUDOLF KOHLRAUSCH (1809–1858) and $c$ is therefore sometimes referred to as *Weber's constant*. In all his works from 1907 and onward, ALBERT EINSTEIN (1879–1955) used $c$ to denote the speed of light in free space.

## 1.1.2 The electrostatic field

Instead of describing the electrostatic interaction in terms of a 'force action at a distance', it turns out that for many purposes it is useful to introduce the concept of a field. Thus we describe the electrostatic interaction in terms of a static vectorial *electric field* $\mathbf{E}^{\text{stat}}$ defined by the limiting process

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \lim_{q\to 0}\frac{\mathbf{F}^{\text{es}}(\mathbf{x})}{q} \tag{1.2}$$

where $\mathbf{F}^{\text{es}}$ is the electrostatic force, as defined in equation (1.1) above, from a net electric charge $q'$ on the test particle with a small net electric charge $q$.[7] In the SI system of units, electric fields are therefore measured in $\text{NC}^{-1}$ or, equivalently, in $\text{Vm}^{-1}$. Since the purpose of the limiting process is to ascertain that the test charge $q$ does not distort the field set up by $q'$, the expression for $\mathbf{E}^{\text{stat}}$ does not depend explicitly on $q$ but only on the charge $q'$ and the relative position vector $\mathbf{x}-\mathbf{x}'$. This means that we can say that any net electric charge produces

[7] If we picture the test charge as an electrically charged particle, the charge of such a particle cannot tend smoothly to 0 simply because the lowest allowable amount of charge is that of an individual quark, namely $1/3\,|e|$ where $e$ is the elementary charge

an electric field in the space that surrounds it, regardless of the existence of a second charge anywhere in this space.[8] However, in order to experimentally detect a charge, a second (test) charge that senses the presence of the first one, must be introduced.

Using equations (1.1) and (1.2) on the previous page, and formula (F.114) on page 220, we find that the electrostatic field $\mathbf{E}^{\text{stat}}$ at the *observation point* $\mathbf{x}$ (also known as the *field point*), due to a field-producing electric charge $q'$ at the source point $\mathbf{x}'$, is given by

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.3}$$

In the presence of several field producing discrete electric charges $q'_i$, located at the points $\mathbf{x}'_i$, $i = 1, 2, 3, \ldots$, respectively, in otherwise empty space, the assumption of linearity of vacuum[9] allows us to superimpose their individual electrostatic fields into a total electrostatic field

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_i q'_i \frac{\mathbf{x}-\mathbf{x}'_i}{|\mathbf{x}-\mathbf{x}'_i|^3} \tag{1.4}$$

If the discrete electric charges are small and numerous enough, we can, in a continuum limit, assume that the total charge $q'$ from an extended volume to be built up by local infinitesimal elemental charges $\mathrm{d}q'$, each producing an elemental electric field

$$\mathrm{d}\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\,\mathrm{d}q'\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.5}$$

By introducing the *electric charge density* $\rho$, measured in $\mathrm{Cm}^{-3}$ in SI units, at the point $\mathbf{x}'$ within the volume element $\mathrm{d}^3x' = \mathrm{d}x'_1\mathrm{d}x'_2\mathrm{d}x'_3$ (measured in $\mathrm{m}^3$), the elemental charge can be expressed as $\mathrm{d}q' = \mathrm{d}^3x'\,\rho(\mathbf{x}')$, and the elemental electrostatic field as

$$\mathrm{d}\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\,\mathrm{d}^3x'\,\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.6}$$

Integrating this over the entire source volume $V'$, we obtain

$$\begin{aligned}\mathbf{E}^{\text{stat}}(\mathbf{x}) &= \int \mathrm{d}\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.7}$$

where we used formula (F.114) on page 220 and the fact that $\rho(\mathbf{x}')$ does not depend on the unprimed (field point) coordinates on which $\boldsymbol{\nabla}$ operates.

[8] In the preface to the first edition of the first volume of his book *A Treatise on Electricity and Magnetism*, first published in 1873, Maxwell describes this in the following almost poetic manner:

> 'For instance, Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.'

[9] In fact, a vacuum exhibits a *quantum mechanical non-linearity* due to *vacuum polarisation* effects, manifesting themselves in the momentary creation and annihilation of electron-positron pairs, but classically this non-linearity is negligible.

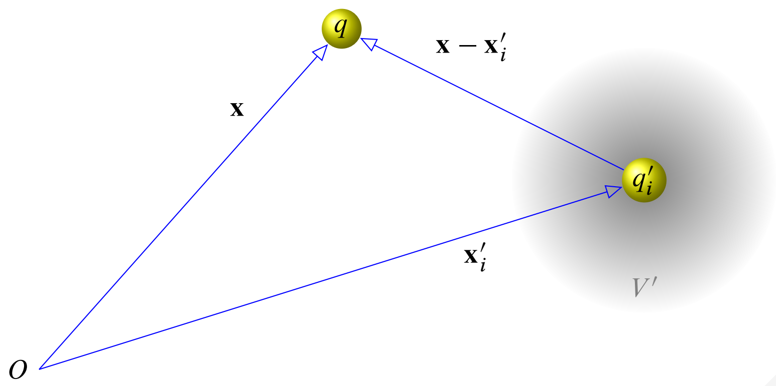


Figure 1.2: Coulomb's law for a continuous charge density $\rho(\mathbf{x}')$ within a volume $V'$ of limited extent. In particular, a charge density $\rho(\mathbf{x}') = \sum_i^N q'_i\,\delta(\mathbf{x}'-\mathbf{x}'_i)$ would represent a source distribution consisting of $N$ discrete charges $q'_i$ located at $\mathbf{x}'_i$, where $i = 1, 2, 3, \ldots, N$.

We emphasise that under the assumption of linear superposition, equation (1.7) on the facing page is valid for an arbitrary distribution of electric charges, including discrete charges, in which case $\rho$ is expressed in terms of Dirac delta distributions:[10]

$$\rho(\mathbf{x}') = \sum_i q'_i\,\delta(\mathbf{x}'-\mathbf{x}'_i) \tag{1.8}$$

as illustrated in figure 1.2. Inserting this expression into expression (1.7) on the facing page we recover expression (1.4) on the preceding page, as we should.

According to *Helmholtz's theorem*, discussed in subsection M.3.7, any well-behaved vector field is completely known once we know its divergence and curl at all points $\mathbf{x}$ in 3D space.[11] Taking the divergence of the general $\mathbf{E}^{\text{stat}}$ expression for an arbitrary electric charge distribution, equation (1.7) on the facing page, and applying formula (F.126) on page 222 [see also equation (M.75) on page 246], we obtain

$$\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{1.9a}$$

which is the differential form of *Gauss's law of electrostatics*. Since, according to formula (F.100) on page 220, $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha(\mathbf{x}) \equiv \mathbf{0}$ for any $\mathbb{R}^3$ scalar field $\alpha(\mathbf{x})$, we immediately find that in electrostatics

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \mathbf{0} \tag{1.9b}$$

*i.e.* that $\mathbf{E}^{\text{stat}}$ is a purely *irrotational* field.

To summarise, electrostatics can be described in terms of two vector partial differential equations

$$\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{1.10a}$$

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0} \tag{1.10b}$$

[10] Since, by definition, the integral

$$\int_{V'}\mathrm{d}^3x'\,\delta(\mathbf{x}'-\mathbf{x}'_i) \equiv \int_{V'}\mathrm{d}^3x'\,\delta(x'-x'_i) \times \delta(y'-y'_i)\delta(z'-z'_i) = 1$$

is dimensionless, and $x$ has the SI dimension m, the 3D Dirac delta distribution $\delta(\mathbf{x}'-\mathbf{x}'_i)$ must have the SI dimension $\mathrm{m}^{-3}$.

[11] HERMANN LUDWIG FERDINAND VON HELMHOLTZ (1821–1894) was a physicist, physician and philosopher who contributed to wide areas of science, ranging from electrodynamics to ophthalmology.

representing four scalar partial differential equations.

## 1.2 Magnetostatics

Whereas electrostatics deals with static electric charges (electric charges that do not move), and the interaction between these charges, *magnetostatics* deals with static electric currents (electric charges moving with constant speeds), and the interaction between these currents. Here we shall discuss the theory of magnetostatics in some detail.

### 1.2.1 Ampère's law

Experiments on the force interaction between two small loops that carry static electric currents $I$ and $I'$ (*i.e.* the currents $I$ and $I'$ do not vary in time) have shown that the loops interact via a mechanical force, much the same way that static electric charges interact. Let $\mathbf{F}^{\text{ms}}(\mathbf{x})$ denote the magnetostatic force on a loop $C$, with tangential line vector element $\mathrm{d}\mathbf{l}$, located at $\mathbf{x}$ and carrying a current $I$ in the direction of $\mathrm{d}\mathbf{l}$, due to the presence of a loop $C'$, with tangential line element $\mathrm{d}\mathbf{l}'$, located at $\mathbf{x}'$ and carrying a current $I'$ in the direction of $\mathrm{d}\mathbf{l}'$ in otherwise empty space. This spatial configuration is illustrated in graphical form in figure 1.3 on the facing page.

According to *Ampère's law* the magnetostatic force in question is given by the expression[12]

$$\begin{aligned}\mathbf{F}^{\text{ms}}(\mathbf{x}) &= \frac{\mu_0 I I'}{4\pi} \oint_C \mathrm{d}\mathbf{l} \times \oint_{C'} \mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\frac{\mu_0 I I'}{4\pi} \oint_C \mathrm{d}\mathbf{l} \times \oint_{C'} \mathrm{d}\mathbf{l}' \times \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{1.11}$$

[12] ANDRÉ-MARIE AMPÈRE (1775–1836) was a French mathematician and physicist who, only a few days after he learned about the findings by the Danish physicist and chemist HANS CHRISTIAN ØRSTED (1777–1851) regarding the magnetic effects of electric currents, presented a paper to the Académie des Sciences in Paris, postulating the force law that now bears his name.

In SI units, $\mu_0 = 4\pi \times 10^{-7}$ Henry per metre ($\mathrm{Hm}^{-1}$) is the *permeability of free space*. From the definition of $\varepsilon_0$ and $\mu_0$ (in SI units) we observe that

$$\varepsilon_0 \mu_0 = \frac{10^7}{4\pi c^2}\ (\mathrm{Fm}^{-1}) \times 4\pi \times 10^{-7}\ (\mathrm{Hm}^{-1}) = \frac{1}{c^2}\ (\mathrm{s}^2\mathrm{m}^{-2}) \tag{1.12}$$

which is a most useful relation.

At first glance, equation (1.11) above may appear asymmetric in terms of the loops and therefore be a force law that does not obey Newton's third law. However, by applying the vector triple product 'bac-cab' formula (F.53) on page 218,

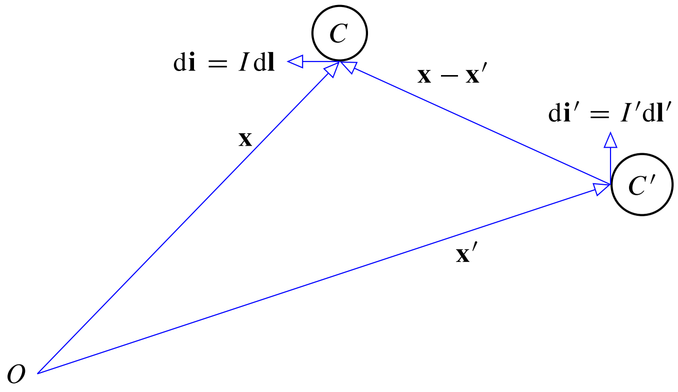


Figure 1.3: Ampère's law postulates how a small loop $C$, carrying a static electric current $I$ directed along the line element $\mathrm{d}\mathbf{l}$ at $\mathbf{x}$, experiences a magnetostatic force $\mathbf{F}^{\mathrm{ms}}(\mathbf{x})$ from a small loop $C'$, carrying a static electric current $I'$ directed along the line element $\mathrm{d}\mathbf{l}'$ located at $\mathbf{x}'$.

we can rewrite (1.11) as

$$\begin{aligned}\mathbf{F}^{\mathrm{ms}}(\mathbf{x}) = &-\frac{\mu_0 I I'}{4\pi}\oint_{C'} \mathrm{d}\mathbf{l}' \otimes \oint_{C} \mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &-\frac{\mu_0 I I'}{4\pi}\oint_{C} \mathrm{d}\mathbf{l} \cdot \oint_{C'} \mathrm{d}\mathbf{l}' \otimes \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\end{aligned} \tag{1.13}$$

Since the integrand in the first integral over $C$ is an exact differential, this integral vanishes and we can rewrite the force expression, formula (1.11) on the preceding page, in the following symmetric way

$$\mathbf{F}^{\mathrm{ms}}(\mathbf{x}) = -\frac{\mu_0 I I'}{4\pi}\oint_{C} \mathrm{d}\mathbf{l} \cdot \oint_{C'} \mathrm{d}\mathbf{l}' \otimes \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1.14}$$

which clearly exhibits the expected interchange symmetry between loops $C$ and $C'$.

### 1.2.2 The magnetostatic field

In analogy with the electrostatic case, we may attribute the magnetostatic force interaction to a static vectorial *magnetic field* $\mathbf{B}^{\mathrm{stat}}$. The small elemental static magnetic field $\mathrm{d}\mathbf{B}^{\mathrm{stat}}(\mathbf{x})$ at the field point $\mathbf{x}$ due to a line current element $\mathrm{d}\mathbf{i}'(\mathbf{x}') = I'\mathrm{d}\mathbf{l}'(\mathbf{x}') = \mathrm{d}^3x'\,\mathbf{j}(\mathbf{x}')$ of static current $I'$ with *electric current density* $\mathbf{j}$, measured in $\mathrm{Am}^{-2}$ in SI units, directed along the local line element $\mathrm{d}\mathbf{l}'$ of the loop at $\mathbf{x}'$, is

$$\begin{aligned}\mathrm{d}\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) &\stackrel{\mathrm{def}}{\equiv} \lim_{I\to 0}\frac{\mathrm{d}\mathbf{F}^{\mathrm{ms}}(\mathbf{x})}{I} = \frac{\mu_0}{4\pi}\mathrm{d}\mathbf{i}'(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\\ &= \frac{\mu_0}{4\pi}\,\mathrm{d}^3x'\,\mathbf{j}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\end{aligned} \tag{1.15}$$

which is the differential form of the *Biot-Savart law*.

The elemental field vector $\mathrm{d}\mathbf{B}^{\mathrm{stat}}(\mathbf{x})$ at the field point $\mathbf{x}$ is perpendicular to the plane spanned by the line current element vector $\mathrm{d}\mathbf{i}'(\mathbf{x}')$ at the source point $\mathbf{x}'$, and the relative position vector $\mathbf{x}-\mathbf{x}'$. The corresponding local elemental force $\mathrm{d}\mathbf{F}^{\mathrm{ms}}(\mathbf{x})$ is directed perpendicular to the local plane spanned by $\mathrm{d}\mathbf{B}^{\mathrm{stat}}(\mathbf{x})$ and the line current element $\mathrm{d}\mathbf{i}(\mathbf{x})$. The SI unit for the magnetic field, sometimes called the *magnetic flux density* or *magnetic induction*, is Tesla (T).

If we integrate expression (1.15) on the preceding page around the entire loop at $\mathbf{x}$, we obtain

$$\begin{aligned}\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) &= \int \mathrm{d}\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) \\ &= \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\,\mathbf{j}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\,\mathbf{j}(\mathbf{x}')\times\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.16}$$

where we used formula (F.114) on page 220, formula (F.87) on page 219, and the fact that $\mathbf{j}(\mathbf{x}')$ does not depend on the unprimed coordinates on which $\boldsymbol{\nabla}$ operates. Comparing equation (1.7) on page 4 with equation (1.16) above, we see that there exists an analogy between the expressions for $\mathbf{E}^{\mathrm{stat}}$ and $\mathbf{B}^{\mathrm{stat}}$ but that they differ in their vectorial characteristics. With this definition of $\mathbf{B}^{\mathrm{stat}}$, equation (1.11) on page 6 may be written

$$\mathbf{F}^{\mathrm{ms}}(\mathbf{x}) = I\oint_C \mathrm{d}\mathbf{l}\times\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) = \oint_C \mathrm{d}\mathbf{i}\times\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) \tag{1.17}$$

In order to assess the properties of $\mathbf{B}^{\mathrm{stat}}$, we determine its divergence and curl. Taking the divergence of both sides of equation (1.16) above and utilising formula (F.99) on page 220, we obtain

$$\boldsymbol{\nabla}\cdot\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = 0 \tag{1.18}$$

since, according to formula (F.99) on page 220, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{a})$ vanishes for any vector field $\mathbf{a}(\mathbf{x})$.

With the use of formula (F.128) on page 222, the curl of equation (1.16) above can be written

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B}^{\mathrm{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &= \mu_0\mathbf{j}(\mathbf{x}) - \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\,[\boldsymbol{\nabla}'\cdot\mathbf{j}(\mathbf{x}')]\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{1.19}$$

assuming that $\mathbf{j}(\mathbf{x}')$ falls off sufficiently fast at large distances. For the stationary currents of magnetostatics, $\nabla \cdot \mathbf{j} = 0$ since there cannot be any charge accumulation in space. Hence, the last integral vanishes and we can conclude that

$$\nabla \times \mathbf{B}^{\mathrm{stat}}(\mathbf{x}) = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.20}$$

We se that the static magnetic field $\mathbf{B}^{\mathrm{stat}}$ is purely *rotational*.

## 1.3 Electrodynamics

As we saw in the previous sections, the laws of electrostatics and magnetostatics can be summarised in two pairs of time-independent, uncoupled partial differential equations, namely the *equations of classical electrostatics*

$$\nabla \cdot \mathbf{E}^{\mathrm{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{1.21a}$$

$$\nabla \times \mathbf{E}^{\mathrm{stat}}(\mathbf{x}) = \mathbf{0} \tag{1.21b}$$

and the *equations of classical magnetostatics*

$$\nabla \cdot \mathbf{B}^{\mathrm{stat}}(\mathbf{x}) = 0 \tag{1.21c}$$

$$\nabla \times \mathbf{B}^{\mathrm{stat}}(\mathbf{x}) = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.21d}$$

Since there is nothing *a priori* that connects $\mathbf{E}^{\mathrm{stat}}$ directly with $\mathbf{B}^{\mathrm{stat}}$, we must consider classical electrostatics and classical magnetostatics as two separate and mutually independent physical theories.

However, when we include time-dependence, these theories are unified into *Classical Electrodynamics*. This unification of the theories of electricity and magnetism can be inferred from two empirically established facts:

1. Electric charge is a conserved quantity and electric current is a transport of electric charge. As we shall see, this fact manifests itself in the equation of continuity and, as a consequence, in *Maxwell's displacement current*.

2. A change in the magnetic flux through a loop will induce an electromotive force electric field in the loop. This is the celebrated *Faraday's law of induction*.

### 1.3.1 The indestructibility of electric charge

Let $\mathbf{j}(t,\mathbf{x})$ denote the time-dependent electric current density. In the simplest case it can be defined as $\mathbf{j} = \mathbf{v}\rho$ where $\mathbf{v}$ is the velocity of the electric charge density $\rho$.[13]

[13] A more accurate model is to assume that the individual charge and current elements obey some distribution function that describes their local variation of velocity in space and time. For instance, $\mathbf{j}$ can be defined in statistical mechanical terms as $\mathbf{j}(t,\mathbf{x}) = \sum_\alpha q_\alpha \int \mathrm{d}^3v\, \mathbf{v}\, f_\alpha(t,\mathbf{x},\mathbf{v})$ where $f_\alpha(t,\mathbf{x},\mathbf{v})$ is the (normalised) distribution function for particle species $\alpha$ carrying an electric charge $q_\alpha$.

The *electric charge conservation law* can be formulated in the *equation of continuity for electric charge*

$$\frac{\partial\rho(t,\mathbf{x})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}(t,\mathbf{x}) = 0 \tag{1.22}$$

or

$$\frac{\partial\rho(t,\mathbf{x})}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{j}(t,\mathbf{x}) \tag{1.23}$$

which states that the time rate of change of electric charge $\rho(t,\mathbf{x})$ is balanced by a negative divergence in the electric current density $\mathbf{j}(t,\mathbf{x})$, *i.e.* an influx of charge. Conservation laws will be studied in more detail in chapter 4.

### 1.3.2 Maxwell's displacement current

We recall from the derivation of equation (1.20) on the previous page that there we used the fact that in magnetostatics $\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{x}) = 0$. In the case of non-stationary sources and fields, we must, in accordance with the continuity equation (1.22) above, set $\boldsymbol{\nabla}\cdot\mathbf{j}(t,\mathbf{x}) = -\partial\rho(t,\mathbf{x})/\partial t$. Doing so, and formally repeating the steps in the derivation of equation (1.20) on the previous page, we would obtain the result

$$\boldsymbol{\nabla}\times\mathbf{B}(t,\mathbf{x}) = \mu_0\mathbf{j}(t,\mathbf{x}) + \frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\frac{\partial\rho(t,\mathbf{x}')}{\partial t}\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.24}$$

If we assume that equation (1.7) on page 4 can be generalised to time-varying fields, we can make the identification[14]

[14] Later, we will need to consider this generalisation and formal identification further.

$$\begin{aligned}
&\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\int_{V'}\mathrm{d}^3x'\,\rho(t,\mathbf{x}')\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\
&\quad= \frac{\partial}{\partial t}\left[-\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\rho(t,\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\\
&\quad= \frac{\partial}{\partial t}\left[-\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right] = \frac{\partial}{\partial t}\mathbf{E}(t,\mathbf{x})
\end{aligned} \tag{1.25}$$

The result is Maxwell's source equation for the $\mathbf{B}$ field

$$\boldsymbol{\nabla}\times\mathbf{B}(t,\mathbf{x}) = \mu_0\left(\mathbf{j}(t,\mathbf{x}) + \frac{\partial}{\partial t}\varepsilon_0\mathbf{E}(t,\mathbf{x})\right) = \mu_0\mathbf{j}(t,\mathbf{x}) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}(t,\mathbf{x}) \tag{1.26}$$

where $\varepsilon_0 \partial \mathbf{E}(t, \mathbf{x}) / \partial t$ is the famous *displacement current*. This, at the time, unobserved current was introduced by Maxwell, in a stroke of genius, in order to make also the right-hand side of this equation divergence-free when $\mathbf{j}(t, \mathbf{x})$ is assumed to represent the density of the total electric current. This total current can be split up into 'ordinary' conduction currents, polarisation currents and magnetisation currents. This will be discussed in subsection 9.1 on page 190.

The displacement current behaves like a current density flowing in free space. As we shall see later, its existence has far-reaching physical consequences as it predicts that such physical observables as electromagnetic energy, linear momentum, and angular momentum can be transmitted over very long distances, even through empty space.

### 1.3.3 Electromotive force

If an electric field $\mathbf{E}(t, \mathbf{x})$ is applied to a conducting medium, a current density $\mathbf{j}(t, \mathbf{x})$ will be set up in this medium. But also mechanical, hydrodynamical and chemical processes can give rise to electric currents. Under certain physical conditions, and for certain materials, one can assume that a linear relationship exists between the electric current density $\mathbf{j}$ and $\mathbf{E}$. This approximation is called *Ohm's law*:[15]

$$\mathbf{j}(t, \mathbf{x}) = \sigma \mathbf{E}(t, \mathbf{x}) \tag{1.27}$$

where $\sigma$ is the *electric conductivity* measured in Siemens per metre ($\mathrm{Sm}^{-1}$).

We can view Ohm's law equation (1.27) as the first term in a Taylor expansion of a general law $\mathbf{j}[\mathbf{E}(t, \mathbf{x})]$. This general law incorporates *non-linear effects* such as *frequency mixing* and *frequency conversion*. Examples of media that are highly non-linear are semiconductors and plasma. We draw the attention to the fact that even in cases when the linear relation between $\mathbf{E}$ and $\mathbf{j}$ is a good approximation, we still have to use Ohm's law with care. The conductivity $\sigma$ is, in general, time-dependent (*temporal dispersive media*) but then it is often the case that equation (1.27) above is valid for each individual temporal Fourier (spectral) component of the field. In some media, such as *magnetised plasma* and certain material, the conductivity is different in different directions. For such *electromagnetically anisotropic media* (*spatial dispersive media*) the scalar electric conductivity $\sigma$ in Ohm's law equation (1.27) has to be replaced by a *conductivity tensor*. If the response of the medium is not only anisotropic but also non-linear, higher-order tensorial terms have to be included.

If the current is caused by an applied electric field $\mathbf{E}(t, \mathbf{x})$, this electric field will exert work on the charges in the medium and, unless the medium is superconducting, there will be some energy loss. The time rate at which this energy is

[15] In semiconductors this approximation is in general applicable only for a limited range of $\mathbf{E}$. This property is used in semiconductor diodes for rectifying alternating currents.

expended is $\mathbf{j} \cdot \mathbf{E}$ per unit volume ($\mathrm{Wm}^{-3}$). If $\mathbf{E}$ is irrotational (conservative), $\mathbf{j}$ will decay away with time. Stationary currents therefore require that an electric field due to an *electromotive force* (*EMF*) is present. In the presence of such a field $\mathbf{E}^{\mathrm{emf}}$, Ohm's law, equation (1.27) on the previous page, takes the form

$$\mathbf{j} = \sigma(\mathbf{E}^{\mathrm{stat}} + \mathbf{E}^{\mathrm{emf}}) \tag{1.28}$$

The electromotive force is defined as

$$\mathcal{E} = \oint_C \mathrm{d}\mathbf{l} \cdot (\mathbf{E}^{\mathrm{stat}} + \mathbf{E}^{\mathrm{emf}}) \tag{1.29}$$

where $\mathrm{d}\mathbf{l}$ is a tangential line element of the closed loop $C$.[16]

[16] The term 'electromotive force' is something of a misnomer since $\mathcal{E}$ represents a voltage, *i.e.* its SI dimension is V.

## 1.3.4 Faraday's law of induction

In subsection 1.1.2 we derived the differential equations for the electrostatic field. Specifically, on page 5 we derived equation (1.9b) stating that $\boldsymbol{\nabla} \times \mathbf{E}^{\mathrm{stat}} = \mathbf{0}$ and hence that $\mathbf{E}^{\mathrm{stat}}$ is a *conservative field* (it can be expressed as a gradient of a scalar field). This implies that the closed line integral of $\mathbf{E}^{\mathrm{stat}}$ in equation (1.29) above vanishes and that this equation becomes

$$\mathcal{E} = \oint_C \mathrm{d}\mathbf{l} \cdot \mathbf{E}^{\mathrm{emf}} \tag{1.30}$$

It has been established experimentally that a non-conservative EMF field is produced in a closed circuit $C$ at rest if the magnetic flux through this circuit varies with time. This is formulated in *Faraday's law* which, in Maxwell's generalised form, reads

$$\begin{aligned}\mathcal{E}(t) &= \oint_C \mathrm{d}\mathbf{l} \cdot \mathbf{E}(t,\mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathrm{m}}(t) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{B}(t,\mathbf{x}) = -\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x})\end{aligned} \tag{1.31}$$

where $\Phi_{\mathrm{m}}$ is the *magnetic flux* and $S$ is the surface encircled by $C$, interpreted as a generic stationary 'loop' and not necessarily as a conducting circuit. Application of Stokes' theorem on this integral equation, transforms it into the differential equation

$$\boldsymbol{\nabla} \times \mathbf{E}(t,\mathbf{x}) = -\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \tag{1.32}$$

that is valid for arbitrary variations in the fields and constitutes the Maxwell equation which explicitly connects electricity with magnetism.

Any change of the magnetic flux $\Phi_{\mathrm{m}}$ will induce an EMF. Let us therefore consider the case, illustrated in figure 1.4 on the facing page, when the 'loop' is

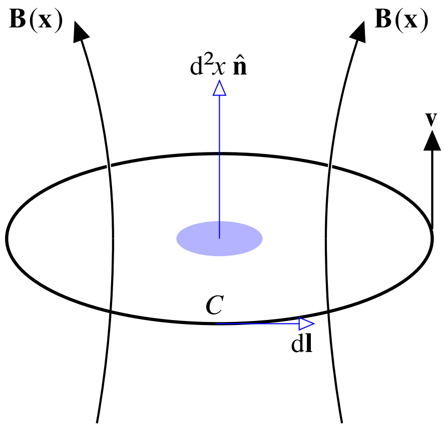


Figure 1.4: A loop $C$ that moves with velocity $\mathbf{v}$ in a spatially varying magnetic field $\mathbf{B}(\mathbf{x})$ will sense a varying magnetic flux during the motion.

moved in such a way that it encircles a magnetic field which varies during the movement. The total time derivative is evaluated according to the well-known operator formula

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \boldsymbol{\nabla} \tag{1.33}$$

This follows immediately from the multivariate chain rule for the differentiation of an arbitrary differentiable function $f(t, \mathbf{x}(t))$. Here, $\mathrm{d}\mathbf{x}/\mathrm{d}t$ describes a chosen path in space. We shall choose the flow path which means that $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{v}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{1.34}$$

where, in a continuum picture, $\mathbf{v}$ is the fluid velocity. For this particular choice, the *convective derivative* $\mathrm{d}\mathbf{x}/\mathrm{d}t$ is usually referred to as the *material derivative* and is denoted $\mathrm{D}\mathbf{x}/\mathrm{D}t$.

Applying the rule (1.34) to Faraday's law, equation (1.31) on the preceding page, we obtain

$$\mathcal{E}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{B} = -\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{1.35}$$

Furthermore, taking the divergence of equation (1.32) on the facing page, we see that

$$\boldsymbol{\nabla} \cdot \frac{\partial}{\partial t}\mathbf{B}(t, \mathbf{x}) = \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{B}(t, \mathbf{x}) = -\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \times \mathbf{E}(t, \mathbf{x})] = 0 \tag{1.36}$$

where in the last step formula (F.99) on page 220 was used. Since this is true for all times $t$, we conclude that

$$\boldsymbol{\nabla} \cdot \mathbf{B}(t, \mathbf{x}) = 0 \tag{1.37}$$

also for time-varying fields; this is in fact one of the Maxwell equations. Using this result and formula (F.89) on page 219, we find that

$$\boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{1.38}$$

since, during spatial differentiation, $\mathbf{v}$ is to be considered as constant, This allows us to rewrite equation (1.35) on the previous page in the following way:

$$\begin{aligned}\mathcal{E}(t) &= \oint_C \mathrm{d}\mathbf{l} \cdot \mathbf{E}^{\mathrm{emf}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{B} \\ &= -\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v})\end{aligned} \tag{1.39}$$

With Stokes' theorem applied to the last integral, we finally get

$$\mathcal{E}(t) = \oint_C \mathrm{d}\mathbf{l} \cdot \mathbf{E}^{\mathrm{emf}} = -\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \oint_C \mathrm{d}\mathbf{l} \cdot (\mathbf{B} \times \mathbf{v}) \tag{1.40}$$

or, rearranging the terms,

$$\oint_C \mathrm{d}\mathbf{l} \cdot (\mathbf{E}^{\mathrm{emf}} - \mathbf{v} \times \mathbf{B}) = -\int_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{1.41}$$

where $\mathbf{E}^{\mathrm{emf}}$ is the field induced in the 'loop', *i.e.* in the moving system. The application of Stokes' theorem 'in reverse' on equation (1.41) above yields

$$\boldsymbol{\nabla} \times (\mathbf{E}^{\mathrm{emf}} - \mathbf{v} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.42}$$

An observer in a fixed frame of reference measures the electric field

$$\mathbf{E} = \mathbf{E}^{\mathrm{emf}} - \mathbf{v} \times \mathbf{B} \tag{1.43}$$

and an observer in the moving frame of reference measures the following *Lorentz force* on a charge $q$

$$\mathbf{F} = q\mathbf{E}^{\mathrm{emf}} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \tag{1.44}$$

corresponding to an 'effective' electric field in the 'loop' (moving observer)

$$\mathbf{E}^{\mathrm{emf}} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{1.45}$$

Hence, we conclude that for a stationary observer, the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.46}$$

is indeed valid even if the 'loop' is moving.

## 1.3.5 The microscopic Maxwell equations

We are now able to collect the results from the above considerations and formulate the equations of classical electrodynamics, valid for arbitrary variations in time and space of the coupled electric and magnetic fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$. The equations are, in *SI units*,[17]

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.47a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.47b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{1.47c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{1.47d}$$

In these equations $\rho = \rho(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric charge density, with contributions from free as well as induced (polarisation) charges. Likewise, $\mathbf{j} = \mathbf{j}(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric current density, with contributions from conduction currents (motion of free charges) as well as all atomistic (polarisation and magnetisation) currents. As they stand, the equations therefore incorporate the classical interaction between all electric charges and currents, free or bound, in the system and are called *Maxwell's microscopic equations*. They were first formulated by Lorentz and therefore another name frequently used for them is the *Maxwell-Lorentz equations* and the name we shall use. Together with the appropriate *constitutive relations* that relate $\rho$ and $\mathbf{j}$ to the fields, and the initial and boundary conditions pertinent to the physical situation at hand, they form a system of well-posed partial differential equations that completely determine $\mathbf{E}$ and $\mathbf{B}$.

## 1.3.6 Dirac's symmetrised Maxwell equations

If we look more closely at the microscopic Maxwell equations (1.47), we see that they exhibit a certain, albeit not complete, symmetry. Dirac therefore made the *ad hoc* assumption that there exist *magnetic monopoles* represented by a *magnetic charge density*, which we denote by $\rho^{\mathrm{m}} = \rho^{\mathrm{m}}(t, \mathbf{x})$, and a *magnetic current density*, which we denote by $\mathbf{j}^{\mathrm{m}} = \mathbf{j}^{\mathrm{m}}(t, \mathbf{x})$.[18]

With these new hypothetical physical entities included in the theory, and with the electric charge density denoted $\rho^{\mathrm{e}}$ and the electric current density denoted $\mathbf{j}^{\mathrm{e}}$, the Maxwell-Lorentz equations will be symmetrised into the following two

[17] In *CGS units* the Maxwell-Lorentz equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$$
$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}$$

in *Heaviside-Lorentz units* (one of several *natural units*)

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$$
$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\mathbf{j}$$

and in *Planck units* (another set of natural units)

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$$
$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j}$$

[18] JULIAN SEYMOUR SCHWINGER (1918–1994) once put it:

> '...there are strong theoretical reasons to believe that magnetic charge exists in nature, and may have played an important role in the development of the Universe. Searches for magnetic charge continue at the present time, emphasising that electromagnetism is very far from being a closed object'.

The magnetic monopole was first postulated by PIERRE CURIE (1859–1906) and inferred from experiments in 2009.

scalar and two coupled vectorial partial differential equations (SI units):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{1.48a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} \tag{1.48b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{j}^{\mathrm{m}} \tag{1.48c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}^{\mathrm{e}} \tag{1.48d}$$

We shall call these equations *Dirac's symmetrised Maxwell equations* or the *electromagnetodynamic equations*.

Taking the divergence of (1.48c), we find that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mu_0 \boldsymbol{\nabla} \cdot \mathbf{j}^{\mathrm{m}} \equiv 0 \tag{1.49}$$

where we used the fact that the divergence of a curl always vanishes. Using (1.48b) to rewrite this relation, we obtain the *equation of continuity for magnetic charge*

$$\frac{\partial \rho^{\mathrm{m}}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}^{\mathrm{m}} = 0 \tag{1.50}$$

which has the same form as that for the electric charges (electric monopoles) and currents, equation (1.22) on page 10.

## 1.4 Examples

EXAMPLE 1.1 ▷Faraday's law derived from the assumed conservation of magnetic charge

POSTULATE 1.1 (INDESTRUCTIBILITY OF MAGNETIC CHARGE) *Magnetic charge exists and is indestructible in the same way that electric charge exists and is indestructible.*

In other words, we *postulate* that there exists an equation of continuity for magnetic charges:

$$\frac{\partial \rho^{\mathrm{m}}(t,\mathbf{x})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}^{\mathrm{m}}(t,\mathbf{x}) = 0 \tag{1.51}$$

Use this postulate and Dirac's symmetrised form of Maxwell's equations to derive Faraday's law.

The assumption of the existence of magnetic charges suggests a Coulomb-like law for magnetic fields:

$$\begin{aligned}\mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \rho^{\text{m}}(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \rho^{\text{m}}(\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x'\, \frac{\rho^{\text{m}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.52}$$

[*cf.* equation (1.7) on page 4 for $\mathbf{E}^{\text{stat}}$] and, if magnetic currents exist, a Biot-Savart-like law for electric fields [*cf.* equation (1.16) on page 8 for $\mathbf{B}^{\text{stat}}$]:

$$\begin{aligned}\mathbf{E}^{\text{stat}}(\mathbf{x}) &= -\frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}^{\text{m}}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}^{\text{m}}(\mathbf{x}') \times \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}^{\text{m}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{1.53}$$

Taking the curl of the latter and using formula (F.128) on page 222

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\mu_0\mathbf{j}^{\text{m}}(\mathbf{x}) - \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x'\, [\boldsymbol{\nabla}'\cdot\mathbf{j}^{\text{m}}(\mathbf{x}')]\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.54}$$

assuming that $\mathbf{j}^{\text{m}}$ falls off sufficiently fast at large distances. Stationarity means that $\boldsymbol{\nabla}\cdot\mathbf{j}^{\text{m}} = 0$ so the last integral vanishes and we can conclude that

$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\mu_0\mathbf{j}^{\text{m}}(\mathbf{x}) \tag{1.55}$$

It is intriguing to note that if we assume that formula (1.53) above is valid also for time-varying magnetic currents, then, with the use of the representation of the Dirac delta function, equation (F.116) on page 220, the equation of continuity for magnetic charge, equation (1.50) on the preceding page, and the assumption of the generalisation of equation (1.52) above to time-dependent magnetic charge distributions, we obtain, at least formally,

$$\boldsymbol{\nabla}\times\mathbf{E}(t,\mathbf{x}) = -\mu_0\mathbf{j}^{\text{m}}(t,\mathbf{x}) - \frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \tag{1.56}$$

[*cf.* equation (1.24) on page 10] which we recognise as equation (1.48c) on the facing page. A transformation of this electromagnetodynamic result by rotating into the 'electric realm' of charge space, thereby letting $\mathbf{j}^{\text{m}}$ tend to zero, yields the electrodynamic equation (1.47c) on page 15, *i.e.* the Faraday law in the ordinary Maxwell equations. This process would also provide an alternative interpretation of the term $\partial\mathbf{B}/\partial t$ as a *magnetic displacement current*, dual to the *electric displacement current* [*cf.* equation (1.26) on page 10].

By postulating the indestructibility of a hypothetical magnetic charge, and assuming a direct extension of results from statics to dynamics, we have been able to replace Faraday's experimental results on electromotive forces and induction in loops as a foundation for the Maxwell equations by a more fundamental one. At first sight, this result seems to be in conflict with the concept of retardation. Therefore a more detailed analysis of it is required. This analysis is left to the reader.

End of example 1.1 ◁

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2

# ELECTROMAGNETIC FIELDS AND WAVES

As a first step in the study of the dynamical properties of the classical electromagnetic field, we shall in this chapter, as an alternative to the first-order Maxwell-Lorentz equations, derive a set of second-order differential equations for the fields **E** and **B**. It turns out that these second-order equations are *wave equations* for the field vectors **E** and **B**, indicating that electromagnetic vector wave modes are very natural and common manifestations of classical electrodynamics.[1]

But before deriving these alternatives to the Maxwell-Lorentz equations, we shall discuss the mathematical techniques of making use of complex variables to represent physical observables in order to simplify the mathematical treatment. In this chapter we will also describe how to make use of the single spectral component (Fourier component) technique, which simplifies the algebra, at the same time as it clarifies the physical content.

[1] In 1864, in a lecture at the Royal Society of London, JAMES CLERK MAXWELL (1831–1879) himself said:

> 'We have strong reason to conclude that light itself — including radiant heat and other radiation, if any — is an electromagnetic disturbance in the form of waves propagated through the electro-magnetic field according to electro-magnetic laws.'

## 2.1 Axiomatic classical electrodynamics

In chapter 1 we described the historical route which led to the formulation of the microscopic Maxwell equations. From now on we shall consider these equations as *postulates*, *i.e.* as the *axiomatic foundation of classical electrodynamics*.[2] As such, these equations postulate, in scalar and vector differential equation form, the behaviour in time $t \in \mathbb{R}$ and in space $\mathbf{x} \in \mathbb{R}^3$ of the relation between the electric and magnetic fields $\mathbf{E}(t,\mathbf{x}) \in \mathbb{R}^3$ and $\mathbf{B}(t,\mathbf{x}) \in \mathbb{R}^3$, respectively, and the charge density $\rho(t,\mathbf{x}) \in \mathbb{R}$ and current density $\mathbf{j}(t,\mathbf{x}) \in \mathbb{R}^3$ [*cf.* equations (1.47) on page 15]

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss's law)} \tag{2.1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \text{(No magnetic charges)} \tag{2.1b}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0} \qquad \text{(Faraday's law)} \tag{2.1c}$$

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{j} \qquad \text{(Maxwell's law)} \tag{2.1d}$$

[2] FRITZ ROHRLICH writes in *Classical Charged Particles* that

> 'A physical theory, in the narrow sense of the word, is a logical structure based on assumptions and definitions which permits one to predict the outcome of a maximum number of different experiments on the basis of a minimum number of postulates. One usually desires the postulates (or axioms) to be as self-evident as possible; one demands simplicity, beauty, and even elegance.'

and are *not* to be viewed as equations that only describe how the fields $\mathbf{E}$ and $\mathbf{B}$ are generated by $\rho$ and $\mathbf{j}$, but also the other way around.

We reiterate that in these equations $\rho(t,\mathbf{x})$ and $\mathbf{j}(t,\mathbf{x})$ are the *total* charge and current densities, respectively. Hence, these equations are considered microscopic in the sense that *all* charges and currents, including the intrinsic ones in matter, such as bound charges in atoms and molecules as well as magnetisation currents in magnetic material, are included, but macroscopic in the sense that quantum effects are neglected. Despite the fact that the charge and current densities may not only be considered as the sources of the fields, but may equally well be considered being generated by the fields, we shall follow the convention and refer to them as the *source terms* of the microscopic Maxwell equations. Analogously, the two equations where they appear will be referred to as the *Maxwell-Lorentz source equations*.

I we allow for magnetic charge and current densities $\rho^{\mathrm{m}}$ and $\mathbf{j}^{\mathrm{m}}$, respectively, in addition to electric charge and current densities $\rho^{\mathrm{e}} \equiv \rho$ and $\mathbf{j}^{\mathrm{e}} \equiv \mathbf{j}$, we will have to replace the Maxwell-Lorentz equations by Dirac's symmetrised equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{2.2a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} = \frac{\rho^{\mathrm{m}}}{c^2 \varepsilon_0} \tag{2.2b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{j}^{\mathrm{m}} \tag{2.2c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}^{\mathrm{e}} \tag{2.2d}$$

and consider them to be the postulates of *electromagnetodynamics*.

## 2.2 Complex notation and physical observables

In order to simplify the mathematical treatment, we shall frequently allow the mathematical variables that represent the fields, the charge and current densities, and other physical quantities be analytically continued into the complex domain. However, when we use such a *complex notation* we must be very careful how to interpret the results derived within this notation. This is because every *physical observable* is, by definition, real-valued.[3] Consequently, the mathematical expression for the observable under consideration must also be real-valued to be physically meaningful.

[3] A physical observable is something that can, one way or another, be ultimately reduced to an input to the human sensory system. In other words, physical observables quantify (our perception of) the physical reality and as such they must, of course, be described by real-valued quantities.

If a physical scalar variable, or a component of a physical vector or tensor, is represented mathematically by the complex-valued number $\psi$, *i.e.* if $\psi \in \mathbb{C}$, then in classical electrodynamics (in fact, in classical physics as a whole), one makes the identification $\psi^{\text{observable}} = \operatorname{Re}\left\{\psi^{\text{mathematical}}\right\}$. Therefore, it is always understood that one must take the real part of a complex mathematical variable in order for it to represent a classical physical observable, *i.e.* something that is observed in Nature or measured in an experiment.[4]

For mathematical convenience and ease of calculation, we shall in the following regularly use — and tacitly assume — complex notation, stating explicitly when we do not. One convenient property of the complex notation is that differentiations often become trivial to perform. However, care must always be exercised. A typical situation being when products of two or more quantities are calculated since, for instance, for two complex-valued variables $\psi_1$ and $\psi_2$ we know that $\operatorname{Re}\{\psi_1\psi_2\} \neq \operatorname{Re}\{\psi_1\}\operatorname{Re}\{\psi_2\}$. On the other hand, $(\psi_1\psi_2)^* = \psi_1{}^*\psi_2{}^*$.

[4] This is at variance with quantum physics, where $\psi^{\text{observable}} = \left|\psi^{\text{mathematical}}\right|$. Letting $^*$ denote complex conjugation, the real part can be written $\operatorname{Re}\{\psi\} = \frac{1}{2}(\psi + \psi^*)$, *i.e.* as the arithmetic mean of $\psi$ and its complex conjugate $\psi^*$. Similarly, the magnitude can be written $|\psi| = (\psi\psi^*)^{1/2}$, *i.e.* as the geometric mean of $\psi$ and $\psi^*$. Under certain conditions, also the imaginary part corresponds to a physical observable.

### 2.2.1 Physical observables and averages

As just mentioned, it is important to be aware of the limitations of the mathematical technique of using a complex representation for physical observables, particularly when evaluating products of complex mathematical variables that are to represent physical observables.

Let us, for example, consider two physical vector fields $\mathbf{a}(t,\mathbf{x})$ and $\mathbf{b}(t,\mathbf{x})$ represented by their Fourier components $\mathbf{a}_0(\mathbf{x})\exp(-\mathrm{i}\omega t)$ and $\mathbf{b}_0(\mathbf{x})\exp(-\mathrm{i}\omega t)$, *i.e.* by vectors in (a domain of) 3D complex space $\mathbb{C}^3$. Furthermore, let $\circ$ be a binary infix operator for these vectors, representing either the scalar product operator $\cdot$, the vector product operator $\times$, or the dyadic product operator $\otimes$. To ensure that any products of $\mathbf{a}$ and $\mathbf{b}$ represent an observable in classical physics, we must make the interpretation

$$\begin{aligned} \mathbf{a}(t,\mathbf{x}) \circ \mathbf{b}(t,\mathbf{x}) &\equiv \operatorname{Re}\{\mathbf{a}\} \circ \operatorname{Re}\{\mathbf{b}\} \\ &= \operatorname{Re}\left\{\mathbf{a}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \circ \operatorname{Re}\left\{\mathbf{b}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \end{aligned} \tag{2.3}$$

We can express the real part of the complex vectors $\mathbf{a}$ and $\mathbf{b}$ as

$$\operatorname{Re}\{\mathbf{a}\} = \operatorname{Re}\left\{\mathbf{a}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} = \frac{1}{2}[\mathbf{a}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathbf{a}_0{}^*(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}] \tag{2.4a}$$

and

$$\operatorname{Re}\{\mathbf{b}\} = \operatorname{Re}\left\{\mathbf{b}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} = \frac{1}{2}[\mathbf{b}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathbf{b}_0{}^*(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}] \tag{2.4b}$$

respectively. Hence, the physically acceptable interpretation of the scalar product of two complex vectors, representing classical physical observables, is

$$
\begin{aligned}
\mathbf{a}(t,\mathbf{x}) \circ \mathbf{b}(t,\mathbf{x}) &= \mathrm{Re}\left\{\mathbf{a}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \circ \mathrm{Re}\left\{\mathbf{b}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \\
&= \frac{1}{2}[\mathbf{a}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t} + {\mathbf{a}_0}^*(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}] \circ \frac{1}{2}[\mathbf{b}_0(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t} + {\mathbf{b}_0}^*(\mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}] \\
&= \frac{1}{4}\left(\mathbf{a}_0 \circ {\mathbf{b}_0}^* + {\mathbf{a}_0}^* \circ \mathbf{b}_0 + \mathbf{a}_0 \circ \mathbf{b}_0\,\mathrm{e}^{-2\mathrm{i}\omega t} + {\mathbf{a}_0}^* \circ {\mathbf{b}_0}^*\,\mathrm{e}^{2\mathrm{i}\omega t}\right) \\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_0 \circ {\mathbf{b}_0}^*\right\} + \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_0 \circ \mathbf{b}_0\,\mathrm{e}^{-2\mathrm{i}\omega t}\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_0\,\mathrm{e}^{-\mathrm{i}\omega t} \circ {\mathbf{b}_0}^*\,\mathrm{e}^{\mathrm{i}\omega t}\right\} + \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_0\mathrm{e}^{-\mathrm{i}\omega t} \circ \mathbf{b}_0\,\mathrm{e}^{-\mathrm{i}\omega t}\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}(t,\mathbf{x}) \circ \mathbf{b}^*(t,\mathbf{x})\right\} + \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}(t,\mathbf{x}) \circ \mathbf{b}(t,\mathbf{x})\right\}
\end{aligned}
\tag{2.5}
$$

In physics, we are often forced to measure the *temporal average* (*cycle average*) of a physical observable. We use the notation $\langle\cdots\rangle_t$ for such an average and find that the average of the product of the two physical quantities represented by **a** and **b** can be expressed as

$$
\begin{aligned}
\langle \mathbf{a} \circ \mathbf{b}\rangle_t &\equiv \langle \mathrm{Re}\left\{\mathbf{a}\right\} \circ \mathrm{Re}\left\{\mathbf{b}\right\}\rangle_t = \frac{1}{2}\mathrm{Re}\left\{\mathbf{a} \circ \mathbf{b}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}^* \circ \mathbf{b}\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_0 \circ {\mathbf{b}_0}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{{\mathbf{a}_0}^* \circ \mathbf{b}_0\right\}
\end{aligned}
\tag{2.6}
$$

This is because the oscillating function $\exp(-2\mathrm{i}\omega t)$ in equation (2.5) above vanishes when averaged in time over a complete cycle $2\pi/\omega$ (or over infinitely many cycles), and, therefore, $\langle \mathbf{a}(t,\mathbf{x}) \circ \mathbf{b}(t,\mathbf{x})\rangle_t$ gives no contribution.

### 2.2.2 Maxwell equations in Majorana representation

It is often convenient to express electrodynamics in terms of the *complex-field six-vector*, also known as the *Riemann-Silberstein vector*,

$$
\mathbf{G}(t,\mathbf{x}) = \mathbf{E}(t,\mathbf{x}) + \mathrm{i}c\mathbf{B}(t,\mathbf{x}) \tag{2.7}
$$

where $\mathbf{G} \in \mathbb{C}^3$ even if $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$. If we use this vector, the Maxwell equations (2.1) on page 19 can be written

$$
\boldsymbol{\nabla} \cdot \mathbf{G} = \frac{\rho}{\varepsilon_0} \tag{2.8a}
$$

$$
\boldsymbol{\nabla} \times \mathbf{G} - \frac{\mathrm{i}}{c}\frac{\partial \mathbf{G}}{\partial t} = \mathrm{i}c\mu_0\mathbf{j} \tag{2.8b}
$$

In regions where $\rho = 0$ and $\mathbf{j} = \mathbf{0}$ these equations reduce to

$$
\boldsymbol{\nabla} \cdot \mathbf{G} = 0 \tag{2.9a}
$$

$$
\boldsymbol{\nabla} \times \mathbf{G} = \frac{\mathrm{i}}{c}\frac{\partial \mathbf{G}}{\partial t} \tag{2.9b}
$$

which, with the help of the *linear momentum operator*

$$\hat{\mathsf{p}} = -\mathrm{i}\hbar\boldsymbol{\nabla} \tag{2.10}$$

can be rewritten

$$\hat{\mathsf{p}} \cdot \mathbf{G} = 0 \tag{2.11a}$$

$$\mathrm{i}\hbar\frac{\partial \mathbf{G}}{\partial t} = c\mathrm{i}\hat{\mathsf{p}} \times \mathbf{G} \tag{2.11b}$$

The first equation is the *transversality condition* $\mathbf{p} = \hbar\mathbf{k} \perp \mathbf{G}$ where we anticipate the usual quantal relation between the *field momentum* $\mathbf{p}$ and the *wave vector* $\mathbf{k}$,[5] whereas the second equation describes the dynamics.

Using formula (F.105) on page 220, we can rewrite equation (2.11b) above as

$$\mathrm{i}\hbar\frac{\partial \mathbf{G}}{\partial t} = c(\mathsf{S} \cdot \hat{\mathsf{p}})\mathbf{G} \tag{2.12}$$

where

$$\mathsf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \hat{\mathbf{x}}_1 + \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \hat{\mathbf{x}}_2 + \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{\mathbf{x}}_3 \tag{2.13}$$

Then by introducing the Hamiltonian-like operator

$$\hat{\mathsf{H}} = c\mathsf{S} \cdot \hat{\mathsf{p}} = -\mathrm{i}c\hbar\mathsf{S} \cdot \boldsymbol{\nabla} \tag{2.14}$$

we can write the Maxwell-Lorentz curl equations as

$$\mathrm{i}\hbar\frac{\partial \mathbf{G}}{\partial t} = \hat{\mathsf{H}}\mathbf{G} \tag{2.15}$$

*i.e.* as a Schrödinger/Pauli/Dirac-like equation; sometimes these are referred to as *neutrino equations*. This formulation of the free-space electromagnetic field equations is known as the *Majorana representation* of the Maxwell-Lorentz equations or the *Majorana formalism*.[6]

[5] The scalar quantity $\hbar = h/(2\pi)$ is the *reduced Planck constant* where the *Planck constant* proper $h \approx 6.62606957 \times 10^{-34}$ Js.

[6] It so happens that ETTORE MAJORANA (1906-1938) used the definition $\mathbf{G} = \mathbf{E} - \mathrm{i}c\mathbf{B}$, but this is no essential difference from the definition (2.7). One may say that Majorana used the other branch of $\sqrt{-1}$ as the imaginary unit.

## 2.3 The wave equations for **E** and **B**

The Maxwell-Lorentz equations (2.1) on page 19 are four first-order *coupled differential equations* (both **E** and **B** appear in the same equations). Two of the equations are scalar [equations (2.1a) and (2.1b)], and two are in 3D Euclidean vector form [equations (2.1c) and (2.1d)], representing three scalar equations each. Hence, the Maxwell equations represent eight $(1 + 1 + 3 + 3 = 8)$ scalar coupled first-order partial differential equations. However, it is well known from the theory of differential equations that a set of first-order, coupled partial differential equations can be transformed into a smaller set of second-order partial differential equations that sometimes become decoupled in the process. It turns out that in our case we will obtain one second-order differential equation for **E** and one second-order differential equation for **B**. These second-order partial differential equations are, as we shall see, *wave equations*, and we shall discuss their properties and implications. In certain propagation media, the **B** wave field can be easily obtained from the solution of the **E** wave equation but in general this is not the case.

To bring the first-order differential equations (2.1) on page 19 into second order one needs, of course, to operate on them with first-order differential operators. If we apply the curl vector operator $(\boldsymbol{\nabla}\times)$ to both sides of the two Maxwell-Lorentz vector equations (2.1c) and (2.1d) on page 19, assuming that the fields vary in such a regular way that temporal and spatial differentiation commute, we obtain the second-order differential equations

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})+\boldsymbol{\nabla}\times\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})+\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B})=\mathbf{0} \tag{2.16a}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})-\frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{E})=\mu_0\boldsymbol{\nabla}\times\mathbf{j} \tag{2.16b}$$

As they stand, these second-order partial differential equations still appear to be coupled. However, by using the Maxwell-Lorentz equations (2.1) on page 19 once again we can formally decouple them into

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})=-\mu_0\frac{\partial\mathbf{j}}{\partial t} \tag{2.17a}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{B})=\mu_0\boldsymbol{\nabla}\times\mathbf{j} \tag{2.17b}$$

If we use the operator triple product 'bac-cab' formula (F.96) on page 220, which gives

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})-\nabla^2\mathbf{E} \tag{2.18}$$

when applied to **E** and similarly to **B**, Gauss's law equation (2.1a) on page 19, and then rearrange the terms, we obtain the two inhomogeneous *vector wave equations*

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}-\nabla^2\mathbf{E}=\Box^2\mathbf{E}=-\frac{\boldsymbol{\nabla}\rho}{\varepsilon_0}-\mu_0\frac{\partial\mathbf{j}}{\partial t} \tag{2.19a}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}-\nabla^2\mathbf{B}=\Box^2\mathbf{B}=\mu_0\boldsymbol{\nabla}\times\mathbf{j} \tag{2.19b}$$

where $\square^2$ is the *d'Alembert operator*, defined by expression (F.182) on page 228. These are the general wave equations for the electromagnetic fields in regions where there exist sources $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$ of any kind. Simple everyday examples of such regions are electric conductors (*e.g.* radio and TV transmitter antennas) or plasma (*e.g.* the Sun and its surrounding corona). In principle, the sources $\rho$ and $\mathbf{j}$ can still cause the wave equations to be coupled, but in many important situations this is not the case.[7]

We notice that outside the source region, *i.e.* in free space where $\rho = 0$ and $\mathbf{j} = \mathbf{0}$, the inhomogeneous wave equations (2.19) on the preceding page simplify to the well-known uncoupled, homogeneous wave equations

$$\square^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = \mathbf{0} \tag{2.20a}$$

$$\square^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = \mathbf{0} \tag{2.20b}$$

These equations describe how the fields that were generated in the source region, propagate as *vector waves* through free space. Once these waves impinge upon another region that can sustain charges and/or currents for a long enough time, such as a receiving antenna or other electromagnetic sensors, the fields interact with the charges and the currents in this second region in accordance with equations (2.19) on the facing page.

[7] Clearly, if the current density in the RHS ('right-hand side') of equation (2.19b) is a function of $\mathbf{E}$, as is the case if, for instance, Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ is applicable, the coupling is not removed.

## 2.3.1 The time-independent wave equations for E and B

Often one can assume that the temporal dependency of $\mathbf{E}$ and $\mathbf{B}$ and of the sources $\rho$ and $\mathbf{j}$ is well-behaved enough that it can be represented by the sum of a finite number $N$ of *temporal spectral components* (*temporal Fourier components*), or, in other words, in the form of a *temporal Fourier series*. In such situations it is sufficient to study the properties of one arbitrary member of this set of spectral components $\{\omega_n : n = 1, 2, 3, \ldots, N\}$, *i.e.*

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_n(\mathbf{x})\cos(\omega_n t) = \mathbf{E}_n(\mathbf{x})\mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}\omega_n t}\right\} \equiv \mathbf{E}_n(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{2.21a}$$

$$\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_n(\mathbf{x})\cos(\omega_n t) = \mathbf{B}_n(\mathbf{x})\mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}\omega_n t}\right\} \equiv \mathbf{B}_n(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{2.21b}$$

where $\mathbf{E}_n, \mathbf{B}_n \in \mathbb{R}^3$ and $\omega_n, t \in \mathbb{R}$ is assumed. This is because the Maxwell-Lorentz equations are linear, implying that the general solution is obtained by a weighted linear superposition (summation) of the result for each such spectral component, where the weight of the spectral component in question is given by its *Fourier amplitude*, $\mathbf{E}_n(\mathbf{x})$, and $\mathbf{B}_n(\mathbf{x})$, respectively.

In a physical system, a temporal spectral component is identified uniquely by its *angular frequency* $\omega_n$. A wave containing only a finite number of temporal spectral components is called a *time-harmonic wave*. In the limit when only one single frequency is present, we talk about a *monochromatic wave*. Strictly speaking, purely monochromatic waves do not exist.[8]

By inserting the temporal spectral component equation (2.21a) on the previous page into equation (2.19a) on page 24 one finds that for an arbitrary component the following equation is obtained:

$$\nabla^2 \mathbf{E}_n \mathrm{e}^{-\mathrm{i}\omega_n t} + \frac{\omega_n^2}{c^2}\mathbf{E}_n \mathrm{e}^{-\mathrm{i}\omega_n t} + \mathrm{i}\omega_n \mu_0 \mathbf{j}_n \mathrm{e}^{-\mathrm{i}\omega_n t} = \frac{\boldsymbol{\nabla}\rho_n}{\varepsilon_0}\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{2.22}$$

After dividing out the common factor $\exp(-\mathrm{i}\omega_n t)$, we obtain the *time-independent wave equation*

$$\nabla^2 \mathbf{E}_n + \frac{\omega_n^2}{c^2}\mathbf{E}_n + \mathrm{i}\omega_n \mu_0 \mathbf{j}_n = \frac{\boldsymbol{\nabla}\rho_n}{\varepsilon_0} \tag{2.23}$$

and similarly for $\mathbf{B}$. Solving this equation, multiplying the solution obtained by $\exp(-\mathrm{i}\omega_n t)$ and summing over all $N$ such Fourier (spectral) components with frequencies $\omega_n, n = 1, 2, 3, \ldots, N$ present in the sources, and hence in the fields, the complete solution of the original wave equation is obtained. This is a consequence of the *superposition principle* which is valid as long as nonlinear effects can be neglected.

In the limit of very many frequency components, the Fourier sum goes over into a *Fourier integral*. To illustrate this generic case, let us introduce the *Fourier transform* of $\mathbf{E}(t, \mathbf{x})$

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{E}(t, \mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \tag{2.24a}$$

and the corresponding *inverse Fourier transform*

$$\mathbf{E}(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{E}_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{2.24b}$$

where the amplitude $\mathbf{E}_\omega(\mathbf{x}) \in \mathbb{C}^3$ is a continuous function of (angular) frequency $\omega \in \mathbb{R}$ and of $\mathbf{x} \in \mathbb{R}^3$.

We see that the Fourier transform of $\partial\mathbf{E}(t, \mathbf{x})/\partial t$ becomes

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \left(\frac{\partial \mathbf{E}(t, \mathbf{x})}{\partial t}\right) \mathrm{e}^{\mathrm{i}\omega t} \\ &\quad = \frac{1}{2\pi}\underbrace{\left[\mathbf{E}(t, \mathbf{x})\,\mathrm{e}^{\mathrm{i}\omega t}\right]_{-\infty}^{\infty}}_{=\mathbf{0}\text{ since }\mathbf{E}\to\mathbf{0},\, t\to\pm\infty} - \mathrm{i}\omega\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{E}(t, \mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \\ &\quad = -\mathrm{i}\omega\, \mathbf{E}_\omega(\mathbf{x}) \end{aligned} \tag{2.25}$$

and that, consequently,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \left(\frac{\partial^2 \mathbf{E}(t, \mathbf{x})}{\partial t^2}\right) \mathrm{e}^{\mathrm{i}\omega t} = -\omega^2 \mathbf{E}_\omega(\mathbf{x}) \tag{2.26}$$

[8] When subtle classical and quantum radiation effects are taken into account, one finds that all emissions suffer an unavoidable, intrinsic *line broadening*. Also, simply because the Universe has existed for only about 13.5 billion years, which is a finite time, no signals in the Universe can be observed to have a spectral width that is smaller than the inverse of this age.

Fourier transforming equation (2.19a) on page 24, and using formulæ (2.25) and (2.26) on the facing page, we thus obtain

$$\nabla^2 \mathbf{E}_\omega + \frac{\omega^2}{c^2}\mathbf{E}_\omega + \mathrm{i}\omega\mu_0 \mathbf{j}_\omega = \frac{\boldsymbol{\nabla}\rho_\omega}{\varepsilon_0} \tag{2.27}$$

which is mathematically identical to equation (2.23) on the preceding page. A subsequent inverse Fourier transformation of the solution $\mathbf{E}_\omega$ of this equation leads to the same result as is obtained from the solution of equation (2.23) on the facing page. Hence, by considering just one temporal Fourier component we obtain results which are identical to those that we would have obtained by employing the machinery of Fourier transforms and Fourier integrals. Hence, under the assumption of linearity (superposition principle) there is usually no need for the formal forward and inverse Fourier transform technique.

What was said above in general terms about temporal spectral components is true also for *spatial spectral components* (*spatial Fourier components*) only that we must use a three-dimensional Fourier representation

$$\mathbf{E}_\mathbf{k}(t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathrm{d}^3x\, \mathbf{E}(t,\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.28a}$$

$$\mathbf{E}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}^3k\, \mathbf{E}_\mathbf{k}(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.28b}$$

Since we always assume that the real part shall be taken (if necessary), we can pick any pair of the spatial amplitudes in equations (2.21a) and (2.21b) on page 25, denote the members of this pair by $\mathbf{E}_0$ and $\mathbf{B}_0$, respectively, and then represent them as the Fourier modes

$$\mathbf{E}_0(\mathbf{x}) = \mathbf{e}_0 \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}}\right\} = \mathbf{E}_{\omega,\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.29a}$$

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{b}_0 \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{x}}\right\} = \mathbf{B}_{\omega,\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.29b}$$

respectively, where $\mathbf{k}_0$ is the *wave vector* (measured in $\mathrm{m}^{-1}$) of mode 0; in the last step we introduced a complex notation and also dropped the mode number since the formulæ are valid for any mode $n$.

Now, since

$$\frac{\partial}{\partial t}\mathrm{e}^{-\mathrm{i}\omega t} = -\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega t} \tag{2.30a}$$

and

$$\boldsymbol{\nabla}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \hat{\mathbf{x}}_i\frac{\partial}{\partial x_i}\mathrm{e}^{\mathrm{i}k_j x_j} = \mathrm{i}\hat{\mathbf{x}}_i k_j \delta_{ij}\mathrm{e}^{\mathrm{i}k_j x_j} = \mathrm{i}\hat{\mathbf{x}}_i k_i \mathrm{e}^{\mathrm{i}k_j x_j} = \mathrm{i}\mathbf{k}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{2.30b}$$

we see that for each spectral component in equations (2.29) above, temporal and spatial differential operators turn into algebraic operations according to the following scheme:

$$\frac{\partial}{\partial t} \mapsto -\mathrm{i}\omega \tag{2.31a}$$
$$\boldsymbol{\nabla} \mapsto \mathrm{i}\mathbf{k} \tag{2.31b}$$
$$\boldsymbol{\nabla}\cdot \mapsto \mathrm{i}\mathbf{k}\cdot \tag{2.31c}$$
$$\boldsymbol{\nabla}\times \mapsto \mathrm{i}\mathbf{k}\times \tag{2.31d}$$

We note that

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \mathrm{i}\mathbf{k}\cdot\mathbf{E} = \mathrm{i}\mathbf{k}\cdot\mathbf{E}_\parallel \tag{2.32a}$$
$$\boldsymbol{\nabla}\times\mathbf{E} = \mathrm{i}\mathbf{k}\times\mathbf{E} = \mathrm{i}\mathbf{k}\times\mathbf{E}_\perp \tag{2.32b}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = \mathrm{i}\mathbf{k}\cdot\mathbf{B} = \mathrm{i}\mathbf{k}\cdot\mathbf{B}_\parallel \tag{2.32c}$$
$$\boldsymbol{\nabla}\times\mathbf{B} = \mathrm{i}\mathbf{k}\times\mathbf{B} = \mathrm{i}\mathbf{k}\times\mathbf{B}_\perp \tag{2.32d}$$
$$\boldsymbol{\nabla}\rho = \mathrm{i}\mathbf{k}\rho \tag{2.32e}$$
$$\boldsymbol{\nabla}\cdot\mathbf{j} = \mathrm{i}\mathbf{k}\cdot\mathbf{j} = \mathrm{i}\mathbf{k}\cdot\mathbf{j}_\parallel \tag{2.32f}$$
$$\boldsymbol{\nabla}\times\mathbf{j} = \mathrm{i}\mathbf{k}\times\mathbf{j} = \mathrm{i}\mathbf{k}\times\mathbf{j}_\perp \tag{2.32g}$$

Hence, with respect to the wave vector $\mathbf{k}$, the $\boldsymbol{\nabla}\cdot$ operator projects out the spatially longitudinal component, and the $\boldsymbol{\nabla}\times$ operator projects out the spatially transverse component of the field vector in question. Put in another way,

$$\boldsymbol{\nabla}\cdot\mathbf{E}_\perp = 0 \tag{2.33a}$$
$$\boldsymbol{\nabla}\times\mathbf{E}_\parallel = \mathbf{0} \tag{2.33b}$$

and so on for the other observables. This can be viewed as a instance in $\mathbf{k}$ space of the *Helmholtz's theorem*, discussed in subsection M.3.7 on page 247, saying that if $\mathbf{E}$ falls off suitably rapidly at infinity, it can be written as the sum of a rotational part $\mathbf{E}^{\text{rotat}}$ and an irrotational part $\mathbf{E}^{\text{irrot}}$:

$$\mathbf{E} = \mathbf{E}^{\text{rotat}} + \mathbf{E}^{\text{irrot}} \tag{2.34}$$

where, according to equations (M.85) on page 248,

$$\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{rotat}} = 0 \tag{2.35a}$$
$$\boldsymbol{\nabla}\times\mathbf{E}^{\text{irrot}} = \mathbf{0} \tag{2.35b}$$

by making the *formal* identification[9]

$$\mathbf{E}_\perp = \mathbf{E}^{\text{rotat}} \tag{2.36a}$$
$$\mathbf{E}_\parallel = \mathbf{E}^{\text{irrot}} \tag{2.36b}$$

[9] Strictly speaking, the spatial Fourier component is a plane wave and plane waves do not fall off at all at infinity as required for the Helmholtz decomposition to be applicable. However, only wave packets made up of a sum of plane waves are physically acceptable and such packages can be localised well enough.

For the magnetic field

$$\mathbf{B} = \mathbf{B}^{\text{rotat}} + \mathbf{B}^{\text{irrot}} \tag{2.37}$$

where

$$\nabla \cdot \mathbf{B}^{\text{rotat}} = 0 \tag{2.38a}$$
$$\nabla \times \mathbf{B}^{\text{irrot}} = \mathbf{0} \tag{2.38b}$$

we make the analogous formal identification

$$\mathbf{B}_{\perp} = \mathbf{B}^{\text{rotat}} \tag{2.39}$$
$$\mathbf{B}_{\parallel} = \mathbf{B}^{\text{irrot}} \tag{2.40}$$

As we see from equations (2.31) on the preceding page, the Fourier transform of a function of time $t$ is a function of angular frequency $\omega$, and the Fourier transform of a function of the position vector $\mathbf{x}$ is a function of the wave vector $\mathbf{k}$. One says that $\omega$ is a *reciprocal space* to $t$ and that $\mathbf{k}$ spans a space that is reciprocal to $\mathbf{x}$. In the reciprocal $\omega$ and $\mathbf{k}$ space the Maxwell-Lorentz equations are

$$\mathrm{i}\mathbf{k} \cdot \mathbf{E}_{\parallel} = \frac{\rho}{\varepsilon_0} \tag{2.41a}$$
$$\mathrm{i}\mathbf{k} \cdot \mathbf{B}_{\parallel} = 0 \tag{2.41b}$$
$$\mathrm{i}\mathbf{k} \times \mathbf{E}_{\perp} - \mathrm{i}\omega\mathbf{B} = \mathbf{0} \tag{2.41c}$$
$$\mathrm{i}\mathbf{k} \times \mathbf{B}_{\perp} + \mathrm{i}\frac{\omega}{c^2}\mathbf{E} = \mu_0 \mathbf{j} \tag{2.41d}$$

Applying the Helmholtz decomposition, the Maxwell-Lorentz equations become

$$\nabla \cdot \mathbf{E}^{\text{irrot}} = \frac{\rho}{\varepsilon_0} \tag{2.42a}$$
$$\nabla \cdot \mathbf{B}^{\text{irrot}} = 0 \tag{2.42b}$$
$$\nabla \times \mathbf{E}^{\text{rotat}} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{2.42c}$$
$$\nabla \times \mathbf{B}^{\text{rotat}} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{2.42d}$$

## 2.4 Examples

EXAMPLE 2.1 ▷Products of Riemann-Silberstein vectors with themselves for $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$

One fundamental property of the 3D complex vector space $\mathbb{C}^3$, to which the Riemann-Silberstein vector $\mathbf{G} = \mathbf{E} + \mathrm{i}c\mathbf{B}$ belongs, is that inner (scalar) products in this space are invariant under rotations just as they are in $\mathbb{R}^3$. However, as discussed in example M.4 on page 255, products in $\mathbb{C}^3$ can be defined in two different ways. Considering the special case of the scalar product of $\mathbf{G}$ with itself, assuming that $\mathbf{E} \in \mathbb{R}^3$ and $\mathbf{B} \in \mathbb{R}^3$, we have the following two possibilities:

1. The inner (scalar) product defined as $\mathbf{G}$ scalar multiplied with itself

$$\mathbf{G}\cdot\mathbf{G} = (\mathbf{E} + \mathrm{i}c\mathbf{B})\cdot(\mathbf{E} + \mathrm{i}c\mathbf{B}) = \mathbf{E}\cdot\mathbf{E} - c^2\mathbf{B}\cdot\mathbf{B} + 2\mathrm{i}c\mathbf{E}\cdot\mathbf{B} \tag{2.43}$$

   Since 'length' is a scalar quantity that is invariant under rotations, we find that

$$E^2 - c^2B^2 = Const \tag{2.44a}$$
$$\mathbf{E}\cdot\mathbf{B} = Const \tag{2.44b}$$

2. The inner (scalar) product defined as $\mathbf{G}$ scalar multiplied with the complex conjugate of itself

$$\mathbf{G}\cdot\mathbf{G}^* = (\mathbf{E} + \mathrm{i}c\mathbf{B})\cdot(\mathbf{E} - \mathrm{i}c\mathbf{B}) = \mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B} \equiv E^2 + c^2B^2 \tag{2.45}$$

   is also an invariant scalar quantity. As we shall see in chapter chapter 4, this quantity is proportional to the *electromagnetic field energy density*.

3. As with any 3D vector, the cross product of $\mathbf{G}$ with itself vanishes:

$$\begin{aligned}\mathbf{G}\times\mathbf{G} &= (\mathbf{E} + \mathrm{i}c\mathbf{B})\times(\mathbf{E} + \mathrm{i}c\mathbf{B}) \\ &= \mathbf{E}\times\mathbf{E} - c^2\mathbf{B}\times\mathbf{B} + \mathrm{i}c(\mathbf{E}\times\mathbf{B}) + \mathrm{i}c(\mathbf{B}\times\mathbf{E}) \\ &= \mathbf{0} + \mathbf{0} + \mathrm{i}c(\mathbf{E}\times\mathbf{B}) - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) = \mathbf{0}\end{aligned} \tag{2.46}$$

4. The cross product of $\mathbf{G}$ with the complex conjugate of itself does, however, not vanish. Instead it is

$$\begin{aligned}\mathbf{G}\times\mathbf{G}^* &= (\mathbf{E} + \mathrm{i}c\mathbf{B})\times(\mathbf{E} - \mathrm{i}c\mathbf{B}) \\ &= \mathbf{E}\times\mathbf{E} + c^2\mathbf{B}\times\mathbf{B} - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) + \mathrm{i}c(\mathbf{B}\times\mathbf{E}) \\ &= \mathbf{0} + \mathbf{0} - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) - \mathrm{i}c(\mathbf{E}\times\mathbf{B}) = -2\mathrm{i}c(\mathbf{E}\times\mathbf{B})\end{aligned} \tag{2.47}$$

   which is proportional to the electromagnetic energy flux (the so called *Poynting vector* or the *electromagnetic linear momentum density*), to be introduced in chapter chapter 4.

5. The dyadic product of $\mathbf{G}$ with itself is

$$\mathbf{G}\otimes\mathbf{G} = (\mathbf{E} + \mathrm{i}c\mathbf{B})(\mathbf{E} + \mathrm{i}c\mathbf{B}) = \mathbf{E}\otimes\mathbf{E} - c^2\mathbf{B}\otimes\mathbf{B} + \mathrm{i}c(\mathbf{E}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{E}) \tag{2.48}$$

   or, in component form,

$$(\mathbf{G}\otimes\mathbf{G})_{ij} = E_iE_j - c^2B_iB_j + \mathrm{i}c(E_iB_j + B_iE_j) \tag{2.49}$$

6. The dyadic product of $\mathbf{G}^*$ with itself is

$$\mathbf{G}^* \otimes \mathbf{G}^* = (\mathbf{E} - \mathrm{i}c\mathbf{B})(\mathbf{E} - \mathrm{i}c\mathbf{B}) = \mathbf{E} \otimes \mathbf{E} - c^2\mathbf{B} \otimes \mathbf{B} - \mathrm{i}c(\mathbf{E} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{E}) = (\mathbf{G} \otimes \mathbf{G})^* \tag{2.50}$$

7. The dyadic product of $\mathbf{G}$ with its own complex conjugate from the right is

$$\mathbf{G} \otimes \mathbf{G}^* = (\mathbf{E} + \mathrm{i}c\mathbf{B})(\mathbf{E} - \mathrm{i}c\mathbf{B}) = \mathbf{E} \otimes \mathbf{E} + c^2\mathbf{B} \otimes \mathbf{B} - \mathrm{i}c(\mathbf{E} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{E}) \tag{2.51}$$

and from the left it is

$$\mathbf{G}^* \otimes \mathbf{G} = (\mathbf{E} - \mathrm{i}c\mathbf{B})(\mathbf{E} + \mathrm{i}c\mathbf{B}) = \mathbf{E} \otimes \mathbf{E} + c^2\mathbf{B} \otimes \mathbf{B} + \mathrm{i}c(\mathbf{E} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{E}) = (\mathbf{G} \otimes \mathbf{G}^*)^* \tag{2.52}$$

End of example 2.1 ◁

▷ Wave polarisation

EXAMPLE 2.2

Since electromagnetic waves are vector waves they exhibit *wave polarisation*. Let us consider a single *plane wave* that propagates in free space,[10] *i.e.* a wave where the electric and magnetic field vectors are restricted to a two-dimensional plane that is perpendicular to the propagation direction. Let us choose this plane to be the $x_1x_2$ plane and the *propagation vector* (wave vector) $\mathbf{k}$ to be along the $x_3$ axis: $\mathbf{k} = k\hat{\mathbf{x}}_3$. A generic temporal Fourier mode of the electric field vector $\mathbf{E}$ with (angular) frequency $\omega$ is therefore described by the real-valued expression

$$\mathbf{E}(t, \mathbf{x}) = E_1 \cos(\omega t - kx_3 + \delta_1)\,\hat{\mathbf{x}}_1 + E_2 \cos(\omega t - kx_3 + \delta_2)\,\hat{\mathbf{x}}_2 \tag{2.53}$$

where the amplitudes $E_i$ and phases $\delta_i$, can take any value. In complex notation we can write this as

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &= E_1 \mathrm{e}^{\mathrm{i}\delta_1}\, \mathrm{e}^{\mathrm{i}(kx_3 - \omega t)}\, \hat{\mathbf{x}}_1 + E_2 \mathrm{e}^{\mathrm{i}\delta_2}\, \mathrm{e}^{\mathrm{i}(kx_3 - \omega t)}\, \hat{\mathbf{x}}_2 \\ &= \left(E_1 \mathrm{e}^{\mathrm{i}\delta_1} \hat{\mathbf{x}}_1 + E_2 \mathrm{e}^{\mathrm{i}\delta_2} \hat{\mathbf{x}}_2\right) \mathrm{e}^{\mathrm{i}(kx_3 - \omega t)} \\ &= \left(E_1 \hat{\mathbf{x}}_1 + E_2 \mathrm{e}^{\mathrm{i}\delta_0} \hat{\mathbf{x}}_2\right) \mathrm{e}^{\mathrm{i}(kx_3 - \omega t + \delta_1)}\end{aligned} \tag{2.54}$$

where $\delta_0 = \delta_2 - \delta_1$. When this phase difference $\delta_0$ vanishes, the electric field oscillates along a line directed at an angle $\arctan(E_2/E_1)$ relative to the $\mathbf{x}_1$ axis. This is called *linear wave polarisation*. When $\delta_0 \neq 0$ the wave is in general in a state of *elliptical wave polarisation*. Later, in subsection 4.2.5, we will show that wave polarisation is a manifestation of the fact that the electromagnetic field carries angular momentum.

For the special cases $\delta_0 = \pm\pi/2$ and $E_1 = E_2 = E_0$ the wave can, in complex notation, be described as

$$\mathbf{E}(t, \mathbf{x}) = E_0\, \mathrm{e}^{\mathrm{i}(kx_3 - \omega t + \delta_1)} (\hat{\mathbf{x}}_1 \pm \mathrm{i}\hat{\mathbf{x}}_2) \tag{2.55}$$

As discussed in example M.1 on page 250, this shows that the field vector $\mathbf{E}$ rotates around the $x_3$ axis as it propagates along this axis. This rotation is called *circular wave polarisation*.

For $\delta_1 = 0$ in equation (2.55), the linear superpostion $1/2(\mathbf{E}_+ + \mathbf{E}_-)$ represents a wave that is *linearly polarised* along $\hat{\mathbf{x}}_1$ and for $\delta_1 = \pi/2$ the superposition $1/2(\mathbf{E}_+ - \mathbf{E}_-)$ represents a wave that is linearly polarised along $\hat{\mathbf{x}}_2$.

The *helical base vectors*

$$\hat{\mathbf{h}}_\pm = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}_1 \pm \mathrm{i}\hat{\mathbf{x}}_2) \tag{2.56}$$

which are fixed unit vectors, allow us to write equation (2.55) above

[10] A single plane wave is a mathematical idealisation. In reality, plane waves appear as building blocks of *wave packets*, *i.e.* superpositions of a (possibly infinite) number of individual plane waves with different properties (frequencies, directions,...). *E.g.* a radio beam from a transmitting antenna is a superposition (Fourier sum or integral) of many plane waves with slightly different angles relative to a fixed, given axis or a plane.

$$\mathbf{E}(t, \mathbf{x}) = \sqrt{2}E_0 \mathrm{e}^{\mathrm{i}(kx_3 - \omega t + \delta_1)} \hat{\mathbf{h}}_\pm \tag{2.57}$$

We use the convention that $\hat{\mathbf{h}}_+$ represents *left-hand circular polarisation* and $\hat{\mathbf{h}}_-$ *right-hand circular polarisation*.[11] Left-hand (right-hand) circular polarised waves are said to have *positive helicity* (*negative helicity*) with respect to the direction of propagation (along $\mathbf{k}$).

[11] In physics, two different conventions are used. The handedness refers to the rotation in space of the electric field vector, either when viewed as the wave propagates away from the observer or toward the observer.

End of example 2.2◁

EXAMPLE 2.3 ▷Wave equations expressed in terms of Riemann-Silberstein vectors

If we use the Maxwell-Lorentz equations expressed in the Riemann-Silberstein vector $\mathbf{G} = \mathbf{E} + \mathrm{i}c\mathbf{B}$, *i.e.* equations (2.8) on page 22, we can perform similar steps as we did when deriving equations (2.19) on page 24. We then find that

$$\Box^2 \mathbf{G} = -\frac{\boldsymbol{\nabla}\rho}{\varepsilon_0} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \mathrm{i}\mu_0 c \boldsymbol{\nabla} \times \mathbf{j} \tag{2.58}$$

Taking the real and imaginary parts of this equation, assuming that $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$, we recover the wave equations (2.17) on page 24, as expected.

End of example 2.3◁

## 2.5 Bibliography

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# 3

# ELECTROMAGNETIC POTENTIALS AND GAUGES

As described in chapter 1 the concepts of electric and magnetic fields were introduced such that they are intimately related to the mechanical forces between (static) charges and currents given by Coulomb's law and Ampère's law, respectively. Just as in mechanics, it turns out that in electrodynamics it is often more convenient to express the theory in terms of potentials rather then in terms of the electric and magnetic fields (Coulomb and Ampère forces) themselves. This is particularly true for problems related to radiation and relativity. As we shall see in chapter 7, the potentials play a central rôle in the formulation of relativistically covariant electromagnetism. And at the quantum level, electrodynamics is almost exclusively formulated in terms of potentials rather than electric and magnetic fields.

In this chapter we introduce and study the properties of such potentials and find that they exhibit some remarkable properties that elucidate the fundamental aspects of electromagnetism, lead naturally to the special theory of relativity, and pave the way for gauge field theories.

## 3.1 The electrostatic scalar potential

As we saw in equation (1.9b) on page 5, the time-independent electric (electrostatic) field $\mathbf{E}^{\text{stat}}(\mathbf{x})$ is irrotational. According to formula (F.100) on page 220 we may therefore express it in terms of the gradient of a scalar field. If we denote this scalar field by $-\Phi^{\text{stat}}(\mathbf{x})$, we get

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\boldsymbol{\nabla}\Phi^{\text{stat}}(\mathbf{x}) \tag{3.1}$$

Taking the divergence of this and using equation (1.9a) on page 5, we obtain *Poisson's equation*

$$\nabla^2\Phi^{\text{stat}}(\mathbf{x}) = -\boldsymbol{\nabla}\cdot\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\varepsilon_0} \tag{3.2}$$

If we compare with the definition of $\mathbf{E}^{\text{stat}}$, namely equation (1.7) on page 4, we see that this equation has the solution

$$\Phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.3}$$

where the integration is taken over all source points $\mathbf{x}'$ at which the charge density $\rho(\mathbf{x}')$ is non-zero. The scalar function $\Phi^{\text{stat}}(\mathbf{x})$ in equation (3.3) above is called the *electrostatic scalar potential*.

## 3.2 The magnetostatic vector potential

Let us consider the equations of magnetostatics, equations (1.21c) on page 9. According to formula (F.99) on page 220 any vector field $\mathbf{a}$ has the property that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) \equiv 0$ and in the derivation of equation (1.18) on page 8 in magnetostatics we found that $\boldsymbol{\nabla}\cdot\mathbf{B}^{\text{stat}}(\mathbf{x}) = 0$. We therefore realise that we can always write

$$\mathbf{B}^{\text{stat}}(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}^{\text{stat}}(\mathbf{x}) \tag{3.4}$$

where $\mathbf{A}^{\text{stat}}(\mathbf{x})$ is called the *magnetostatic vector potential*. In the magnetostatic case, we may start from Biot-Savart's law as expressed by equation (1.16) on page 8. Identifying this expression with equation (3.4) above allows us to define the static vector potential as

$$\mathbf{A}^{\text{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.5}$$

From equations (3.1) and (3.4) on pages 33–34 we conclude that if we transform equations equations (3.3) and (3.5) above in the following way

$$\Phi^{\text{stat}}(\mathbf{x}) \mapsto \Phi^{\text{stat}\prime}(\mathbf{x}) = \Phi^{\text{stat}}(\mathbf{x}) + \alpha(\mathbf{x}) \tag{3.6a}$$

$$\mathbf{A}^{\text{stat}}(\mathbf{x}) \mapsto \mathbf{A}^{\text{stat}\prime}(\mathbf{x}) = \mathbf{A}^{\text{stat}}(\mathbf{x}) + \mathbf{a}(\mathbf{x}) \tag{3.6b}$$

the fields $\mathbf{E}^{\text{stat}}$ and $\mathbf{B}^{\text{stat}}$ will be unaffected provided

$$\boldsymbol{\nabla}\alpha(\mathbf{x}) = \mathbf{0} \tag{3.7a}$$

$$\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x}) = \mathbf{0} \tag{3.7b}$$

*i.e.* if $\alpha$ is an arbitrary scalar function that is not dependent on $\mathbf{x}$, *e.g.* a constant, and $\mathbf{a}(\mathbf{x})$ is an arbitrary vector field whose curl vanishes. According to formula (F.100) on page 220 such a vector field can always be written as the gradient of a scalar field. In other words, the fields are unaffected by the transformation (3.6) if

$$\alpha(\mathbf{x}) = Const \tag{3.8a}$$

$$\mathbf{a}(\mathbf{x}) = \boldsymbol{\nabla}\beta(\mathbf{x}) \tag{3.8b}$$

where $\beta$ is an arbitrary, at least twice continuously differentiable function of $\mathbf{x}$.

## 3.3 The electrodynamic potentials

Let us now generalise the static analysis above to the electrodynamic case, *i.e.* the case with temporal and spatial dependent sources $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$, and the pertinent fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$, as described by the Maxwell-Lorentz equations (2.1) on page 19. In other words, let us study the *electrodynamic potentials* $\Phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$.

According to the non-source Maxwell-Lorentz equation (2.1b), the magnetic field $\mathbf{B}(t, \mathbf{x})$ is divergence-free also in electrodynamics (if magnetic charges are not included). Because of this divergence-free nature of the time- and space-dependent magnetic field, we can express it as the curl of an *electromagnetic vector potential*:

$$\mathbf{B}(t, \mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x}) \tag{3.9}$$

Inserting this expression into the other non-source Maxwell-Lorentz equation (2.1c) on page 19, we obtain

$$\boldsymbol{\nabla} \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial}{\partial t}\left[\boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x})\right] = -\boldsymbol{\nabla} \times \frac{\partial}{\partial t}\mathbf{A}(t, \mathbf{x}) \tag{3.10}$$

or, rearranging the terms,

$$\boldsymbol{\nabla} \times \left(\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t}\mathbf{A}(t, \mathbf{x})\right) = \mathbf{0} \tag{3.11}$$

As before we utilise the vanishing curl of a vector expression to write this vector expression as the gradient of a scalar function. If, in analogy with the electrostatic case, we introduce the *electromagnetic scalar potential* function $-\Phi(t, \mathbf{x})$, equation (3.11) above becomes equivalent to

$$\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t}\mathbf{A}(t, \mathbf{x}) = -\boldsymbol{\nabla}\Phi(t, \mathbf{x}) \tag{3.12}$$

This means that in electrodynamics, $\mathbf{E}(t, \mathbf{x})$ is calculated from the potentials according to the formula

$$\mathbf{E}(t, \mathbf{x}) = -\boldsymbol{\nabla}\Phi(t, \mathbf{x}) - \frac{\partial}{\partial t}\mathbf{A}(t, \mathbf{x}) \tag{3.13}$$

and $\mathbf{B}(t,\mathbf{x})$ from formula (3.9) on the preceding page. Hence, it is a matter of convention (or taste) whether we want to express the laws of electrodynamics in terms of the potentials $\Phi(t,\mathbf{x})$ and $\mathbf{A}(t,\mathbf{x})$, or in terms of the fields $\mathbf{E}(t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x})$. However, there is an important difference between the two approaches: in classical electrodynamics the only directly observable quantities are the fields themselves (and quantities derived from them) and not the potentials. On the other hand, the treatment becomes significantly simpler if we use the potentials in our calculations and then, at the final stage, use equation (3.9) on the previous page and equation (3.13) on the preceding page to calculate the fields or physical quantities expressed in the fields. This is the strategy we shall follow.

## 3.4 Gauge conditions

Inserting (3.13) and (3.9) on the previous page into Maxwell's equations (2.1) on page 19, we obtain, after some simple algebra and the use of equation (1.12) on page 6 and formula (F.96) on page 220, the equations

$$-\nabla^2\Phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = \frac{\rho(t,\mathbf{x})}{\varepsilon_0} \tag{3.14a}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) = \mu_0\,\mathbf{j}(t,\mathbf{x}) - \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t} \tag{3.14b}$$

Subtracting $(1/c^2)\partial^2\Phi/\partial t^2$ from both sides of the first equation and rearranging, the above two equations turn into the following *general inhomogeneous wave equations*

$$\Box^2\Phi = \frac{\rho(t,\mathbf{x})}{\varepsilon_0} + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \tag{3.15a}$$

$$\Box^2\mathbf{A} = \mu_0\,\mathbf{j}(t,\mathbf{x}) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \tag{3.15b}$$

where $\Box^2$ is the *d'Alembert operator*, defined by formula (F.182) on page 228. These two second-order, coupled, partial differential equations, representing in all four scalar equations (one for $\Phi$ and one each for the three components $A_i, i = 1,2,3$ of $\mathbf{A}$), are completely equivalent to the formulation of electrodynamics in terms of Maxwell's equations, which represent eight scalar first-order, coupled, partial differential equations.

As they stand, equations (3.14) and (3.15) look complicated and may seem to be of limited use. However, if we write equation (3.9) on the previous page in the form $\boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) = \mathbf{B}(t,\mathbf{x})$ we can consider this as a specification of $\boldsymbol{\nabla}\times\mathbf{A}$. But we know from *Helmholtz's theorem* [see subsection M.3.7 on page 247] that in order to determine the (spatial) behaviour of $\mathbf{A}$ completely, we must also specify $\boldsymbol{\nabla} \cdot \mathbf{A}$. Since this divergence does not enter the derivation above, we are free to choose $\boldsymbol{\nabla} \cdot \mathbf{A}$ in whatever way we like and still obtain the same physical results. This illustrates the power of formulating electrodynamics in terms of potentials.

## 3.4.1 Lorenz-Lorentz gauge

If we choose $\boldsymbol{\nabla} \cdot \mathbf{A}$ to fulfil the so called *Lorenz-Lorentz gauge condition*[1]

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{3.16}$$

the coupled inhomogeneous wave equations (3.15) on page 36 simplify to the following set of *uncoupled inhomogeneous wave equations*:

$$\Box^2 \Phi = \frac{\rho(t, \mathbf{x})}{\varepsilon_0} \tag{3.17a}$$

$$\Box^2 \mathbf{A} = \mu_0 \mathbf{j}(t, \mathbf{x}) \tag{3.17b}$$

Each of these four scalar equations is an *inhomogeneous wave equation* of the following form:

$$\Box^2 \Psi(t, \mathbf{x}) = s(t, \mathbf{x}) \tag{3.18}$$

where $\Psi$ denotes for either $\Phi$ or one of the components $A_i, i = 1, 2, 3$ of the vector potential $\mathbf{A}$, and $s$ is a shorthand for the pertinent source component, $\rho(t, \mathbf{x})/\varepsilon_0$ or $\mu_0 j_i(t, \mathbf{x}), i = 1, 2, 3$, respectively.

We assume that our sources are well-behaved enough in time $t$ so that the *Fourier transform* pair for the generic source function $s$

$$s(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, s_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{3.19a}$$

$$s_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, s(t, \mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \tag{3.19b}$$

exists, and that the same is true for the generic potential component $\Psi$:

$$\Psi(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \Psi_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{3.20a}$$

$$\Psi_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, \Psi(t, \mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \tag{3.20b}$$

Inserting the Fourier representations (3.19) and (3.20) into equation (3.18) above, and using the vacuum dispersion relation for electromagnetic waves relating the

[1] In fact, the Dutch physicist HENDRIK ANTOON LORENTZ (1853-1928), who in 1903 demonstrated the covariance of Maxwell's equations, was not the original discoverer of the gauge condition (3.16). It had been discovered by the Danish physicist LUDVIG VALENTIN LORENZ (1829-1891) already in 1867. This fact has sometimes been overlooked in the literature and the condition was earlier referred to as the *Lorentz gauge condition*. Prior to that, BERNHARD RIEMANN, had discussed this condition in a lecture in 1858.

*angular frequency* $\omega$, the speed of light $c$, and the *wave number* $k = (2\pi)/\lambda$ where $\lambda$ is the *vacuum wavelength*,

$$\omega = ck \tag{3.21}$$

the generic 3D inhomogeneous wave equation (3.18) on the preceding page, turns into

$$\nabla^2 \Psi_\omega(\mathbf{x}) + k^2 \Psi_\omega(\mathbf{x}) = -s_\omega(\mathbf{x}) \tag{3.22}$$

which is the 3D *inhomogeneous time-independent wave equation*, often called the 3D *inhomogeneous Helmholtz equation*.

As postulated by *Huygens's principle*, each point on a wave front acts as a point source for spherical wavelets of varying amplitude (weight). A new wave front is formed by a linear superposition of the individual weighted wavelets from each of the point sources on the old wave front. The solution of (3.22) can therefore be expressed as a weighted sum of solutions of an equation where the source term has been replaced by a single point source

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') + k^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \tag{3.23}$$

and the solution of equation (3.22) above which corresponds to the frequency $\omega$ is given by the weighted superposition

$$\Psi_\omega(\mathbf{x}) = \int_{V'} \mathrm{d}^3x' \, s_\omega(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \tag{3.24}$$

(plus boundary conditions) where $s_\omega(\mathbf{x}')$ is the wavelet amplitude at the source point $\mathbf{x}'$. The function $G(\mathbf{x}, \mathbf{x}')$ is called the *Green function* or the *propagator*.

Due to translational invariance in space, $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x} - \mathbf{x}')$. Furthermore, in equation (3.23) above, the Dirac generalised function $\delta(\mathbf{x} - \mathbf{x}')$, which represents the point source, depends only on $\mathbf{x} - \mathbf{x}'$ and there is no angular dependence in the equation. Hence, the solution can only be dependent on $r = |\mathbf{x} - \mathbf{x}'|$ and not on the direction of $\mathbf{x} - \mathbf{x}'$. If we interpret $r$ as the radial coordinate in a spherically polar coordinate system, and recall the expression for the Laplace operator in such a coordinate system, equation (3.23) above becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rG) + k^2(rG) = -r\delta(r) \tag{3.25}$$

Away from $r = |\mathbf{x} - \mathbf{x}'| = 0$, *i.e.* away from the source point $\mathbf{x}'$, this equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rG) + k^2(rG) = 0 \tag{3.26}$$

with the well-known general solution

$$G = C_+ \frac{\mathrm{e}^{ikr}}{r} + C_- \frac{\mathrm{e}^{-ikr}}{r} \equiv C_+ \frac{\mathrm{e}^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_- \frac{\mathrm{e}^{-ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{3.27}$$

where $C_\pm$ are constants.

In order to determine the constants $C_\pm$, we insert the general solution, equation (3.27) above, into equation (3.23) on the preceding page and integrate over a small volume around $r = |\mathbf{x}-\mathbf{x}'| = 0$. Since

$$G(|\mathbf{x}-\mathbf{x}'|) \sim C_+ \frac{1}{|\mathbf{x}-\mathbf{x}'|} + C_- \frac{1}{|\mathbf{x}-\mathbf{x}'|}, \quad |\mathbf{x}-\mathbf{x}'| \to 0 \tag{3.28}$$

the volume integrated equation (3.23) on the facing page can be approximated by

$$\begin{aligned} &(C_+ + C_-) \int_{V'} \mathrm{d}^3x' \, \nabla^2 \left( \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \\ &+ k^2 (C_+ + C_-) \int_{V'} \mathrm{d}^3x' \, \frac{1}{|\mathbf{x}-\mathbf{x}'|} = - \int_{V'} \mathrm{d}^3x' \, \delta(|\mathbf{x}-\mathbf{x}'|) \end{aligned} \tag{3.29}$$

In virtue of the fact that the volume element $\mathrm{d}^3x'$ in spherical polar coordinates is proportional to $r^2 = |\mathbf{x}-\mathbf{x}'|^2$ [see formula (F.19) on page 216], the second integral vanishes when $|\mathbf{x}-\mathbf{x}'| \to 0$. Furthermore, from equation (F.116) on page 220, we find that the integrand in the first integral can be written as $-4\pi\delta(|\mathbf{x}-\mathbf{x}'|)$ and, hence, that the two constants $C\pm$ must fulfil the condition

$$C_+ + C_- = \frac{1}{4\pi} \tag{3.30}$$

Now that we have determined the relation between $C_+$ and $C_-$, we insert the general solution equation (3.27) above into equation (3.24) on the preceding page and obtain the general solution in the $\omega$ domain:

$$\Psi_\omega(\mathbf{x}) = C_+ \int_{V'} \mathrm{d}^3x' \, s_\omega(\mathbf{x}') \frac{\mathrm{e}^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_- \int_{V'} \mathrm{d}^3x' \, s_\omega(\mathbf{x}') \frac{\mathrm{e}^{-ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{3.31}$$

In order to find the solution in the $t$ domain, we take the inverse Fourier transform of this by inserting the above expression for $\Psi_\omega(\mathbf{x})$ into equation (3.20) on page 37:

$$\begin{aligned} \Psi(t, \mathbf{x}) = C_+ \int_{V'} \mathrm{d}^3x' \int_{-\infty}^{\infty} \mathrm{d}\omega \, s_\omega(\mathbf{x}') \frac{\exp\left[-\mathrm{i}\omega\left(t - \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \\ + C_- \int_{V'} \mathrm{d}^3x' \int_{-\infty}^{\infty} \mathrm{d}\omega \, s_\omega(\mathbf{x}') \frac{\exp\left[-\mathrm{i}\omega\left(t + \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \end{aligned} \tag{3.32}$$

If we introduce the *retarded time* $t'_{\text{ret}}$ and the *advanced time* $t'_{\text{adv}}$ in the following way [using the fact that $k/\omega = 1/c$ in free space, according to formula (3.21) on page 38]:

$$t'_{\text{ret}} = t'_{\text{ret}}(t, |\mathbf{x}-\mathbf{x}'|) = t - \frac{k\,|\mathbf{x}-\mathbf{x}'(t'_{\text{ret}})|}{\omega} = t - \frac{|\mathbf{x}-\mathbf{x}'(t'_{\text{ret}})|}{c} \tag{3.33a}$$

$$t'_{\text{adv}} = t'_{\text{adv}}(t, |\mathbf{x}-\mathbf{x}'|) = t + \frac{k\,|\mathbf{x}-\mathbf{x}'(t'_{\text{adv}})|}{\omega} = t + \frac{|\mathbf{x}-\mathbf{x}'(t'_{\text{adv}})|}{c} \tag{3.33b}$$

and use equation (3.19) on page 37, we obtain

$$\Psi(t,\mathbf{x}) = C_+ \int_{V'} \mathrm{d}^3x'\, \frac{f(t'_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + C_- \int_{V'} \mathrm{d}^3x'\, \frac{f(t'_{\text{adv}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.34}$$

This is a solution to the generic inhomogeneous wave equation for the potential components equation (3.18) on page 37. We note that the solution at time $t$ at the field point $\mathbf{x}$ is dependent on the behaviour at other times $t'$ of the source at $\mathbf{x}'$ and that both retarded and advanced $t'$ are mathematically acceptable solutions. However, if we assume that causality requires that the potential at $(t, \mathbf{x})$ is set up by the source at an earlier time, *i.e.* at $(t'_{\text{ret}}, \mathbf{x}')$, we must in equation (3.34) above set $C_- = 0$ and therefore, according to equation (3.30) on the previous page, $C_+ = 1/(4\pi)$.[2]

[2] In fact, inspired by ideas put forward by PAUL ADRIEN MAURICE DIRAC (1902–1984), JOHN ARCHIBALD WHEELER (1911–2008) and RICHARD PHILLIPS FEYNMAN (1918–1988) derived, in 1945, a consistent electrodynamics based on both the retarded and the advanced potentials.

From the above discussion about the solution of the inhomogeneous wave equations in the Lorenz-Lorentz gauge we conclude that, if we discard the advanced potentials, the electrodynamic potentials in free space can be written

$$\Phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x'\, \frac{\rho\big(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}})\big)}{|\mathbf{x}(t)-\mathbf{x}'(t'_{\text{ret}})|} \tag{3.35a}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}\big(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}})\big)}{|\mathbf{x}(t)-\mathbf{x}'(t'_{\text{ret}})|} \tag{3.35b}$$

These *retarded potentials* were obtained as solutions to the Lorenz-Lorentz inhomogeneous wave equations (3.17). The expressions (3.35) are therefore valid only in the Lorenz-Lorentz gauge.[3] scalar first-order, coupled, partial differential equations. In other gauges (other choices of $\boldsymbol{\nabla}\cdot\mathbf{A}$) the expressions for the potentials are analytically different but will, of course, yield the very same physical fields $\mathbf{E}$ and $\mathbf{B}$ as the expressions (3.35) do. As they stand, we shall use expressions (3.35) quite frequently in the sequel.

[3] In 1897, TULLIO LEVI-CIVITA (1873–1941) showed that the retarded Lorenz-Lorentz potentials solves equations (3.14).

## 3.4.2 Coulomb gauge

In *Coulomb gauge*, often employed in *quantum electrodynamics*, one chooses $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ so that equations (3.14) or equations (3.15) on page 36 become

$$\nabla^2\Phi = -\frac{\rho(t,\mathbf{x})}{\varepsilon_0} \tag{3.36a}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}(t,\mathbf{x}) + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t} \tag{3.36b}$$

The first of these two is the *time-dependent Poisson's equation* which, in analogy with equation (3.3) on page 34, has the solution

$$\Phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(t,\mathbf{x}'(t))}{|\mathbf{x}(t)-\mathbf{x}'(t)|} \tag{3.37}$$

We see that in the scalar potential expression, the charge density source is evaluated at time $t$. Hence, the scalar potential does not exhibit any retardation, which means that the effect of the charge shows up in the potential *instantaneously*, as if the propagation speed were infinite. This is not in disagreement with the laws of nature since in classical physics potentials are not physical observables.

Since in the Coulomb gauge the scalar potential $\Phi$ does not suffer any retardation (or advancement) but the fields $\mathbf{E}$ and $\mathbf{B}$ themselves must be physical and therefore must exhibit retardation effects, all retardation must occur in the vector potential $\mathbf{A}$, *i.e.* the solution of the inhomogeneous wave equation (3.36b) above. As we see, the last term in the RHS of this equation contains the scalar potential $\Phi$, which, according to equation (3.37) above, in turn depends on the charge density $\rho$. The continuity equation (1.22) on page 10 provides a relation between $\rho$ and $\mathbf{j}$ on which we can apply Helmholtz decomposition and find that

$$\mathbf{j} = \mathbf{j}^{\text{rotat}} + \mathbf{j}^{\text{irrot}} \tag{3.38}$$

where

$$\boldsymbol{\nabla}\cdot\mathbf{j}^{\text{rotat}} = 0 \tag{3.39a}$$

$$\boldsymbol{\nabla}\times\mathbf{j}^{\text{irrot}} = \mathbf{0} \tag{3.39b}$$

[*cf.* equations (2.36) on page 28]. Then the equation of continuity becomes

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}^{\text{irrot}} &= \frac{\partial}{\partial t}\left(-\varepsilon_0\nabla^2\Phi\right) + \boldsymbol{\nabla}\cdot\mathbf{j}^{\text{irrot}} \\ &= \boldsymbol{\nabla}\cdot\left[\left(-\varepsilon_0\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}\right) + \mathbf{j}^{\text{irrot}}\right] = 0\end{aligned} \tag{3.40}$$

Furthermore, since $\boldsymbol{\nabla}\times\boldsymbol{\nabla} = \mathbf{0}$ and $\boldsymbol{\nabla}\times\mathbf{j}^{\text{irrot}} = \mathbf{0}$, one finds that

$$\boldsymbol{\nabla}\times\left[\left(-\varepsilon_0\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}\right) + \mathbf{j}^{\text{irrot}}\right] = \mathbf{0} \tag{3.41}$$

According to Helmholtz's theorem, this implies that

$$\varepsilon_0\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t} = \mathbf{j}^{\text{irrot}} \tag{3.42}$$

The inhomogeneous wave equation (3.36b) on the preceding page thus becomes

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial \Phi}{\partial t} = -\mu_0 \mathbf{j} + \mu_0 \mathbf{j}^{\text{irrot}} = -\mu_0 \mathbf{j}^{\text{rotat}} \tag{3.43}$$

which shows that in Coulomb gauge the source of the vector potential $\mathbf{A}$ is the rotational (transverse) component of the current, $\mathbf{j}^{\text{rotat}}$. The irrotational (longitudinal) component of the current $\mathbf{j}^{\text{irrot}}$ does not contribute to the vector potential. The retarded (particular) solution in Coulomb gauge of equations (3.14) on page 36 is therefore [*cf.* equations (3.35) on page 40]:

$$\Phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \, \frac{\rho\big(t, \mathbf{x}'(t)\big)}{|\mathbf{x}(t) - \mathbf{x}'(t)|} \tag{3.44a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}^{\text{rotat}}\big(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}})\big)}{|\mathbf{x}(t) - \mathbf{x}'(t'_{\text{ret}})|} \tag{3.44b}$$

The Coulomb gauge is also called the *transverse gauge* or the *radiation gauge*.

### 3.4.3 Velocity gauge

If $\boldsymbol{\nabla} \cdot \mathbf{A}$ fulfils the *velocity gauge condition*, sometimes referred to as the *complete* $\alpha$*-Lorenz gauge*,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\alpha}{c^2}\frac{\partial \Phi}{\partial t} = 0, \qquad \alpha = \frac{c^2}{v^2} \tag{3.45}$$

we obtain the Lorenz-Lorentz gauge condition in the limit $v = c$, *i.e.* $\alpha = 1$, and the Coulomb gauge condition in the limit $v \to \infty$, *i.e.* $\alpha = 0$, respectively, where $v$ is the propagation speed of the scalar potential. Hence, the velocity gauge is a generalisation of both these gauges.[4] Inserting equation (3.45) above into the coupled inhomogeneous wave equations (3.15) on page 36 they become

[4] The value $\alpha = -1$, corresponding to an imaginary speed $v = \mathrm{i}c$, gives the *Kirchhoff gauge*, introduced already in 1857 by GUSTAV ROBERT KIRCHHOFF (1824–1884).

$$\Box^2 \Phi = -\frac{\rho(t, \mathbf{x})}{\varepsilon_0} - \frac{1-\alpha}{c^2}\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial t} \tag{3.46a}$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{j}(t, \mathbf{x}) + \frac{1-\alpha}{c^2}\boldsymbol{\nabla}\frac{\partial \Phi}{\partial t} \tag{3.46b}$$

## 3.5 Gauge transformations

We saw in section 3.1 on page 33 and in section 3.2 on page 34 that in electrostatics and magnetostatics we have a certain *mathematical* degree of freedom, up to terms of vanishing gradients and curls, to pick suitable forms for the potentials and still get the same *physical* result. In fact, the way the electromagnetic scalar potential $\Phi(t,\mathbf{x})$ and the vector potential $\mathbf{A}(t,\mathbf{x})$ are related to the physical observables gives leeway for similar manipulation of them also in electrodynamics.

In analogy with equations (3.6) on page 34 we introduce

$$\Phi(t,\mathbf{x}) \mapsto \Phi'(t,\mathbf{x}) = \Phi(t,\mathbf{x}) + \alpha(t,\mathbf{x}) \tag{3.47a}$$

$$\mathbf{A}(t,\mathbf{x}) \mapsto \mathbf{A}'(t,\mathbf{x}) = \mathbf{A}(t,\mathbf{x}) + \mathbf{a}(t,\mathbf{x}) \tag{3.47b}$$

By inserting these transformed potentials into equation (3.13) on page 35 for the electric field $\mathbf{E}(t,\mathbf{x})$ and into equation (3.9) on page 35 for the magnetic field $\mathbf{B}(t,\mathbf{x})$, we see that the fields will be unaffected provided

$$\boldsymbol{\nabla}\alpha(t,\mathbf{x}) + \frac{\partial \mathbf{a}(t,\mathbf{x})}{\partial t} = \mathbf{0} \tag{3.48a}$$

$$\boldsymbol{\nabla} \times \mathbf{a}(t,\mathbf{x}) = \mathbf{0} \tag{3.48b}$$

If we introduce an arbitrary, sufficiently differentiable scalar function $\Gamma(t,\mathbf{x})$, the second condition is fulfilled if we require that

$$\mathbf{a}(t,\mathbf{x}) = \boldsymbol{\nabla}\Gamma(t,\mathbf{x}) \tag{3.49a}$$

which, when inserted into the first condition in turn requires that

$$\alpha(t,\mathbf{x}) = -\frac{\partial \Gamma(t,\mathbf{x})}{\partial t} \tag{3.49b}$$

Hence, if we *simultaneously* transform both $\Phi(t,\mathbf{x})$ and $\mathbf{A}(t,\mathbf{x})$ into new ones $\Phi'(t,\mathbf{x})$ and $\mathbf{A}'(t,\mathbf{x})$ according to the scheme

$$\Phi(t,\mathbf{x}) \mapsto \Phi'(t,\mathbf{x}) = \Phi(t,\mathbf{x}) - \frac{\partial \Gamma(t,\mathbf{x})}{\partial t} \tag{3.50a}$$

$$\mathbf{A}(t,\mathbf{x}) \mapsto \mathbf{A}'(t,\mathbf{x}) = \mathbf{A}(t,\mathbf{x}) + \boldsymbol{\nabla}\Gamma(t,\mathbf{x}) \tag{3.50b}$$

and insert the transformed potentials into equation (3.13) on page 35 for the electric field and into equation (3.9) on page 35 for the magnetic field, we obtain the transformed fields

$$\begin{aligned}\mathbf{E}' &= -\boldsymbol{\nabla}\Phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\left(\frac{\partial \Gamma}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial(\boldsymbol{\nabla}\Gamma)}{\partial t} \\ &= -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial(\boldsymbol{\nabla}\Gamma)}{\partial t} - \frac{\partial(\boldsymbol{\nabla}\Gamma)}{\partial t} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}\end{aligned} \tag{3.51a}$$

$$\mathbf{B}' = \boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Gamma) = \boldsymbol{\nabla} \times \mathbf{A} \tag{3.51b}$$

where, once again equation (F.100) on page 220 was used. This explicit calculation clearly demonstrates that the fields $\mathbf{E}$ and $\mathbf{B}$ are unaffected by the gauge transformation (3.50). The function $\Gamma(t,\mathbf{x})$ is called the *gauge function*.

A transformation of the potentials $\Phi$ and $\mathbf{A}$ which leaves the fields, and hence Maxwell's equations, invariant is called a *gauge transformation*. Any physical law that does not change under a gauge transformation is said to be *gauge invariant*. It is only those quantities (expressions) that are gauge invariant that are observable and therefore have experimental significance. Trivially, the electromagnetic fields and the Maxwell-Lorentz equations themselves are gauge invariant and electrodynamics is therefore a *gauge theory* and as such the prototype for all gauge theories.[5]

[5] A very important extension is the *Yang-Mills theory*, introduced in 1954. This theory has had a profound impact on modern physics.

As just shown, the potentials $\Phi(t,\mathbf{x})$ and $\mathbf{A}(t,\mathbf{x})$ calculated from equations (3.14) on page 36, with an arbitrary choice of $\boldsymbol{\nabla}\cdot\mathbf{A}$, can be gauge transformed according to (3.50) on the previous page. If, in particular, we choose $\boldsymbol{\nabla}\cdot\mathbf{A}$ according to the Lorenz-Lorentz condition, equation (3.16) on page 37, and apply the gauge transformation (3.50) on the resulting Lorenz-Lorentz potential equations (3.17) on page 37, these equations will be transformed into

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial^2\Gamma}{\partial t^2} - \nabla^2\Gamma\right) = \frac{\rho(t,\mathbf{x})}{\varepsilon_0} \tag{3.52a}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} - \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial^2\Gamma}{\partial t^2} - \nabla^2\Gamma\right) = \mu_0\mathbf{j}(t,\mathbf{x}) \tag{3.52b}$$

We notice that if we require that the gauge function $\Gamma(t,\mathbf{x})$ itself be restricted to fulfil the wave equation

$$\frac{1}{c^2}\frac{\partial^2\Gamma}{\partial t^2} - \nabla^2\Gamma = 0 \tag{3.53}$$

these transformed Lorenz-Lorentz equations will keep their original form. The set of potentials which have been gauge transformed according to equation (3.50) on the preceding page with a gauge function $\Gamma(t,\mathbf{x})$ restricted to fulfil equation (3.53) above, or, in other words, those gauge transformed potentials for which the equations (3.17) on page 37 are invariant, comprise the *Lorenz-Lorentz gauge*.

## 3.5.1 Other gauges

Other useful gauges are

- The *Poincaré gauge* (*multipolar gauge*, *radial gauge*)

$$\Phi'(t,\mathbf{x}) = -\int_0^1 \mathrm{d}\alpha\, \mathbf{x}\cdot\mathbf{E}(t,\alpha\mathbf{x}) \tag{3.54a}$$

$$\mathbf{A}'(t,\mathbf{x}) = -\int_0^1 \mathrm{d}\alpha\, \alpha\mathbf{x}\times\mathbf{B}(t,\alpha\mathbf{x}) \tag{3.54b}$$

  where $\alpha$ is a scalar parameter. We note that in Poincaré gauge $\mathbf{A}'$ and $\mathbf{x}$ are orthogonal.

- The *Weyl gauge*, also known as the *temporal gauge* or *Hamilton gauge*, defined by $\Phi' = 0$.
- The *axial gauge*, defined by $A'_3 = 0$.

The process of choosing a particular gauge condition is known as *gauge fixing*.

## 3.6 Examples

EXAMPLE 3.1 ▷Multipole expansion of the electrostatic potential

The integral in the electrostatic potential formula

$$\Phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{3.55}$$

where $\rho$ is the charge density introduced in equation (1.9a) on page 5, is not always possible to evaluate analytically. However, for a charge distribution source $\rho(\mathbf{x}')$ that is well localised in a small volume $V'$ around $\mathbf{x}_0$, a series expansion of the integrand in such a way that the dominant contributions are contained in the first few terms can be made. *E.g.* if we Taylor expand the inverse distance $1/|\mathbf{x}-\mathbf{x}'|$ with respect to the point $\mathbf{x}' = \mathbf{x}_0$ we obtain

$$\begin{aligned}
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|} \\
&= \frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{3}\cdots\sum_{i_n=1}^{3} \\
&\quad\times \frac{\partial^n \frac{1}{|\mathbf{x}-\mathbf{x}_0|}}{\partial x_{i_1}\cdots\partial x_{i_n}}[-(x'_{i_1}-x_{0_{i_1}})]\cdots[-(x'_{i_n}-x_{0_{i_n}})] \\
&= \frac{1}{|\mathbf{x}-\mathbf{x}_0|} + \sum_{n=1}^{\infty}\sum_{\substack{n_1+n_2+n_3=n\\ n_i\geq 0}}\frac{(-1)^n}{n_1!n_2!n_3!} \\
&\quad\times \frac{\partial^n \frac{1}{|\mathbf{x}-\mathbf{x}_0|}}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}}(x'_1-x_{0_1})^{n_1}(x'_2-x_{0_2})^{n_2}(x'_3-x_{0_3})^{n_3}
\end{aligned} \tag{3.56}$$

Inserting this into the integral in formula (3.55), we obtain the expansion

$$\begin{aligned}
\Phi^{\text{stat}}(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\Bigg[\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}') + \sum_{n=1}^{\infty}\sum_{\substack{n_1+n_2+n_3=n\\ n_i\geq 0}}\frac{(-1)^n}{n_1!n_2!n_3!} \\
&\times \frac{\partial^n \frac{1}{|\mathbf{x}-\mathbf{x}_0|}}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}}\int_{V'}\mathrm{d}^3x'\,(x'_1-x_{0_1})^{n_1}(x'_2-x_{0_2})^{n_2}(x'_3-x_{0_3})^{n_3}\rho(\mathbf{x}')\Bigg]
\end{aligned} \tag{3.57}$$

Clearly, the first integral in this expansion is nothing but the *static net charge*

$$q = \int_{V'}\mathrm{d}^3x'\,\rho(\mathbf{x}') \tag{3.58}$$

If we introduce the *electrostatic dipole moment vector*

$$\mathbf{d}(\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\,\rho(\mathbf{x}') \tag{3.59}$$

with components $p_i$, $i = 1, 2, 3$, and the *electrostatic quadrupole moment tensor*

$$\mathbf{Q}(\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\otimes(\mathbf{x}'-\mathbf{x}_0)\,\rho(\mathbf{x}') \tag{3.60}$$

with components $Q_{ij}$, $i, j = 1, 2, 3$, and use the fact that

$$\frac{\partial \frac{1}{|\mathbf{x}-\mathbf{x}_0|}}{\partial x_i} = -\frac{x_i - x_{0_i}}{|\mathbf{x}-\mathbf{x}_0|^3} \tag{3.61}$$

and that

$$\frac{\partial^2 \frac{1}{|\mathbf{x}-\mathbf{x}_0|}}{\partial x_i \partial x_j} = \frac{3(x_i - x_{0_i})(x_j - x_{0_j}) - |\mathbf{x}-\mathbf{x}_0|^2 \delta_{ij}}{|\mathbf{x}-\mathbf{x}_0|^5} \tag{3.62}$$

then we can write the first three terms of the expansion of equation (3.55) on the preceding page as

$$\begin{aligned}\Phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\Bigg[\frac{q}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|^2}\mathbf{d}\cdot\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}\\ + \frac{1}{|\mathbf{x}-\mathbf{x}_0|^3} Q_{ij}\Big(\frac{3}{2}\frac{x_i - x_{0_i}}{|\mathbf{x}-\mathbf{x}_0|}\frac{x_j - x_{0_j}}{|\mathbf{x}-\mathbf{x}_0|} - \frac{1}{2}\delta_{ij}\Big) + \dots\Bigg]\end{aligned} \tag{3.63}$$

where Einstein's summation convention over $i$ and $j$ is implied. We see that at large distances from a localised charge distribution, the electrostatic potential can, to the lowest order, be approximated by the (Coulomb) potential from a single point charge $q$ located at the moment point $\mathbf{x}_0$. We also see that

$$\begin{aligned}\mathbf{d}(\mathbf{x}_0) &= \int_{V'} \mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\,\rho(\mathbf{x}') = \int_{V'} \mathrm{d}^3x'\,\mathbf{x}'\,\rho(\mathbf{x}') - \mathbf{x}_0\int_{V'} \mathrm{d}^3x'\,\rho(\mathbf{x}')\\ &= \int_{V'} \mathrm{d}^3x'\,\mathbf{x}'\,\rho(\mathbf{x}') - \mathbf{x}_0 q\end{aligned} \tag{3.64}$$

from which we draw the conclusion that if $q \neq 0$, it is always possible to choose the moment point $\mathbf{x}_0$ such that $\mathbf{d} = \mathbf{0}$, and if $q = 0$, then $\mathbf{d}$ is independent of the choice of moment point $\mathbf{x}_0$. Furthermore, one can show that

$$\alpha\frac{1}{2}\delta_{ij}\frac{3(x_i - x_{0_i})(x_j - x_{0_j}) - |\mathbf{x}-\mathbf{x}_0|^2 \delta_{ij}}{|\mathbf{x}-\mathbf{x}_0|^5} = 0 \tag{3.65}$$

where $\alpha$ is an arbitrary constant. Choosing it to be

$$\alpha = \frac{1}{3}\int_{V'} \mathrm{d}^3x'\,\left|\mathbf{x}'-\mathbf{x}_0\right|^2 \rho(\mathbf{x}') \tag{3.66}$$

we can transform $Q_{ij}$ into

$$Q'_{ij} = Q_{ij} - \alpha\delta_{ij} = \int_{V'} \mathrm{d}^3x'\,\Big(\big[(x'_i - x_{0_i})(x'_j - x_{0_j}) - \frac{1}{3}\left|\mathbf{x}'-\mathbf{x}_0\right|^2 \delta_{ij}\big]\rho(\mathbf{x}')\Big) \tag{3.67}$$

or

$$\mathbf{Q}' = \mathbf{Q} - \mathbf{1}_3\alpha = \int_{V'} \mathrm{d}^3x'\,\Big(\big[(\mathbf{x}'-\mathbf{x}_0)\otimes(\mathbf{x}'-\mathbf{x}_0) - \mathbf{1}_3\frac{1}{3}\left|\mathbf{x}'-\mathbf{x}_0\right|^2\big]\rho(\mathbf{x}')\Big) \tag{3.68}$$

where $\mathbf{1}_3 = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i$ is the unit tensor. It follows immediately that $Q'_{ii} = 0$ (Einstein summation), *i.e.* that $\mathbf{Q}'$ is *traceless*. Rotating the coordinate system, it is possible to diagonalise the tensors $\mathbf{Q}$ and $\mathbf{Q}'$. For any spherical symmetric distribution of charge, all components of $\mathbf{Q}'$ vanish if the moment point $\mathbf{x}_0$ is chosen as the symmetry centre of the distribution.

If the charge distribution $\rho(\mathbf{x})$ is made up of discrete point charges $q_n$ with coordinates $\mathbf{x}_n$, the definitions above of $q$, $\mathbf{d}$, $\mathbf{Q}$ and $\mathbf{Q}'$ become

$$q = \sum_n q_n \tag{3.69a}$$

$$\mathbf{d} = \sum_n q_n(\mathbf{x}_n - \mathbf{x}_0) \tag{3.69b}$$

$$\mathbf{Q} = \sum_n q_n(\mathbf{x}_n - \mathbf{x}_0) \otimes (\mathbf{x}_n - \mathbf{x}_0) \tag{3.69c}$$

$$\mathbf{Q}' = \sum_n q_n\left[(\mathbf{x}_n - \mathbf{x}_0) \otimes (\mathbf{x}_n - \mathbf{x}_0) - \mathbf{1}_3 \frac{1}{3} |\mathbf{x}_n - \mathbf{x}_0|^2\right] \tag{3.69d}$$

End of example 3.1 ◁

EXAMPLE 3.2 ▷Electromagnetodynamic potentials

In Dirac's symmetrised form of electrodynamics (electromagnetodynamics), Maxwell's equations are replaced by [see also equations (2.2) on page 20]:

$$\nabla \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{3.70a}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} \tag{3.70b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{j}^{\mathrm{m}} \tag{3.70c}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}^{\mathrm{e}} \tag{3.70d}$$

In this theory, one derives the inhomogeneous wave equations for the usual 'electric' scalar and vector potentials $(\Phi^{\mathrm{e}}, \mathbf{A}^{\mathrm{e}})$ and their 'magnetic' counterparts $(\Phi^{\mathrm{m}}, \mathbf{A}^{\mathrm{m}})$ by assuming that the potentials are related to the fields in the following symmetrised form:

$$\mathbf{E} = -\nabla \Phi^{\mathrm{e}}(t, \mathbf{x}) - \frac{\partial}{\partial t}\mathbf{A}^{\mathrm{e}}(t, \mathbf{x}) - \nabla \times \mathbf{A}^{\mathrm{m}} \tag{3.71a}$$

$$\mathbf{B} = -\frac{1}{c^2}\nabla \Phi^{\mathrm{m}}(t, \mathbf{x}) - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{A}^{\mathrm{m}}(t, \mathbf{x}) + \nabla \times \mathbf{A}^{\mathrm{e}} \tag{3.71b}$$

In the absence of magnetic charges, or, equivalently for $\Phi^{\mathrm{m}} \equiv 0$ and $\mathbf{A}^{\mathrm{m}} \equiv \mathbf{0}$, these formulæ reduce to the usual Maxwell theory, formulæ (3.9) and (3.13) on page 35 respectively, as they should.

Inserting the symmetrised expressions (3.71) above into equations (3.70) above, one obtains [*cf.*, equations (3.14) on page 36]

$$\nabla^2 \Phi^{\mathrm{e}} + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}^{\mathrm{e}}\right) = -\frac{\rho^{\mathrm{e}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.72a}$$

$$\nabla^2 \Phi^{\mathrm{m}} + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}^{\mathrm{m}}\right) = -\frac{\rho^{\mathrm{m}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.72b}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{\mathrm{e}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{e}} + \nabla\left(\nabla \cdot \mathbf{A}^{\mathrm{e}} + \frac{1}{c^2}\frac{\partial \Phi^{\mathrm{e}}}{\partial t}\right) = \mu_0 \mathbf{j}^{\mathrm{e}}(t, \mathbf{x}) \tag{3.72c}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{\mathrm{m}}}{\partial t^2} - \nabla^2 \mathbf{A}^{\mathrm{m}} + \nabla\left(\nabla \cdot \mathbf{A}^{\mathrm{m}} + \frac{1}{c^2}\frac{\partial \Phi^{\mathrm{m}}}{\partial t}\right) = \mu_0 \mathbf{j}^{\mathrm{m}}(t, \mathbf{x}) \tag{3.72d}$$

By choosing the conditions on the divergence of the vector potentials as the generalised Lorenz-Lorentz condition [*cf.* equation (3.16) on page 37]

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{\mathrm{e}} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi^{\mathrm{e}} = 0 \tag{3.73a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{\mathrm{m}} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi^{\mathrm{m}} = 0 \tag{3.73b}$$

these coupled wave equations simplify to

$$\Box^2 \Phi^{\mathrm{e}} = \frac{\rho^{\mathrm{e}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.74a}$$

$$\Box^2 \mathbf{A}^{\mathrm{e}} = \mu_0 \mathbf{j}^{\mathrm{e}}(t, \mathbf{x}) \tag{3.74b}$$

$$\Box^2 \Phi^{\mathrm{m}} = \frac{\rho^{\mathrm{m}}(t, \mathbf{x})}{\varepsilon_0} \tag{3.74c}$$

$$\Box^2 \mathbf{A}^{\mathrm{m}} = \mu_0 \mathbf{j}^{\mathrm{m}}(t, \mathbf{x}) \tag{3.74d}$$

exhibiting, once again, the striking properties of Dirac's symmetrised Maxwell theory.

End of example 3.2◁

▷Longitudinal and transverse components in gauge transformations — EXAMPLE 3.3

If we represent the vector potential $\mathbf{A}(t, \mathbf{x})$ in the reciprocal $\mathbf{k}$ space as described at the end of subsection 2.3.1 on page 25, the gauge transformation equation (3.50b) on page 43 becomes

$$\mathbf{A}_{\mathbf{k}}(t) \mapsto \mathbf{A}'_{\mathbf{k}}(t) = \mathbf{A}_{\mathbf{k}}(t) - \mathrm{i}\mathbf{k}\Gamma_{\mathbf{k}}(t) \tag{3.75}$$

we can separate it into its longitudinal and transverse components

$$\mathbf{A}'_{\parallel} = \mathbf{A}_{\parallel} + \mathrm{i}\mathbf{k}\Gamma_k(t) = \mathbf{A}_{\parallel} + \boldsymbol{\nabla}\Gamma(t, \mathbf{x}) = \mathbf{A}_{\parallel} + \mathrm{i}\mathbf{k}\Gamma_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{3.76a}$$

$$\mathbf{A}'_{\perp} = \mathbf{A}_{\perp} \tag{3.76b}$$

A Helmholtz decomposition [see formula (M.83) on page 248]

$$\mathbf{A} = \mathbf{A}^{\mathrm{rotat}} + \mathbf{A}^{\mathrm{irrot}} \tag{3.77}$$

shows that

$$\mathbf{A}'^{\mathrm{rotat}} = \mathbf{A}^{\mathrm{rotat}} \tag{3.78a}$$

$$\mathbf{A}'^{\mathrm{irrot}} = \mathbf{A}^{\mathrm{irrot}} + \boldsymbol{\nabla}\Gamma \tag{3.78b}$$

Hence, a law (expression) that depends on $\mathbf{A}$ only through its transverse/rotational component $\mathbf{A}_{\perp} = \mathbf{A}^{\mathrm{rotat}}$ is gauge invariant, whereas a law that depends on the longitudinal/irrotational component $\mathbf{A}_{\parallel} = \mathbf{A}^{\mathrm{irrot}}$ is in general not gauge invariant and, if so, it does not represent a physical observable.

With the caveat about plane waves mentioned near formulæ (2.36) on page 28, the following then applies for the electric field and the magnetic field $\mathbf{E} = \mathbf{E}_{\perp} + \mathbf{E}_{\parallel}$ and $\mathbf{B} = \mathbf{B}_{\perp} + \mathbf{B}_{\parallel}$, respectively:

$$\mathbf{E}_{\perp} = -\frac{\partial \mathbf{A}_{\perp}}{\partial t} \tag{3.79a}$$

$$\mathbf{E}_{\parallel} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}_{\parallel}}{\partial t} \tag{3.79b}$$

$$\mathbf{B}_{\perp} = \boldsymbol{\nabla} \times \mathbf{A}_{\perp} \tag{3.79c}$$

$$\mathbf{B}_{\parallel} = \mathbf{0} \tag{3.79d}$$

In terms of rotational and irrotational parts, the electric and magnetic fields are given by the sums $\mathbf{E} = \mathbf{E}^{\text{rotat}} + \mathbf{E}^{\text{irrot}}$ and $\mathbf{B} = \mathbf{B}^{\text{rotat}} + \mathbf{B}^{\text{irrot}}$, respectively, where the individual terms are

$$\mathbf{E}^{\text{rotat}} = -\frac{\partial \mathbf{A}^{\text{rotat}}}{\partial t} \tag{3.80a}$$

$$\mathbf{E}^{\text{irrot}} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}^{\text{irrot}}}{\partial t} \tag{3.80b}$$

$$\mathbf{B}^{\text{rotat}} = \boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}} \tag{3.80c}$$

$$\mathbf{B}^{\text{irrot}} = \mathbf{0} \tag{3.80d}$$

End of example 3.3◁

EXAMPLE 3.4 ▷Gauge transformations and quantum mechanics

As discussed in section 2.2 on page 20, quantum theory requires that we take the magnitude rather than the real part of our mathematical variables in order to turn them into quantities that represent physical observables. In non-relativistic quantum mechanics, the physical observable *probability density* is $\psi\psi^* = |\psi|^2$, where the wave function $\psi \in \mathbb{C}$ solves the *Schrödinger equation*

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = \widehat{\mathsf{H}}\psi \tag{3.81}$$

and $\widehat{\mathsf{H}}$ is the *Hamilton operator*.

The non-relativistic *Hamiltonian* for a classical particle with charge $q$ in an electromagnetic field, described by the scalar potential $\Phi$ and vector potential $\mathbf{A}$, is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\Phi \tag{3.82}$$

where $\mathbf{p}$ is the linear momentum. The corresponding quantal Hamilton operator is obtained from the *correspondence principle*, *viz.*, by replacing $\mathbf{p}$ by the operator $\widehat{\mathbf{p}} = -\mathrm{i}\hbar\boldsymbol{\nabla}$, referred to as *minimal coupling*. This gives the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A})^2\,\psi + q\Phi\psi \tag{3.83}$$

The idea is to perform a gauge transformation from the potentials $\Phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ to new potentials

$$\Phi \mapsto \Phi'(t, \mathbf{x}) = \Phi(t, \mathbf{x}) + \frac{\partial \Gamma(t, \mathbf{x})}{\partial t} \tag{3.84a}$$

$$\mathbf{A} \mapsto \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) - \boldsymbol{\nabla}\Gamma(t, \mathbf{x}) \tag{3.84b}$$

and then find a $\theta(t, \mathbf{x})$, expressed in the gauge function $\Gamma(t, \mathbf{x})$, so that the transformed Schrödinger equation can be written

$$\mathrm{i}\hbar\frac{\partial(\mathrm{e}^{\mathrm{i}\theta}\psi)}{\partial t} = \frac{1}{2m}(-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A})^2\,\mathrm{e}^{\mathrm{i}\theta}\psi + q\Phi\mathrm{e}^{\mathrm{i}\theta}\psi \tag{3.85}$$

Under the gauge transformation equation (3.84) above, the Schrödinger equation (3.83) transforms into

$$\mathrm{i}\hbar\frac{\partial \psi'}{\partial t} = \frac{1}{2m}[-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]^2\,\psi' + q\Phi\psi' + q\frac{\partial \Gamma}{\partial t}\psi' \tag{3.86}$$

Now, setting

$$\psi'(t,\mathbf{x}) = \mathrm{e}^{\mathrm{i}\theta(t,\mathbf{x})}\psi(t,\mathbf{x}) \tag{3.87}$$

we see that

$$\begin{aligned}
&\big[-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)\big]^2\psi' \\
&\quad= [-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]\,[-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]\,\mathrm{e}^{\mathrm{i}\theta}\psi \\
&\quad= [-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)] \\
&\qquad\times\Big[-\mathrm{i}\hbar\mathrm{e}^{\mathrm{i}\theta}(\boldsymbol{\nabla}\psi) - \mathrm{i}\hbar\mathrm{e}^{\mathrm{i}\theta}(\boldsymbol{\nabla}\mathrm{i}\theta)\psi - q\mathbf{A}\mathrm{e}^{\mathrm{i}\theta}\psi + (q\boldsymbol{\nabla}\Gamma)\mathrm{e}^{\mathrm{i}\theta}\psi\Big] \\
&\quad= [-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]\,\mathrm{e}^{\mathrm{i}\theta}\,[-\mathrm{i}\hbar\boldsymbol{\nabla} - \mathrm{i}\hbar(\boldsymbol{\nabla}\mathrm{i}\theta) - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]\,\psi \\
&\quad= \Big[-\mathrm{i}\hbar(\boldsymbol{\nabla}\mathrm{i}\theta)\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}\hbar\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{\nabla} - q\mathbf{A}\mathrm{e}^{\mathrm{i}\theta} + (q\boldsymbol{\nabla}\Gamma)\mathrm{e}^{\mathrm{i}\theta}\Big] \\
&\qquad\times[-\mathrm{i}\hbar\boldsymbol{\nabla} - \mathrm{i}\hbar(\boldsymbol{\nabla}\mathrm{i}\theta) - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]\,\psi \\
&\quad= \mathrm{e}^{\mathrm{i}\theta}\,[-\mathrm{i}\hbar\boldsymbol{\nabla} - \mathrm{i}\hbar(\boldsymbol{\nabla}\mathrm{i}\theta) - q\mathbf{A} + (q\boldsymbol{\nabla}\Gamma)]^2\,\psi \\
&\quad= \mathrm{e}^{\mathrm{i}\theta}\,[-\mathrm{i}\hbar\boldsymbol{\nabla} - q\mathbf{A} + \hbar(\boldsymbol{\nabla}\theta) + (q\boldsymbol{\nabla}\Gamma)]^2\,\psi
\end{aligned} \tag{3.88}$$

Clearly, the gauge transformed Hamilton operator is unchanged iff $\hbar(\boldsymbol{\nabla}\theta) = -q(\boldsymbol{\nabla}\Gamma)$, or, equivalently, iff $\theta(t,\mathbf{x}) = -q\Gamma(t,\mathbf{x})/\hbar$. This has as a consequence that

$$\begin{aligned}
&\mathrm{i}\hbar\frac{\partial\psi'}{\partial t} - q\frac{\partial\Gamma}{\partial t}\psi' = \mathrm{i}\hbar\frac{\partial\psi'}{\partial t} + \hbar\frac{\partial\theta}{\partial t}\psi' = \mathrm{i}\hbar\frac{\partial}{\partial t}(\mathrm{e}^{\mathrm{i}\theta}\psi) + \hbar\frac{\partial\theta}{\partial t}\mathrm{e}^{\mathrm{i}\theta}\psi \\
&\quad= \mathrm{i}^2\hbar\frac{\partial\theta}{\partial t}\mathrm{e}^{\mathrm{i}\theta}\psi + \mathrm{i}\hbar\mathrm{e}^{\mathrm{i}\theta}\frac{\partial\psi}{\partial t} + \hbar\frac{\partial\theta}{\partial t}e^{\mathrm{i}\theta}\psi = \mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\hbar\frac{\partial\psi}{\partial t}
\end{aligned} \tag{3.89}$$

Inserting this into the transformed Schrödinger equation (3.86) on the preceding page, we recover the untransformed Schrödinger equation (3.83).

We conclude that under a gauge transformation of the potentials $\Phi$ and $\mathbf{A}$ and with minimal coupling as in equation (3.83) on the facing page, the wave function changes from $\psi$ to $\mathrm{e}^{\mathrm{i}\theta}\psi$ where the phase angle $\theta$ is real-valued. Hence, even if the wavefunction is not invariant, the quantum physical observable $|\psi|^2$ is unaffected by a gauge transformation of the classical potentials $\Phi$ and $\mathbf{A}$ that appear in the Hamilton operator. The fact that $\theta(t,\mathbf{x})$ is coordinate dependent, means that we are dealing with a *local gauge transformation*.

Since the probability density $\psi\psi^* = |\psi|^2$ and the probability current $-\mathrm{i}\hbar(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)/(2m) - q\mathbf{A}\,|\psi|^2/m$ do not change under a gauge transformation of $\Phi$ and $\mathbf{A}$, the charge density $\rho = q\,|\psi|^2$ and therefore also the charge $\int_{V'}\mathrm{d}^3x'\rho$ are conserved. In other words, electromagnetic gauge symmetry corresponds to electric charge conservation and *vice versa*.

For the gauge transformation given by formulæ (3.84) on the preceding page, WOLFGANG PAULI introduced the notation *gauge transformation of the second kind* whereas he called a wavefunction phase change, expression (3.87) above, a *gauge transformation of the first kind*.

End of example 3.4◁

## 3.7 Bibliography

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# 4

# FUNDAMENTAL PROPERTIES OF THE ELECTROMAGNETIC FIELD

In this chapter we shall explore a number of fundamental properties of the electromagnetic fields themselves as well as of the physical observables constructed from them. Of particular interest are *symmetries* ('principles of simplicity'), since they have a striking predictive power and are essential ingredients of the physics. This includes both discrete and continuous geometric symmetries (conjugation, reflection, reversion, translation, rotation) and intrinsic symmetries (duality, reciprocity). Intimately related to symmetries are *conserved quantities* (*constants of motion*) of which our primary interest will be the electromagnetic energy, linear momentum, centre of energy, and angular momentum. These ten conserved quantities, one scalar, two vectors and one pseudovector,[1] can carry information over large distances and are all more or less straightforwardly related to their counterparts in classical mechanics (indeed in all field theories). But we will also consider other conserved quantities where the relation classical mechanics is less straightforward.

[1] The concomitant symmetries comprise the ten-parameter *Poincaré group* $P(10)$.

To derive useful mathematical expressions for the physical observables that we want to study, we will take the microscopic Maxwell-Lorentz equations (2.1) on page 19 as our axiomatic starting point.

## 4.1 Discrete symmetries

An analysis of the discrete (non-continuous) symmetries of a physical system provides deep insight into the system's most fundamental characteristics. In this section we find out how the electromagnetic fields behave under certain discrete symmetry transformations.

### 4.1.1 Charge conjugation, spatial inversion, and time reversal

Let us first investigate the transformation properties of the charge density $\rho$, the current density $\mathbf{j}$, and the associated fields $\mathbf{E}$ and $\mathbf{B}$ under *charge conjugation*, *i.e.* a change of sign of charge $q \mapsto q' = -q$, called C symmetry; under *spatial inversion*, *i.e.* a change of sign of the space coordinates $\mathbf{x} \mapsto \mathbf{x}' = -\mathbf{x}$, called *parity transformation* or P symmetry; and under *time reversal*, *i.e.* a change of sign of the time coordinate $t \mapsto t' = -t$, called T symmetry.

Let us study the C, P, and T symmetries one by one, following the standard convention that $t$, $q$, $c$ and $\varepsilon_0$ are ordinary scalars (not pseudoscalars) and are therefore unaffected by coordinate changes, and that the position vector $\mathbf{x}$ is, by definition, the prototype of all ordinary (polar) vectors against which all other vectors are benchmarked. In our study, we use the fact that

$$\rho \propto q \tag{4.1a}$$

and that

$$\mathbf{j} = \rho \mathbf{v}^{\text{mech}} = \rho \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \tag{4.1b}$$

The transformation properties follow directly from the Maxwell-Lorentz equations (2.1) on page 19.

## 4.1.2 C symmetry

A charge conjugation $q \mapsto q' = -q$ results in the following changes:

$$\rho \mapsto \rho' = -\rho \tag{4.2a}$$

$$\mathbf{j} \mapsto \mathbf{j}' = \rho' \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} = -\rho \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} = -\rho \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\mathbf{j} \tag{4.2b}$$

$$\boldsymbol{\nabla} \mapsto \boldsymbol{\nabla}' = \boldsymbol{\nabla} \tag{4.2c}$$

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \tag{4.2d}$$

When we apply them to the Maxwell-Lorentz equation (2.1a) we see that

$$\boldsymbol{\nabla}' \cdot \mathbf{E}'(t', \mathbf{x}') = \frac{\rho'}{\varepsilon_0} = \frac{(-\rho)}{\varepsilon_0} = -\frac{\rho}{\varepsilon_0} = -\boldsymbol{\nabla} \cdot \mathbf{E}(t, \mathbf{x}) \tag{4.3a}$$

Since $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$, and $\varepsilon_0$ is unaffected by the changes described by equations (4.2), we conclude that that $\boldsymbol{\nabla} \cdot \mathbf{E}' = -\boldsymbol{\nabla} \cdot \mathbf{E}$, as postulated by the Maxwell-Lorentz equations, can be true only if $\mathbf{E}' = -\mathbf{E}$. In other words, the Maxwell-Lorentz equations postulate that the electric field must change direction if the sign of the charge is changed, in agreement with Coulomb's law in electrostatics.

If we use this when the transformation is applied to equation (2.1c) on page 19, we obtain

$$\boldsymbol{\nabla}' \times \mathbf{E}'(t', \mathbf{x}') = -\frac{\partial \mathbf{B}'(t', \mathbf{x}')}{\partial t'} = \boldsymbol{\nabla} \times [-\mathbf{E}(t, \mathbf{x})] = \frac{\partial \mathbf{B}(t, \mathbf{x})}{\partial t} \tag{4.3b}$$

implying that under charge conjugation, the fields transform as

$$\mathbf{E}'(t',\mathbf{x}') = -\mathbf{E}(t,\mathbf{x}) \tag{4.4a}$$

$$\mathbf{B}'(t',\mathbf{x}') = -\mathbf{B}(t,\mathbf{x}) \tag{4.4b}$$

Consequently, all terms in the Maxwell-Lorentz equations change sign under charge conjugation. This means that these equations are invariant under the C symmetry. No breaking of the charge conjugation symmetry has sofar been observed in Nature.

### 4.1.3 P symmetry

A 3D spatial inversion $\mathbf{x} \mapsto \mathbf{x}' = -\mathbf{x}$ results in the following changes:

$$\rho \mapsto \rho' = \rho \tag{4.5a}$$

$$\mathbf{j} \mapsto \mathbf{j}' = \rho' \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} = \rho \frac{\mathrm{d}(-\mathbf{x})}{\mathrm{d}t} = -\rho \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\mathbf{j} \tag{4.5b}$$

$$\boldsymbol{\nabla} \mapsto \boldsymbol{\nabla}' = -\boldsymbol{\nabla} \tag{4.5c}$$

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \tag{4.5d}$$

When applied to the Maxwell-Lorentz equations (2.1a) and (2.1c) on page 19, we see that

$$\boldsymbol{\nabla}' \cdot \mathbf{E}'(t',\mathbf{x}') = -\boldsymbol{\nabla} \cdot \mathbf{E}'(t',-\mathbf{x}) = \frac{\rho'}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} = -\boldsymbol{\nabla} \cdot [-\mathbf{E}(t,\mathbf{x})] \tag{4.6a}$$

$$\boldsymbol{\nabla}' \times \mathbf{E}'(t',\mathbf{x}') = -\frac{\partial \mathbf{B}'(t',\mathbf{x}')}{\partial t'} = -\boldsymbol{\nabla} \times [-\mathbf{E}(t,\mathbf{x})] = -\frac{\partial \mathbf{B}(t,\mathbf{x})}{\partial t} \tag{4.6b}$$

implying that under parity transformation the fields transform as

$$\mathbf{E}(t',\mathbf{x}') = -\mathbf{E}(t,\mathbf{x}) \tag{4.7a}$$

$$\mathbf{B}'(t',\mathbf{x}') = \mathbf{B}(t,\mathbf{x}) \tag{4.7b}$$

Hence, the Maxwell-Lorentz postulates imply that $\mathbf{E}$ is an ordinary vector (polar vector), as it should be, and that $\mathbf{B}$ is a *pseudovector* (axial vector) with properties as described in subsection M.2.2 on page 239.

### 4.1.4 T symmetry

A time reversal $t \mapsto t' = -t$ results in the following changes:

$$\rho \mapsto \rho' = \rho \tag{4.8a}$$

$$\mathbf{j} \mapsto \mathbf{j}' = -\mathbf{j} \tag{4.8b}$$

$$\boldsymbol{\nabla} \mapsto \boldsymbol{\nabla}' = \boldsymbol{\nabla} \tag{4.8c}$$

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t'} = -\frac{\partial}{\partial t} \tag{4.8d}$$

When we apply them to the Maxwell-Lorentz equations equation (2.1a) and equation (2.1c) on page 19, we see that

$$\boldsymbol{\nabla}' \cdot \mathbf{E}'(t', \mathbf{x}') = \boldsymbol{\nabla} \cdot \mathbf{E}'(-t, \mathbf{x}) = \frac{\rho'}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} = \boldsymbol{\nabla} \cdot \mathbf{E}(t, \mathbf{x}) \tag{4.9a}$$

$$\begin{aligned}\boldsymbol{\nabla}' \times \mathbf{E}'(t', \mathbf{x}') &= -\frac{\partial \mathbf{B}'(t', \mathbf{x}')}{\partial t'} = -\frac{\partial \mathbf{B}'(-t, \mathbf{x}')}{\partial(-t)} = \frac{\partial \mathbf{B}'(-t, \mathbf{x}')}{\partial t} \\ &= \boldsymbol{\nabla} \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial \mathbf{B}(t, \mathbf{x})}{\partial t}\end{aligned} \tag{4.9b}$$

implying that under time reversal

$$\mathbf{E}'(t', \mathbf{x}') = \mathbf{E}(t, \mathbf{x}) \tag{4.10a}$$

$$\mathbf{B}'(t', \mathbf{x}') = -\mathbf{B}(t, \mathbf{x}) \tag{4.10b}$$

We see that **E** is even and **B** odd under the T symmetry. The Universe as a whole is asymmetric under time reversal. On quantum scales this is manifested by the *uncertainty principle* and on classical scales by the *arrow of time* which is related to the increase of *thermodynamic entropy* in a closed system.

The CPT *theorem* states that the combined CPT symmetry must hold for all physical phenomena. No violation of this law has been observed to date. However, in 1964 it was experimentally discovered that the combined CP symmetry is violated in *neutral kaon decays*.[2]

[2] This discovery led to the Nobel Prize in Physics 1980.

## 4.2 Continuous symmetries

It is well established that a deeper understanding of a physical system can be obtained if one finds the system's *conserved quantities* (*constants of motion*), *i.e.* those observables of the system that do not change with time. According to *Noether's theorem*,[3] a system's conserved quantities is closely related to its *continuous symmetries*.

[3] AMALIE EMMY NOETHER (1882–1935) made important contributions to mathematics and theoretical physics. Her (first) theorem, which states that any differentiable symmetry of (the action of) a physical system has a corresponding *conservation law*, is considered to be a fundamental tool of theoretical physics.

### 4.2.1 General conservation laws

Consider a certain physical substance, quantity, property or other such entity that flows, in a time-dependent way, in 3D space. This can, for instance, be a fluid, electric charge, or quantum mechanical probability. Let us denote the volume density of this entity by $\varrho(t, \mathbf{x})$ and its flow velocity by $\mathbf{v}(t, \mathbf{x})$. Let us also introduce a closed volume $V$, fixed in space and enclosed by a perfectly permeable surface $S$ with an infinitesimally small directed area element $\mathrm{d}^2x\, \hat{\mathbf{n}} = \mathrm{d}x_i \mathrm{d}x_j \hat{\mathbf{n}}$, where $\mathrm{d}x_i$ and $\mathrm{d}x_j$ are two coordinates that span the local tangent plane of the surface and $\hat{\mathbf{n}}$ is an outward directed unit vector, orthogonal to the tangent plane. The closed volume $V$ is called the *control volume*. In the special case that $V$ is spherical, it is usually referred to as the *control sphere*.

If the velocity of the entity in question is $\mathbf{v}$, it flows with a velocity across the surface $S$ of the volume $V$ at the rate $\varrho\mathbf{v}$ per unit time and unit area. Clearly, an inward flow (antiparallel to $\hat{\mathbf{n}}$) will increase the amount of entity in $V$. We also have to allow for an increase due to the production of the entity inside $V$. This increase is quantified by the source density $s$. Recalling that the normal vector $\hat{\mathbf{n}}$ points outward, the following balance equation must then hold:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}^3x\,\varrho}_{\text{Total change within } V} = \underbrace{-\oint_S \mathrm{d}^2x\,\hat{\mathbf{n}}\cdot\varrho\mathbf{v}}_{\text{Flow into } V} + \underbrace{\int_V \mathrm{d}^3x\,s}_{\text{Production inside } V} \tag{4.11}$$

With the help of the divergence theorem, identity (F.121b) on page 221, this balance equation can be written

$$\int_V \mathrm{d}^3x\left(\frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) - s\right) = 0 \tag{4.12}$$

where $\varrho\mathbf{v}$ is the flux density of the entity under consideration. Since this balance equation must hold for any volume $V$, the integrand must vanish and we obtain the *continuity equation*

$$\frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) = s \tag{4.13}$$

This inhomogeneous, linear partial differential equation expresses the local balance between the explicit temporal change of the density of the entity, the flow of it across the surface of a control volume, and the local production of the entity within the control volume. In the absence of such a production, *i.e.* if $s = 0$, equation shows that the amount of entity $\int_V \mathrm{d}^3x\,\varrho$ in $V$ is only changed if the entity flows in or out of $V$.

In fact, since, according to formula (F.99) on page 220, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{a}) = 0$ for any arbitrary, continuously differentiable vector field $\mathbf{a}$, we can generalise equation (4.13) above to

$$\frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot(\varrho\mathbf{v} + \boldsymbol{\nabla}\times\mathbf{a}) = s \tag{4.14}$$

Furthermore, since the time derivative $\mathrm{d}/\mathrm{d}t$ operating on a scalar, vector or tensor field, dependent on $t$ and $\mathbf{x}(t)$, is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \tag{4.15}$$

according to the chain rule [*cf.* equation (1.34) on page 13], and since [*cf.*. identity (F.81) on page 219]

$$\nabla \cdot (\varrho \mathbf{v}) = \varrho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \varrho \tag{4.16}$$

we can rewrite the equation of continuity as

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho \nabla \cdot (\mathbf{v} + \nabla \times \mathbf{a}) = s \tag{4.17}$$

where, again, $\mathbf{a}$ is an arbitrary, differentiable pseudovector field with dimension $\mathrm{m^2 s^{-1}}$, *e.g.* the *moment of velocity* with respect to $\mathbf{x}_0$, *i.e.* $(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}$.

### 4.2.2 Conservation of electric charge

If we multiply both members of the Maxwell-Lorentz equation (2.1a) on page 19 by $\varepsilon_0$ and apply partial differentiation with respect to time $t$, we get

$$\frac{\partial \rho(t,\mathbf{x})}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}(t,\mathbf{x}) \tag{4.18}$$

and if we divide both members of the Maxwell-Lorentz equation (2.1d) by $\mu_0$ and take the divergence, we get

$$\nabla \cdot \mathbf{j}(t,\mathbf{x}) = \frac{1}{\mu_0} \underbrace{\nabla \cdot \big(\nabla \times \mathbf{B}(t,\mathbf{x})\big)}_{=0} - \underbrace{\frac{1}{\mu_0 c^2}}_{=\varepsilon_0} \nabla \cdot \frac{\partial \mathbf{E}(t,\mathbf{x})}{\partial t} = -\varepsilon_0 \nabla \cdot \frac{\partial \mathbf{E}(t,\mathbf{x})}{\partial t} \tag{4.19}$$

Since the electromagnetic fields are assumed to be well-behaved so that differentiation with respect to time and space commute when they operate on $\mathbf{E}(t,\mathbf{x})$, we see that it follows from the two Maxwell-Lorentz equations used that

$$\frac{\partial \rho(t,\mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}(t,\mathbf{x}) = 0 \tag{4.20}$$

This is a differential (local) balance equation of the same type as equation (4.13) on the previous page where now the entity density $\varrho$ represents the electric charge density $\rho$,[4] and where $s = 0$. Equation (4.20) above shows that the total electric charge within the control volume $V$ at time $t$, $q = \int_V \mathrm{d}^3x\, \rho(t,\mathbf{x})$, changes if and only if a *charge density flux*, or, equivalently, an *electric current density* (amount of current per unit area), $\mathbf{j}$ passes across the surface enclosing the control volume $V$ at time $t$. Hence, it postulates that electric charge can neither be created nor destroyed. Consequently, the Maxwell-Lorentz equations postulate that electric charge is *indestructible*.[5]

This is the first *conservation law* that we have derived from the microscopic Maxwell equations (2.1) on page 19 and it can be shown to be a manifestation

[4] Note that the density of *charge* and the (number) density of *charges* are two different things!

[5] That electricity is indestructible was postulated already 1747 by BENJAMIN FRANKLIN (1706–1770), printer, scientist, inventor, philosopher, statesman, and one of the founding fathers of the United States of America.

of the gauge symmetry of electrodynamics. This result is, of course, consistent *a posteriori* with the fact that these axiomatic laws were formulated under the *a priori* assumption that the continuity equation (1.22) on page 10 was valid. As shown in example 3.4 on page 50, the symmetry that is associated with charge conservation is the *gauge symmetry*.

### 4.2.3 Conservation of energy

The continuity equation (4.20) on the facing page contains the divergence of the first-order vector quantity $\mathbf{j}$. As an example of the divergence of a second order vector quantity, let us study the divergence of $\mathbf{E} \times \mathbf{B}$. Using the Maxwell equations (2.1c) and (2.1d) on page 19, we obtain the balance equation

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) &= \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \\ &= -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{E} \cdot \mathbf{j} - \varepsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= -\mu_0 \left( \frac{\partial}{\partial t} \frac{\varepsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) + \mathbf{j} \cdot \mathbf{E} \right)\end{aligned} \tag{4.21}$$

where formula (F.83) on page 219 was used.

Let us define the *electromagnetic field energy density*[6]

$$u^{\text{field}}(t, \mathbf{x}) = \frac{\varepsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \quad (\mathrm{J m^{-3}}) \tag{4.22}$$

and the *electromagnetic energy flux* (also known as the *Poynting vector*)[7]

$$\mathbf{S}(t, \mathbf{x}) = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B} \quad (\mathrm{W m^{-2}}) \tag{4.23}$$

which can also be viewed as the *electromagnetic energy current density*.

With the help of these two definitions we can write the balance equation (4.21) above as

$$\frac{\partial u^{\text{field}}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \tag{4.24}$$

*i.e.* as a continuity equation with a source density

$$s = -\mathbf{j} \cdot \mathbf{E} = -\rho \mathbf{v}^{\text{mech}} \cdot \mathbf{E} \tag{4.25}$$

If we compare equation (4.24) with the generic continuity equation (4.13) on page 57, we see that if we let $\varrho = u^{\text{field}}$ and $\mathbf{S} = u^{\text{field}} \mathbf{v}^{\text{field}}$, we see that we can interpret the quantity

$$\mathbf{v}^{\text{field}} = \frac{\mathbf{S}}{u^{\text{field}}} = \varepsilon c^2 \frac{\mathbf{E} \times \mathbf{B}}{u^{\text{field}}} \tag{4.26}$$

[6] This expression was derived in 1853 by WILLIAM THOMSON, knighted LORD KELVIN, (1824–1927), Scottish mathematical physicist and engineer.

[7] This expression was derived in 1884 by the English physicist JOHN HENRY POYNTING (1882–1914) and in 1885 by OLIVER HEAVISIDE (1850–1925), a self-taught English mathematician, physicist and electrical engineer who was in many ways ahead of his contemporaries and had a remarkable impact on how we look at physics today.

as the *energy density velocity*,

From the definition of $\mathbf{E}$, we conclude that $\rho\mathbf{E}$ represents force per unit volume ($\mathrm{Nm^{-3}}$) and that $\rho\mathbf{E}\cdot\mathbf{v}^{\mathrm{mech}}$ therefore represents work per unit volume or, in other words, power density ($\mathrm{Wm^{-3}}$). It is known as the *Lorentz power density* and is equivalent to the time rate of change of the *mechanical kinetic energy density* ($\mathrm{Jm^{-3}}$) of the current carrying particles

$$\frac{\partial u^{\mathrm{mech}}}{\partial t} = \mathbf{j}\cdot\mathbf{E} = \rho\mathbf{v}^{\mathrm{mech}}\cdot\mathbf{E} \tag{4.27}$$

Equation (4.24) on the previous page can therefore be written

$$\frac{\partial u^{\mathrm{mech}}}{\partial t} + \frac{\partial u^{\mathrm{field}}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0 \tag{4.28}$$

This is the *energy density balance equation* in local (differential) form.

Expressing the *Lorentz power*, $\int_V \mathrm{d}^3x\;\mathbf{j}\cdot\mathbf{E}$, as the time rate of change of the *mechanical energy*:

$$\frac{\mathrm{d}U^{\mathrm{mech}}}{\mathrm{d}t} = \int_V \mathrm{d}^3x\,\frac{\partial}{\partial t}u^{\mathrm{mech}}(t,\mathbf{x}) = \int_V \mathrm{d}^3x\;\mathbf{j}\cdot\mathbf{E} \tag{4.29}$$

and introducing the *electromagnetic field energy*

$$U^{\mathrm{field}} = U^{\mathrm{e}} + U^{\mathrm{m}} \tag{4.30}$$

where, as follows from formula (4.22) on the preceding page,

$$U^{\mathrm{e}}(t) = \frac{\varepsilon_0}{2}\int_V \mathrm{d}^3x\;\mathbf{E}\cdot\mathbf{E} \tag{4.31}$$

is the *electric field energy* and

$$U^{\mathrm{m}}(t) = \frac{\varepsilon_0}{2}\int_V \mathrm{d}^3x\;c^2\mathbf{B}\cdot\mathbf{B} \tag{4.32}$$

is the *magnetic field energy*, we can write the integral version of the balance equation (4.28) above as

$$\frac{\mathrm{d}U^{\mathrm{mech}}}{\mathrm{d}t} + \frac{\mathrm{d}U^{\mathrm{field}}}{\mathrm{d}t} + \oint_S \mathrm{d}^2x\;\hat{\mathbf{n}}\cdot\mathbf{S} = 0 \tag{4.33}$$

This is the *energy theorem in Maxwell's theory*, also known as *Poynting's theorem*.

Allowing for an EMF and assuming that Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}^{\mathrm{emf}}) \tag{4.34}$$

is a valid approximation, or, equivalently, that

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{E}^{\mathrm{emf}} \tag{4.35}$$

we obtain the relation

$$\int_V d^3x\, \mathbf{j} \cdot \mathbf{E} = \int_V d^3x\, \frac{\mathbf{j} \cdot \mathbf{j}}{\sigma} - \int_V d^3x\, \mathbf{j} \cdot \mathbf{E}^{\text{emf}} \tag{4.36}$$

which, when inserted into equation (4.33) on the facing page and use is made of equation (4.29) on the preceding page, yields

$$\underbrace{\int_V d^3x\, \mathbf{j} \cdot \mathbf{E}^{\text{emf}}}_{\text{Supplied electric power}} = \underbrace{\frac{dU^{\text{field}}}{dt}}_{\text{Field power}} + \underbrace{\oint_S d^2x\, \hat{\mathbf{n}} \cdot \mathbf{S}}_{\text{Radiated power}} + \underbrace{\int_V d^3x\, \frac{\mathbf{j} \cdot \mathbf{j}}{\sigma}}_{\text{Joule heat}} \tag{4.37}$$

This shows how the supplied power (left-hand side, LHS) is expelled in the form of a time rate change of *electric and magnetic field energy*, *radiated electromagnetic power*, and *Joule heat power* (*Ohmic losses*) in the system (right-hand side, RHS).

The conservation of energy is a manifestation of the *temporal translational invariance* of the Maxwell-Lorentz equations.

### 4.2.4 Conservation of linear (translational) momentum

The derivation of the energy conservation formula (4.24) on page 59 started with a study of the divergence of $\mathbf{E} \times \mathbf{B}$. We now seek a balance equation involving the time derivative of $\mathbf{E} \times \mathbf{B}$ and find, using the Maxwell-Lorentz equations (2.1c) and (2.1d) on page 19, that

$$\begin{aligned}
\frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t} &= \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} = \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} - \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} \\
&= -\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) - c^2 \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \frac{1}{\varepsilon_0} \overbrace{\mathbf{B} \times \mathbf{j}}^{= -\mathbf{j} \times \mathbf{B}}
\end{aligned} \tag{4.38}$$

A combination of the identities (F.79) and (F.86) on page 219 yields

$$\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \tfrac{1}{2} \boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{E}) - \boldsymbol{\nabla} \cdot (\mathbf{E} \otimes \mathbf{E}) + (\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{E} \tag{4.39a}$$

$$\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \tfrac{1}{2} \boldsymbol{\nabla}(\mathbf{B} \cdot \mathbf{B}) - \boldsymbol{\nabla} \cdot (\mathbf{B} \otimes \mathbf{B}) + (\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{B} \tag{4.39b}$$

Using the Maxwell-Lorentz equations (2.1a) and (2.1b) on page 19, and the identity (F.102) on page 220 allows us to write

$$\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla} \cdot \left[\tfrac{1}{2}(\mathbf{E} \cdot \mathbf{E})\mathbf{1}_3 - \mathbf{E} \otimes \mathbf{E}\right] + \frac{\rho}{\varepsilon_0}\mathbf{E} \tag{4.40a}$$

$$\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \cdot \left[\tfrac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbf{1}_3 - \mathbf{B} \otimes \mathbf{B}\right] + \mathbf{0} \tag{4.40b}$$

Let us introduce the *electromagnetic linear momentum flux tensor*, also known as (the negative of) the *Maxwell stress tensor* or the *electromagnetic linear momentum current density*,

$$\mathsf{T} = \tfrac{1}{2}\varepsilon_0(\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B})\mathbf{1}_3 - (\mathbf{E} \otimes \mathbf{E} + c^2 \mathbf{B} \otimes \mathbf{B}) \tag{4.41}$$

measured in Pa, *i.e.* $\mathrm{Nm}^{-2}$ in SI units. In tensor component form, $\mathbf{T}$ is

$$T_{ij} = u^{\mathrm{field}}\delta_{ij} - \varepsilon_0 E_i E_j - \varepsilon_0 c^2 B_i B_j \tag{4.42}$$

where $u^{\mathrm{field}}$ is the electromagnetic energy density, defined in formula (4.22) on page 59. The component $T_{ij}$ is the electromagnetic linear momentum flux in the $i$th direction that passes across a surface element in the $j$th direction per unit time, per unit area. The tensor $\mathbf{T}$ has the properties

$$T_{ij} = T_{ji} \tag{4.43a}$$

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = u^{\mathrm{field}} \tag{4.43b}$$

$$\det(\mathbf{T}) = u^{\mathrm{field}}\big[(c g^{\mathrm{field}})^2 - (u^{\mathrm{field}})^2\big] \tag{4.43c}$$

We can now rewrite equation (4.38) on the preceding page as

$$\frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t} = -\frac{1}{\varepsilon_0}\big[\boldsymbol{\nabla} \cdot \mathbf{T} + \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}\big] \tag{4.44}$$

If we introduce the *electromagnetic linear momentum density* $\mathbf{g}^{\mathrm{field}}$ by making the identification[8]

[8] This follows from *Planck's relation* $\mathbf{S} = \mathbf{g}^{\mathrm{field}}c^2$. Since the LHS of this equation is (energy density) × (velocity) and the RHS is (mass density) × (velocity) $\times c^2$, we see that this relation forebodes the relativistic relation $E = mc^2$, where $E$ is energy and $m$ is mass.

$$\mathbf{g}^{\mathrm{field}}(t, \mathbf{x}) = \varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{\mathbf{S}}{c^2} \tag{4.45}$$

we can therefore write equation (4.38) on the previous page as

$$\frac{\partial \mathbf{g}^{\mathrm{field}}}{\partial t} + \mathbf{f} + \boldsymbol{\nabla} \cdot \mathbf{T} = \mathbf{0} \tag{4.46}$$

where

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{4.47}$$

This polar vector $\mathbf{f}$ has the dimension $\mathrm{Nm}^{-3}$ and we therefore identify it as a force density and call it the *Lorentz force density*.

According to classical mechanics (*Newton's second law*, *Euler's first law*), the mechanical force density $\mathbf{f}(t, \mathbf{x})$ must equal the time rate change of the *mechanical linear momentum density*

$$\mathbf{g}^{\mathrm{mech}}(t, \mathbf{x}) = \varrho_{\mathrm{m}}(\mathbf{x})\mathbf{v}^{\mathrm{mech}}(t, \mathbf{x}) \tag{4.48}$$

where $\varrho_{\mathrm{m}}$ is the (volumetric) *mass density* ($\mathrm{kgm}^{-3}$). We are therefore able to write the *linear momentum density* balance equation (4.38) on the preceding page as a local (differential) continuity equation in the standard form

$$\frac{\partial \mathbf{g}^{\mathrm{mech}}}{\partial t} + \frac{\partial \mathbf{g}^{\mathrm{field}}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} = \mathbf{0} \tag{4.49}$$

Integration of equation (4.49) on the preceding page over the control volume $V$, enclosed by the surface $S$, yields, with the help of the divergence theorem, the *conservation law for linear momentum*

$$\frac{\mathrm{d}\mathbf{p}^{\text{mech}}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{p}^{\text{field}}}{\mathrm{d}t} + \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathsf{T} = \mathbf{0} \tag{4.50}$$

where

$$\mathbf{p}^{\text{mech}}(t) = \int_V \mathrm{d}^3x\, \mathbf{g}^{\text{mech}} = \int_V \mathrm{d}^3x\, \varrho_{\text{m}} \mathbf{v}^{\text{mech}} \tag{4.51a}$$

and

$$\mathbf{p}^{\text{field}}(t) = \int_V \mathrm{d}^3x\, \mathbf{g}^{\text{field}} = \int_V \mathrm{d}^3x\, \varepsilon_0 (\mathbf{E} \times \mathbf{B}) \tag{4.51b}$$

or

$$\underbrace{\int_V \mathrm{d}^3x\, \mathbf{f}}_{\text{Force on the matter}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathrm{d}^3x\, \varepsilon_0 (\mathbf{E} \times \mathbf{B})}_{\text{EM field momentum}} + \underbrace{\oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathsf{T}}_{\text{Linear momentum flow}} = \mathbf{0} \tag{4.52}$$

This is the *linear momentum theorem in Maxwell's theory* which shows that not only the mechanical particles (charges) but also the electromagnetic field itself carries *linear momentum* (*translational momentum*) and can thus be assumed to be particle- or fluid-like.

If we assume that we have a single localised charge $q$, such that the charge density is given in terms of a Dirac distribution as in equation (1.8) on page 5, with the summation running over one particle only, the evaluation of the first integral in equation (4.52) above shows that the force on this charge is

$$\mathbf{F} = \int_V \mathrm{d}^3x\, \mathbf{f} = \int_V \mathrm{d}^3x\, (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}) = q(\mathbf{E} + \mathbf{v}^{\text{mech}} \times \mathbf{B}) \tag{4.53}$$

which is the *Lorentz force*; see also equation (1.44) on page 14. Note that equation (4.53) above follows directly from a conservation law, and therefore is a consequence of a symmetry of the Maxwell-Lorentz postulates equations (2.1). Hence, the Lorentz force does not have to be separately postulated.

#### 4.2.4.1 Gauge-invariant operator formalism

Taking equation (4.51b) above as a starting point and using the fact that the **E** and **B** fields are assumed to behave in such a way that the can be Helmholtz decomposed as [*cf.* formula (F.129) on page 222]

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}^{\text{irrot}}(t, \mathbf{x}) + \mathbf{E}^{\text{rotat}}(t, \mathbf{x}) \tag{4.54a}$$

$$\mathbf{B}(t, \mathbf{x}) = \mathbf{B}^{\text{rotat}}(t, \mathbf{x}) \tag{4.54b}$$

we see that we can write the field momentum in the following manifestly gauge-invariant form:

$$\mathbf{p}^{\text{field}}(t) = \varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E}^{\text{irrot}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) + \varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E}^{\text{rotat}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) \tag{4.55}$$

Let us first evaluate the first integral in the RHS of this expression. Applying identities (F.79) and (F.86) on page 219, with $\mathbf{a} \mapsto \mathbf{E}^{\text{irrot}}$ and $\mathbf{b} \mapsto \mathbf{A}^{\text{rotat}}$, recalling from the Helmholtz decomposed Maxwell-Lorentz equations (2.42) on page 29 that $\boldsymbol{\nabla} \cdot \mathbf{E}^{\text{irrot}} = \rho/\varepsilon_0$ and that $\boldsymbol{\nabla} \times \mathbf{E}^{\text{irrot}} = \mathbf{0}$, the first integrand becomes

$$\mathbf{E}^{\text{irrot}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) = \frac{\rho}{\varepsilon_0}\mathbf{A}^{\text{rotat}} - \mathbf{A}^{\text{rotat}} \cdot (\boldsymbol{\nabla} \otimes \mathbf{E}^{\text{irrot}}) \tag{4.56}$$

$$+ \boldsymbol{\nabla}(\mathbf{E}^{\text{irrot}} \cdot \mathbf{A}^{\text{rotat}}) - \boldsymbol{\nabla} \cdot \mathbf{E}^{\text{irrot}} \otimes \mathbf{A}^{\text{rotat}} \tag{4.57}$$

Performing the integration and using identities (F.121) on page 221, we find that the first integral can be expressed as the sum of four integrals:

$$\int_V \mathrm{d}^3x\, \mathbf{E}^{\text{irrot}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) = \frac{1}{\varepsilon_0} \int_V \mathrm{d}^3x\, \rho \mathbf{A}^{\text{rotat}} - \int_V \mathrm{d}^3x\, \mathbf{A}^{\text{rotat}} \cdot (\boldsymbol{\nabla} \otimes \mathbf{E}^{\text{irrot}})$$

$$+ \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} (\mathbf{E}^{\text{irrot}} \cdot \mathbf{A}^{\text{rotat}}) - \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot (\mathbf{E}^{\text{irrot}} \otimes \mathbf{A}^{\text{rotat}}) \tag{4.58}$$

The second integral in the RHS of this equation can be integrated by parts by using identity (F.86) on page 219, with $\mathbf{a} \mapsto \mathbf{A}^{\text{rotat}}$ and $\mathbf{b} \mapsto \mathbf{E}^{\text{irrot}}$. The result is

$$\int_V \mathrm{d}^3x\, \mathbf{A}^{\text{rotat}} \cdot (\boldsymbol{\nabla} \otimes \mathbf{E}^{\text{irrot}}) = \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} (\mathbf{E}^{\text{irrot}} \cdot \mathbf{A}^{\text{rotat}}) - \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla} \cdot \mathbf{A}^{\text{rotat}}) \mathbf{E}^{\text{irrot}} \tag{4.59}$$

Inserting this, using the fact that, by definition, $\boldsymbol{\nabla} \cdot \mathbf{A}^{\text{rotat}} = 0$, we find that the first integral in equation (4.55) is

$$\varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E}^{\text{irrot}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) = \int_V \mathrm{d}^3x\, \rho \mathbf{A}^{\text{rotat}} - \varepsilon_0 \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{E}^{\text{irrot}} \otimes \mathbf{A}^{\text{rotat}} \tag{4.60}$$

We evaluate the second integral in equation (4.55) exactly analogously and find that

$$\varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E}^{\text{rotat}} \times (\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}) = -\,\varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{A}^{\text{rotat}} \times (\boldsymbol{\nabla} \times \mathbf{E}^{\text{rotat}}) - \varepsilon_0 \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{E}^{\text{rotat}} \otimes \mathbf{A}^{\text{rotat}} \tag{4.61}$$

Putting all of this together, we find that the electromagnetic linear (translational) momentum is given by the exact,[9] manifestly gauge-invariant formula

[9] Strictly speaking, in the expression for $\mathbf{E}^{\text{rotat}}$ and $\mathbf{A}^{\text{rotat}}$, we have neglected the respective surface integrals that must be retained if $V \subset \mathbb{R}^3$.

$$
\begin{aligned}
\mathbf{p}^{\text{field}}(t) = &\int_V \mathrm{d}^3x\, \rho \mathbf{A}^{\text{rotat}} - \varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{A}^{\text{rotat}} \times (\boldsymbol{\nabla} \times \mathbf{E}^{\text{rotat}}) \\
&- \varepsilon_0 \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{E} \otimes \mathbf{A}^{\text{rotat}}
\end{aligned}
\tag{4.62}
$$

If we restrict ourselves to consider a single temporal Fourier component of the rotational ('transverse') component of $\mathbf{E}$ in a region where $\rho = 0$, and use the results in example 3.3 on page 49, we find that in complex representation

$$
\mathbf{E}^{\text{rotat}} = -\frac{\partial \mathbf{A}^{\text{rotat}}}{\partial t} = \mathrm{i}\omega \mathbf{A}^{\text{rotat}} \tag{4.63}
$$

which allows us to replace $\mathbf{A}^{\text{rotat}}$ by $-\mathrm{i}\mathbf{E}^{\text{rotat}}/\omega$, yielding, after applying identity (F.93) on page 220 and equation (2.6) on page 22,

$$
\begin{aligned}
\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = &\operatorname{Re}\left\{ -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla} \otimes \mathbf{E}^{\text{rotat}}) \cdot (\mathbf{E}^{\text{rotat}})^* \right\} \\
&+ \operatorname{Re}\left\{ \mathrm{i}\frac{\varepsilon_0}{2\omega} \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{E} \otimes (\mathbf{E}^{\text{rotat}})^* \right\}
\end{aligned}
\tag{4.64}
$$

If the tensor (dyadic) $\mathbf{E} \otimes (\mathbf{E}^{\text{rotat}})^*$ is regular and falls off sufficiently rapidly at large distances (or if $\hat{\mathbf{n}} \cdot \mathbf{E} = 0$), we can discard the surface integral term and find that the cycle averaged linear momentum carried by the rotational components of the fields, $\mathbf{E}^{\text{rotat}}$ and $\mathbf{B}^{\text{rotat}} \equiv \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}$, is

$$
\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = \operatorname{Re}\left\{ -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla} \otimes \mathbf{E}^{\text{rotat}}) \cdot (\mathbf{E}^{\text{rotat}})^* \right\} \tag{4.65}
$$

In complex tensor notation, with Einstein's summation convention applied, this can be written

$$
\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x\, (E_j^{\text{rotat}})^* \left( \hat{\mathbf{x}}_i\, \partial_i E_j^{\text{rotat}} \right) = -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x\, (E_j^{\text{rotat}})^* \boldsymbol{\nabla} E_j^{\text{rotat}} \tag{4.66}
$$

Making use of the quantal *linear momentum operator*

$$
\hat{\mathbf{p}} = -\mathrm{i}\hbar \boldsymbol{\nabla} \tag{4.67}
$$

we see that we can write the expression for the linear momentum of the electromagnetic field in terms of this operator as

$$
\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = \frac{\varepsilon_0}{2\hbar\omega} \sum_{i=1}^{3} \int_V \mathrm{d}^3x\, (E_i^{\text{rotat}})^*\, \hat{\mathbf{p}}\, E_i^{\text{rotat}} \tag{4.68}
$$

If we introduce the vector

$$\boldsymbol{\Psi} = \Psi_i \hat{\mathbf{x}}_i = \sqrt{\frac{\varepsilon_0}{2\hbar\omega}} \mathbf{E}^{\text{rotat}} = \sqrt{\frac{\varepsilon_0}{2\hbar\omega}} E_i^{\text{rotat}} \hat{\mathbf{x}}_i \tag{4.69}$$

we can write

$$\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = \sum_{i=1}^{3} \int_V \mathrm{d}^3x \, \Psi_i^* \, \hat{\mathbf{p}} \, \Psi_i \tag{4.70}$$

or, if we assume Einstein's summation convention,

$$\left\langle \mathbf{p}^{\text{field}} \right\rangle_t = \langle \Psi_i | \, \hat{\mathbf{p}} \, | \Psi_i \rangle \tag{4.71}$$

That is, we can represent the cycle (temporal) averaged linear momentum carried by a monochromatic electromagnetic field as a sum of diagonal quantal matrix elements (expectation values) where the rotational ('transverse') component of the (scaled) electric field vector behaves as a vector wavefunction.

The conservation of linear (translational) momentum is a manifestation of the *spatial translational invariance* of the Maxwell-Lorentz equations.

### 4.2.5 Conservation of angular (rotational) momentum

Euler[10] showed in 1775 that the most general dynamical state of a mechanical system is the sum of its translational motion, described by the system's *mechanical linear momentum* $\mathbf{p}^{\text{mech}}$, and its rotational motion, described by the system's *mechanical moment of momentum*, or *mechanical angular momentum* $\mathbf{J}^{\text{mech}}(\mathbf{x}_0)$ about a moment point $\mathbf{x}_0$, and that these two momenta are in general independent of each other.[11]

In the special case of a classical rigid body for which the contribution from internal angular momenta cancel,[12] the angular momentum around $\mathbf{x}_0$ is given by the pseudovector $\mathbf{J}^{\text{mech}}(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{p}^{\text{mech}}$. For a closed system of rotating and orbiting bodies, *e.g.* a spinning planet orbiting a (non-rotating) star, the *total mechanical angular momentum* of the system is the vectorial sum of two contributions:

$$\mathbf{J}^{\text{mech}}(\mathbf{x}_0) = \boldsymbol{\Sigma}^{\text{mech}} + \mathbf{L}^{\text{mech}}(\mathbf{x}_0) \tag{4.72}$$

where $\boldsymbol{\Sigma}^{\text{mech}}$ is the intrinsic *mechanical spin angular momentum*, describing the spin of the planet around its own axis, and $\mathbf{L}^{\text{mech}}$ is the extrinsic *mechanical orbital angular momentum*, describing the motion of the planet in an orbit around the star. As is well known from mechanics, $\mathbf{J}^{\text{mech}}$ of a closed mechanical system is conserved and to change it one has to apply a *mechanical torque*

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{J}^{\text{mech}}}{\mathrm{d}t} \tag{4.73}$$

[10] LEONHARD EULER, 1707–1783, Swiss mathematician and physicist and one of the most prolific and influential mathematicians in history.

[11] Rational mechanist, mathematician and physics historian CLIFFORD AMBROSE TRUESDELL III (1919–2000) wrote several excellent accounts on how this result was arrived at by Euler in the latter half of the 18th century.

[12] This 'fact' is really an empirical law that cannot in general be derived from Newton's laws.

Starting from the definition of the mechanical linear momentum density $\mathbf{g}^{\text{mech}}$, formula (4.48) on page 62, we define the *mechanical angular momentum density* about a fixed moment point $\mathbf{x}_0$ as

$$\mathbf{h}^{\text{mech}}(t, \mathbf{x}, \mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{mech}}(t, \mathbf{x}) \tag{4.74}$$

Analogously, we define the *electromagnetic moment of momentum density* or *electromagnetic angular momentum density* about a moment point $\mathbf{x}_0$ as the pseudovector

$$\mathbf{h}^{\text{field}}(t, \mathbf{x}, \mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{field}}(t, \mathbf{x}) \tag{4.75}$$

where $\mathbf{g}^{\text{field}}$ is given by equation (4.45) on page 62. Partial differentiation with respect to time yields

$$\begin{aligned}\frac{\partial \mathbf{h}^{\text{field}}(t, \mathbf{x}, \mathbf{x}_0)}{\partial t} &= \frac{\partial}{\partial t}\left[(\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{field}}\right] \\ &= \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times \mathbf{g}^{\text{field}} + (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial \mathbf{g}^{\text{field}}}{\partial t} \\ &= (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial \mathbf{g}^{\text{field}}(t, \mathbf{x})}{\partial t}\end{aligned} \tag{4.76}$$

where we used the fact that $\mathrm{d}\mathbf{x}/\mathrm{d}t$ is the energy density velocity $\mathbf{v}^{\text{field}}$ and, hence, is parallel to $\mathbf{g}^{\text{field}}$; see equation (4.26) on page 59. We use equation (4.46) on page 62 to immediately find that

$$\frac{\partial \mathbf{h}^{\text{field}}(t, \mathbf{x}, \mathbf{x}_0)}{\partial t} + (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}(t, \mathbf{x}) + (\mathbf{x} - \mathbf{x}_0) \times \boldsymbol{\nabla} \cdot \mathsf{T}(t, \mathbf{x}) = \mathbf{0} \tag{4.77}$$

Identifying

$$\frac{\partial \mathbf{h}^{\text{mech}}(t, \mathbf{x}, \mathbf{x}_0)}{\partial t} = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}(t, \mathbf{x}) \tag{4.78}$$

as the *Lorentz torque density* and introducing the *electromagnetic angular momentum flux tensor*

$$\mathsf{K}(t, \mathbf{x}, \mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathsf{T}(t, \mathbf{x}) \tag{4.79}$$

where $\mathsf{T}$ is the electromagnetic linear momentum flux tensor given by expression (4.41) on page 61, we see that the local (differential) form of the balance equation for angular momentum density can be written (discarding the arguments $t$ and $\mathbf{x}$ from now on)

$$\frac{\partial \mathbf{h}^{\text{mech}}(\mathbf{x}_0)}{\partial t} + \frac{\partial \mathbf{h}^{\text{field}}(\mathbf{x}_0)}{\partial t} + \boldsymbol{\nabla} \cdot \mathsf{K}(\mathbf{x}_0) = \mathbf{0} \tag{4.80}$$

where the symmetric pseudotensor $\mathsf{K}(\mathbf{x}_0)$ represents the *electromagnetic angular momentum current density* around $\mathbf{x}_0$ and we used the fact that $(\mathbf{x} - \mathbf{x}_0) \times [\boldsymbol{\nabla} \cdot \mathsf{T}(t, \mathbf{x})] = \boldsymbol{\nabla} \cdot [(\mathbf{x} - \mathbf{x}_0) \times \mathsf{T}(t, \mathbf{x})] = \boldsymbol{\nabla} \cdot \mathsf{K}(t, \mathbf{x}, \mathbf{x}_0)$ [see example 4.4 on page 76].

Integration over the entire volume $V$, enclosed by the surface $S$, yields the *conservation law for angular momentum*

$$\frac{\mathrm{d}\mathbf{J}^{\,\mathrm{mech}}(\mathbf{x}_0)}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{J}^{\,\mathrm{field}}(\mathbf{x}_0)}{\mathrm{d}t} + \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathsf{K}(\mathbf{x}_0) = \mathbf{0} \tag{4.81}$$

where the mechanical and electromagnetic angular momentum pseudovectors are

$$\mathbf{J}^{\,\mathrm{mech}}(\mathbf{x}_0) = \int_V \mathrm{d}^3x\, \mathbf{h}^{\mathrm{mech}}(\mathbf{x}_0) \tag{4.82a}$$

and

$$\begin{aligned}\mathbf{J}^{\,\mathrm{field}}(\mathbf{x}_0) &= \int_V \mathrm{d}^3x\, \mathbf{h}^{\mathrm{field}}(\mathbf{x}_0) = \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\mathrm{field}} \\ &= \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{E} \times \mathbf{B})\end{aligned} \tag{4.82b}$$

respectively. We can formulate — and interpret — this conservation law in the following way:

$$\begin{aligned}&\underbrace{\int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}}_{\text{Torque on the matter}} + \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{E} \times \mathbf{B})}_{\text{Field angular momentum}} \\ &\qquad + \underbrace{\oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathsf{K}(\mathbf{x}_0)}_{\text{Angular momentum flow}} = \mathbf{0}\end{aligned} \tag{4.83}$$

This *angular momentum theorem* is the angular analogue of the linear momentum theorem, equation (4.52) on page 63. It shows that the electromagnetic field, like any physical field, can carry *angular momentum*, also known as *rotational momentum*.

For a single localised charge $q$, *i.e.* for a charge density given by equation (1.8) on page 5 with summation over one particle only, the evaluation of the first integral in equation (4.83) above shows that the mechanical torque on this charge is

$$\begin{aligned}\mathbf{N}(\mathbf{x}_0) &= \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f} = \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}) \\ &= (\mathbf{x} - \mathbf{x}_0) \times q(\mathbf{E} + \mathbf{v}^{\mathrm{mech}} \times \mathbf{B}) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{F}\end{aligned} \tag{4.84}$$

where $\mathbf{F}$ is the *Lorentz force* given by expression (4.53) on page 63. The physical observable $\mathbf{N}(\mathbf{x}_0)$ is known as the *Lorentz torque*.

#### 4.2.5.1 Gauge-invariant operator formalism

Using the fact that we can always express the magnetic field as $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}$ if $\mathbf{A}$ is regular enough and falls off sufficiently fast at large distances [*cf.* example 3.3 on page 49], we can write

$$\begin{aligned}\mathbf{J}^{\text{field}}(t, \mathbf{x}_0) &= \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times [\mathbf{E}(t, \mathbf{x}) \times \mathbf{B}(t, \mathbf{x}))] \\ &= \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \big(\mathbf{E}(t, \mathbf{x}) \times [(\boldsymbol{\nabla} \times \mathbf{A}^{\text{rotat}}(t, \mathbf{x})]\big)\end{aligned} \tag{4.85}$$

Employing the vector identity (F.92) on page 219, we can rewrite this as

$$\begin{aligned}\mathbf{J}^{\text{field}}(t, \mathbf{x}_0) = \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times [(\boldsymbol{\nabla} \otimes \mathbf{A}^{\text{rotat}}) \cdot \mathbf{E}] \\ - \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times [\mathbf{E} \cdot (\boldsymbol{\nabla} \otimes \mathbf{A}^{\text{rotat}})]\end{aligned} \tag{4.86}$$

Partial integration of this yields the manifestly gauge invariant expression

$$\begin{aligned}\mathbf{J}^{\text{field}}(t, \mathbf{x}_0) = \varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E} \times \mathbf{A}^{\text{rotat}} \\ + \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times [(\boldsymbol{\nabla} \otimes \mathbf{A}^{\text{rotat}}) \cdot \mathbf{E}] \\ - \varepsilon_0 \int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \cdot (\mathbf{E} \otimes (\mathbf{x} - \mathbf{x}_0) \times \mathbf{A}^{\text{rotat}}) \\ + \varepsilon_0 \int_V \mathrm{d}^3x\, [(\mathbf{x} - \mathbf{x}_0) \times \mathbf{A}^{\text{rotat}}](\boldsymbol{\nabla} \cdot \mathbf{E})\end{aligned} \tag{4.87}$$

which consists of one *intrinsic* term (a term that is not dependent on the choice of moment point $\mathbf{x}_0$):

$$\boldsymbol{\Sigma}^{\text{field}}(t) = \varepsilon_0 \int_V \mathrm{d}^3x\, \mathbf{E}(t, \mathbf{x}) \times \mathbf{A}^{\text{rotat}}(t, \mathbf{x}) \tag{4.88a}$$

and three *extrinsic* terms that do depend on the choice of $\mathbf{x}_0$:

$$\begin{aligned}\mathbf{L}^{\text{field}}(t, \mathbf{x}_0) = \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \rho \mathbf{A}^{\text{rotat}}(t, \mathbf{x}) \\ + \varepsilon_0 \int_V \mathrm{d}^3x\, (\mathbf{x} - \mathbf{x}_0) \times \big([\boldsymbol{\nabla} \otimes \mathbf{A}^{\text{rotat}}(t, \mathbf{x})] \cdot \mathbf{E}(t, \mathbf{x})\big) \\ - \varepsilon_0 \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot [\mathbf{E}(t, \mathbf{x}) \otimes (\mathbf{x} - \mathbf{x}_0) \times \mathbf{A}^{\text{rotat}}(t, \mathbf{x})]\end{aligned} \tag{4.88b}$$

where use was made of the Maxwell-Lorentz equation (2.1a) on page 19. Hence, the total field angular momentum $\mathbf{J}^{\text{field}}$ is the sum of two manifestly gauge invariant components, one intrinsic ($\boldsymbol{\Sigma}^{\text{field}}$) and one extrinsic ($\mathbf{L}^{\text{field}}$):

$$\mathbf{J}^{\text{field}}(t, \mathbf{x}_0) = \boldsymbol{\Sigma}^{\text{field}}(t) + \mathbf{L}^{\text{field}}(t, \mathbf{x}_0) \tag{4.89}$$

in perfect analogy with the total mechanical angular momentum; *cf.* equation (4.82a) on page 68.

If there is no net electric charge density $\rho$ in the integration volume, the first integral in the RHS of equation (4.88b) on the previous page vanishes, and if $\mathbf{E} \otimes (\mathbf{x} - \mathbf{x}_0) \times \mathbf{A}^{\text{rotat}}$ is regular and falls off sufficiently rapidly with $|\mathbf{x}|$, the contribution from the surface integral in equation (4.88b) on the preceding page can be neglected if we make the enclosing volume $V$ large enough. Furthermore, for a single temporal Fourier component and in complex notation, we obtain the expressions

$$\left\langle \mathbf{\Sigma}^{\text{field}} \right\rangle_t = -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x \, (\mathbf{E}^* \times \mathbf{E}) \tag{4.90a}$$

$$\begin{aligned} \left\langle \mathbf{L}^{\text{field}}(\mathbf{x}_0) \right\rangle_t &= -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x \, E_i^* [(\mathbf{x} - \mathbf{x}_0) \times \boldsymbol{\nabla}] E_i \\ &= -\mathrm{i}\frac{\varepsilon_0}{2\omega} \int_V \mathrm{d}^3x \, E_i^* (\mathbf{x} \times \boldsymbol{\nabla}) E_i \\ &\quad + \mathrm{i}\frac{\varepsilon_0}{2\omega} \mathbf{x}_0 \times \int_V \mathrm{d}^3x \, E_i^* \boldsymbol{\nabla} E_i \\ &= \left\langle \mathbf{L}^{\text{field}}(0) \right\rangle_t - \mathbf{x}_0 \times \left\langle \mathbf{p}^{\text{field}} \right\rangle_t \end{aligned} \tag{4.90b}$$

where $\left\langle \mathbf{p}^{\text{field}} \right\rangle_t$ is the cycle averaged *EM field linear momentum* given by expression (4.66) on page 65.

Recalling that in quantum mechanics the *spin angular momentum operator* is

$$\widehat{\mathbf{\Sigma}}_{jk} = -\mathrm{i}\hbar\epsilon_{ijk}\hat{\mathbf{x}}_i \tag{4.91}$$

which, with the help of the matrix vector expression (M.26) on page 238 can be written

$$\widehat{\mathbf{\Sigma}} = -\hbar\mathsf{S} \tag{4.92}$$

and the quantal *orbital angular momentum operator* is

$$\hat{\mathsf{L}} = -\mathrm{i}\hbar\mathbf{x} \times \boldsymbol{\nabla} = -\mathrm{i}\hbar\epsilon_{ijk}x_j\partial_k\hat{\mathbf{x}}_i \tag{4.93}$$

we can write (Einstein summation convention assumed)

$$\left\langle \mathbf{\Sigma}^{\text{field}} \right\rangle_t = \frac{\varepsilon_0}{2\hbar\omega} \int_V \mathrm{d}^3x \, E_j^* \widehat{\mathbf{\Sigma}}_{jk} E_k \tag{4.94a}$$

$$\left\langle \mathbf{L}^{\text{field}}(\mathbf{0}) \right\rangle_t = \frac{\varepsilon_0}{2\hbar\omega} \int_V \mathrm{d}^3x \, E_i^* \, \hat{\mathsf{L}} \, E_i \tag{4.94b}$$

or, with the use of the vector $\mathbf{\Psi}$ introduced in equation (4.69) on page 66,

$$\left\langle \mathbf{\Sigma}^{\text{field}} \right\rangle_t = \int_V \mathrm{d}^3x \, \Psi_i^* \, \widehat{\mathbf{\Sigma}} \, \Psi_i = \langle \Psi_i | \, \widehat{\mathbf{\Sigma}} \, | \Psi_i \rangle \tag{4.95a}$$

$$\left\langle \mathbf{L}^{\text{field}}(\mathbf{0}) \right\rangle_t = \int_V \mathrm{d}^3x \, \Psi_i^* \, \hat{\mathsf{L}} \, \Psi_i = \langle \Psi_i | \, \hat{\mathsf{L}} \, | \Psi_i \rangle \tag{4.95b}$$

Hence, under the assumptions made above, we can interpret $\boldsymbol{\Sigma}^{\text{field}}$ as the *electromagnetic spin angular momentum* and $\mathbf{L}^{\text{field}}$ as the *electromagnetic orbital angular momentum*.

The conservation of angular (rotational) momentum is a manifestation of the *rotational invariance* of the Maxwell-Lorentz equations.

### 4.2.6 Electromagnetic duality

We notice that Dirac's symmetrised Maxwell-Lorentz equations (2.2) on page 20 have the following transformation property (recall that $\varepsilon_0\mu_0 = 1/c^2$):

$$\mathbf{E} \mapsto c\mathbf{B} \tag{4.96a}$$

$$c\mathbf{B} \mapsto -\mathbf{E} \tag{4.96b}$$

$$c\rho^{\text{e}} \mapsto \rho^{\text{m}} \tag{4.96c}$$

$$\rho^{\text{m}} \mapsto -c\rho^{\text{e}} \tag{4.96d}$$

$$c\,\mathbf{j}^{\text{e}} \mapsto \mathbf{j}^{\text{m}} \tag{4.96e}$$

$$\mathbf{j}^{\text{m}} \mapsto -c\,\mathbf{j}^{\text{e}} \tag{4.96f}$$

which is a particular case ($\theta = \pi/2$) of the general *duality transformation*, also known as the *Heaviside-Larmor-Rainich transformation* (indicated by the *Hodge star operator* $\star$ in the upper left-hand corner of the symbol in question)

$$^{\star}\mathbf{E} = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta \tag{4.97a}$$

$$c\,^{\star}\mathbf{B} = -\mathbf{E}\sin\theta + c\mathbf{B}\cos\theta \tag{4.97b}$$

$$c\,^{\star}\rho^{\text{e}} = c\rho^{\text{e}}\cos\theta + \rho^{\text{m}}\sin\theta \tag{4.97c}$$

$$^{\star}\rho^{\text{m}} = -c\rho^{\text{e}}\sin\theta + \rho^{\text{m}}\cos\theta \tag{4.97d}$$

$$c\,^{\star}\mathbf{j}^{\text{e}} = c\,\mathbf{j}^{\text{e}}\cos\theta + \mathbf{j}^{\text{m}}\sin\theta \tag{4.97e}$$

$$^{\star}\mathbf{j}^{\text{m}} = -c\,\mathbf{j}^{\text{e}}\sin\theta + \mathbf{j}^{\text{m}}\cos\theta \tag{4.97f}$$

This transformation leaves the symmetrised Maxwell equations, and hence the physics they describe (often referred to as *electromagnetodynamics*), invariant. Since $\mathbf{E}$ and $\mathbf{j}^{\text{e}}$ are true (polar) vectors, $\mathbf{B}$ a pseudovector (axial vector), and $\rho^{\text{e}}$ a (true) scalar, we conclude that the *magnetic charge density* $\rho^{\text{m}}$ as well as the angle $\theta$, which behaves as a *mixing angle* in a two-dimensional *charge space*, must be pseudoscalars[13] and the *magnetic current density* $\mathbf{j}^{\text{m}}$ a pseudovector.

The invariance of Dirac's symmetrised Maxwell equations under the duality transformation (4.97) means that the amount of magnetic monopole density $\rho^{\text{m}}$ is irrelevant for the physics as long as the ratio $\rho^{\text{m}}/\rho^{\text{e}} = \tan\theta$ is kept constant. So whether we assume that charged particles are only electrically charged or also have an amount of magnetic charge with a given, fixed ratio between the

[13] Recall that the Taylor expansion of $\cos\theta$ contains only even powers of $\theta$ and therefor is an ordinary (true) scalar, whereas the expansion of $\sin\theta$ contains only odd powers and therefore is a pseudoscalar.

two types of charges, is a matter of convention, as long as we assume that this fraction is *the same for all charged particles*. Such particles are called *dyons*, a concept introduced by Schwinger. By varying the mixing angle $\theta$ we can change the fraction of magnetic monopoles at will without changing the laws of electrodynamics. For $\theta = 0$ we recover the usual Maxwell electrodynamics.

### 4.2.7 Electromagnetic virial theorem

If instead of vector multiplying the linear momentum densities $\mathbf{g}^{\text{mech}}$ and $\mathbf{g}^{\text{field}}$ by the position vector relative to a fix point $\mathbf{x}_0$ as we did in subsection 4.2.5 on page 66, we scalar multiply, the following differential balance equation is obtained:

$$\frac{\partial\big((\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{g}^{\text{mech}}\big)}{\partial t} + \frac{\partial\big((\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{g}^{\text{field}}\big)}{\partial t} + \boldsymbol{\nabla}\cdot\big((\mathbf{x}-\mathbf{x}_0)\cdot\mathsf{T}\big) = u^{\text{field}} \tag{4.98}$$

This is the *electromagnetic virial theorem*, analogous to the virial theorem of Clausius in mechanics. The quantity $(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{g}^{\text{field}}$ is the *electromagnetic virial density*. When integrated over space and time averaged, this theorem is a statement of the partitioning of energy in electrodynamics and finds use in, *e.g.* *plasma physics*.

## 4.3 Examples

EXAMPLE 4.1 ▷C, P, and T symmetries for the electromagnetic potentials

The CPT symmetries of the electromagnetic potentials $\Phi(t,\mathbf{x})$ and $\mathbf{A}(t,\mathbf{x})$ can be found trivially by using either expressions (3.35) or expressions (3.44) for the potentials. They can also be found by combining the results in subsection 4.1.1 and formulæ (F.141) on page 224. The result is

CHARGE CONJUGATION

$$\mathsf{C}:\Phi\mapsto-\Phi \tag{4.99a}$$
$$\mathsf{C}:\mathbf{A}\mapsto-\mathbf{A} \tag{4.99b}$$

SPACE INVERSION (PARITY)

$$\mathsf{P}:\Phi\mapsto\Phi \tag{4.100a}$$
$$\mathsf{P}:\mathbf{A}\mapsto-\mathbf{A} \tag{4.100b}$$

TIME REVERSAL

$$\mathsf{T}: \Phi \mapsto \Phi \tag{4.101a}$$
$$\mathsf{T}: \mathbf{A} \mapsto -\mathbf{A} \tag{4.101b}$$

End of example 4.1 ◁

▷Conservation of the total energy in a closed system — EXAMPLE 4.2

Show, by explicit calculation, that the total energy $U = U^{\text{mech}} + U^{\text{field}}$ of a closed system comprising $N$ non-relativistic particles of mass $m_i$, speed $v_i$ and charge $q_i$, and pertinent electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$,

$$U = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 + \frac{\varepsilon_0}{2}\int_V \mathrm{d}^3x\,(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}) \tag{4.102}$$

is conserved.

To show that the total energy $U$ of the closed system is conserved, *i.e.* is constant in time, is to show that the derivative of $U$ with respect to time $t$ vanishes. Direct differentiation yields

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} + \varepsilon_0 \int_V \mathrm{d}^3x \left(\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + c^2\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\right) \tag{4.103}$$

From the Lorentz force equation (4.53) on page 63 and Newton's second law we find that

$$m_i\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = q_i\big(\mathbf{E}(t,\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t,\mathbf{x}_i)\big)$$

which means that

$$\sum_{i=1}^{N} m_i\mathbf{v}_i\cdot\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \sum_{i=1}^{N} q_i\mathbf{v}_i\cdot\mathbf{E}(t,\mathbf{x}_i) + \sum_{i=1}^{N} q_i \underbrace{\mathbf{v}_i\cdot\big(\mathbf{v}_i\times\mathbf{B}(t,\mathbf{x}_i)\big)}_{\equiv 0} = \sum_{i=1}^{N} q_i\mathbf{v}_i\cdot\mathbf{E}(t,\mathbf{x}_i)$$

Furthermore, from the Maxwell-Lorentz equations (2.1) on page 19 we find that

$$\frac{\partial\mathbf{E}}{\partial t} = c^2\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{\varepsilon_0}\mathbf{j} \tag{4.104a}$$
$$\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} \tag{4.104b}$$

Substitution into equation (4.103) above yields, after some rearrangement of terms,

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}t} &= \sum_{i=1}^{N} q_i\mathbf{v}_i\cdot\mathbf{E}(t,\mathbf{x}_i) - \int_V \mathrm{d}^3x\,\mathbf{j}\cdot\mathbf{E}(t,\mathbf{x}) \\ &+ \frac{1}{\mu_0}\int_V \mathrm{d}^3x\,\big(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E})\big)\end{aligned} \tag{4.105}$$

Since the current is carried by discrete charged particles $i = 1, 2, \ldots, N$, the current density can be represented as

$$\mathbf{j} = \sum_{i=1}^{N} q_i\mathbf{v}_i\,\delta\big(\mathbf{x} - \mathbf{x}_i(t)\big) \tag{4.106}$$

and therefore the two first terms in equation (4.105) on the preceding page cancel. If we use also formula (F.83) on page 219 we see that equation (4.105) can be written

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{1}{\mu_0}\int_V \mathrm{d}^3x\, \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = -\int_V \mathrm{d}^3x\, \boldsymbol{\nabla}\cdot\mathbf{S} = -\oint_S \mathrm{d}^2x\, \hat{\mathbf{n}}\cdot\mathbf{S} \tag{4.107}$$

where in the last step the divergence theorem, formula (F.121b) on page 221, was used.

This result shows that the rate at which the total energy is lost in a volume equals the amount of energy flux that flows outward across a closed surface enclosing this volume. Of course, this is nothing but what is stated in the energy theorem (Poynting's theorem), formula (4.33) on page 60. Now, if the surface lies entirely outside the boundaries of the system under study, and this system is closed, no energy flux passes through the surface and hence

$$\frac{\mathrm{d}U}{\mathrm{d}t} = 0$$

showing that the total energy $U = U^{\text{mech}} + U^{\text{field}}$ of a closed system is indeed a conserved quantity. QED ■

Put in another way: If we observe a change in the radiated energy from a given system, we can deduce that there has been a similar change, but with opposite sign, of the mechanical energy of the system. This can be useful to know if we want to study the dynamics of a remote system, *e.g.* in the Universe.

End of example 4.2◁

EXAMPLE 4.3 ▷Conservation of the total linear momentum in a closed system

Show, by explicit calculation, that the total linear momentum $\mathbf{p} = \mathbf{p}^{\text{mech}} + \mathbf{p}^{\text{field}}$ of a closed electromechanical system comprising $N$ non-relativistic particles of mass $m_i$, speed $v_i$ and charge $q_i$, and pertinent electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$,

$$\mathbf{p} = \sum_{i=1}^{N} m_i\mathbf{v}_i + \varepsilon_0\int_V \mathrm{d}^3x\,(\mathbf{E}\times\mathbf{B}) \tag{4.108}$$

is conserved.

To show that the total linear momentum $\mathbf{p}$ of the closed system is conserved, *i.e.* is constant in time, is to show that the derivative of $\mathbf{p}$ with respect to time $t$ vanishes. Direct differentiation yields

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \sum_{i=1}^{N} m_i\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} + \varepsilon_0\int_V \mathrm{d}^3x\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\right) \tag{4.109}$$

From the Lorentz force equation (4.53) on page 63 and Newton's second law we find that

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \frac{q_i}{m_i}\big(\mathbf{E}(t,\mathbf{x}_i) + \mathbf{v}_i\times\mathbf{B}(t,\mathbf{x}_i)\big)$$

Substitution of this expression and equations (4.104) on the previous page into equation (4.109) yields, after some rearrangement of terms,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \sum_{i=1}^{N}\left[q_i\mathbf{E}(t,\mathbf{x}_i) + \big(\mathbf{v}_i\times\mathbf{B}(t,\mathbf{x}_i)\big)\right] - \int_V \mathrm{d}^3x\,\mathbf{j}\times\mathbf{B}(t,\mathbf{x})\\ &- \varepsilon_0\int_V \mathrm{d}^3x\,\big(\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) + c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\big)\end{aligned} \tag{4.110}$$

By first applying formula (F.93) on page 220 and then formula (F.86) on page 219 on the last term, we find that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = &\sum_{i=1}^{N}\left(q_i\mathbf{E}(t,\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t,\mathbf{x}_i)\right) - \int_V \mathrm{d}^3x\ \mathbf{j}\times\mathbf{B}(t,\mathbf{x}) \\ &-\varepsilon_0\int_V \mathrm{d}^3x\ (\mathbf{E}\otimes\boldsymbol{\nabla}\cdot\mathbf{E} + c^2\mathbf{B}\otimes\boldsymbol{\nabla}\cdot\mathbf{B}) \\ &-\varepsilon_0\int_V \mathrm{d}^3x\ \left(\tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}) - \boldsymbol{\nabla}\cdot(\mathbf{E}\otimes\mathbf{E}+c^2\mathbf{B}\otimes\mathbf{B})\right)\end{aligned} \tag{4.111}$$

We can now use the Maxwell-Lorentz equations to make the substitutions $\varepsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \rho$ and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ to obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = &\sum_{i=1}^{N}\left(q_i\mathbf{E}(t,\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t,\mathbf{x}_i)\right) - \int_V \mathrm{d}^3x\ \left(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B}(t,\mathbf{x})\right) \\ &-\varepsilon_0\int_V \mathrm{d}^3x\ \boldsymbol{\nabla}\cdot\left(\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B})\mathbf{1}_3 - (\mathbf{E}\otimes\mathbf{E}+c^2\mathbf{B}\otimes\mathbf{B})\right)\end{aligned} \tag{4.112}$$

Since the current is carried by discrete charged particles the current density $\mathbf{j}$ can be represented as formula (4.106) and the charge density as

$$\rho = \sum_{i=1}^{N} q_i\delta(\mathbf{x}-\mathbf{x}_i) \tag{4.113}$$

the two first terms in equation (4.112) above cancel and, after applying the divergence formula (F.121b) on page 221, we are left with

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\varepsilon_0\oint_S \mathrm{d}^2x\ \hat{\mathbf{n}}\cdot\left(\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B})\mathbf{1}_3 - (\mathbf{E}\otimes\mathbf{E}+c^2\mathbf{B}\otimes\mathbf{B})\right) = -\varepsilon_0\oint_S \mathrm{d}^2x\ \hat{\mathbf{n}}\cdot\mathsf{T} \tag{4.114}$$

where the tensor $\mathsf{T}$ is the *electromagnetic linear momentum current density* (the negative of the *Maxwell stress tensor*); *cf.* equation (4.41) on page 61.

This result shows that the rate at which the total linear momentum is lost in a volume equals the amount of linear momentum flux that flows outward across a closed surface enclosing this volume. If the surface lies entirely outside the boundaries of the system under study, and this system is closed, no linear momentum flux passes through the surface and hence

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{0}$$

showing that indeed the total linear momentum $\mathbf{p} = \mathbf{p}^{\text{mech}} + \mathbf{p}^{\text{field}}$ of a closed system is a conserved quantity. QED ■

If our system under study can be considered to be a closed electromechanical system and we observe a change in the electromagnetic linear momentum in this system, we can deduce that there has been a similar change, but with opposite sign, of the mechanical linear moment of the system. This allows us to determine mechanical properties of a system by analysing the radiation (radio, light, ...) from it.

In the quantum picture the linear momentum of a photon is $\hbar\mathbf{k}$ where $k = |\mathbf{k}| = 2\pi/\lambda$ is the wavenumber and $\lambda$ the radiated wavelength. The shift in $\lambda$ is the *translational Doppler shift*. Since $\omega = ck$ for a photon in free space, we experience the change of electromagnetic linear momentum as an associated frequency shift, a *redshift* if it is to the long wavelength (low frequency) side and a *blueshift* if it is to the short wavelength (high frequency) side.

This vocabulary is used regardless of whether the radiation falls in the optical range or not.

End of example 4.3◁

EXAMPLE 4.4 ▷Properties of the angular momentum flux tensor $\mathbf{K}$

In the derivation of the conservation law for the angular momentum, we used the fact that $(\mathbf{x}-\mathbf{x}_0)\times[\nabla\cdot\mathbf{T}(t,\mathbf{x})] = \nabla\cdot[(\mathbf{x}-\mathbf{x}_0)\times\mathbf{T}(t,\mathbf{x})] = \nabla\cdot\mathbf{K}(t,\mathbf{x},\mathbf{x}_0)$. Show this by explicit calculation.

According to identity (F.36) on page 218

$$(\mathbf{x}-\mathbf{x}_0)\times(\nabla\cdot\mathbf{T}) = \epsilon_{klm}\hat{\mathbf{x}}_k(x_l - x_{0_l})(\nabla\cdot\mathbf{T})_m \tag{4.115}$$

In tensor notation

$$\nabla\cdot\mathbf{T} = \hat{\mathbf{x}}_i\partial_i\cdot\hat{\mathbf{x}}_j\hat{\mathbf{x}}_k T_{jk} = \delta_{ij}\hat{\mathbf{x}}_k\partial_i T_{jk} = \hat{\mathbf{x}}_k\partial_i T_{ik} \equiv \hat{\mathbf{x}}_j\partial_i T_{ij} \tag{4.116}$$

where, in the last step, we made a summation (dummy) index replacement $k \mapsto j$ so that we recognise this as identity (F.72) on page 219 for $\mathbf{A} = \mathbf{T}$.

By definition

$$(\nabla\cdot\mathbf{T})_m \equiv (\hat{\mathbf{x}}_j\partial_i T_{ij})_m = (\hat{\mathbf{x}}_j\partial_i T_{ij})\cdot\hat{\mathbf{x}}_m = \delta_{jm}\partial_i T_{ij} = \partial_i T_{im} \tag{4.117}$$

which means that

$$\begin{aligned}(\mathbf{x}-\mathbf{x}_0)\times(\nabla\cdot\mathbf{T}) &= \epsilon_{klm}\hat{\mathbf{x}}_k(x_l - x_{0_l})\partial_i T_{im}\\ &= \epsilon_{klm}\hat{\mathbf{x}}_k\big([\partial_i(x_l - x_{0_l})T_{im}] - \underbrace{\delta_{il}T_{im}}_{T_{lm}}\big)\\ &= \partial_i[\epsilon_{klm}\hat{\mathbf{x}}_k(x_l - x_{0_l})T_{im}] - \hat{\mathbf{x}}_k\underbrace{\epsilon_{klm}T_{lm}}_{=0}\\ &= \nabla\cdot[(\mathbf{x}-\mathbf{x}_0)\times\mathbf{T}] = \nabla\cdot\mathbf{K}\end{aligned} \tag{4.118}$$

where we used the fact that $\mathbf{T}$ is symmetric, *i.e.* that $T_{lm} = T_{ml}$ [see formula (4.43a) on page 62] whereas $\epsilon_{klm} = -\epsilon_{mlk}$ [see formula (M.24) on page 237]. QED ■

End of example 4.4◁

EXAMPLE 4.5 ▷Conservation of the total angular momentum of a closed system

Show, by explicit calculation, that the total angular momentum around a momentum point $\mathbf{x}_0$, $\mathbf{J}(\mathbf{x}_0) = \mathbf{J}^{\text{mech}}(\mathbf{x}_0) + \mathbf{J}^{\text{field}}(\mathbf{x}_0)$, of a closed electromechanical system comprising $N$ non-relativistic particles of mass $m_i$, speed $v_i$ and charge $q_i$, and pertinent electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$,

$$\mathbf{J}(t,\mathbf{x}_0) = \sum_{i=1}^{N}(\mathbf{x}_i - \mathbf{x}_0)\times m_i\mathbf{v}_i + \varepsilon_0\int_V \mathrm{d}^3x\,(\mathbf{x}-\mathbf{x}_0)\times(\mathbf{E}\times\mathbf{B}) \tag{4.119}$$

is conserved.

To show that the total angular momentum $\mathbf{J}$ of the closed system is conserved, *i.e.* is constant in time, is to show that the derivative of $\mathbf{J}$ with respect to time $t$ vanishes.

For simplicity we put the origin of our coordinate system at the moment point $\mathbf{x}_0$ and then follow the same procedure as for the proof of the constancy of linear momentum in example

4.5 on the preceding page. Direct differentiation then yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} &= \sum_{i=1}^{N} m_i \Big( \underbrace{\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} \times \mathbf{v}_i}_{\equiv 0} + \mathbf{x}_i \times \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} \Big) \\ &\quad + \varepsilon_0 \int_V \mathrm{d}^3x \Big[ \underbrace{\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \times (\mathbf{E} \times \mathbf{B})}_{\equiv 0} + \mathbf{x} \times \Big( \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \Big) + \mathbf{x} \times \Big( \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \Big) \Big] \\ &= \sum_{i=1}^{N} m_i \mathbf{x}_i \times \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} + \varepsilon_0 \int_V \mathrm{d}^3x \, \mathbf{x} \times \Big[ \Big( \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \Big) + \Big( \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \Big) \Big]\end{aligned} \tag{4.120}$$

From the Lorentz force equation (4.53) on page 63 and Newton's second law we find that

$$\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \frac{q_i}{m_i} \big( \mathbf{E}(t, \mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t, \mathbf{x}_i) \big)$$

Substitution of this expression and equations (4.104) on page 73 into equation (4.120) yields, after some rearrangement of terms,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} &= \sum_{i=1}^{N} \big[ q_i \mathbf{x}_i \times \mathbf{E}(t, \mathbf{x}_i) + \mathbf{x}_i \times \big( \mathbf{v}_i \times \mathbf{B}(t, \mathbf{x}_i) \big) \big] \\ &\quad - \int_V \mathrm{d}^3x \, \mathbf{x} \times (\mathbf{j} \times \mathbf{B}) - \varepsilon_0 \int_V \mathrm{d}^3x \, \mathbf{x} \times \big( \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) + c^2 \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \big)\end{aligned} \tag{4.121}$$

By first applying formula (F.93) on page 220, and then formula (F.86) on page 219 on the last term, we find that

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} &= \sum_{i=1}^{N} \mathbf{x}_i \times \big( q_i \mathbf{E}(t, \mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t, \mathbf{x}_i) \big) \\ &\quad - \int_V \mathrm{d}^3x \, \mathbf{x} \times (\mathbf{j} \times \mathbf{B}) - \varepsilon_0 \int_V \mathrm{d}^3x \, \mathbf{x} \times (\mathbf{E} \otimes \boldsymbol{\nabla} \cdot \mathbf{E} + c^2 \mathbf{B} \otimes \boldsymbol{\nabla} \cdot \mathbf{B}) \\ &\quad - \varepsilon_0 \int_V \mathrm{d}^3x \, \mathbf{x} \times \big( \tfrac{1}{2} \boldsymbol{\nabla} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) - \boldsymbol{\nabla} \cdot (\mathbf{E} \otimes \mathbf{E} + c^2 \mathbf{B} \otimes \mathbf{B}) \big)\end{aligned} \tag{4.122}$$

We then use the Maxwell-Lorentz equations to make the substitutions $\varepsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and obtain

$$\begin{aligned}\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} &= \sum_{i=1}^{N} \mathbf{x}_i \times \big( q_i \mathbf{E}(t, \mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(t, \mathbf{x}_i) \big) - \int_V \mathrm{d}^3x \, \mathbf{x} \times \big( \rho \mathbf{E} + (\mathbf{j} \times \mathbf{B}) \big) \\ &\quad - \varepsilon_0 \int_V \mathrm{d}^3x \, \mathbf{x} \times \boldsymbol{\nabla} \cdot \big( \tfrac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \mathbf{1}_3 - (\mathbf{E} \otimes \mathbf{E} + c^2 \mathbf{B} \otimes \mathbf{B}) \big)\end{aligned} \tag{4.123}$$

The electric current is carried by discrete charged particles so that the current density $\mathbf{j}$ is given by formula (4.106) on page 73 and the charge density by formula (4.113) on page 75. Hence, the two first terms in equation (4.123) above cancel. After applying the divergence theorem, formula (F.121b) on page 221, on the remaining third term, we are left with

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = -\varepsilon_0 \oint_S \mathrm{d}^2x \, \hat{\mathbf{n}} \cdot \mathsf{K} \tag{4.124}$$

where the tensor $\mathsf{K}$ is the *electromagnetic angular momentum current density*; *cf.* equation (4.79) on page 67.

This result shows that the rate at which the total angular momentum is lost (gained) in a volume is equal to the amount of angular momentum flux that flows outward (inward) across a closed surface enclosing this volume. If the surface lies entirely outside the boundaries of the system under study, and this system is closed, no angular momentum flux passes through the surface and hence

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \mathbf{0}$$

Hence, we have shown that the total angular momentum $\mathbf{J} = \mathbf{J}^{\mathrm{mech}} + \mathbf{J}^{\mathrm{field}}$ of a closed system is conserved.
QED ■

If we observe a change in the electromagnetic angular momentum in a given system, we can deduce that there has been a similar change, but with opposite sign, of the mechanical angular moment of the system.

End of example 4.5◁

EXAMPLE 4.6 ▷Spin angular momentum and wave polarisation

Consider a generic temporal Fourier mode of the electric field vector $\mathbf{E}$ of a circularly polarised wave with (angular) frequency $\omega$. According to equation (2.57) on page 32 it can be written

$$\mathbf{E}(t, \mathbf{x}) = E(t, \mathbf{x})\, \hat{\mathbf{h}}_{\pm} \tag{4.125}$$

where

$$E(t, \mathbf{x}) = \sqrt{2}E_0\, \mathrm{e}^{\mathrm{i}(kx_3 - \omega t + \delta_1)} \tag{4.126}$$

and

$$\hat{\mathbf{h}}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}_1 \pm \mathrm{i}\hat{\mathbf{x}}_2) \tag{4.127}$$

As before, we use the convention that $\hat{\mathbf{h}}_{+}$ represents *left-hand circular polarisation* and $\hat{\mathbf{h}}_{-}$ *right-hand circular polarisation*. Noting that

$$(\hat{\mathbf{h}}_{\pm})^{*} \times \hat{\mathbf{h}}_{\pm} = \pm\mathrm{i}\hat{\mathbf{z}} \tag{4.128}$$

we see that

$$\mathbf{E}^{*} \times \mathbf{E} = \pm\mathrm{i}\,|E|^{2}\,\hat{\mathbf{z}} = \pm\mathrm{i}E^{2}\hat{\mathbf{z}} \tag{4.129}$$

When we insert this into equation (4.90a), we find that the cycle averaged spin angular momentum of a circularly polarised wave is

$$\left\langle \mathbf{\Sigma}^{\mathrm{field}} \right\rangle_t = \pm\frac{\varepsilon_0}{2\omega}\int_V \mathrm{d}^3x\, E^2\,\hat{\mathbf{z}} = \pm\frac{\left\langle U^{\mathrm{field}}\right\rangle_t}{\omega}\,\hat{\mathbf{z}} \tag{4.130}$$

where $U^{\mathrm{field}}$ is the field energy. Considering the fact that the wave-particle duality shows that the wave can be considered to be a gas with $N$ photons so that the kinetic energy of the field is $U^{\mathrm{field}} = N\hbar\omega$. This means that the spin of the wave is

$$\left\langle \mathbf{\Sigma}^{\text{field}} \right\rangle_t = \pm \frac{\langle U^{\text{field}} \rangle_t}{\omega} \hat{\mathbf{z}} = \pm \frac{N\hbar\omega}{\omega} \hat{\mathbf{z}} = \pm N\hbar\hat{\mathbf{z}} \tag{4.131}$$

Hence, each photon of a right-hand or a left-hand circular polarised wave carries a spin angular momentum of $\hbar$ or $-\hbar$, respectively.

End of example 4.6◁

▷Orbital angular momentum —— EXAMPLE 4.7

The Cartesian components of the quantal *orbital angular momentum operator* (OAM) $\hat{\mathbf{L}} = \mathrm{i}\hbar\mathbf{x} \times \boldsymbol{\nabla}$, as given by expression (4.93) on page 70, are

$$\hat{\mathsf{L}}_x = -\mathrm{i}\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{4.132a}$$

$$\hat{\mathsf{L}}_y = -\mathrm{i}\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{4.132b}$$

$$\hat{\mathsf{L}}_z = -\mathrm{i}\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{4.132c}$$

as is well known from Quantum Mechanics.

In cylindrical coordinates $(\rho, \varphi, z)$ the components are

$$\hat{\mathsf{L}}_x = -\mathrm{i}\hbar\left[\sin\varphi\left(z\frac{\partial}{\partial \rho} - \rho\frac{\partial}{\partial z}\right) + \frac{z}{\rho}\cos\varphi\frac{\partial}{\partial \varphi}\right] \tag{4.133a}$$

$$\hat{\mathsf{L}}_y = -\mathrm{i}\hbar\left[\cos\varphi\left(z\frac{\partial}{\partial \rho} - \rho\frac{\partial}{\partial z}\right) - \frac{z}{\rho}\sin\varphi\frac{\partial}{\partial \varphi}\right] \tag{4.133b}$$

$$\hat{\mathsf{L}}_z = -\mathrm{i}\hbar\frac{\partial}{\partial \varphi} \tag{4.133c}$$

and in spherical coordinates $(r, \varphi, \theta)$ they are

$$\hat{\mathsf{L}}_x = -\mathrm{i}\hbar\left(\sin\varphi\frac{\partial}{\partial \theta} + \cos\varphi\cot\theta\frac{\partial}{\partial \varphi}\right) \tag{4.134a}$$

$$\hat{\mathsf{L}}_y = -\mathrm{i}\hbar\left(\cos\varphi\frac{\partial}{\partial \theta} - \sin\varphi\cot\theta\frac{\partial}{\partial \varphi}\right) \tag{4.134b}$$

$$\hat{\mathsf{L}}_z = -\mathrm{i}\hbar\frac{\partial}{\partial \varphi} \tag{4.134c}$$

For an electric field $\mathbf{E}$ that depends on the azimuthal angle $\varphi$ in such a way that

$$\mathbf{E} = \mathbf{E}_0(t, \mathbf{x})\Phi(\varphi) \tag{4.135}$$

we find that

$$\hat{\mathsf{L}}_z\mathbf{E}(t, \mathbf{x}) = -\mathrm{i}\hbar\left(\frac{\partial \mathbf{E}_0(t, \mathbf{x})}{\partial \varphi}\Phi(\varphi) - \mathbf{E}_0(t, \mathbf{x})\frac{\partial \Phi(\varphi)}{\partial \varphi}\right) \tag{4.136}$$

If $\mathbf{E}_0(t, \mathbf{x})$ is rotationally symmetric around the the $z$ axis, so that $\mathbf{E}_0 = \mathbf{E}_0(t, \rho, z)$ in cylindrical coordinates, $\mathbf{E}_0 = \mathbf{E}_0(t, r, \theta)$ in spherical (polar) coordinates *etc.* the first term in the RHS vanishes.

If the azimuthal part is expressed in a Fourier series

$$\Phi(\varphi) = \sum_{m=-\infty}^{\infty} c_m\,\mathrm{e}^{\mathrm{i}m\varphi} \tag{4.137}$$

we see that

$$\hat{\mathsf{L}}_z \mathbf{E}(t, \mathbf{x}) = \sum_{m=-\infty}^{\infty} c_m m\hbar \mathbf{E}_0\, \mathrm{e}^{im\varphi} = \sum_{m=-\infty}^{\infty} c_m m\hbar \mathbf{E}_m \tag{4.138}$$

*i.e.* a weighted superposition of OAM $L_z$ eigenstates $\mathbf{E}_m = \mathbf{E}_0\, \mathrm{e}^{im\varphi}$. Furthermore,

$$\hat{\mathsf{L}}_z \mathbf{E}_m = m\hbar \mathbf{E}_m \tag{4.139}$$

which means that for a rotationally symmetric beam with an *azimuthal phase* dependence given by $\exp(im\varphi)$ one deduces, according to formula (4.90b) on page 70, that the $z$ component of the orbital angular momentum of each photon in this beam is $m\hbar$.

End of example 4.7◁

EXAMPLE 4.8 ▷Duality of the electromagnetodynamic equations

Show that the symmetric, electromagnetodynamic Maxwell-Lorentz equations (2.2) on page 20 (Dirac's symmetrised Maxwell equations) are invariant under the duality transformation (4.97).

Explicit application of the transformation yields

$$\begin{aligned}\boldsymbol{\nabla} \cdot {}^{\star}\mathbf{E} &= \boldsymbol{\nabla} \cdot (\mathbf{E} \cos\theta + c\mathbf{B} \sin\theta) = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \cos\theta + c\mu_0 \rho^{\mathrm{m}} \sin\theta \\ &= \frac{1}{\varepsilon_0}\left(\rho^{\mathrm{e}} \cos\theta + \frac{1}{c}\rho^{\mathrm{m}} \sin\theta\right) = \frac{{}^{\star}\rho^{\mathrm{e}}}{\varepsilon_0}\end{aligned} \tag{4.140a}$$

$$\begin{aligned}\boldsymbol{\nabla} \cdot {}^{\star}\mathbf{B} &= \boldsymbol{\nabla} \cdot (-\frac{1}{c}\mathbf{E} \sin\theta + \mathbf{B} \cos\theta) = -\frac{\rho^{\mathrm{e}}}{c\varepsilon_0} \sin\theta + \mu_0 \rho^{\mathrm{m}} \cos\theta \\ &= \mu_0 \left(-c\rho^{\mathrm{e}} \sin\theta + \rho^{\mathrm{m}} \cos\theta\right) = \mu_0 {}^{\star}\rho^{\mathrm{m}}\end{aligned} \tag{4.140b}$$

$$\begin{aligned}\boldsymbol{\nabla} \times {}^{\star}\mathbf{E} + \frac{\partial {}^{\star}\mathbf{B}}{\partial t} &= \boldsymbol{\nabla} \times (\mathbf{E} \cos\theta + c\mathbf{B} \sin\theta) + \frac{\partial}{\partial t}\left(-\frac{1}{c}\mathbf{E} \sin\theta + \mathbf{B} \cos\theta\right) \\ &= -\mu_0 \mathbf{j}^{\mathrm{m}} \cos\theta - \frac{\partial \mathbf{B}}{\partial t} \cos\theta + c\mu_0 \mathbf{j}^{\mathrm{e}} \sin\theta + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \sin\theta \\ &\quad - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \sin\theta + \frac{\partial \mathbf{B}}{\partial t} \cos\theta = -\mu_0 \mathbf{j}^{\mathrm{m}} \cos\theta + c\mu_0 \mathbf{j}^{\mathrm{e}} \sin\theta \\ &= -\mu_0(-c\,\mathbf{j}^{\mathrm{e}} \sin\theta + \mathbf{j}^{\mathrm{m}} \cos\theta) = -\mu_0 {}^{\star}\mathbf{j}^{\mathrm{m}}\end{aligned} \tag{4.140c}$$

$$\begin{aligned}\boldsymbol{\nabla} \times {}^{\star}\mathbf{B} - \frac{1}{c^2}\frac{\partial {}^{\star}\mathbf{E}}{\partial t} &= \boldsymbol{\nabla} \times (-\frac{1}{c}\mathbf{E} \sin\theta + \mathbf{B} \cos\theta) - \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E} \cos\theta + c\mathbf{B} \sin\theta) \\ &= \frac{1}{c}\mu_0 \mathbf{j}^{\mathrm{m}} \sin\theta + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \cos\theta + \mu_0 \mathbf{j}^{\mathrm{e}} \cos\theta + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \cos\theta \\ &\quad - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \cos\theta - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \sin\theta \\ &= \mu_0 \left(\frac{1}{c}\mathbf{j}^{\mathrm{m}} \sin\theta + \mathbf{j}^{\mathrm{e}} \cos\theta\right) = \mu_0 {}^{\star}\mathbf{j}^{\mathrm{e}}\end{aligned} \tag{4.140d}$$

QED ■

End of example 4.8◁

▷Duality expressed in Riemann-Silberstein formalism —— EXAMPLE 4.9

Expressed in the Riemann-Silberstein complex field vector, introduced in equation (2.7) on page 22, the duality transformation equations (4.97) on page 71 become

$$
\begin{aligned}
{}^{\star}\mathbf{G} &= {}^{\star}\mathbf{E} + \mathrm{i}c\,{}^{\star}\mathbf{B} = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta - \mathrm{i}\mathbf{E}\sin\theta + \mathrm{i}c\mathbf{B}\cos\theta \\
&= \mathbf{E}(\cos\theta - \mathrm{i}\sin\theta) + \mathrm{i}c\mathbf{B}(\cos\theta - \mathrm{i}\sin\theta) = (\mathbf{E} + \mathrm{i}c\mathbf{B})\mathrm{e}^{-\mathrm{i}\theta} = \mathbf{G}\mathrm{e}^{-\mathrm{i}\theta}
\end{aligned}
\tag{4.141}
$$

from which it is easy to see that for $\theta \in \mathbb{R}$

$$
{}^{\star}\mathbf{G} \cdot {}^{\star}\mathbf{G}^{*} = \left|{}^{\star}G\right|^2 = \mathbf{G}\mathrm{e}^{-\mathrm{i}\theta} \cdot \left(\mathbf{G}\mathrm{e}^{-\mathrm{i}\theta}\right)^{*} = \mathbf{G}\mathrm{e}^{-\mathrm{i}\theta} \cdot \mathbf{G}^{*}\mathrm{e}^{\mathrm{i}\theta} = |G|^2 \tag{4.142}
$$

whereas

$$
{}^{\star}\mathbf{G} \cdot {}^{\star}\mathbf{G} = \mathbf{G} \cdot \mathbf{G}\mathrm{e}^{-2\mathrm{i}\theta} \tag{4.143}
$$

If this transformation were local, *i.e.* if $\theta = \theta(t, \mathbf{x})$, we see that spatial and temporal differentiation of ${}^{\star}\mathbf{G}$ would lead to

$$
\frac{\partial\,{}^{\star}\mathbf{G}}{\partial t} = -\mathrm{i}\frac{\partial\theta}{\partial t}\mathrm{e}^{-\mathrm{i}\theta}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\frac{\partial\mathbf{G}}{\partial t} = -\mathrm{i}\frac{\partial\theta}{\partial t}{}^{\star}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\frac{\partial\mathbf{G}}{\partial t} \tag{4.144a}
$$

$$
\boldsymbol{\nabla} \cdot {}^{\star}\mathbf{G} = -\mathrm{i}(\boldsymbol{\nabla}\theta) \cdot \mathrm{e}^{-\mathrm{i}\theta}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\nabla} \cdot \mathbf{G} = -\mathrm{i}(\boldsymbol{\nabla}\theta) \cdot {}^{\star}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\nabla} \cdot \mathbf{G} \tag{4.144b}
$$

$$
\boldsymbol{\nabla} \times {}^{\star}\mathbf{G} = -\mathrm{i}(\boldsymbol{\nabla}\theta) \times \mathrm{e}^{-\mathrm{i}\theta}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\nabla} \times \mathbf{G} = -\mathrm{i}(\boldsymbol{\nabla}\theta) \times {}^{\star}\mathbf{G} + \mathrm{e}^{-\mathrm{i}\theta}\boldsymbol{\nabla} \times \mathbf{G} \tag{4.144c}
$$

However, if we require that the physics be unaffected by a duality transformation, the free-space Maxwell-Lorentz equations (2.9) on page 22 must hold also for ${}^{\star}\mathbf{G}$. *I.e.*

$$
\boldsymbol{\nabla} \cdot {}^{\star}\mathbf{G} = 0 \tag{4.145a}
$$

$$
\boldsymbol{\nabla} \times {}^{\star}\mathbf{G} = \frac{\mathrm{i}}{c}\frac{\partial\,{}^{\star}\mathbf{G}}{\partial t} \tag{4.145b}
$$

Then, using formulæ (4.144), as well as equations (2.9), we find that this would mean that

$$
(\boldsymbol{\nabla}\theta) \cdot {}^{\star}\mathbf{G} = 0 \tag{4.146a}
$$

$$
(\boldsymbol{\nabla}\theta) \times {}^{\star}\mathbf{G} = \frac{\mathrm{i}}{c}\frac{\partial\theta}{\partial t}{}^{\star}\mathbf{G} \tag{4.146b}
$$

which can be fulfilled only if

$$
\frac{\partial\theta}{\partial t} = 0 \tag{4.147a}
$$

$$
\boldsymbol{\nabla}\theta = \mathbf{0} \tag{4.147b}
$$

Hence, the mixing angle $\theta$ must be independent of time and space.

—— End of example 4.9◁

EXAMPLE 4.10 ▷Examples of other conservation laws

In addition to the conservation laws for energy, linear momentum and angular momentum, a large number of other electromagnetic conservation laws can be derived.

### CONSERVATION OF CHIRALITY DENSITY

For instance, as can be derived directly from the Maxwell-Lorentz equations, the following conservation law holds in free space (vacuum):

$$\frac{\partial \chi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{X} = 0 \tag{4.148}$$

where the pseudoscalar

$$\chi \stackrel{\text{def}}{\equiv} \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) + c^2 \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \tag{4.149}$$

is the *chirality density*, and the pseudovector

$$\mathbf{X} \stackrel{\text{def}}{\equiv} \mathbf{E} \times \frac{\partial \mathbf{E}}{\partial t} + c^2 \mathbf{B} \times \frac{\partial \mathbf{B}}{\partial t} \tag{4.150}$$

is the *chirality flow*.

### CONSERVATION OF TOTAL ELECTRIC CURRENT

If we use the notation $\mathbf{j}^{\text{cond}}$ for the conduction current associated with the actual motion of electric charges, both free and bound, *i.e.* $\mathbf{j}^{\text{cond}} = \mathbf{j}^{\text{free}} + \mathbf{j}^{\text{bound}}$, and $\mathbf{j}^{\text{disp}}$ for the displacement current $\varepsilon_0 \partial(\mathbf{E})/\partial t$, the Maxwell-Lorentz equation (2.1d) on page 19 can be written

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j}^{\text{cond}} + \mathbf{j}^{\text{disp}} = \mathbf{j}^{\text{tot}} \tag{4.151}$$

Differentiating this with respect to time $t$ and using the Maxwell equation (2.1c) on page 19 and formula (F.104) on page 220, we obtain the following local *conservation law for the total current*

$$\frac{\partial \mathbf{j}^{\text{tot}}}{\partial t} + \boldsymbol{\nabla} \cdot \left( \mathbf{1}_3 \times \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{E}) \right) = \mathbf{0} \tag{4.152}$$

which is a rather obfuscated way of writing the *wave equation* for the electric field vector $\mathbf{E}$. Normally it is written as in equation (2.19a) on page 24.

We note that the global (*i.e.* volume integrated) version of this wave-equation-turned-conservation-law formula can, with the help of the rank two tensor

$$\mathsf{W} = \mathbf{1}_3 \times (\boldsymbol{\nabla} \times \mathbf{E}) \tag{4.153}$$

be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathrm{d}^3x \, \mathbf{j}^{\text{tot}} + \frac{1}{\mu_0} \oint_S \mathrm{d}^2x \, \hat{\mathbf{n}} \cdot \mathsf{W} = \mathbf{0} \tag{4.154}$$

and hence as a conservation law for the integrated curl of the magnetic field:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathrm{d}^3x \, (\boldsymbol{\nabla} \times \mathbf{B}) + \oint_S \mathrm{d}^2x \, \hat{\mathbf{n}} \cdot \mathsf{W} = \mathbf{0} \tag{4.155}$$

or, using the identity (F.121c) on page 221,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \times \mathbf{B} + \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{W} = \mathbf{0} \tag{4.156}$$

End of example 4.10◁

## 4.4 Bibliography

[25] J. D. JACKSON, *Classical Electrodynamics*, third ed., John Wiley & Sons, Inc., New York, NY ..., 1999, ISBN 0-471-30932-X.

[26] F. MELIA, *Electrodynamics*, Chicago Lectures in Physics. Cambridge University Press, Cambridge and London, 1991, ISBN 0-226-51957-0.

# 5

# FIELDS FROM ARBITRARY CHARGE AND CURRENT DISTRIBUTIONS

The electric and magnetic fields **E** and **B** generated by prescribed charge and current sources $\rho$ and **j** can — at least in principle — be obtained by directly solving the Maxwell-Lorentz differential equations given at the beginning of chapter 2, or the wave equations given later in the same chapter. However, it is often technically easier and physically more lucid to calculate the fields from the electromagnetic potentials $\Phi$ and **A** that we introduced in chapter 3. We saw in that chapter that these potentials can, in a suitable gauge, be readily obtained in the form of a volume integral over the spatial distribution of the source elements, divided by the distance between the actual source element and the observer.

In this chapter we will use electromagnetic potentials to derive exact, closed-form, analytic expressions for the electric and magnetic fields generated by prescribed but completely arbitrary charge and current sources at rest, distributed arbitrarily within a volume of finite extent in otherwise free space. As we shall find, both the electric and magnetic field vectors are actually a sum of several vector composants, each characterized by its particular vectorial property and fall-off behaviour with respect to distance from the source elements. This means that different field vector composants have different magnitudes, directions and phases in different zones. These zones are customarily — and self-explanatorily — referred to as the *near zone*, the *intermediate zone*, and the *far zone*, respectively.

Those composants of the field vectors that have the slowest fall-off with distance from the source and therefore dominate in the far zone, will be referred to as the *far fields*. Because of their importance, a special analysis of these far fields is given at the end of the chapter. Surprising as it may seem, it will be shown in chapter 6 that certain physical observables receive their far-zone contributions *not* from the dominant far fields but from a combination of far fields and sub-dominant near-zone and intermediate-zone fields. For completeness, we therefore include a derivation of approximate expressions for *all* field composants, dominant as well as sub-dominant, valid at large distances from an arbitrary source.

## 5.1 Fourier component method

As discussed earlier, the linearity of the Maxwell-Lorentz equations means that the complete solution of them can be found in terms of a superposition of Fourier components, each of which individually solves these equations; the resulting superposition (the sum of several such Fourier components) is called a *wave-packet*.

We recall from the treatment in chapter 3 that in order to use the Fourier component method to find the solution (3.34) on page 40 for the generic inhomogeneous wave equation (3.18) on page 37, we presupposed the existence of the temporal Fourier transform pair (3.19) for the generic source term. That such transform pairs exist is not always the case, but it is true for reasonably well-behaved, non-erratic physical variables which are neither strictly monotonically increasing nor strictly monotonically decreasing with time. For charge and current densities that vary in time, we can therefore, without loss of generality, work with individual temporal Fourier components $\rho_\omega(\mathbf{x})$ and $\mathbf{j}_\omega(\mathbf{x})$, respectively. Strictly speaking, the existence of a signal represented by a single Fourier component assumes a *monochromatic* source (*i.e.* a source containing only one single frequency component), which requires that this source must have existed for an infinitely long time. However, by taking the proper limits, we may still use this approach even for sources and fields of finite temporal duration.

This is the method we shall utilise in this chapter in order to derive the electric and magnetic fields in vacuum from arbitrary given charge densities $\rho(t,\mathbf{x})$ and current densities $\mathbf{j}(t,\mathbf{x})$, defined by the temporal Fourier transform pairs

$$\rho(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.1a}$$

$$\rho_\omega(\mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \rho(t,\mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \tag{5.1b}$$

and

$$\mathbf{j}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{j}_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.2a}$$

$$\mathbf{j}_\omega(\mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{j}(t,\mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega t} \tag{5.2b}$$

respectively. The derivation will be completely general except we keep only *retarded* potentials and assume that the source region is at rest (no bulk motion) relative to the observer.

The temporal Fourier transform pair for the retarded scalar potential can then be written

$$\Phi(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \Phi_\omega(\mathbf{x}) \, \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.3a}$$

$$\Phi_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, \Phi(t, \mathbf{x}) \, \mathrm{e}^{\mathrm{i}\omega t} = \frac{1}{\varepsilon_0} \int_{V'} \mathrm{d}^3x' \, \rho_\omega(\mathbf{x}') G(|\mathbf{x} - \mathbf{x}'|) \tag{5.3b}$$

where, in the last step, we made use of the explicit expression for the temporal Fourier transform of the generic potential component $\Psi_\omega(\mathbf{x})$, equation (3.31) on page 39, and introduced the *Green function*

$$G(|\mathbf{x} - \mathbf{x}'|) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{5.4}$$

Similarly, we must require that the following Fourier transform pair for the vector potential exists:

$$\mathbf{A}(t, \mathbf{x}) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \mathbf{A}_\omega(\mathbf{x}) \, \mathrm{e}^{-\mathrm{i}\omega t} \tag{5.5a}$$

$$\mathbf{A}_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, \mathbf{A}(t, \mathbf{x}) \, \mathrm{e}^{\mathrm{i}\omega t} = \frac{1}{\varepsilon_0 c^2} \int_{V'} \mathrm{d}^3x' \, \mathbf{j}_\omega(\mathbf{x}') G(|\mathbf{x} - \mathbf{x}'|) \tag{5.5b}$$

Analogous transform pairs must exist for the fields themselves.

In the limit that the sources can be considered monochromatic, containing one single frequency $\omega_0$ only, we can safely assume that $\rho_\omega = \rho_0 \delta(\omega - \omega_0)$, $\mathbf{j}_\omega = \mathbf{j}_0 \delta(\omega - \omega_0)$ *etc.*. Our Fourier integrals then become trivial and we obtain the simpler expressions

$$\rho(t, \mathbf{x}) = \rho_0(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{5.6a}$$

$$\mathbf{j}(t, \mathbf{x}) = \mathbf{j}_0(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{5.6b}$$

$$\Phi(t, \mathbf{x}) = \Phi_0(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{5.6c}$$

$$\mathbf{A}(t, \mathbf{x}) = \mathbf{A}_0(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{5.6d}$$

where the real-valuedness of all these quantities is implied. As discussed above, all formulæ derived for a general temporal Fourier representation of the source (general distribution of frequencies in the source) are valid in these limiting cases. In this context, we can therefore, without any essential loss of stringency, formally identify $\rho_0$ with the Fourier amplitude $\rho_\omega$ and so on.

In order to simplify the computations, we will work in $\omega$ space and, at the final stage, inverse Fourier transform back to ordinary $t$ space. We shall be using the Lorenz-Lorentz gauge and note that, in $\omega$ space, the Lorenz-Lorentz condition, equation (3.16) on page 37, takes the form

$$\boldsymbol{\nabla} \cdot \mathbf{A}_\omega - \mathrm{i}\frac{k}{c} \, \Phi_\omega = 0 \tag{5.7}$$

This provides a relation between (the temporal Fourier transforms of) the vector and scalar potentials $\mathbf{A}_\omega$ and $\Phi_\omega$.

## 5.2 The retarded electric field

In order to calculate the retarded electric field, we use the temporally Fourier transformed version of formula (3.13) on page 35, with equations (5.3) on the previous page and equations (5.5) on the preceding page as the explicit expressions for the Fourier transforms of $\Phi$ and $\mathbf{A}$, respectively:

$$\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) &= -\boldsymbol{\nabla}\Phi_\omega(\mathbf{x}) + \mathrm{i}\omega\mathbf{A}_\omega(\mathbf{x}) \\
&= -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\rho_\omega(\mathbf{x}')G(|\mathbf{x}-\mathbf{x}'|) \\
&\quad + \frac{\mathrm{i}\omega}{\varepsilon_0 c^2}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')G(|\mathbf{x}-\mathbf{x}'|) \\
&= \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\
&\quad - \frac{\mathrm{i}k}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \\
&\quad + \frac{\mathrm{i}k}{4\pi\varepsilon_0 c}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned} \tag{5.8}$$

Taking the inverse Fourier transform of this expression, the following expression for the retarded electric field is obtained:

$$\begin{aligned}
\mathbf{E}(t,\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho(t'_{\text{ret}},\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \\
&\quad + \frac{1}{4\pi\varepsilon_0 c}\int_{V'}\mathrm{d}^3x'\,\frac{\dot\rho(t'_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \\
&\quad - \frac{1}{4\pi\varepsilon_0 c^2}\int_{V'}\mathrm{d}^3x'\,\frac{\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned} \tag{5.9}$$

Letting $\mathrm{d}q' = \mathrm{d}^3x'\rho(t'_{\text{ret}},\mathbf{x}')$ and $\mathrm{d}\mathbf{i}' = I'\mathrm{d}\mathbf{l}' = \mathrm{d}^3x'\,\mathbf{j}(t'_{\text{ret}},\mathbf{x}')$, the corresponding formula in infinitesimal differential form becomes

$$\mathrm{d}\mathbf{E}(t,x) = \frac{1}{4\pi\varepsilon_0}\left(\mathrm{d}q'\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} + \frac{1}{c}\mathrm{d}\dot q'\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} - \frac{1}{c^2}\mathrm{d}\dot{\mathbf{i}}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{5.10}$$

which in the static limit reduces to the infinitesimal Coulomb law, formula (1.5) on page 4.

We shall now further expand equation (5.9) above. To this end we first note that the Fourier transform of the continuity equation (4.20) on page 58

$$\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}') - \mathrm{i}\omega\rho_\omega(\mathbf{x}') = 0 \tag{5.11}$$

can be used to express $\rho_\omega$ in terms of $\mathbf{j}_\omega$ as follows

$$\rho_\omega(\mathbf{x}') = -\frac{\mathrm{i}}{\omega}\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}') \tag{5.12}$$

Doing so in the last term of equation (5.8) on the facing page, and also using the fact that $k = \omega/c$, we can rewrite this equation as

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\Bigg[\int_{V'}\mathrm{d}^3x'\,\frac{\rho_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\ -\frac{1}{c}\underbrace{\int_{V'}\mathrm{d}^3x'\left(\frac{[\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}k\,\mathbf{j}_\omega(\mathbf{x}')\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}}_{\mathbf{K}_\omega}\Bigg] \tag{5.13}$$

The last vector-valued integral can be further rewritten in the following way (where $l$ and $m$ are summation indices and Einstein's summation convention is assumed):

$$\begin{aligned}\mathbf{K}_\omega &= \int_{V'}\mathrm{d}^3x'\left(\frac{[\boldsymbol{\nabla}'\cdot\mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}k\,\mathbf{j}_\omega(\mathbf{x}')\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\ &= \int_{V'}\mathrm{d}^3x'\left(\frac{\partial j_{\omega,l}}{\partial x'_l}\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|}-\mathrm{i}kj_{\omega m}(\mathbf{x}')\right)\hat{\mathbf{x}}_m\,\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{5.14}$$

But, since

$$\begin{aligned}\frac{\partial}{\partial x'_l}\left(j_{\omega,l}\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|^2}\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right) &= \left(\frac{\partial j_{\omega,l}}{\partial x'_l}\right)\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|^2}\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \\ &\quad + j_{\omega,l}\frac{\partial}{\partial x'_l}\left(\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|^2}\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{5.15}$$

we can rewrite $\mathbf{K}_\omega$ as

$$\begin{aligned}\mathbf{K}_\omega &= -\int_{V'}\mathrm{d}^3x'\left[j_{\omega,l}\frac{\partial}{\partial x'_l}\left(\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|^2}\hat{\mathbf{x}}_m\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)+\mathrm{i}k\,\mathbf{j}_\omega\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right] \\ &\quad +\int_{V'}\mathrm{d}^3x'\,\frac{\partial}{\partial x'_l}\left(j_{\omega,l}\frac{x_m-x'_m}{|\mathbf{x}-\mathbf{x}'|^2}\hat{\mathbf{x}}_m\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right) \\ &= -\int_{V'}\mathrm{d}^3x'\left[\mathbf{j}_\omega(\mathbf{x}')\cdot\boldsymbol{\nabla}'\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)+\mathrm{i}k\,\mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right] \\ &\quad +\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\nabla}'\cdot\left(\mathbf{j}_\omega(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\,\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{5.16}$$

where, according to identity (F.121e) on page 221, the last term vanishes if the dyadic inside the big parentheses is regular and tends to zero at large distances.

Further evaluation of the derivative in the first term makes it possible to write

$$
\begin{aligned}
\mathbf{K}_\omega = & -2\int_{V'} \mathrm{d}^3x' \left[\mathbf{j}_\omega(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')\right](\mathbf{x}-\mathbf{x}')\,\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^4} \\
& + \mathrm{i}k\int_{V'} \mathrm{d}^3x' \left[\mathbf{j}_\omega(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')\right](\mathbf{x}-\mathbf{x}')\,\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^3} \\
& + \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^2} \\
& - \mathrm{i}k\int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned}
\tag{5.17}
$$

Using the triple product 'bac-cab' formula (F.53) on page 218 backwards, and inserting the resulting expression for $\mathbf{K}_\omega$ into equation (5.13) on the previous page, we arrive at the following final expression for the temporal Fourier transform of the total $\mathbf{E}$ field:

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) = & -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x'\, \rho_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\
& + \frac{\mathrm{i}\omega}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\
= & \frac{1}{4\pi\varepsilon_0}\Bigg[\int_{V'} \mathrm{d}^3x'\, \frac{\rho_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\
& + \frac{1}{c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\cdot(\mathbf{x}-\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4} \\
& + \frac{1}{c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4} \\
& - \frac{\mathrm{i}k}{c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\Bigg]
\end{aligned}
\tag{5.18}
$$

Taking the inverse Fourier transform of equation (5.18) above, once again using the vacuum relation $\omega = kc$, we find, at last, the expression in time domain for the total electric field:

$$
\begin{aligned}
\mathbf{E}(t,\mathbf{x}) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{E}_\omega(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\omega t} \\
&= \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x'\, \frac{\rho(t'_{\mathrm{ret}},\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Retarded Coulomb field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4}}_{\text{Intermediate field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x'\, \frac{[\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4}}_{\text{Intermediate field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x'\, \frac{[\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')]\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Far field}}
\end{aligned}
\tag{5.19}
$$

where, as before,

$$
\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}') \stackrel{\text{def}}{\equiv} \left(\frac{\partial \mathbf{j}}{\partial t}\right)_{t=t'_{\mathrm{ret}}}
\tag{5.20}
$$

Here, the first term represents the *retarded Coulomb field* and the last term represents the *far field* which dominates at very large distances. The other two terms represent the *intermediate field* which contributes significantly only to the fields themselves in the *near zone* and must be properly taken into account there.

## 5.3 The retarded magnetic field

Let us now compute the magnetic field from the vector potential, defined by equation (5.5) and equation (5.5b) on page 87, and formula (3.9) on page 35:

$$
\mathbf{B}(t,\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x})
\tag{5.21}
$$

Using the Fourier transformed version of this equation and equation (5.5b) on page 87, we obtain

$$
\mathbf{B}_\omega(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}_\omega(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}
\tag{5.22}
$$

Utilising formula (F.87) on page 219 and recalling that $\mathbf{j}_\omega(\mathbf{x}')$ does not depend on $\mathbf{x}$, we can rewrite this as

$$\begin{aligned}
\mathbf{B}_\omega(\mathbf{x}) &= -\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}') \times \left[\boldsymbol{\nabla}\left(\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right)\right] \\
&= -\frac{1}{4\pi\varepsilon_0 c^2}\left[\int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}') \times \left(-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right. \\
&\quad \left. + \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}') \times \left(\mathrm{i}k\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] \\
&= \frac{1}{4\pi\varepsilon_0 c^2}\left[\int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\
&\quad \left. + \int_{V'} \mathrm{d}^3x'\, \frac{(-\mathrm{i}k)\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right]
\end{aligned} \tag{5.23}$$

From this expression for the magnetic field in the frequency ($\omega$) domain, we finally obtain the total magnetic field in the temporal ($t$) domain by taking the inverse Fourier transform (using the identity $-\mathrm{i}k = -\mathrm{i}\omega/c$):

$$\begin{aligned}
\mathbf{B}(t,\mathbf{x}) &= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{B}_\omega(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega t} \\
&= \frac{1}{4\pi\varepsilon_0 c^2}\left\{\int_{V'} \mathrm{d}^3x'\, \frac{\left[\int_{-\infty}^{\infty}\mathrm{d}\omega\, \mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}(\omega t - k|\mathbf{x}-\mathbf{x}'|)}\right] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\
&\quad \left. + \frac{1}{c}\int_{V'} \mathrm{d}^3x'\, \frac{\left[\int_{-\infty}^{\infty}\mathrm{d}\omega\,(-\mathrm{i}\omega)\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}(\omega t - k|\mathbf{x}-\mathbf{x}'|)}\right] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right\}
\end{aligned} \tag{5.24}$$

Comparing with equations (3.33) on page 40, we can identify the exponents of the exponentials in the integrands as $-\mathrm{i}\omega t'_{\mathrm{ret}}$ and find that the total retarded magnetic field can be written as the sum of two terms[1]

[1] Equation (5.9) and equation (5.25) seem to have been first introduced by Panofsky and Phillips. Later they were given by Jefimenko and they are therefore sometimes called the *Jefimenko equations*.

$$\begin{aligned}
\mathbf{B}(t,\mathbf{x}) &= \underbrace{\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Retarded induction field}} \\
&\quad + \underbrace{\frac{1}{4\pi\varepsilon_0 c^3}\int_{V'} \mathrm{d}^3x'\, \frac{\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}}_{\text{Far field}}
\end{aligned} \tag{5.25}$$

where

$$\dot{\mathbf{j}}(t'_{\mathrm{ret}},\mathbf{x}') \stackrel{\text{def}}{\equiv} \left(\frac{\partial \mathbf{j}}{\partial t}\right)_{t=t'_{\mathrm{ret}}} \tag{5.26}$$

The first term, the *retarded induction field* that dominates near the current source but falls off rapidly with distance from it, is the electrodynamic version of the Biot-Savart law in electrostatics, formula (1.16) on page 8. The second term, the *far field*, dominates at large distances. The spatial derivatives ($\boldsymbol{\nabla}$) gave rise to a time derivative ($\dot{}$) so this term represents the part of the magnetic field that is generated by the time rate of change of the current, *i.e.* accelerated charges, at the retarded time.

In infinitesimal differential form, formula (5.25) on the preceding page becomes

$$\mathrm{d}\mathbf{B}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\left(\mathrm{d}\mathbf{i}'(t',\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} + \frac{1}{c}\mathrm{d}\dot{\mathbf{i}}'(t,\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\right) \tag{5.27}$$

where

$$\mathrm{d}\mathbf{i}'(t',\mathbf{x}') = \mathrm{d}^3x'\,\mathbf{j}(t'_{\mathrm{ret}},\mathbf{x}') \tag{5.28a}$$

and

$$\mathrm{d}\dot{\mathbf{i}}'(t',\mathbf{x}') \stackrel{\mathrm{def}}{\equiv} \left(\frac{\partial(\mathrm{d}\mathbf{i})}{\partial t}\right)_{t=t'_{\mathrm{ret}}} = \mathrm{d}^3x'\left(\frac{\partial\mathbf{j}}{\partial t}\right)_{t=t'_{\mathrm{ret}}} \tag{5.28b}$$

Equation (5.27) is the dynamic generalisation of the static infinitesimal Biot-Savart law, equation (1.15) on page 7, to which it reduces in the static limit.

With this we have achieved our goal of finding closed-form analytic expressions for the electric and magnetic fields when the sources of the fields are completely arbitrary, prescribed distributions of charges and currents. The only assumption made is that the advanced potentials have been discarded [recall the discussion following equation (3.34) on page 40 in chapter 3].

## 5.4 The total electric and magnetic fields at large distances from the sources

For many purposes it is convenient to have access to approximate expressions for the electric and magnetic fields that are valid in the *far zone*, *i.e.* very far away from the source region. Let us therefore derive such expressions.

As illustrated in figure 5.1 on the following page we assume that the sources $\rho$ and $\mathbf{j}$ are located at stationary points $\mathbf{x}'$ near a fixed point $\mathbf{x}_0$ inside a volume $V'$ that is not moving relative to the observer. Hence the distance from each source point $\mathbf{x}'$ to the observation point (field point) $\mathbf{x}$ is assumed to be constant in time. The non-moving source volume $V'$ is located in free space and has

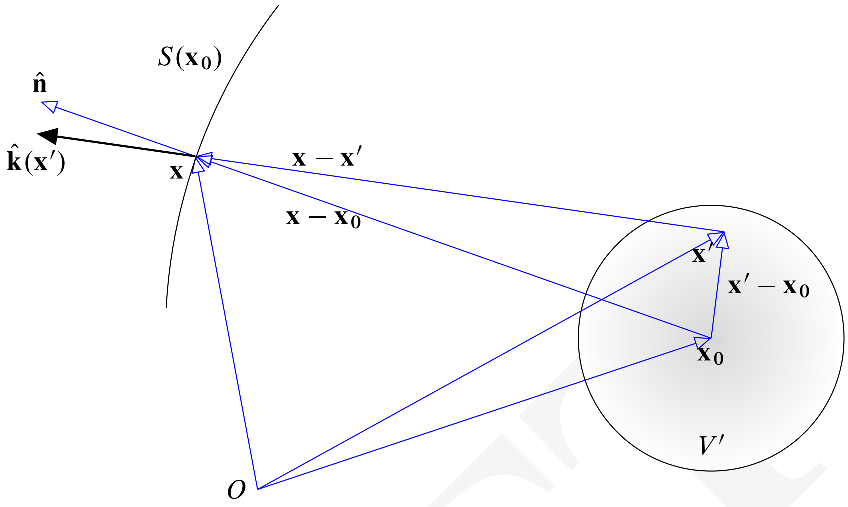


Figure 5.1: Relation between the unit vector $\hat{\mathbf{n}}$, anchored at the observation point $\mathbf{x}$ and directed along $\mathbf{x}-\mathbf{x}_0$, and hence normal to the surface $S(\mathbf{x}_0)$ which has its centre at $\mathbf{x}_0$ and passes through $\mathbf{x}$, and the wave vector $\mathbf{k}(\mathbf{x}')$, directed along $\mathbf{x}-\mathbf{x}'$, of fields generated at the source point $\mathbf{x}'$ near the point $\mathbf{x}_0$ in the source volume $V'$. At distances $|\mathbf{x}-\mathbf{x}'|$ much larger than the extent of $V'$, the unit vector $\hat{\mathbf{k}}(\mathbf{x}')$ and the unit vector $\hat{\mathbf{n}} \stackrel{\text{def}}{\equiv} \hat{\mathbf{k}}(\mathbf{x}_0)$ are nearly coincident.

such a limited spatial extent that $\sup|\mathbf{x}'-\mathbf{x}_0| \ll \inf|\mathbf{x}-\mathbf{x}'|$, and the integration surface $S(\mathbf{x}_0)$, centred on $\mathbf{x}_0$ and with an outward pointing normal unit vector $\hat{\mathbf{n}} = \widehat{\mathbf{x}-\mathbf{x}_0}$, has a large enough radius $|\mathbf{x}-\mathbf{x}_0| \gg \sup|\mathbf{x}'-\mathbf{x}_0|$.

The exact wave vector of fields generated at $\mathbf{x}'$ reaching an observer at $\mathbf{x}$ can be written

$$\mathbf{k}(\mathbf{x}') = k\hat{\mathbf{k}}(\mathbf{x}') \equiv k\widehat{\mathbf{x}-\mathbf{x}'} = k\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \tag{5.29}$$

expressing the fact that its magnitude $k = \omega/c$ is constant but its direction is along $\mathbf{x}-\mathbf{x}'$ and thus is dependent on the location of the source element at $\mathbf{x}'$ in $V'$. Now,

$$\begin{aligned}
\left|\mathbf{x}-\mathbf{x}'\right| &\equiv \left|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)\right| \\
&= \sqrt{\left|\mathbf{x}-\mathbf{x}_0\right|^2 - 2(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0) + \left|\mathbf{x}'-\mathbf{x}_0\right|^2} \\
&= |\mathbf{x}-\mathbf{x}_0|\left(1 - \frac{2(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^2} + \frac{|\mathbf{x}'-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{x}_0|^2}\right)^{\frac{1}{2}} \\
&\approx |\mathbf{x}-\mathbf{x}_0|\left[1 + \hat{\mathbf{n}}\cdot\frac{\mathbf{x}'-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{2}\frac{|\mathbf{x}'-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{x}_0|^2} + \frac{1}{8}\left(\frac{2\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|}\right)^2 + \dots\right] \\
&= |\mathbf{x}-\mathbf{x}_0|\left(1 + \frac{|\mathbf{x}'-\mathbf{x}_0|}{|\mathbf{x}-\mathbf{x}_0|}\cos\Theta + \frac{1}{2}\frac{|\mathbf{x}'-\mathbf{x}_0|^2}{|\mathbf{x}-\mathbf{x}_0|^2}\sin^2\Theta + \dots\right)
\end{aligned} \tag{5.30}$$

where we made a *binomial expansion*. We can, for the geometry just described, make the magnitude approximation

$$k\left|\mathbf{x}-\mathbf{x}'\right| \approx k\left|\mathbf{x}-\mathbf{x}_0\right| - \mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0) \tag{5.31}$$

in the phases (exponentials) and the cruder approximation

$$\left|\mathbf{x}-\mathbf{x}'\right| \approx \left|\mathbf{x}-\mathbf{x}_0\right| \tag{5.32}$$

in the amplitudes (denominators) in equation (5.18) on page 90 and in equation (5.23) on page 92. We then get the following approximate expressions for the Fourier amplitudes of the electric and magnetic fields, valid at sufficiently large distances from the bounded source volume:

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) \approx{}& \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,\rho_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\,\hat{\mathbf{k}}(\mathbf{x}')\\
&+\frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\cdot\hat{\mathbf{k}}(\mathbf{x}')]\,\hat{\mathbf{k}}(\mathbf{x}')\\
&+\frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}(\mathbf{x}')]\times\hat{\mathbf{k}}(\mathbf{x}')\\
&-\frac{\mathrm{i}k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}(\mathbf{x}')]\times\hat{\mathbf{k}}(\mathbf{x}')\end{aligned} \tag{5.33a}$$

$$\begin{aligned}\mathbf{B}_\omega(\mathbf{x}) \approx{}& \frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}(\mathbf{x}')\\
&-\frac{\mathrm{i}k}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}(\mathbf{x}')\end{aligned} \tag{5.33b}$$

At a field (observation) point $\mathbf{x}$ located sufficiently far away from the source volume $V'$ such that $\inf|\mathbf{x}-\mathbf{x}'| \gg \sup|\mathbf{x}'-\mathbf{x}_0|$, and $|\mathbf{x}-\mathbf{x}_0| \gg \sup|\mathbf{x}'-\mathbf{x}_0|$, we can assume that the direction of all wave vectors from the sources in $V'$ are parallel to each other. *I.e.* we can make the *paraxial approximation*

$$\hat{\mathbf{k}}(\mathbf{x}') = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} = \widehat{\mathbf{x}-\mathbf{x}'} \approx \hat{\mathbf{k}}(\mathbf{x}_0) = \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} = \widehat{\mathbf{x}-\mathbf{x}_0} \equiv \hat{\mathbf{n}} \tag{5.34}$$

where $\hat{\mathbf{n}}$ is the constant unit vector normal to the surface $S(\mathbf{x}_0)$ of a large sphere centred on $\mathbf{x}_0$ and passing through the (fixed) field point $\mathbf{x}$ (see figure 5.1 on the preceding page). Then formulæ (5.33) on page 95 can be further approximated as

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) \approx\ & \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,\rho_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\,\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\cdot\hat{\mathbf{n}}]\,\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}} \\
&- \frac{\mathrm{i}k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.35a}
$$

$$
\begin{aligned}
\mathbf{B}_\omega(\mathbf{x}) \approx\ & \frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{n}} \\
&- \frac{\mathrm{i}k}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.35b}
$$

which, after reordering of scalar and vector products, using the fact that $\hat{\mathbf{n}}$ is a constant unit vector, can be written

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) \approx\ & \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\left(\int_{V'}\mathrm{d}^3x'\,\rho_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\left(\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\cdot\hat{\mathbf{n}}\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\left[\left(\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\times\hat{\mathbf{n}}\right]\times\hat{\mathbf{n}} \\
&- \frac{\mathrm{i}k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\left[\left(\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\times\hat{\mathbf{n}}\right]\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.36a}
$$

$$
\begin{aligned}
\mathbf{B}_\omega(\mathbf{x}) \approx\ & \frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\left(\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\times\hat{\mathbf{n}} \\
&- \frac{\mathrm{i}k}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\left(\int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right)\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.36b}
$$

or

$$
\begin{aligned}
\mathbf{E}_\omega(\mathbf{x}) \approx\ & \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\,\mathcal{Q}_\omega(\mathbf{x}_0)\,\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\,\mathcal{I}_\omega(\mathbf{x}_0)\cdot\hat{\mathbf{n}}\hat{\mathbf{n}} \\
&+ \frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\,(\mathcal{I}_\omega(\mathbf{x}_0)\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} \\
&- \frac{\mathrm{i}k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,(\mathcal{I}_\omega(\mathbf{x}_0)\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.37a}
$$

$$
\begin{aligned}
\mathbf{B}_\omega(\mathbf{x}) \approx & \frac{1}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2} \mathcal{I}_\omega(\mathbf{x}_0) \times \hat{\mathbf{n}} \\
& - \frac{\mathrm{i}k}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \mathcal{I}_\omega(\mathbf{x}_0) \times \hat{\mathbf{n}}
\end{aligned}
\tag{5.37b}
$$

where

$$
\begin{aligned}
\mathcal{Q}_\omega(\mathbf{x}_0) & \stackrel{\text{def}}{\equiv} \int_{V'} \mathrm{d}^3x'\, \rho_\omega(\mathbf{x}')\, \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)} \\
& = \mathrm{e}^{\mathrm{i}k\hat{\mathbf{n}}\cdot\mathbf{x}_0} \int_{V'} \mathrm{d}^3x'\, \rho_\omega(\mathbf{x}')\, \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot\mathbf{x}'}
\end{aligned}
\tag{5.38a}
$$

and

$$
\begin{aligned}
\mathcal{I}_\omega(\mathbf{x}_0) & \stackrel{\text{def}}{\equiv} \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\, \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)} \\
& = \mathrm{e}^{\mathrm{i}k\hat{\mathbf{n}}\cdot\mathbf{x}_0} \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}')\, \mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot\mathbf{x}'}
\end{aligned}
\tag{5.38b}
$$

Inverse Fourier transforming these expressions we find that the total retarded **E** and **B** fields very far away from a source are, to a good approximation, given by

$$
\begin{aligned}
\mathbf{E}(t,\mathbf{x}) \approx & \frac{1}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}_0|^2} \int_{V'} \mathrm{d}^3x'\, \rho(t',\mathbf{x}')\hat{\mathbf{n}} \\
& + \frac{1}{4\pi\varepsilon_0 c\, |\mathbf{x}-\mathbf{x}_0|^2} \int_{V'} \mathrm{d}^3x'\, [\mathbf{j}(t',\mathbf{x}')\cdot\hat{\mathbf{n}}]\hat{\mathbf{n}} \\
& + \frac{1}{4\pi\varepsilon_0 c\, |\mathbf{x}-\mathbf{x}_0|^2} \int_{V'} \mathrm{d}^3x'\, [\mathbf{j}(t',\mathbf{x}')\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}} \\
& + \frac{1}{4\pi\varepsilon_0 c^2\, |\mathbf{x}-\mathbf{x}_0|} \int_{V'} \mathrm{d}^3x'\, [\dot{\mathbf{j}}(t',\mathbf{x}')\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.39a}
$$

and

$$
\begin{aligned}
\mathbf{B}(t,\mathbf{x}) \approx & \frac{1}{4\pi\varepsilon_0 c^2\, |\mathbf{x}-\mathbf{x}_0|^2} \int_{V'} \mathrm{d}^3x'\, \mathbf{j}(t',\mathbf{x}')\times\hat{\mathbf{n}} \\
& - \frac{1}{4\pi\varepsilon_0 c^3\, |\mathbf{x}-\mathbf{x}_0|} \int_{V'} \mathrm{d}^3x'\, \dot{\mathbf{j}}(t',\mathbf{x}')\times\hat{\mathbf{n}}
\end{aligned}
\tag{5.39b}
$$

In these expressions we can use equation (5.31) to consistently approximate $t'$ as follows (the fields propagate in free space where $k/\omega = 1/c$):

$$
\begin{aligned}
t'(\mathbf{x}') & \approx t - \frac{k\,|\mathbf{x}-\mathbf{x}_0|}{\omega} + \frac{\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)}{\omega} = t - \frac{|\mathbf{x}-\mathbf{x}_0|}{c} + \frac{|\mathbf{x}'-\mathbf{x}_0|\cos\theta'_0}{c} \\
& = t - \frac{|\mathbf{x}-\mathbf{x}_0|}{c}\left(1 - \frac{|\mathbf{x}'-\mathbf{x}_0|}{|\mathbf{x}-\mathbf{x}_0|}\cos\theta'_0\right)
\end{aligned}
\tag{5.40}
$$

where $\theta'_0$ is the angle between $\hat{\mathbf{k}}(\mathbf{x}')$ and $\mathbf{x}'-\mathbf{x}_0$.

### 5.4.1 The far fields

When $|\mathbf{x}-\mathbf{x}_0| \to \infty$ and the source region is of finite extent, the only surviving composants in expression (5.19) and expression (5.25) are the *far fields*

$$\mathbf{E}^{\text{far}}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x' \, \frac{[\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{5.41a}$$

$$\mathbf{B}^{\text{far}}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^3}\int_{V'} \mathrm{d}^3x' \, \frac{\dot{\mathbf{j}}(t'_{\text{ret}},\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \tag{5.41b}$$

In the frequency (temporal Fourier) domain, these far fields are represented *exactly* by

$$\begin{aligned}\mathbf{E}^{\text{far}}_\omega(\mathbf{x}) & \\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x' \, \frac{[\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \times (\mathbf{x}-\mathbf{x}')] \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c}\int_{V'} \mathrm{d}^3x' \, \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}[\mathbf{j}_\omega(\mathbf{x}') \times \hat{\mathbf{k}}(\mathbf{x}')] \times \hat{\mathbf{k}}(\mathbf{x}')\end{aligned} \tag{5.42a}$$

$$\begin{aligned}\mathbf{B}^{\text{far}}_\omega(\mathbf{x}) & \\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c^2}\int_{V'} \mathrm{d}^3x' \, \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{j}_\omega(\mathbf{x}') \times \hat{\mathbf{k}}(\mathbf{x}')\end{aligned} \tag{5.42b}$$

respectively.

Within the approximation (5.31), the expressions (5.42) above for the far fields can be simplified to

$$\begin{aligned}\mathbf{E}^{\text{far}}_\omega(\mathbf{x}) \approx &-\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \\ &\times \int_{V'} \mathrm{d}^3x' \, [\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)} \times \hat{\mathbf{k}}(\mathbf{x}')] \times \hat{\mathbf{k}}(\mathbf{x}')\end{aligned} \tag{5.43a}$$

$$\begin{aligned}\mathbf{B}^{\text{far}}_\omega(\mathbf{x}) \approx &-\mathrm{i}k\frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \\ &\times \int_{V'} \mathrm{d}^3x' \, [\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}(\mathbf{x}')\cdot(\mathbf{x}'-\mathbf{x}_0)} \times \hat{\mathbf{k}}(\mathbf{x}')]\end{aligned} \tag{5.43b}$$

Assuming that also the paraxial approximation (5.34) is applicable, the approximate expressions (5.43) above for the far fields can be further simplified to

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) \approx -\mathrm{i}k \frac{1}{4\pi\varepsilon_0 c} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \left[\left(\int_{V'} \mathrm{d}^3x' \, [\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right) \times \hat{\mathbf{n}}\right] \times \hat{\mathbf{n}} \tag{5.44a}$$

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) \approx -\mathrm{i}k \frac{1}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \left(\int_{V'} \mathrm{d}^3x' \, [\mathbf{j}_\omega(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\right) \times \hat{\mathbf{n}} \tag{5.44b}$$

We see that at very large distances $r = |\mathbf{x} - \mathbf{x_0}|$ the fields fall off as $1/r$ are, are purely transverse (perpendicular to $\hat{\mathbf{n}}$) and mutually orthogonal.

## 5.5 Examples

EXAMPLE 5.1 ▷Alternative expressions for **E** and **B**

One can express the fields **E** and **B** directly in integrals of arbitrary source terms in several ways. For instance, by the *electromagnetic field density vector*

$$\begin{aligned}\boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) &= \frac{\rho(t',\mathbf{x}')}{4\pi\varepsilon_0\,|\mathbf{x}-\mathbf{x}'|^2}\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\\ &+ \frac{1}{4\pi\varepsilon_0 c\,|\mathbf{x}-\mathbf{x}'|^2}\left(\mathbf{j}(t',\mathbf{x}')\boldsymbol{\cdot}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right)\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\\ &+ \frac{1}{4\pi\varepsilon_0 c\,|\mathbf{x}-\mathbf{x}'|^2}\left(\mathbf{j}(t',\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right)\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\\ &+ \frac{1}{4\pi\varepsilon_0 c^2\,|\mathbf{x}-\mathbf{x}'|}\left(\dot{\mathbf{j}}(t',\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right)\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{5.45}$$

where

$$t' = t'_{\mathrm{ret}} = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{5.46}$$

one can, for non-moving sources, express the infinitesimal differential fields as

$$\mathrm{d}\mathbf{E}(t,\mathbf{x}) = \boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) \tag{5.47a}$$

$$\mathrm{d}\mathbf{B}(t,\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) \tag{5.47b}$$

The fields themselves are in this case given by

$$\mathbf{E}(t,\mathbf{x}) = \int_{V'}\mathrm{d}^3x'\,\boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) \tag{5.48a}$$

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{c}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times\boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) \tag{5.48b}$$

Denoting $\mathbf{x}-\mathbf{x}' = \mathbf{r}' = r'\hat{\mathbf{r}}'$, formula (5.45) can be written

$$\begin{aligned}\boldsymbol{\Theta}(t',\mathbf{x}',\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0 r'^2}\left(\rho(t',\mathbf{x}') + \frac{1}{c}\mathbf{j}(t',\mathbf{x}')\boldsymbol{\cdot}\hat{\mathbf{r}}'\right)\hat{\mathbf{r}}'\\ &+ \frac{1}{4\pi\varepsilon_0 c r'}\left(\frac{1}{r'}\mathbf{j}(t',\mathbf{x}')\times\hat{\mathbf{r}}' + \frac{1}{c}\dot{\mathbf{j}}(t',\mathbf{x}')\times\hat{\mathbf{r}}'\right)\times\hat{\mathbf{r}}'\end{aligned} \tag{5.49}$$

where

$$t' = t'_{\mathrm{ret}} = t - \frac{r'}{c} \tag{5.50}$$

End of example 5.1◁

## 5.6 Bibliography

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# 6

# RADIATION AND RADIATING SYSTEMS

In chapter 3 we were able to derive general expressions for the scalar and vector potentials from which we later (in chapter 5) derived exact analytic expressions of general validity for the total electric and magnetic fields generated by completely arbitrary distributions of charge and current sources that are located within a certain region in space. The only limitation in the calculation of the fields was that the advanced potentials were discarded on — admittedly not totally convincing — physical grounds.

In chapter 4 we showed that the electromagnetic energy, linear momentum, and angular momentum are all conserved quantities and in this chapter we will show that these quantities can be radiated all the way to infinity and therefore be used in wireless communications over long distances and for observing very remote objects in Nature, including electromagnetic radiation sources in the Universe. Radiation processes are irreversible in that the radiation does not return to the radiator but is lost from it forever.[1] However, the radiation can, of course, be sensed by other charges and currents that are located in free space, possibly very far away from the sources. This is precisely what makes it possible for our eyes to observe light, and even more so our our telescope to observe and analyse optical and radio signals, from extremely distant stars and galaxies. This consequence of Maxwell's equations, with the displacement current included, was verified experimentally by HEINRICH RUDOLF HERTZ about twenty years after Maxwell had made his theoretical predictions. Hertz's experimental and theoretical studies paved the way for radio and TV broadcasting, radar, wireless communications, radio astronomy and a host of other applications and technologies.

[1] This is referred to as *time arrow of radiation*.

Thus, one can, at least in principle, calculate the radiated fields, flux of energy, linear momentum and angular momentum, as well as other electromagnetic observables at any time at any point in space generated by an arbitrary charge and current density of the source. However, in practice it is often difficult to evaluate the source integrals, at least analytically, unless the charge and current densities have a simple distribution in space. In the general case, one has to resort to approximations. We shall consider both these situations in this chapter.

## 6.1 Radiation of linear momentum and energy

Let us consider an electromagnetic field $(\mathbf{E}, \mathbf{B})$ that during a finite time interval $\Delta t$ is emitted from a localised source distribution in a volume $V$ around a point $\mathbf{x}_0$ into surrounding free space. After a certain time $t_0$ this *electromagnetic pulse* (signal) has propagated such a long distance radially outward from $\mathbf{x}_0$ that the field is wholly located within two concentric spheres of radii $r_0 = ct_0$ and $r_0 + \Delta r = ct_0 + c\Delta t$, respectively. The total linear momentum carried by this electromagnetic pulse (signal) is the volume integral of the linear momentum density $\mathbf{g}^{\text{field}} = \mathbf{S}/c^2 = \varepsilon_0 \mathbf{E} \times \mathbf{B}$ [*cf.* equation (4.45) on page 62]. In spherical polar coordinates with the origin chosen at $\mathbf{x}_0$ this volume integral becomes

$$\mathbf{p}^{\text{field}} = \int_V \mathrm{d}r\,\mathrm{d}\Omega\, r^2 \mathbf{g}^{\text{field}} = \varepsilon_0 \int_0^{\pi} \mathrm{d}\theta\, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi \int_{r_0}^{r_0+\Delta r} \mathrm{d}r\, r^2 (\mathbf{E} \times \mathbf{B}) \tag{6.1}$$

In chapter 5 we derived the two expressions (5.44) on page 99 for the $\mathbf{E}$ and $\mathbf{B}$ fields, respectively. These expressions show that at large distances $r' = |\mathbf{x}' - \mathbf{x}_0|$ from the source, the leading order contributions to these fields are purely transverse, mutually orthogonal, and fall off as $1/r$. As a result, at large distances $r$, the dominating component of the linear momentum density $\mathbf{g}^{\text{field}} = \mathbf{S}^{\text{far}}/c^2 = \varepsilon_0 \mathbf{E}^{\text{far}} \times \mathbf{B}^{\text{far}}$, which is purely radial and falls off as $1/r^2$.

The total linear momentum $\mathbf{p}^{\text{field}}$ is obtained when $\mathbf{g}^{\text{field}}$ is integrated over a large spherical shell (centred on the source) of width $\mathrm{d}r = c\mathrm{d}t$, where $\mathrm{d}t$ is the short duration of the signal, and for which the directed area element is $\mathrm{d}^2x\, \hat{\mathbf{n}} = r^2 \mathrm{d}\Omega\, \hat{\mathbf{r}} = r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi\, \hat{\mathbf{r}}$ [*cf.* formula (F.18) on page 215]. But the total integrated power, as given by the surface integral $\oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{S}$ in equation (4.33) on page 60, tends to a constant at infinity, showing that energy $U^{\text{field}}$ and electromagnetic linear momentum $\mathbf{p}^{\text{field}}$ is carried all the way to infinity and is irreversibly lost there. This is the physical foundation of the well-known fact that $\mathbf{p}^{\text{field}}$ and $U^{\text{field}}$ can be transmitted over extremely long distances. The force action on charges in one region in space can therefore cause a force action on charges in a another region in space; see equation (4.50) on page 63. Today's wireless communication technology, be it classical or quantal, is based almost exclusively on the utilisation of this *translational degree of freedom* of the charges (currents) and the fields.

## 6.1.1 Monochromatic signals

If the source is strictly monochromatic, we can obtain the temporal average of the radiated power $P$ directly, simply by averaging over one period so that

$$\begin{aligned}\langle \mathbf{S} \rangle_t &= \frac{1}{\mu_0} \langle \mathbf{E} \times \mathbf{B} \rangle_t = \frac{1}{2\mu_0} \operatorname{Re}\{\mathbf{E} \times \mathbf{B}^*\} \\ &= \frac{1}{2\mu_0} \operatorname{Re}\left\{\mathbf{E}_\omega \mathrm{e}^{-\mathrm{i}\omega t} \times (\mathbf{B}_\omega \mathrm{e}^{-\mathrm{i}\omega t})^*\right\} = \frac{1}{2\mu_0} \operatorname{Re}\{\mathbf{E}_\omega \times \mathbf{B}_\omega^*\}\end{aligned} \tag{6.2}$$

From formula (F.17) and formula (F.18) on page 215 we see that

$$\mathrm{d}^2x = r^2 \mathrm{d}\Omega = |\mathbf{x} - \mathbf{x}_0|^2 \,\mathrm{d}\Omega = |\mathbf{x} - \mathbf{x}_0|^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

We also note from figure 5.1 on page 94 that $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ are nearly parallel. Hence, we can approximate

$$\frac{\hat{\mathbf{k}} \cdot \mathrm{d}^2x \hat{\mathbf{n}}}{|\mathbf{x} - \mathbf{x}_0|^2} \equiv \frac{\mathrm{d}^2x}{|\mathbf{x} - \mathbf{x}_0|^2} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \equiv \mathrm{d}\Omega \,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \approx \mathrm{d}\Omega \tag{6.3}$$

Using the far-field approximations (**??**) for the fields and the fact that $1/c = \sqrt{\varepsilon_0 \mu_0}$, and also introducing the *characteristic impedance of vacuum*

$$R_0 \stackrel{\text{def}}{\equiv} \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 376.7\,\Omega \tag{6.4}$$

we obtain

$$\left\langle \mathbf{S}^{\text{far}} \right\rangle_t = \frac{1}{32\pi^2} R_0 \frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} \left| \int_{V'} \mathrm{d}^3x' \,(\mathbf{j}_\omega \times \mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 \hat{\mathbf{n}} \tag{6.5}$$

Consequently, the amount of power per unit solid angle $\mathrm{d}\Omega$ that flows across an infinitesimal surface element $r^2 \mathrm{d}\Omega = |\mathbf{x} - \mathbf{x}_0|^2 \,\mathrm{d}\Omega$ of a large spherical shell with its origin at $\mathbf{x}_0$ and enclosing all sources, is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{32\pi^2} R_0 \left| \int_{V'} \mathrm{d}^3x' \,(\mathbf{j}_\omega \times \mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 \tag{6.6}$$

This formula is valid far away from the sources and shows that the radiated power is given by an expression which is resistance ($R_0$) times the square of the supplied current (the integrated current density $\mathbf{j}_\omega$), as expected. We note that the emitted power is independent of distance $r = |\mathbf{x} - \mathbf{x}_0|$ and is therefore carried all the way to infinity. The possibility to transmit electromagnetic power over large distances, even in empty space, is the physical foundation for the extremely important wireless communications technology. Besides determining the strength of the radiated power, the integral in formula (6.6) also determines its angular distribution.

### 6.1.2 Finite bandwidth signals

A signal with finite pulse width in time ($t$) domain has a certain spread in frequency ($\omega$) domain. To calculate the total radiated energy we need to integrate over the whole bandwidth. The total energy transmitted through a unit area is the time integral of the Poynting vector:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{S}(t) &= \frac{1}{\mu_0}\int_{-\infty}^{\infty} \mathrm{d}t\, (\mathbf{E} \times \mathbf{B}) \\ &= \frac{1}{\mu_0}\int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}\omega' \int_{-\infty}^{\infty} \mathrm{d}t\, (\mathbf{E}_\omega \times \mathbf{B}_{\omega'})\, \mathrm{e}^{-\mathrm{i}(\omega+\omega')t}\end{aligned} \tag{6.7}$$

If we carry out the temporal integration first and use the fact that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(\omega+\omega')t} = 2\pi\delta(\omega+\omega') \tag{6.8}$$

equation (6.7) above can be written

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{S}(t) &= \frac{2\pi}{\mu_0}\int_{-\infty}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) \\ &= \frac{2\pi}{\mu_0}\left(\int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) + \int_{-\infty}^{0} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega})\right) \\ &= \frac{2\pi}{\mu_0}\left(\int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) - \int_{0}^{-\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega})\right) \\ &= \frac{2\pi}{\mu_0}\left(\int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) + \int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_{-\omega} \times \mathbf{B}_{\omega})\right) \\ &= \frac{2\pi}{\mu_0}\int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}_{-\omega} + \mathbf{E}_{-\omega} \times \mathbf{B}_{\omega}) \\ &= \frac{2\pi}{\mu_0}\int_{0}^{\infty} \mathrm{d}\omega\, (\mathbf{E}_\omega \times \mathbf{B}^{*}_{\omega} + \mathbf{E}^{*}_{\omega} \times \mathbf{B}_{\omega})\end{aligned} \tag{6.9}$$

where the last step follows from physical requirement of real-valuedness of $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$. We insert the Fourier transforms of the field components which dominate at large distances, *i.e.* the far fields (**??**) and (**??**). The result, after integration over the area $S$ of a large sphere which encloses the source volume $V'$, is

$$U = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \int_0^{\infty} \mathrm{d}\omega \left| \int_{V'} \mathrm{d}^3x'\, \frac{\mathbf{j}_\omega \times \mathbf{k}}{|\mathbf{x}-\mathbf{x}'|} \mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|} \right|^2 \hat{\mathbf{k}} \tag{6.10}$$

Inserting the approximations (5.31) and (6.3) into equation (6.10) above, introducing the *spectral energy density* $U_\omega$ via the definition

$$U \stackrel{\text{def}}{\equiv} \int_0^{\infty} \mathrm{d}\omega U_\omega \tag{6.11}$$

and recalling the definition (6.4) on page 105, we obtain

$$\frac{\mathrm{d}U_\omega}{\mathrm{d}\Omega}\,\mathrm{d}\omega \approx \frac{1}{4\pi} R_0 \left| \int_{V'} \mathrm{d}^3x'\,(\mathbf{j}_\omega \times \mathbf{k})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)} \right|^2 \mathrm{d}\omega \tag{6.12}$$

which, at large distances, is a good approximation to the energy that is radiated per unit solid angle $\mathrm{d}\Omega$ in a frequency band $\mathrm{d}\omega$. It is important to notice that Formula (6.12) includes only source coordinates. This means that the total amount of energy that is being radiated is independent on the distance to the source (as long as it is large).

## 6.2 Radiation of angular momentum

Not only electromagnetic linear momentum (Poynting vector) can be radiated from a source and transmitted over very long distances, but the same is also true for electromagnetic angular momentum $\mathbf{J}^{\mathrm{field}}$. Then torque action (the time rate of change of $\mathbf{J}^{\mathrm{field}}$) in one region causes torque action on charges. The use of this *rotational degree of freedom* of the fields has only recently been put to practical use even if it has been known for more than a century.

After straightforward calculations, based on the results obtained in chapter 5, one finds that the complete cycle averaged far-zone expression for a frequency component $\omega$ of the electromagnetic angular momentum density generated by arbitrary charge and current sources can be approximated by

$$\left\langle \mathbf{h}^{\mathrm{field}}(\mathbf{x}_0) \right\rangle_t = \frac{1}{32\pi^2\varepsilon_0 c^3} \left( \frac{\hat{\mathbf{n}} \times \mathrm{Re}\left\{(cq + I_n)\dot{\mathbf{I}}^*\right\}}{c\,|\mathbf{x}-\mathbf{x}_0|^2} + \frac{\hat{\mathbf{n}} \times \mathrm{Re}\left\{(cq + I_n)\mathbf{I}^*\right\}}{|\mathbf{x}-\mathbf{x}_0|^3} \right) \tag{6.13}$$

where, in complex notation,

$$\mathbf{I}(t') \approx \int_{V'} \mathrm{d}^3x'\,\mathbf{j}(t',\mathbf{x}') \tag{6.14}$$

and

$$\dot{\mathbf{I}}(t') \approx \int_{V'} \mathrm{d}^3x'\,\dot{\mathbf{j}}(t',\mathbf{x}') \tag{6.15}$$

We see that at very large distances $r$, the angular momentum density $\mathbf{h}^{\mathrm{field}}$ falls off as $1/r^2$, *i.e.* it has precisely the same behaviour in the far zone as the linear momentum density and can therefore also transfer information wirelessly over large distances. The only difference is that while the direction of the linear momentum (Poynting vector) becomes purely radial at infinity, the angular momentum becomes perpendicular to the linear momentum, *i.e.* purely transverse, there.

## 6.3 Radiation from a localised source at rest

In the general case, and when we are interested in evaluating the radiation far from a source at rest and which is localised in a small volume, we can introduce an approximation which leads to a *multipole expansion* where individual terms can be evaluated analytically. Here we use *Hertz's method*, which focuses on the physics rather than on the mathematics, to obtain this expansion.

### 6.3.1 Electric multipole moments

Let us assume that the charge distribution $\rho$ determining the potential in equation (3.35a) on page 40 has such a small extent that all the source points $\mathbf{x}'$ can be assumed to be located very near a point $\mathbf{x}_0$. At a large distance $|\mathbf{x}-\mathbf{x}_0|$, one can then, to a good approximation, approximate the retarded potential by the Taylor expansion (Einstein's summation convention over $i$ and $j$ is implied); *cf.* example 3.1 on page 46

$$\Phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q(t'_{\text{ret}})}{|\mathbf{x}-\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|^2}\,\mathbf{d}(t'_{\text{ret}},\mathbf{x}_0)\boldsymbol{\cdot}\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}\right.$$
$$\left. + \frac{1}{|\mathbf{x}-\mathbf{x}_0|^3}\,Q_{ij}(t'_{\text{ret}},\mathbf{x}_0)\left(\frac{3}{2}\frac{x_i - x_{0_i}}{|\mathbf{x}-\mathbf{x}_0|}\frac{x_j - x_{0_j}}{|\mathbf{x}-\mathbf{x}_0|} - \frac{1}{2}\delta_{ij}\right) + \dots\right] \tag{6.16}$$

where

$$q(t'_{\text{ret}}) = \int_{V'} \mathrm{d}^3x'\,\rho(t'_{\text{ret}},\mathbf{x}') \tag{6.17a}$$

is the *total charge* or *electric monopole moment*,

$$\mathbf{d}(t'_{\text{ret}},\mathbf{x}_0) = \int_{V'} \mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\,\rho(t'_{\text{ret}},\mathbf{x}') \tag{6.17b}$$

with components $d_i$, $i = 1,2,3$ is the *electric dipole moment vector*, and

$$\mathbf{Q}(t'_{\text{ret}},\mathbf{x}_0) = \int_{V'} \mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\otimes(\mathbf{x}'-\mathbf{x}_0)\,\rho(t'_{\text{ret}},\mathbf{x}') \tag{6.17c}$$

with components $Q_{ij}$, $i,j = 1,2,3$, is the *electric quadrupole moment tensor*.

The source volume is at rest and is so small that internal retardation effects can be neglected, *i.e.* that we can set $t'_{\text{ret}} \approx t - |\mathbf{x}-\mathbf{x}_0|/c$. Then

$$t = t(t'_{\text{ret}}) \approx t'_{\text{ret}} + Const \tag{6.18}$$

where

$$Const = \frac{|\mathbf{x}-\mathbf{x}_0|}{c} \tag{6.19}$$

Hence the transformation between $t$ and $t'_{\text{ret}}$ is a trivial. In the subsequent analysis in this subsection we shall use $t'$ to denote this approximate $t'_{\text{ret}}$.

For a normal medium, the major contributions to the electrostatic interactions come from the net charge and the lowest order electric multipole moments induced by the polarisation due to an applied electric field. Particularly important is the dipole moment. Let $\mathbf{P}$ denote the electric dipole moment density (electric dipole moment per unit volume; unit: $\mathrm{C\,m^{-2}}$), also known as the *electric polarisation*, in some medium. In analogy with the second term in the expansion equation (6.16) on the facing page, the electric potential from this volume distribution $\mathbf{P}(t, \mathbf{x}')$ of electric dipole moments $\mathbf{d}$ at the source point $\mathbf{x}'$ can be written

$$\begin{aligned}\Phi_{\mathbf{d}}(t,\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\mathbf{P}(t',\mathbf{x}')\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\\ &= -\frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\mathbf{P}(t',\mathbf{x}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\mathbf{P}(t',\mathbf{x}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{6.20}$$

Using expression (M.155a) on page 258 and applying the divergence theorem, we can rewrite this expression for the potential as follows:

$$\begin{aligned}\Phi_{\mathbf{d}}(t,\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\left[\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{P}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)-\int_{V'}\mathrm{d}^3x'\,\frac{\boldsymbol{\nabla}'\cdot\mathbf{P}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]\\ &= \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'}\mathrm{d}^2x'\,\hat{\mathbf{n}}'\cdot\frac{\mathbf{P}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\int_{V'}\mathrm{d}^3x'\,\frac{\boldsymbol{\nabla}'\cdot\mathbf{P}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]\end{aligned} \tag{6.21}$$

where the first term, which describes the effects of the induced, non-cancelling dipole moment on the surface of the volume, can be neglected, unless there is a discontinuity in $\hat{\mathbf{n}}\cdot\mathbf{P}$ at the surface. Doing so, we find that the contribution from the electric dipole moments to the potential is given by

$$\Phi_{\mathbf{d}}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{-\boldsymbol{\nabla}'\cdot\mathbf{P}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{6.22}$$

Comparing this expression with expression equation (3.35a) on page 40 for the potential from a charge distribution $\rho(t,\mathbf{x})$, we see that $-\boldsymbol{\nabla}\cdot\mathbf{P}(t,\mathbf{x})$ has the characteristics of a charge density and that, to the lowest order, the effective charge density becomes $\rho(t,\mathbf{x}) - \boldsymbol{\nabla}\cdot\mathbf{P}(t,\mathbf{x})$, in which the second term is a polarisation term that we call $\rho^{\text{pol}}(t,\mathbf{x})$.

## 6.3.2 The Hertz potential

In section 6.3.1 on page 108 we introduced the electric polarisation $\mathbf{P}(t,\mathbf{x})$ such that the polarisation charge density

$$\rho^{\text{pol}} = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{6.23}$$

If we adopt the same idea for the 'true' charge density due to free charges and introduce a vector field $\boldsymbol{\pi}(t, \mathbf{x})$, analogous to $\mathbf{P}(t, \mathbf{x})$, but such that

$$\rho^{\text{true}} \stackrel{\text{def}}{\equiv} -\boldsymbol{\nabla} \cdot \boldsymbol{\pi} \tag{6.24a}$$

which means that the associated 'polarisation current' now is the true current:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = \mathbf{j}^{\text{true}} \tag{6.24b}$$

As a consequence, the equation of continuity for 'true' charges and currents [*cf.* expression (1.22) on page 10] is satisfied:

$$\frac{\partial \rho^{\text{true}}(t, \mathbf{x})}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}^{\text{true}}(t, \mathbf{x}) = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{\pi} + \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\pi}}{\partial t} = 0 \tag{6.25}$$

The vector $\boldsymbol{\pi}$ is called the *polarisation vector* because, formally, it treats also the 'true' (free) charges as polarisation charges. Since in the microscopic Maxwell-Lorentz equation (2.1a) on page 19, the charge density $\rho$ must include all charges, we can write this equation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho^{\text{true}} + \rho^{\text{pol}}}{\varepsilon_0} = \frac{-\boldsymbol{\nabla} \cdot \boldsymbol{\pi} - \boldsymbol{\nabla} \cdot \mathbf{P}}{\varepsilon_0} \tag{6.26}$$

*i.e.* in a form where all the charges are considered to be polarisation charges.

We now introduce a further potential $\boldsymbol{\Pi}^{\text{e}}$ with the following property

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}^{\text{e}} = -\Phi \tag{6.27a}$$

$$\frac{1}{c^2} \frac{\partial \boldsymbol{\Pi}^{\text{e}}}{\partial t} = \mathbf{A} \tag{6.27b}$$

where $\Phi$ and $\mathbf{A}$ are the electromagnetic scalar and vector potentials, respectively. As we see, $\boldsymbol{\Pi}^{\text{e}}$ acts as a '*super-potential*' in the sense that it is a potential from which we can obtain other potentials. It is called the *Hertz vector* or *polarisation potential*. Requiring that the scalar and vector potentials $\Phi$ and $\mathbf{A}$, respectively, satisfy their inhomogeneous wave equations, equations (3.15) on page 36, one finds, using (6.24) and (6.27), that the Hertz vector must satisfy the inhomogeneous wave equation

$$\Box^2 \boldsymbol{\Pi}^{\text{e}} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{\Pi}^{\text{e}} - \nabla^2 \boldsymbol{\Pi}^{\text{e}} = \frac{\boldsymbol{\pi}}{\varepsilon_0} \tag{6.28}$$

This equation is of the same type as equation (3.18) on page 37, and has therefore the retarded solution

$$\boldsymbol{\Pi}^{\text{e}}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\pi}(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{6.29}$$

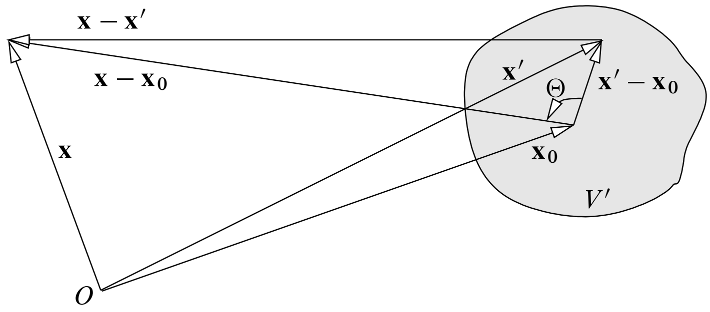


Figure 6.1: Geometry of a typical multipole radiation problem where the field point $\mathbf{x}$ is located some distance away from the finite source volume $V'$ centred around $\mathbf{x}_0$. If $k\,|\mathbf{x}'-\mathbf{x}_0| \ll 1 \ll k\,|\mathbf{x}-\mathbf{x}_0|$, then the radiation at $\mathbf{x}$ is well approximated by a few terms in the multipole expansion.

with Fourier components

$$\mathbf{\Pi}^{\mathrm{e}}_{\omega}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\pi}_{\omega}(\mathbf{x}')\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{6.30}$$

Assume that the source region is a limited volume around some central point $\mathbf{x}_0$ far away from the field (observation) point $\mathbf{x}$ illustrated in figure 6.1. Under these assumptions, we can expand the Hertz vector, expression (6.30) above, due to the presence of non-vanishing $\boldsymbol{\pi}(t'_{\mathrm{ret}},\mathbf{x}')$ in the vicinity of $\mathbf{x}_0$, in a formal series. For this purpose we recall from *potential theory* that

$$\begin{aligned}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &\equiv \frac{\mathrm{e}^{\mathrm{i}k|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|}}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|} \\ &= \mathrm{i}k\sum_{n=0}^{\infty}(2n+1)P_n(\cos\Theta)\,j_n(k\,|\mathbf{x}'-\mathbf{x}_0|)h^{(1)}_n(k\,|\mathbf{x}-\mathbf{x}_0|)\end{aligned} \tag{6.31}$$

where (see figure 6.1)

$\dfrac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ is a *Green function* or *propagator*

$\Theta$ is the angle between $\mathbf{x}'-\mathbf{x}_0$ and $\mathbf{x}-\mathbf{x}_0$

$P_n(\cos\Theta)$ is the *Legendre polynomial* of order $n$

$j_n(k\,|\mathbf{x}'-\mathbf{x}_0|)$ is the *spherical Bessel function of the first kind* of order $n$

$h^{(1)}_n(k\,|\mathbf{x}-\mathbf{x}_0|)$ is the *spherical Hankel function of the first kind* of order $n$

According to the addition theorem for Legendre polynomials

$$P_n(\cos\Theta) = \sum_{m=-n}^{n}(-1)^m P^m_n(\cos\theta)P^{-m}_n(\cos\theta')\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')} \tag{6.32}$$

where $P^m_n$ is an *associated Legendre polynomial of the first kind*, related to the *spherical harmonic* $Y^m_n$ as

$$Y^m_n(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P^m_n(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}$$

and, in spherical polar coordinates,

$$\mathbf{x}' - \mathbf{x}_0 = (|\mathbf{x}' - \mathbf{x}_0|, \theta', \varphi') \tag{6.33a}$$

$$\mathbf{x} - \mathbf{x}_0 = (|\mathbf{x} - \mathbf{x}_0|, \theta, \varphi) \tag{6.33b}$$

If we introduce the *help vector* $\mathbf{C}$ such that

$$\mathbf{C} = \boldsymbol{\nabla} \times \boldsymbol{\Pi}^{\mathrm{e}} \tag{6.34}$$

we see that we can calculate the magnetic and electric fields, respectively, as follows

$$\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{C}}{\partial t} \tag{6.35a}$$

$$\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{C} \tag{6.35b}$$

Clearly, the last equation is valid only if $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ *i.e.* if we are outside the source volume. Since we are mainly interested in the fields in the far zone, a long distance away from the source region, this is no essential limitation.

Inserting equation (6.31) on the previous page, together with formula (6.32) on the preceding page, into equation (6.30) on the previous page, we can in a formally exact way expand the Fourier component of the Hertz vector as

$$\boldsymbol{\Pi}^{\mathrm{e}}_{\omega} = \frac{\mathrm{i}k}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1)(-1)^m h_n^{(1)}(k\,|\mathbf{x} - \mathbf{x}_0|)\, P_n^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi}$$
$$\times \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}_{\omega}(\mathbf{x}')\, j_n(k\,|\mathbf{x}' - \mathbf{x}_0|)\, P_n^{-m}(\cos\theta')\, \mathrm{e}^{-\mathrm{i}m\varphi'} \tag{6.36}$$

We notice that there is no dependence on $\mathbf{x} - \mathbf{x}_0$ inside the integral; the integrand is only dependent on the relative source vector $\mathbf{x}' - \mathbf{x}_0$.

We are interested in the case where the field point is many wavelengths away from the well-localised sources, *i.e.* when the following inequalities

$$k\,|\mathbf{x}' - \mathbf{x}_0| \ll 1 \ll k\,|\mathbf{x} - \mathbf{x}_0| \tag{6.37}$$

hold. Then we may to a good approximation replace $h_n^{(1)}$ with the first term in its asymptotic expansion:

$$h_n^{(1)}(k\,|\mathbf{x} - \mathbf{x}_0|) \approx (-\mathrm{i})^{n+1} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{k\,|\mathbf{x} - \mathbf{x}_0|} \tag{6.38}$$

and replace $j_n$ with the first term in its power series expansion:

$$j_n(k\,|\mathbf{x}' - \mathbf{x}_0|) \approx \frac{2^n n!}{(2n+1)!} \left(k\,|\mathbf{x}' - \mathbf{x}_0|\right)^n \tag{6.39}$$

Inserting these expansions into equation (6.36) on the facing page, we obtain the multipole expansion of the Fourier component of the Hertz vector

$$\mathbf{\Pi}^{\mathrm{e}}_{\omega} \approx \sum_{n=0}^{\infty} \mathbf{\Pi}^{\mathrm{e}\,(n)}_{\omega} \tag{6.40a}$$

where

$$\mathbf{\Pi}^{\mathrm{e}\,(n)}_{\omega} = (-\mathrm{i})^n \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \frac{2^n n!}{(2n)!} \times \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}_{\omega}(\mathbf{x}')\, (k\,|\mathbf{x}'-\mathbf{x}_0|)^n\, P_n(\cos\Theta) \tag{6.40b}$$

This expression is approximately correct only if certain care is exercised; if many $\mathbf{\Pi}^{\mathrm{e}\,(n)}_{\omega}$ terms are needed for an accurate result, the expansions of the spherical Hankel and Bessel functions used above may not be consistent and must be replaced by more accurate expressions. Furthermore, asymptotic expansions as the one used in equation (6.38) on page 112 are not unique.

Taking the inverse Fourier transform of $\mathbf{\Pi}^{\mathrm{e}}_{\omega}$ will yield the Hertz vector in time domain, which inserted into equation (6.34) on the preceding page will yield $\mathbf{C}$. The resulting expression can then in turn be inserted into equations (6.35) on the facing page in order to obtain the radiation fields.

For a linear source distribution along the polar axis, $\Theta = \theta$ in expression (6.40b) above, and $P_n(\cos\theta)$ gives the angular distribution of the radiation. In the general case, however, the angular distribution must be computed with the help of formula (6.32) on page 111. Let us now study the lowest order contributions to the expansion of the Hertz vector.

### 6.3.3 Electric dipole radiation

Choosing $n = 0$ in expression (6.40b) above, we obtain

$$\mathbf{\Pi}^{\mathrm{e}\,(0)}_{\omega} = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi\varepsilon_0\,|\mathbf{x}-\mathbf{x}_0|} \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}_{\omega}(\mathbf{x}') = \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\, \mathbf{d}_{\omega} \tag{6.41}$$

Since $\boldsymbol{\pi}$ represents a dipole moment density for the 'true' charges (in the same vein as $\mathbf{P}$ does so for the polarised charges), $\mathbf{d}_{\omega} = \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}_{\omega}(\mathbf{x}')$ is, by definition, the Fourier component of the *electric dipole moment*

$$\mathbf{d}(t, \mathbf{x}_0) = \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}(t', \mathbf{x}') = \int_{V'} \mathrm{d}^3x'\, (\mathbf{x}' - \mathbf{x}_0)\rho(t', \mathbf{x}') \tag{6.42}$$

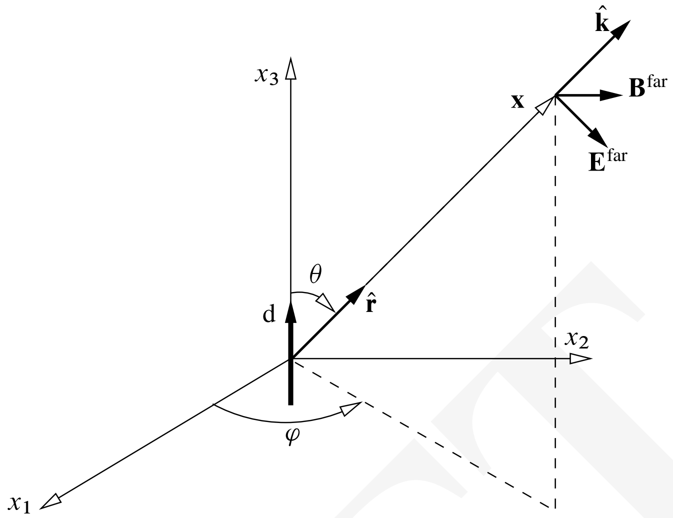


Figure 6.2: If a spherical polar coordinate system $(r, \theta, \varphi)$ is chosen such that the electric dipole moment $\mathbf{d}$ (and thus its Fourier transform $\mathbf{d}_\omega$) is located at the origin and directed along the polar axis, the calculations are simplified.

[*cf.* equation (6.17b) on page 108]. If a spherical coordinate system is chosen with its polar axis along $\mathbf{d}_\omega$ as in figure 6.2, the components of $\mathbf{\Pi}^{\mathrm{e}\,(0)}_\omega$ are

$$\Pi^{\mathrm{e}}_r \stackrel{\text{def}}{\equiv} \mathbf{\Pi}^{\mathrm{e}\,(0)}_\omega \cdot \hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \cos\theta \tag{6.43a}$$

$$\Pi^{\mathrm{e}}_\theta \stackrel{\text{def}}{\equiv} \mathbf{\Pi}^{\mathrm{e}\,(0)}_\omega \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{4\pi\varepsilon_0} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \sin\theta \tag{6.43b}$$

$$\Pi^{\mathrm{e}}_\varphi \stackrel{\text{def}}{\equiv} \mathbf{\Pi}^{\mathrm{e}\,(0)}_\omega \cdot \hat{\boldsymbol{\varphi}} = 0 \tag{6.43c}$$

Evaluating formula (6.34) on page 112 for the help vector $\mathbf{C}$, with the spherically polar components (6.43) of $\mathbf{\Pi}^{\mathrm{e}\,(0)}_\omega$ inserted, we obtain

$$\mathbf{C}_\omega = C^{(0)}_{\omega,\varphi}\,\hat{\boldsymbol{\varphi}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - \mathrm{i}k\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \sin\theta\,\hat{\boldsymbol{\varphi}} \tag{6.44}$$

Applying this to equations (6.35) on page 112, we obtain directly the Fourier components of the fields

$$\mathbf{B}_\omega = -\mathrm{i}\frac{\omega\mu_0}{4\pi}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - \mathrm{i}k\right)\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \sin\theta\,\hat{\boldsymbol{\varphi}} \tag{6.45a}$$

$$\begin{aligned}\mathbf{E}_\omega = \frac{1}{4\pi\varepsilon_0}\Bigg[&2\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{\mathrm{i}k}{|\mathbf{x}-\mathbf{x}_0|}\right)\cos\theta\,\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \\ &+\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|^2} - \frac{\mathrm{i}k}{|\mathbf{x}-\mathbf{x}_0|} - k^2\right)\sin\theta\,\hat{\boldsymbol{\theta}}\Bigg]\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega\end{aligned} \tag{6.45b}$$

Keeping only those parts of the fields which dominate at large distances (the radiation fields) and recalling that the wave vector $\mathbf{k} = k(\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$

where $k = \omega/c$, we can now write down the Fourier components of the radiation parts of the magnetic and electric fields from the dipole:

$$\mathbf{B}_\omega^{\text{far}} = -\frac{\omega\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega k \sin\theta\,\hat{\boldsymbol{\varphi}} = -\frac{\omega\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}(\mathbf{d}_\omega \times \mathbf{k}) \tag{6.46a}$$

$$\mathbf{E}_\omega^{\text{far}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega k^2 \sin\theta\,\hat{\boldsymbol{\theta}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}[(\mathbf{d}_\omega \times \mathbf{k}) \times \mathbf{k}] \tag{6.46b}$$

These fields constitute the *electric dipole radiation*, also known as *E1 radiation*.

## 6.3.4 Magnetic dipole radiation

The next term in the expression (6.40b) on page 113 for the expansion of the Fourier transform of the Hertz vector is for $n = 1$:

$$\begin{aligned}\boldsymbol{\Pi}_\omega^{\mathrm{e}(1)} &= -\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,k\left|\mathbf{x}'-\mathbf{x}_0\right|\boldsymbol{\pi}_\omega(\mathbf{x}')\cos\Theta \\ &= -\mathrm{i}k\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}\int_{V'}\mathrm{d}^3x'\,[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')\end{aligned} \tag{6.47}$$

Here, the term $[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')$ can be rewritten

$$[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}') = (x_i - x_{0,i})(x'_i - x_{0,i})\,\boldsymbol{\pi}_\omega(\mathbf{x}') \tag{6.48}$$

and introducing

$$\eta_i = x_i - x_{0,i} \tag{6.49a}$$

$$\eta'_i = x'_i - x_{0,i} \tag{6.49b}$$

the $j$th component of the integrand in $\boldsymbol{\Pi}_\omega^{\mathrm{e}(1)}$ can be broken up into

$$\begin{aligned}\{[(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}'-\mathbf{x}_0)]\,\boldsymbol{\pi}_\omega(\mathbf{x}')\}_j &= \frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i + \pi_{\omega,i}\eta'_j\right) \\ &\quad + \frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i - \pi_{\omega,i}\eta'_j\right)\end{aligned} \tag{6.50}$$

*i.e.* as the sum of two parts, the first being symmetric and the second antisymmetric in the indices $i, j$. We note that the antisymmetric part can be written as

$$\begin{aligned}\frac{1}{2}\eta_i\left(\pi_{\omega,j}\eta'_i - \pi_{\omega,i}\eta'_j\right) &= \frac{1}{2}[\pi_{\omega,j}(\eta_i\eta'_i) - \eta'_j(\eta_i\pi_{\omega,i})] \\ &= \frac{1}{2}[\boldsymbol{\pi}_\omega(\boldsymbol{\eta}\cdot\boldsymbol{\eta}') - \boldsymbol{\eta}'(\boldsymbol{\eta}\cdot\boldsymbol{\pi}_\omega)]_j \\ &= \frac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_0)\times[\boldsymbol{\pi}_\omega\times(\mathbf{x}'-\mathbf{x}_0)]\right\}_j\end{aligned} \tag{6.51}$$

The utilisation of equations (6.24) on page 110, and the fact that we are considering a single Fourier component,

$$\boldsymbol{\pi}(t,\mathbf{x}) = \boldsymbol{\pi}_\omega \mathrm{e}^{-\mathrm{i}\omega t} \tag{6.52}$$

allow us to express $\boldsymbol{\pi}_\omega$ in $\mathbf{j}_\omega$ as

$$\boldsymbol{\pi}_\omega = \mathrm{i}\,\frac{\mathbf{j}_\omega}{\omega} \tag{6.53}$$

Hence, we can write the antisymmetric part of the integral in formula (6.47) on the preceding page as

$$\begin{aligned}
&\frac{1}{2}(\mathbf{x}-\mathbf{x}_0) \times \int_{V'} \mathrm{d}^3x'\, \boldsymbol{\pi}_\omega(\mathbf{x}') \times (\mathbf{x}'-\mathbf{x}_0) \\
&\quad = \mathrm{i}\frac{1}{2\omega}(\mathbf{x}-\mathbf{x}_0) \times \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_\omega(\mathbf{x}') \times (\mathbf{x}'-\mathbf{x}_0) \\
&\quad = -\mathrm{i}\frac{1}{\omega}(\mathbf{x}-\mathbf{x}_0) \times \mathbf{m}_\omega
\end{aligned} \tag{6.54}$$

where we introduced the Fourier transform of the *magnetic dipole moment*

$$\mathbf{m}_\omega = \frac{1}{2}\int_{V'} \mathrm{d}^3x'\, (\mathbf{x}'-\mathbf{x}_0) \times \mathbf{j}_\omega(\mathbf{x}') \tag{6.55}$$

The final result is that the antisymmetric, magnetic dipole, part of $\boldsymbol{\Pi}^{\mathrm{e}\,(1)}_\omega$ can be written

$$\boldsymbol{\Pi}^{\mathrm{e,\,antisym}\,(1)}_\omega = -\frac{k}{4\pi\varepsilon_0\omega}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|^2}(\mathbf{x}-\mathbf{x}_0) \times \mathbf{m}_\omega \tag{6.56}$$

In analogy with the electric dipole case, we insert this expression into equation (6.34) on page 112 to evaluate $\mathbf{C}$, with which equations (6.35) on page 112 then gives the $\mathbf{B}$ and $\mathbf{E}$ fields. Discarding, as before, all terms belonging to the near fields and transition fields and keeping only the terms that dominate at large distances, we obtain

$$\mathbf{B}^{\mathrm{far}}_\omega(\mathbf{x}) = -\frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}(\mathbf{m}_\omega \times \mathbf{k}) \times \mathbf{k} \tag{6.57a}$$

$$\mathbf{E}^{\mathrm{far}}_\omega(\mathbf{x}) = \frac{k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\mathbf{m}_\omega \times \mathbf{k} \tag{6.57b}$$

which are the fields of the *magnetic dipole radiation* (*M1 radiation*).

### 6.3.5 Electric quadrupole radiation

The symmetric part $\boldsymbol{\Pi}^{\mathrm{e,\,sym}\,(1)}_\omega$ of the $n = 1$ contribution in the equation (6.40b) on page 113 for the expansion of the Hertz vector can be expressed in terms of the *electric quadrupole tensor*, which is defined in accordance with equation (6.17c) on page 108:

$$\mathbf{Q}(t, \mathbf{x}_0) = \int_{V'} \mathrm{d}^3x' \, (\mathbf{x}' - \mathbf{x}_0) \otimes (\mathbf{x}' - \mathbf{x}_0) \rho(t'_{\mathrm{ret}}, \mathbf{x}') \tag{6.58}$$

Again we use this expression in equation (6.34) on page 112 to calculate the fields via equations (6.35) on page 112. Tedious, but fairly straightforward algebra (which we will not present here), yields the resulting fields. The components of the fields that dominate in the far field zone (wave zone) are given by

$$\mathbf{B}^{\mathrm{far}}_{\omega}(\mathbf{x}) = \frac{\mathrm{i}\mu_0\omega}{8\pi} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} (\mathbf{k} \cdot \mathbf{Q}_{\omega}) \times \mathbf{k} \tag{6.59a}$$

$$\mathbf{E}^{\mathrm{far}}_{\omega}(\mathbf{x}) = \frac{\mathrm{i}}{8\pi\varepsilon_0} \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} [(\mathbf{k} \cdot \mathbf{Q}_{\omega}) \times \mathbf{k}] \times \mathbf{k} \tag{6.59b}$$

This type of radiation is called *electric quadrupole radiation* or *E2 radiation*.

## 6.4 Radiation from an extended source volume at rest

Certain radiating systems have a symmetric geometry or are in any other way simple enough that a direct (semi-)analytic calculation of the radiated fields and energy is possible. This is for instance the case when the radiating current flows in a finite, conducting medium of simple geometry at rest such as in a stationary *antenna*.

### 6.4.1 Radiation from a one-dimensional current distribution

We describe the radiation in a standard spherical polar coordinate system as in figure 6.4 on page 120, *i.e.* a system where the polar axis coincides with the $z$ axis, the polar angle $\theta$ is calculated relative to the direction of the positive $z$ axis, and the azimuthal angle $\varphi$ is calculated relative to the direction of positive $x$ axis toward the positive $y$ axis in a right-handed sense. In this polar coordinate system the observation (field) point is located at $\mathbf{x} = (r, \theta, \varphi)$ with $r = |\mathbf{x} - \mathbf{x}_0|$. The origin is chosen at $\mathbf{x}_0$ so that $\mathbf{x} - \mathbf{x}_0 = \mathbf{x} = \mathbf{r} = r\hat{\mathbf{r}}$ and the *wave vector* $\mathbf{k} = k\hat{\mathbf{k}} = k\hat{\mathbf{r}}$.

The Fourier amplitude of the far-zone electric field generated by the *antenna current density*

$$\mathbf{j}(t', \mathbf{x}') = \mathbf{j}_{\omega}(\mathbf{x}')\mathrm{e}^{-\mathrm{i}\omega t'} \tag{6.60}$$

is

$$
\begin{aligned}
\mathbf{E}_{\omega}^{\text{far}}(\mathbf{x}) &= -\mathrm{i}\frac{1}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\hat{\mathbf{r}} \times \int_{V'} \mathrm{d}^3x'\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\,\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}(\mathbf{x}') \\
\mathbf{B}_{\omega}^{\text{far}}(\mathbf{x}) &= -\mathrm{i}\frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'} \mathrm{d}^3x'\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\,\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}(\mathbf{x}')
\end{aligned}
\tag{6.61}
$$

In the paraxial approximation described by equation (5.34) on page 95 one assumes that the distant observer views all parts of the antenna under one and the same fixed polar angle $\theta$, *i.e.* that all wave vectors has the same constant direction as that coming from the midpoint $\mathbf{x}_0 = \mathbf{0}$. In this approximation

$$
\mathbf{E}_{\omega}^{\text{far}} = -\frac{\mathrm{i}}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\hat{\mathbf{r}} \times \left(\mathbf{k} \times \int_{V'} \mathrm{d}^3x'\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathbf{j}_{\omega}(\mathbf{x}')\right) \tag{6.62}
$$

where $\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k} = -\mathbf{k} \times \mathbf{j}_{\omega}(\mathbf{x}')$ was used. In the chosen geometry, the 1D antenna current flows along $\hat{\mathbf{z}}$ :

Let us apply equation (6.6) on page 105 to calculate the radiated EM power from a one-dimensional, time-varying current. Such a current can be set up by feeding the EMF of a generator (*e.g.* a transmitter) onto a stationary, linear, straight, thin, conducting wire across a very short gap at its centre. Due to the applied EMF, the charges in this thin wire of finite length $L$ are set into linear motion to produce a time-varying *antenna current* which is the source of the EM radiation. Linear antennas of this type are called *dipole antennas*. For simplicity, we assume that the conductor resistance and the energy loss due to the electromagnetic radiation are negligible.

Choosing our coordinate system such that the $x_3$ axis is along the antenna axis, the antenna current density can be represented, in complex notation, by $\mathbf{j}(t',\mathbf{x}') = \delta(x_1')\delta(x_2')J(t',x_3')\,\hat{\mathbf{x}}_3$ (measured in Am$^-$2) where $J(t',x_3')$ is the current (measured in A) along the antenna wire. Since we can assume that the antenna wire is infinitely thin, the antenna current must vanish at the endpoints $-L/2$ and $L/2$. At the midpoint, where the antenna is fed across a very short gap in the conducting wire, the antenna current is, of course, equal to the supplied current.

For each Fourier frequency component $\omega_0$, the antenna current $J(t',x_3')$ can be written as $I(x_3')\exp(-\mathrm{i}\omega_0 t')$ so that the antenna current density can be written as $\mathbf{j}(t',\mathbf{x}') = \mathbf{j}_0(\mathbf{x}')\exp(-\mathrm{i}\omega_0 t')$ [*cf.* equations (5.6) on page 87] where

$$
\mathbf{j}_0(\mathbf{x}') = \delta(x_1')\delta(x_2')I(x_3')\,\hat{\mathbf{x}}_3 \tag{6.63}
$$

and where the spatially varying Fourier amplitude $I(x_3')$ of the antenna current fulfils the *time-independent wave equation* (*Helmholtz equation*)

$$
\frac{\mathrm{d}^2 I}{\mathrm{d}x_3'^2} + k^2 I(x_3') = 0\,, \quad I(-L/2) = I(L/2) = 0\,, \quad I(0) = I_0 \tag{6.64}
$$

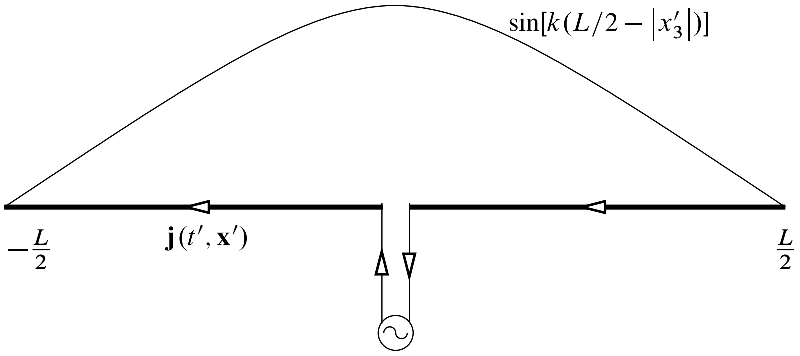


Figure 6.3: A linear antenna used for transmission. The current in the feeder and the antenna wire is set up by the EMF of the generator (the transmitter). At the ends of the wire, the current is reflected back with a $180°$ phase shift to produce a antenna current in the form of a standing wave.

This second-order ordinary differential equation with constant coefficients has the well-known solution

$$I(x_3') = I_0 \frac{\sin[k(L/2 - |x_3'|)]}{\sin(kL/2)} \tag{6.65}$$

where $I_0$ is the amplitude of the antenna current (measured in A), assumed to be constant and supplied, via a non-radiating *transmission line* by the generator/transmitter at the *antenna feed point* (in our case the midpoint of the antenna wire) and $1/\sin(kL/2)$ is a normalisation factor. The antenna current forms a *standing wave* as indicated in figure 6.3.[2]

When $L$ is much smaller than the wavelength $\lambda$, we can approximate the current distribution formula (6.65) by the first term in its Taylor expansion:

$$I(x_3') \approx I_0(1 - 2|x_3'|/L), kL \ll 1. \tag{6.66}$$

Hence, in the most general case of a straight, infinitely thin antenna of finite, arbitrary length $L$ directed along the $x_3'$ axis, the Fourier amplitude of the antenna current density is

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x_1')\delta(x_2') \frac{\sin[k(L/2 - |x_3'|)]}{\sin(kL/2)} \hat{\mathbf{x}}_3 \tag{6.67}$$

For a half-wave dipole antenna ($L = \lambda/2$), the antenna current density is

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x_1')\delta(x_2') \cos(kx_3')\hat{\mathbf{x}}_3 \tag{6.68}$$

while for a short antenna ($L \ll \lambda$) it can be approximated by

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x_1')\delta(x_2')(1 - 2|x_3'|/L)\hat{\mathbf{x}}_3 \tag{6.69}$$

In the case of a travelling wave antenna, in which one end of the antenna is connected to ground via a resistance so that the current at this end does not vanish, the Fourier amplitude of the antenna current density is

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x_1')\delta(x_2') \exp(kx_3')\hat{\mathbf{x}}_3 \tag{6.70}$$

[2] This rather accurate model of the antenna current was introduced in 1987 by HENRY CABOURN POCKLINGTON (1870–1952).

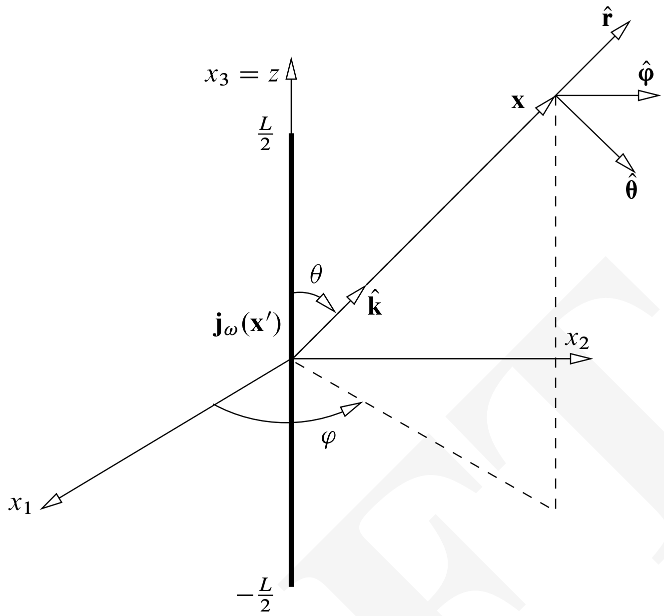


Figure 6.4: We choose a spherical polar coordinate system $(r = |\mathbf{x}|, \theta, \varphi)$ and arrange it so that the linear electric dipole antenna axis (and thus the antenna current density $\mathbf{j}_\omega$) is along the polar axis with the feed point at the origin.

In order to evaluate formula (6.6) on page 105 with the explicit monochromatic current (6.67) inserted, we use a spherical polar coordinate system as in figure 6.4 to evaluate the source integral

$$
\begin{aligned}
\mathcal{I} &= \left| \int_{V'} \mathrm{d}^3x'\, \mathbf{j}_0 \times \mathbf{k}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)} \right|^2 \\
&= \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x_3'\, I_0 \frac{\sin[k(L/2-|x_3'|)]}{\sin(kL/2)}\, k \sin\theta \mathrm{e}^{-\mathrm{i}kx_3'\cos\theta} \mathrm{e}^{\mathrm{i}kx_0\cos\theta_0} \right|^2 \\
&= I_0^2 \frac{k^2 \sin^2\theta}{\sin^2(k\frac{L}{2})} \left| \mathrm{e}^{\mathrm{i}kx_0\cos\theta_0} \right|^2 \left| 2\int_0^{\frac{L}{2}} \mathrm{d}x_3' \sin(kL/2 - kx_3') \cos(kx_3'\cos\theta) \right|^2 \\
&= 4I_0^2 \left( \frac{\cos[(kL/2)\cos\theta] - \cos(kL/2)}{\sin\theta \sin(kL/2)} \right)^2
\end{aligned}
\tag{6.71}
$$

Inserting this expression and $\mathrm{d}\Omega = 2\pi \sin\theta\, \mathrm{d}\theta$ into formula (6.6) on page 105 and integrating over $\theta$, we find that the total radiated power from the antenna is

$$
P(L) = R_0 I_0^2 \frac{1}{4\pi} \int_0^\pi \mathrm{d}\theta \left( \frac{\cos[(kL/2)\cos\theta] - \cos(kL/2)}{\sin\theta \sin(kL/2)} \right)^2 \sin\theta \tag{6.72}
$$

One can show that

$$\lim_{kL\to 0} P(L) = \frac{\pi}{12}\left(\frac{L}{\lambda}\right)^2 R_0 I_0^2 \tag{6.73}$$

where $\lambda$ is the vacuum wavelength.

The quantity

$$R^{\text{rad}}(L) = \frac{P(L)}{I_{\text{eff}}^2} = \frac{P(L)}{\frac{1}{2}I_0^2} = R_0\frac{\pi}{6}\left(\frac{L}{\lambda}\right)^2 \approx 197\left(\frac{L}{\lambda}\right)^2 \;\Omega \tag{6.74}$$

is called the *radiation resistance*. For the technologically important case of a half-wave antenna, *i.e.* for $L = \lambda/2$ or $kL = \pi$, formula (6.72) on the facing page reduces to

$$P(\lambda/2) = R_0 I_0^2 \frac{1}{4\pi}\int_0^{\pi} \mathrm{d}\theta\, \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \tag{6.75}$$

The integral in (6.75) can always be evaluated numerically. But, it can in fact also be evaluated analytically as follows:

$$\begin{aligned}
\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\mathrm{d}\theta = [\cos\theta \mapsto u] = \int_{-1}^{1} \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1-u^2}\,\mathrm{d}u =& \\
\left[\cos^2\left(\frac{\pi}{2}u\right) = \frac{1+\cos(\pi u)}{2}\right]& \\
= \frac{1}{2}\int_{-1}^{1}\frac{1+\cos(\pi u)}{(1+u)(1-u)}\,\mathrm{d}u& \\
= \frac{1}{4}\int_{-1}^{1}\frac{1+\cos(\pi u)}{(1+u)}\,\mathrm{d}u + \frac{1}{4}\int_{-1}^{1}\frac{1+\cos(\pi u)}{(1-u)}\,\mathrm{d}u& \\
= \frac{1}{2}\int_{-1}^{1}\frac{1+\cos(\pi u)}{(1+u)}\,\mathrm{d}u = \left[1+u \mapsto \frac{v}{\pi}\right]& \\
= \frac{1}{2}\int_0^{2\pi}\frac{1-\cos v}{v}\,\mathrm{d}v = \frac{1}{2}[\gamma + \ln 2\pi - \mathrm{Ci}(2\pi)]& \\
\approx 1.22&
\end{aligned} \tag{6.76}$$

where in the last step the *Euler-Mascheroni constant* $\gamma = 0.5772\ldots$ and the *cosine integral* $\mathrm{Ci}(x)$ were introduced. Inserting this into the expression equation (6.75) above we obtain the value $R^{\text{rad}}(\lambda/2) \approx 73\ \Omega$.

## 6.5 Radiation from a localised charge in arbitrary motion

The derivation of the radiation fields for the case of the source moving relative to the observer is considerably more complicated than the stationary cases that

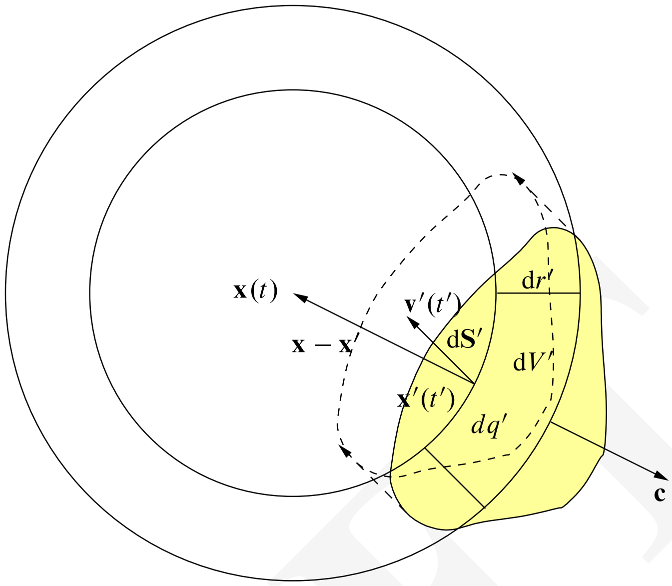


Figure 6.5: Signals that are observed at time $t$ at the field point $\mathbf{x}$ were generated at time $t'$ at source points $\mathbf{x}'$ on a sphere, centred on $\mathbf{x}$ and expanding, as time increases, with velocity $\mathbf{c} = -c(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$ outward from the centre. The source charge element moves with an arbitrary velocity $\mathbf{v}'$ and gives rise to a source 'leakage' out of the volume $\mathrm{d}V' = \mathrm{d}^3x'$.

we have studied so far. In order to handle this non-stationary situation, we start from the retarded potentials (3.35) on page 40 in chapter 3

$$\Phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\mathrm{d}^3x'\,\frac{\rho\big(t'_{\mathrm{ret}},\mathbf{x}'(t'_{\mathrm{ret}})\big)}{|\mathbf{x}(t)-\mathbf{x}'(t'_{\mathrm{ret}})|} \tag{6.77a}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{j}\big(t'_{\mathrm{ret}},\mathbf{x}(t'_{\mathrm{ret}})\big)}{|\mathbf{x}(t)-\mathbf{x}'(t'_{\mathrm{ret}})|} \tag{6.77b}$$

and consider a source region with such a limited spatial extent that the charges and currents are well localised. Specifically, we consider a charge $q'$, for instance an electron, which, classically, can be thought of as a localised, unstructured and rigid 'charge distribution' with a small, finite radius. The part of this 'charge distribution' $\mathrm{d}q'$ which we are considering is located in $\mathrm{d}V' = \mathrm{d}^3x'$ in the sphere in figure 6.5. Since we assume that the electron (or any other similar electric charge) moves with a velocity $\mathbf{v}'$ whose direction is arbitrary and whose magnitude can even be comparable to the speed of light, we cannot say that the charge and current to be used in (6.77) is $\int_{V'}\mathrm{d}^3x'\,\rho(t'_{\mathrm{ret}},\mathbf{x}')$ and $\int_{V'}\mathrm{d}^3x'\,\mathbf{v}\rho(t'_{\mathrm{ret}},\mathbf{x}')$, respectively, because in the finite time interval during which the observed signal is generated, part of the charge distribution will 'leak' out of the volume element $\mathrm{d}^3x'$.

### 6.5.1 The Liénard-Wiechert potentials

In figure 6.5 on page 122 the charge distribution that contributes to the field at $\mathbf{x}(t)$ is located at $\mathbf{x}'(t')$ on a sphere with radius $r = |\mathbf{x} - \mathbf{x}'| = c(t - t')$. The radius interval of this sphere from which radiation is received at the field point $\mathbf{x}$ during the time interval $(t', t' + \mathrm{d}t')$ is $(r', r' + \mathrm{d}r')$ and the net amount of charge in this radial interval is

$$\mathrm{d}q' = \rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}S'\,\mathrm{d}r' - \rho(t'_{\mathrm{ret}}, \mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}S'\,\mathrm{d}t' \tag{6.78}$$

where the last term represents the amount of 'source leakage' due to the fact that the charge distribution moves with velocity $\mathbf{v}'(t') = \mathrm{d}\mathbf{x}'/\mathrm{d}t'$. Since $\mathrm{d}t' = \mathrm{d}r'/c$ and $\mathrm{d}S'\,\mathrm{d}r' = \mathrm{d}^3x'$ we can rewrite the expression for the net charge as

$$\begin{aligned}\mathrm{d}q' &= \rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}^3x' - \rho(t'_{\mathrm{ret}}, \mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c\,|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x' \\ &= \rho(t'_{\mathrm{ret}}, \mathbf{x}')\left(1 - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c\,|\mathbf{x} - \mathbf{x}'|}\right)\mathrm{d}^3x'\end{aligned} \tag{6.79}$$

or

$$\rho(t'_{\mathrm{ret}}, \mathbf{x}')\,\mathrm{d}^3x' = \frac{\mathrm{d}q'}{1 - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c|\mathbf{x}-\mathbf{x}'|}} \tag{6.80}$$

which leads to the expression

$$\frac{\rho(t'_{\mathrm{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3x' = \frac{\mathrm{d}q'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c}} \tag{6.81}$$

This is the expression to be used in the formulæ (6.77) on the facing page for the retarded potentials. The result is (recall that $\mathbf{j} = \rho\mathbf{v}$)

$$\Phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathrm{d}q'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c}} \tag{6.82a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\mathbf{v}'\,\mathrm{d}q'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c}} \tag{6.82b}$$

where we used the relation $\varepsilon_0\mu_0 = 1/c^2$.

For a sufficiently small and well localised charge distribution we can, assuming that the integrands do not change sign in the integration volume, use the mean value theorem of calculus to evaluate these expressions to become

$$\Phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c}}\int \mathrm{d}q' = \frac{q'}{4\pi\varepsilon_0}\frac{1}{s} \tag{6.83a}$$

$$\begin{aligned}\mathbf{A}(t, \mathbf{x}) &= \frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}'}{c}}\int \mathrm{d}q' = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}'}{s} \\ &= \frac{\mathbf{v}'}{c^2}\Phi\end{aligned} \tag{6.83b}$$

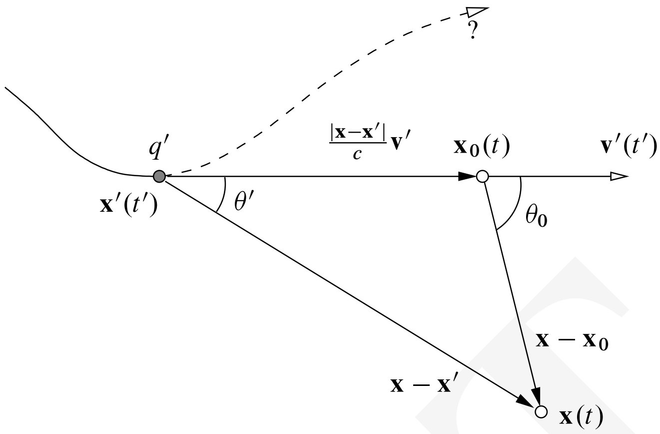


Figure 6.6: Signals that are observed at time $t$ at the field point $\mathbf{x}$ were generated at time $t'$ at the source point $\mathbf{x}'(t')$. After this time $t'$ the particle, which moves with nonuniform velocity, has followed a so far (*i.e.* at $t$) unknown trajectory. Extrapolating tangentially the trajectory from $\mathbf{x}'(t')$, based on the velocity $\mathbf{v}'(t') = \mathrm{d}\mathbf{x}'/\mathrm{d}t'$, defines the virtual simultaneous virtual simultaneous coordinate $\mathbf{x}_0(t)$.

where

$$s = s(t', \mathbf{x}) = \left|\mathbf{x} - \mathbf{x}'(t')\right| - \frac{[\mathbf{x} - \mathbf{x}'(t')] \cdot \mathbf{v}'(t')}{c} \tag{6.84a}$$

$$= \left|\mathbf{x} - \mathbf{x}'(t')\right| \left(1 - \frac{\mathbf{x} - \mathbf{x}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|} \cdot \frac{\mathbf{v}'(t')}{c}\right) \tag{6.84b}$$

$$= [\mathbf{x} - \mathbf{x}'(t')] \cdot \left(\frac{\mathbf{x} - \mathbf{x}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|} - \frac{\mathbf{v}'(t')}{c}\right) \tag{6.84c}$$

is the *retarded relative distance*. The potentials (6.83) are the *Liénard-Wiechert potentials*.[3] In section 7.3.2 on page 165 we shall derive them in a more elegant and general way by using a relativistically covariant formalism.

It should be noted that in the complicated derivation presented above, the observer is in a coordinate system that has an 'absolute' meaning and the velocity $\mathbf{v}'$ is that of the localised charge $q'$, whereas, as we shall see later in the covariant derivation, two reference frames of equal standing are moving relative to each other with $\mathbf{v}'$.

The Liénard-Wiechert potentials are applicable to all problems where a spatially localised charge in arbitrary motion emits electromagnetic radiation, and we shall now study such emission problems. The electric and magnetic fields are calculated from the potentials in the usual way:

$$\mathbf{B}(t, \mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(t, \mathbf{x}) \tag{6.85a}$$

$$\mathbf{E}(t, \mathbf{x}) = -\boldsymbol{\nabla}\Phi(t, \mathbf{x}) - \frac{\partial \mathbf{A}(t, \mathbf{x})}{\partial t} \tag{6.85b}$$

[3] These results were derived independently by ALFRED-MARIE LIÉNARD (1869–1958) in 1898 and EMIL JOHANN WIECHERT (1861–1928) in 1900. When $\mathbf{v}' \parallel (\mathbf{x} - \mathbf{x}')$ and $v \to c$, the potentials become singular. This was first pointed out by ARNOLD JOHANNES WILHELM SOMMERFELD (1868–1951) in 1904.

## 6.5.2 Radiation from an accelerated point charge

Consider a localised charge $q'$ and assume that its trajectory is known experimentally as a function of *retarded time*

$$\mathbf{x}' = \mathbf{x}'(t') \tag{6.86}$$

(in the interest of simplifying our notation, we drop the subscript 'ret' on $t'$ from now on). This means that we know the trajectory of the charge $q'$, *i.e.* $\mathbf{x}'$, for all times up to the time $t'$ at which a signal was emitted in order to precisely arrive at the field point $\mathbf{x}$ at time $t$. Because of the finite speed of propagation of the fields, the trajectory at times later than $t'$ cannot be known at time $t$.

The retarded velocity and acceleration at time $t'$ are given by

$$\mathbf{v}'(t') = \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t'} \tag{6.87a}$$

$$\mathbf{a}'(t') == \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t'} = \frac{\mathrm{d}^2\mathbf{x}'}{\mathrm{d}t'^2} \tag{6.87b}$$

As for the charge coordinate $\mathbf{x}'$ itself, we have in general no knowledge of the velocity and acceleration at times later than $t'$, and definitely not at the time of observation $t$! If we choose the field point $\mathbf{x}$ as fixed, the application of (6.87) to the relative vector $\mathbf{x} - \mathbf{x}'$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t'}[\mathbf{x} - \mathbf{x}'(t')] = -\mathbf{v}'(t') \tag{6.88a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t'^2}[\mathbf{x} - \mathbf{x}'(t')] = -\mathbf{a}'(t') \tag{6.88b}$$

The retarded time $t'$ can, at least in principle, be calculated from the implicit relation

$$t' = t'(t, \mathbf{x}) = t - \frac{|\mathbf{x} - \mathbf{x}'(t')|}{c} \tag{6.89}$$

and we shall see later how this relation can be taken into account in the calculations.

According to formulæ (6.85) on the facing page, the electric and magnetic fields are determined via differentiation of the retarded potentials at the observation time $t$ and at the observation point $\mathbf{x}$. In these formulæ the unprimed $\boldsymbol{\nabla}$, *i.e.* the spatial derivative differentiation operator $\boldsymbol{\nabla} = \hat{\mathbf{x}}_i \partial/\partial x_i$ means that we differentiate with respect to the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ while keeping $t$ fixed, and the unprimed time derivative operator $\partial/\partial t$ means that we differentiate with respect to $t$ while keeping $\mathbf{x}$ fixed. But the Liénard-Wiechert potentials $\Phi$ and $\mathbf{A}$, equations (6.83) on page 123, are expressed in the charge velocity $\mathbf{v}'(t')$ given by equation (6.87a) on page 125 and the retarded relative distance $s(t', \mathbf{x})$ given by equation (6.84) on page 124. This means that the expressions for the potentials $\Phi$ and $\mathbf{A}$ contain terms that are expressed explicitly in $t'$, which in turn is expressed implicitly in $t$ via equation (6.89) on the previous page. Despite this complication, it is possible, as we shall see below, to determine the electric and magnetic fields and associated quantities at the time of observation $t$. To this end, we need to investigate meticulously the action of differentiation on the potentials.

#### 6.5.2.1 The differential operator method

We introduce the convention that a differential operator embraced by parentheses with an index $\mathbf{x}$ or $t$ means that the operator in question is applied at constant $\mathbf{x}$ and $t$, respectively. With this convention, we find that

$$\begin{aligned}\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \left|\mathbf{x} - \mathbf{x}'(t')\right|^p &= p \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{2-p}} \cdot \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} (\mathbf{x} - \mathbf{x}'(t')) \\ &= -p \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'|^{2-p}}\end{aligned} \tag{6.90}$$

Furthermore, by applying the operator $(\partial/\partial t)_{\mathbf{x}}$ to equation (6.89) on the preceding page we find that

$$\begin{aligned}\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} &= 1 - \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'(t'(t, \mathbf{x}))|}{c} \\ &= 1 - \left[\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right] \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} \\ &= 1 + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{c\, |\mathbf{x} - \mathbf{x}'|} \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}}\end{aligned} \tag{6.91}$$

This is an algebraic equation in $(\partial t'/\partial t)_{\mathbf{x}}$ that we can solve to obtain

$$\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'| - (\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')/c} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \tag{6.92}$$

where $s = s(t', \mathbf{x})$ is the retarded relative distance given by equation (6.84) on page 124. Making use of equation (6.92) above, we obtain the following useful operator identity

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \tag{6.93}$$

Likewise, by applying $(\boldsymbol{\nabla})_t$ to equation (6.89) on the previous page we obtain

$$\begin{aligned}(\boldsymbol{\nabla} t')_t &= -\left(\boldsymbol{\nabla} \frac{|\mathbf{x} - \mathbf{x}'(t'(t, \mathbf{x}))|}{c}\right)_t = -\frac{\mathbf{x} - \mathbf{x}'}{c\, |\mathbf{x} - \mathbf{x}'|} \cdot \left(\boldsymbol{\nabla}(\mathbf{x} - \mathbf{x}')\right)_t \\ &= -\frac{\mathbf{x} - \mathbf{x}'}{c\, |\mathbf{x} - \mathbf{x}'|} + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{c\, |\mathbf{x} - \mathbf{x}'|} (\boldsymbol{\nabla} t')_t\end{aligned} \tag{6.94}$$

This is an algebraic equation in $(\boldsymbol{\nabla}t')_t$ with the solution

$$(\boldsymbol{\nabla}t')_t = -\frac{\mathbf{x}-\mathbf{x}'}{cs} \tag{6.95}$$

that gives the following operator relation when $(\boldsymbol{\nabla})_t$ is acting on an arbitrary function of $t'$ and $\mathbf{x}$:

$$(\boldsymbol{\nabla})_t = (\boldsymbol{\nabla}t')_t\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\boldsymbol{\nabla})_{t'} = -\frac{\mathbf{x}-\mathbf{x}'}{cs}\left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\boldsymbol{\nabla})_{t'} \tag{6.96}$$

With the help of the rules (6.96) and (6.93) we are now able to replace $t$ by $t'$ in the operations that we need to perform. We find, for instance, that

$$\begin{aligned}\boldsymbol{\nabla}\Phi \equiv (\boldsymbol{\nabla}\Phi)_t &= \boldsymbol{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\frac{q'}{s}\right)_t \\ &= -\frac{q'}{4\pi\varepsilon_0 s^2}\left[\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} - \frac{\mathbf{v}'(t')}{c} - \frac{\mathbf{x}-\mathbf{x}'}{cs}\left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}}\right]\end{aligned} \tag{6.97a}$$

and, from equation (6.83b) on page 123 and equation (6.93) on the preceding page,

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &\equiv \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\mathbf{x}} = \left[\frac{\partial}{\partial t}\left(\frac{\mu_0}{4\pi}\frac{q'\mathbf{v}'(t')}{s}\right)\right]_{\mathbf{x}} \\ &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3}\left[\left|\mathbf{x}-\mathbf{x}'\right| s\mathbf{a}'(t') - \left|\mathbf{x}-\mathbf{x}'\right|\mathbf{v}'(t')\left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}}\right]\end{aligned} \tag{6.97b}$$

Utilising these relations in the calculation of the $\mathbf{E}$ field from the Liénard-Wiechert potentials, equations (6.83) on page 123, we obtain

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) &= -\boldsymbol{\nabla}\Phi(t,\mathbf{x}) - \frac{\partial}{\partial t}\mathbf{A}(t,\mathbf{x}) \\ &= \frac{q'}{4\pi\varepsilon_0 s^2(t',\mathbf{x})}\left[\frac{[\mathbf{x}-\mathbf{x}'(t')] - |\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}'(t')/c}{|\mathbf{x}-\mathbf{x}'(t')|}\right. \\ &\quad - \frac{[\mathbf{x}-\mathbf{x}'(t')] - |\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}'(t')/c}{cs(t',\mathbf{x})}\left(\frac{\partial s(t',\mathbf{x})}{\partial t'}\right)_{\mathbf{x}} \\ &\quad \left. - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{a}'(t')}{c^2}\right]\end{aligned} \tag{6.98}$$

Starting from expression (6.84a) on page 124 for the retarded relative distance $s(t',\mathbf{x})$, we see that we can evaluate $(\partial s/\partial t')_{\mathbf{x}}$ in the following way

$$\begin{aligned}\left(\frac{\partial s}{\partial t'}\right)_{\mathbf{x}} &= \left[\frac{\partial}{\partial t'}\left(\left|\mathbf{x}-\mathbf{x}'\right| - \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'(t')}{c}\right)\right]_{\mathbf{x}} \\ &= \left(\frac{\partial}{\partial t'}\left|\mathbf{x}-\mathbf{x}'(t')\right|\right)_{\mathbf{x}} \\ &\quad - \frac{1}{c}\left[\left(\frac{\partial[\mathbf{x}-\mathbf{x}'(t')]}{\partial t'}\right)_{\mathbf{x}}\cdot\mathbf{v}'(t') + [\mathbf{x}-\mathbf{x}'(t')]\cdot\left(\frac{\partial \mathbf{v}'(t')}{\partial t'}\right)_{\mathbf{x}}\right] \\ &= -\frac{(\mathbf{x}-\mathbf{x}'(t'))\cdot\mathbf{v}'(t')}{|\mathbf{x}-\mathbf{x}'(t')|} + \frac{v'^2(t')}{c} - \frac{(\mathbf{x}-\mathbf{x}'(t'))\cdot\mathbf{a}'(t')}{c}\end{aligned} \tag{6.99}$$

where equation (6.90) on page 126 and equations (6.87) on page 125, respectively, were used. Hence, the electric field generated by an arbitrarily moving localised charge at $\mathbf{x}'(t')$ is given by the expression

$$\mathbf{E}(t,\mathbf{x}) = \underbrace{\frac{q'}{4\pi\varepsilon_0 s^3(t',\mathbf{x})}\left([\mathbf{x}-\mathbf{x}'(t')] - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}'(t')}{c}\right)\left(1-\frac{v'^2(t')}{c^2}\right)}_{\text{Velocity field (tends to the Coulomb field when } v \to 0)}$$
$$+ \underbrace{\frac{q'}{4\pi\varepsilon_0 s^3(t',\mathbf{x})}\left\{\frac{\mathbf{x}-\mathbf{x}'(t')}{c^2}\times\left[\left([\mathbf{x}-\mathbf{x}'(t')] - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}'(t')}{c}\right)\times\mathbf{a}'(t')\right]\right\}}_{\text{Acceleration (radiation) field}} \tag{6.100}$$

The first part of the field, the *velocity field*, tends to the ordinary Coulomb field when $v' \to 0$ and does not contribute to the radiation. The second part of the field, the *acceleration field*, is radiated into the far zone and is therefore also called the *radiation field*.

From figure 6.6 on page 124 we see that the position the charged particle would have had if at $t'$ all external forces would have been switched off so that the trajectory from then on would have been a straight line in the direction of the tangent at $\mathbf{x}'(t')$ is $\mathbf{x}_0(t)$, the *virtual simultaneous coordinate*. During the arbitrary motion, we interpret $\mathbf{x} - \mathbf{x}_0(t)$ as the coordinate of the field point $\mathbf{x}$ relative to the virtual simultaneous coordinate $\mathbf{x}_0(t)$. Since the time it takes for a signal to propagate (in the assumed free space) from $\mathbf{x}'(t')$ to $\mathbf{x}$ is $|\mathbf{x}-\mathbf{x}'|/c$, this relative vector is given by

$$\mathbf{x}-\mathbf{x}_0(t) = \mathbf{x}-\mathbf{x}'(t') - \frac{|\mathbf{x}-\mathbf{x}'(t')|\,\mathbf{v}'(t')}{c} \tag{6.101}$$

This allows us to rewrite equation (6.100) above in the following way

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 s^3}\Bigg[&\big(\mathbf{x}-\mathbf{x}_0(t)\big)\left(1-\frac{v'^2(t')}{c^2}\right)\\ &+ \big(\mathbf{x}-\mathbf{x}'(t')\big)\times\frac{\big(\mathbf{x}-\mathbf{x}_0(t)\big)\times\mathbf{a}'(t')}{c^2}\Bigg]\end{aligned} \tag{6.102}$$

The magnetic field can be computed in a similar manner:

$$\begin{aligned}\mathbf{B}(t,\mathbf{x}) &= \boldsymbol{\nabla}\times\mathbf{A}(t,\mathbf{x}) \equiv \big(\boldsymbol{\nabla}\times\mathbf{A}\big)_t = \big(\boldsymbol{\nabla}\times\mathbf{A}\big)_{t'} - \frac{\mathbf{x}-\mathbf{x}'}{cs}\times\left(\frac{\partial\mathbf{A}}{\partial t'}\right)_{\mathbf{x}}\\ &= -\frac{q'}{4\pi\varepsilon_0 c^2 s^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{v}' - \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|}\times\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{x}}\end{aligned} \tag{6.103}$$

where we made use of equation (6.83) on page 123 and formula (6.93) on page 126. But, according to (6.97a),

$$\frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|} \times (\boldsymbol{\nabla}\Phi)_t = \frac{q'}{4\pi\varepsilon_0 c^2 s^2} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \times \mathbf{v}' \tag{6.104}$$

so that

$$\begin{aligned}\mathbf{B}(t,\mathbf{x}) &= \frac{\mathbf{x}-\mathbf{x}'}{c\,|\mathbf{x}-\mathbf{x}'|} \times \left[-(\boldsymbol{\nabla}\Phi)_t - \left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{x}}\right] \\ &= \frac{\mathbf{x}-\mathbf{x}'(t')}{c\,|\mathbf{x}-\mathbf{x}'(t')|} \times \mathbf{E}(t,\mathbf{x})\end{aligned} \tag{6.105}$$

The electric far field is obtained from the acceleration field in formula (6.100) on the facing page as

$$\begin{aligned}\mathbf{E}^{\text{far}}(t,\mathbf{x}) &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3} (\mathbf{x}-\mathbf{x}') \times \left[\left((\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}'}{c}\right) \times \mathbf{a}'\right] \\ &= \frac{q'}{4\pi\varepsilon_0 c^2 s^3} [\mathbf{x}-\mathbf{x}'(t')] \times \{[\mathbf{x}-\mathbf{x}_0(t)] \times \mathbf{a}'(t')\}\end{aligned} \tag{6.106}$$

where in the last step we again used formula (6.101) on the preceding page. Combining this formula and formula (6.105), the radiation part of the magnetic field can be written

$$\mathbf{B}^{\text{far}}(t,\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}'(t')}{c\,|\mathbf{x}-\mathbf{x}'(t')|} \times \mathbf{E}^{\text{far}}(t,\mathbf{x}) \tag{6.107}$$

#### 6.5.2.2 The direct method

An alternative to the differential operator transformation technique just described is to try to express all quantities in the potentials directly in $t$ and $\mathbf{x}$. An example of such a quantity is the retarded relative distance $s(t',\mathbf{x})$. According to equation (6.84) on page 124, the square of this retarded relative distance can be written

$$\begin{aligned}s^2(t',\mathbf{x}) = {}&\left|\mathbf{x}-\mathbf{x}'(t')\right|^2 - 2\left|\mathbf{x}-\mathbf{x}'(t')\right| \frac{[\mathbf{x}-\mathbf{x}'(t')]\cdot\mathbf{v}'(t')}{c} \\ &+ \left(\frac{[\mathbf{x}-\mathbf{x}'(t')]\cdot\mathbf{v}'(t')}{c}\right)^2\end{aligned} \tag{6.108}$$

If we use identity (F.37) on page 218 we find that

$$\left(\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c}\right)^2 = \frac{|\mathbf{x}-\mathbf{x}'|^2\,v'^2}{c^2} - \left(\frac{(\mathbf{x}-\mathbf{x}')\times\mathbf{v}'}{c}\right)^2 \tag{6.109}$$

Furthermore, from equation (6.101) on the preceding page, we obtain the identity

$$[\mathbf{x}-\mathbf{x}'(t')] \times \mathbf{v}' = [\mathbf{x}-\mathbf{x}_0(t)] \times \mathbf{v}' \tag{6.110}$$

which, when inserted into equation (6.109) on the preceding page, yields the relation

$$\left(\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c}\right)^2 = \frac{|\mathbf{x}-\mathbf{x}'|^2 v'^2}{c^2} - \left(\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}'}{c}\right)^2 \tag{6.111}$$

Inserting the above into expression (6.108) on the previous page for $s^2$, this expression becomes

$$\begin{aligned}
s^2 &= |\mathbf{x}-\mathbf{x}'|^2 - 2\,|\mathbf{x}-\mathbf{x}'|\,\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c} + \frac{|\mathbf{x}-\mathbf{x}'|^2 v'^2}{c^2} - \left(\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}'}{c}\right)^2 \\
&= \left((\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}'}{c}\right)^2 - \left(\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}'}{c}\right)^2 \\
&= (\mathbf{x}-\mathbf{x}_0)^2 - \left(\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{v}'}{c}\right)^2 \\
&\equiv |\mathbf{x}-\mathbf{x}_0(t)|^2 - \left(\frac{[\mathbf{x}-\mathbf{x}_0(t)]\times\mathbf{v}'(t')}{c}\right)^2
\end{aligned} \tag{6.112}$$

where in the penultimate step we used equation (6.101) on page 128.

What we have just demonstrated is that if the particle velocity at time $t$ can be calculated or projected from its value at the retarded time $t'$, the retarded distance $s$ in the Liénard-Wiechert potentials (6.83) can be expressed in terms of the virtual simultaneous coordinate $\mathbf{x}_0(t)$, *viz.*, the point at which the particle will have arrived at time $t$, *i.e.* when we obtain the first knowledge of its existence at the source point $\mathbf{x}'$ at the retarded time $t'$, and in the field coordinate $\mathbf{x} = \mathbf{x}(t)$, where we make our observations. We have, in other words, shown that all quantities in the definition of $s$, and hence $s$ itself, can, when the motion of the charge is somehow known, be expressed in terms of the time $t$ alone. *I.e.* in this special case we are able to express the retarded relative distance as $s = s(t, \mathbf{x})$ and we do not have to involve the retarded time $t'$ or any transformed differential operators in our calculations.

Taking the square root of both sides of equation (6.112) above, we obtain the following alternative final expressions for the retarded relative distance $s$ in terms of the charge's virtual simultaneous coordinate $\mathbf{x}_0(t)$ and velocity $\mathbf{v}'(t')$:

$$s(t', \mathbf{x}) = \sqrt{|\mathbf{x}-\mathbf{x}_0(t)|^2 - \left(\frac{[\mathbf{x}-\mathbf{x}_0(t)]\times\mathbf{v}'(t')}{c}\right)^2} \tag{6.113a}$$

$$= |\mathbf{x}-\mathbf{x}_0(t)|\sqrt{1 - \frac{v'^2(t')}{c^2}\sin^2\theta_0(t)} \tag{6.113b}$$

$$= \sqrt{|\mathbf{x}-\mathbf{x}_0(t)|^2\left(1 - \frac{v'^2(t')}{c^2}\right) + \left(\frac{[\mathbf{x}-\mathbf{x}_0(t)]\cdot\mathbf{v}'(t')}{c}\right)^2} \tag{6.113c}$$

If we know what velocity the particle will have at time $t$, expression (6.113) on the facing page for $s$ will not be dependent on $t'$.

Using equation (6.113c) on the preceding page and standard vector analytic formulae, we obtain

$$
\begin{aligned}
\boldsymbol{\nabla} s^2 &= \boldsymbol{\nabla}\left[|\mathbf{x}-\mathbf{x}_0|^2\left(1-\frac{v'^2}{c^2}\right)+\left(\frac{(\mathbf{x}-\mathbf{x}_0)\boldsymbol{\cdot}\mathbf{v}'}{c}\right)^2\right] \\
&= 2\left[(\mathbf{x}-\mathbf{x}_0)\left(1-\frac{v'^2}{c^2}\right)+\frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2}\boldsymbol{\cdot}(\mathbf{x}-\mathbf{x}_0)\right] \\
&= 2\left[(\mathbf{x}-\mathbf{x}_0)+\frac{\mathbf{v}'}{c}\times\left(\frac{\mathbf{v}'}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]
\end{aligned}
\tag{6.114}
$$

We shall use this result in example 6.3 on page 148 for a uniform, unaccelerated motion of the charge.

#### 6.5.2.3 Small velocities

If the charge moves at such low speeds that $v'/c \ll 1$, formula (6.84) on page 124 simplifies to

$$
s = \left|\mathbf{x}-\mathbf{x}'\right| - \frac{(\mathbf{x}-\mathbf{x}')\boldsymbol{\cdot}\mathbf{v}'}{c} \approx \left|\mathbf{x}-\mathbf{x}'\right|, \quad v' \ll c \tag{6.115}
$$

and formula (6.101) on page 128

$$
\mathbf{x}-\mathbf{x}_0 = (\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}'}{c} \approx \mathbf{x}-\mathbf{x}', \quad v' \ll c \tag{6.116}
$$

so that the radiation field equation (6.106) on page 129 can be approximated by

$$
\mathbf{E}^{\text{far}}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2\,|\mathbf{x}-\mathbf{x}'|^3}(\mathbf{x}-\mathbf{x}')\times[(\mathbf{x}-\mathbf{x}')\times\mathbf{a}'], \quad v' \ll c \tag{6.117}
$$

from which we obtain, with the use of formula (6.105) on page 129, the magnetic field

$$
\mathbf{B}^{\text{far}}(t,\mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^3\,|\mathbf{x}-\mathbf{x}'|^2}[\mathbf{a}'\times(\mathbf{x}-\mathbf{x}')], \quad v' \ll c \tag{6.118}
$$

It is interesting to note the close correspondence that exists between the non-relativistic fields (6.117) and (6.118) and the electric dipole field equations (6.46) on page 115 if we introduce the electric dipole moment for a localised charge [*cf.* formula (6.42) on page 113]

$$
\mathbf{d} = q'\mathbf{x}'(t') \tag{6.119}
$$

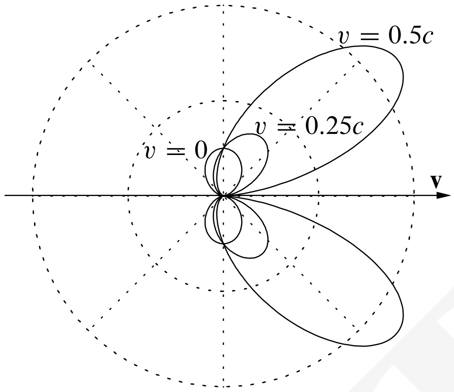


Figure 6.7: Polar diagram of the energy loss angular distribution factor $\sin^2\theta/(1 - v\cos\theta/c)^5$ during bremsstrahlung for particle speeds $v' = 0$, $v' = 0.25c$, and $v' = 0.5c$.

and at the same time make the transitions

$$q'\mathbf{a}' = \ddot{\mathbf{d}} \mapsto -\omega^2\mathbf{d}_\omega \tag{6.120a}$$

$$\mathbf{x} - \mathbf{x}' = \mathbf{x} - \mathbf{x}_0 \tag{6.120b}$$

The energy flux in the far zone is described by the Poynting vector as a function of $\mathbf{E}^{\text{far}}$ and $\mathbf{B}^{\text{far}}$. We use the close correspondence with the dipole case to find that it becomes

$$\mathbf{S}^{\text{far}} = \frac{\mu_0 q'^2 (\mathbf{a}')^2}{16\pi^2 c\,|\mathbf{x}-\mathbf{x}'|^2}\sin^2\theta\,\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \tag{6.121}$$

where $\theta$ is the angle between $\mathbf{a}'$ and $\mathbf{x}-\mathbf{x}_0$. The total radiated power (integrated over a closed spherical surface) becomes

$$P = \frac{\mu_0 q'^2 (\mathbf{a}')^2}{6\pi c} = \frac{q'^2 a'^2}{6\pi\varepsilon_0 c^3} \tag{6.122}$$

which is the *Larmor formula for radiated power* from an accelerated charge. Note that here we are treating a charge with $v' \ll c$ but otherwise *totally unspecified motion* while we compare with formulæ derived for a *stationary oscillating dipole*. The electric and magnetic fields, equation (6.117) on the previous page and equation (6.118) on the preceding page, respectively, and the expressions for the Poynting flux and power derived from them, are here *instantaneous* values, dependent on the instantaneous position of the charge at $\mathbf{x}'(t')$. The angular distribution is that which is 'frozen' to the point from which the energy is radiated.

### 6.5.3 Bremsstrahlung

An important special case of radiation is when the velocity $\mathbf{v}'$ and the acceleration $\mathbf{a}'$ are collinear (parallel or anti-parallel) so that $\mathbf{v}' \times \mathbf{a}' = \mathbf{0}$. This condition (for an arbitrary magnitude of $\mathbf{v}'$) inserted into expression (6.106) on page 129 for the radiation field, yields

$$\mathbf{E}^{\text{far}}(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2 s^3}(\mathbf{x} - \mathbf{x}') \times [(\mathbf{x} - \mathbf{x}') \times \mathbf{a}'], \quad \mathbf{v}' \parallel \mathbf{a}' \tag{6.123}$$

from which we obtain, with the use of formula (6.105) on page 129, the magnetic field

$$\mathbf{B}^{\text{far}}(t, \mathbf{x}) = \frac{q'\,|\mathbf{x} - \mathbf{x}'|}{4\pi\varepsilon_0 c^3 s^3}[\mathbf{a}' \times (\mathbf{x} - \mathbf{x}')], \quad \mathbf{v}' \parallel \mathbf{a}' \tag{6.124}$$

The difference between this case and the previous case of $v' \ll c$ is that the approximate expression (6.115) on page 131 for $s$ is no longer valid; instead we must use the correct expression (6.84) on page 124. The angular distribution of the energy flux (Poynting vector) far away from the source therefore becomes

$$\mathbf{S}^{\text{far}} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c\,|\mathbf{x} - \mathbf{x}'|^2} \frac{\sin^2\theta}{\left(1 - \frac{v'}{c}\cos\theta\right)^6} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tag{6.125}$$

It is interesting to note that the magnitudes of the electric and magnetic fields are the same whether $\mathbf{v}'$ and $\mathbf{a}'$ are parallel or anti-parallel.

We must be careful when we compute the energy ($\mathbf{S}$ integrated over time). The Poynting vector is related to the time $t$ when it is measured and to a *fixed* surface in space. The radiated power into a solid angle element $\mathrm{d}\Omega$, measured relative to the particle's retarded position, is given by the formula

$$\frac{\mathrm{d}U^{\text{rad}}(\theta)}{\mathrm{d}t}\,\mathrm{d}\Omega = \mathbf{S}\cdot(\mathbf{x} - \mathbf{x}')\left|\mathbf{x} - \mathbf{x}'\right|\,\mathrm{d}\Omega = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1 - \frac{v'}{c}\cos\theta\right)^6}\,\mathrm{d}\Omega \tag{6.126}$$

On the other hand, the radiation loss due to radiation from the charge at retarded time $t'$:

$$\frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t'}\,\mathrm{d}\Omega = \frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t}\left(\frac{\partial t}{\partial t'}\right)_{\mathbf{x}}\mathrm{d}\Omega \tag{6.127}$$

Using formula (6.92) on page 126, we obtain

$$\frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t'}\,\mathrm{d}\Omega = \frac{\mathrm{d}U^{\text{rad}}}{\mathrm{d}t}\frac{s}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\Omega = \mathbf{S}\cdot(\mathbf{x} - \mathbf{x}')s\,\mathrm{d}\Omega \tag{6.128}$$

Inserting equation (6.125) above for $\mathbf{S}$ into (6.128), we obtain the explicit expression for the energy loss due to radiation evaluated at the retarded time

$$\frac{\mathrm{d}U^{\text{rad}}(\theta)}{\mathrm{d}t'}\,\mathrm{d}\Omega = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1 - \frac{v'}{c}\cos\theta\right)^5}\,\mathrm{d}\Omega \tag{6.129}$$

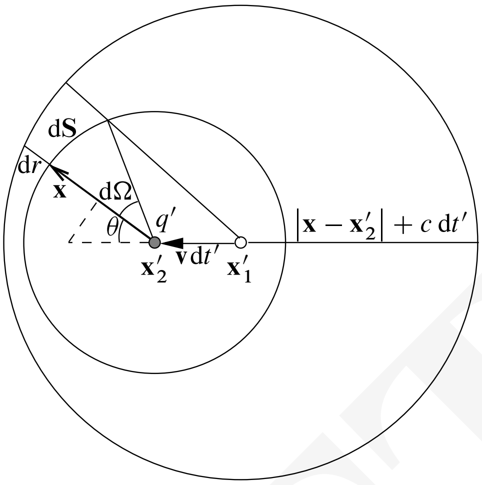


Figure 6.8: Location of radiation between two spheres as the charge moves with velocity $\mathbf{v}'$ from $\mathbf{x}'_1$ to $\mathbf{x}'_2$ during the time interval $(t', t'+\mathrm{d}t')$. The observation point (field point) is at the fixed location $\mathbf{x}$.

The angular factors of this expression, for three different particle speeds, are plotted in figure 6.7 on page 132.

Comparing expression (6.126) on the preceding page with expression (6.129) on the previous page, we see that they differ by a factor $1-v'\cos\theta/c$ that comes from the extra factor $s/\left|\mathbf{x}-\mathbf{x}'\right|$ introduced in (6.128). Let us explain this in geometrical terms.

During the interval $(t', t'+\mathrm{d}t')$ and within the solid angle element $\mathrm{d}\Omega$ the particle radiates an energy $[\mathrm{d}U^{\mathrm{rad}}(\theta)/\mathrm{d}t']\,\mathrm{d}t'\mathrm{d}\Omega$. As shown in figure 6.8 this energy is at time $t$ located between two spheres, one outer with its origin at $\mathbf{x}'_1(t')$ and radius $c(t-t')$, and one inner with its origin at $\mathbf{x}'_2(t'+\mathrm{d}t') = \mathbf{x}'_1(t')+\mathbf{v}'\,\mathrm{d}t'$ and radius $c[t-(t'+\mathrm{d}t')] = c(t-t'-\mathrm{d}t')$.

From Figure 6.8 we see that the volume element subtending the solid angle element

$$\mathrm{d}\Omega = \frac{\mathrm{d}^2x}{\left|\mathbf{x}-\mathbf{x}'_2\right|^2} \tag{6.130}$$

is

$$\mathrm{d}^3x = \mathrm{d}^2x\,\mathrm{d}r = \left|\mathbf{x}-\mathbf{x}'_2\right|^2\,\mathrm{d}\Omega\,\mathrm{d}r \tag{6.131}$$

Here, $\mathrm{d}r$ denotes the differential distance between the two spheres and can be evaluated in the following way

$$
\begin{aligned}
\mathrm{d}r &= \left|\mathbf{x}-\mathbf{x}'_2\right| + c\,\mathrm{d}t' - \left|\mathbf{x}-\mathbf{x}'_2\right| - \underbrace{\frac{\mathbf{x}-\mathbf{x}'_2}{\left|\mathbf{x}-\mathbf{x}'_2\right|}\cdot\mathbf{v}'}_{v'\cos\theta}\,\mathrm{d}t' \\
&= \left(c - \frac{\mathbf{x}-\mathbf{x}'_2}{\left|\mathbf{x}-\mathbf{x}'_2\right|}\cdot\mathbf{v}'\right)\mathrm{d}t' = \frac{cs}{\left|\mathbf{x}-\mathbf{x}'_2\right|}\,\mathrm{d}t'
\end{aligned}
\tag{6.132}
$$

where formula (6.84) on page 124 was used in the last step. Hence, the volume element under consideration is

$$
\mathrm{d}^3x = \mathrm{d}^2x\,\mathrm{d}r = \frac{s}{\left|\mathbf{x}-\mathbf{x}'_2\right|}\,\mathrm{d}^2x\,c\mathrm{d}t'
\tag{6.133}
$$

We see that the energy that is radiated per unit solid angle during the time interval $(t', t' + \mathrm{d}t')$ is located in a volume element whose size is $\theta$ dependent. This explains the difference between expression (6.126) on page 133 and expression (6.129) on page 133.

Let the radiated energy, integrated over $\Omega$, be denoted $\tilde{U}^{\mathrm{rad}}$. After tedious, but relatively straightforward integration of formula (6.129) on page 133, one obtains

$$
\frac{\mathrm{d}\tilde{U}^{\mathrm{rad}}}{\mathrm{d}t'} = \frac{\mu_0 q'^2 a'^2}{6\pi c}\,\frac{1}{\left(1-\frac{v'^2}{c^2}\right)^3} = \frac{2}{3}\,\frac{q'^2 a'^2}{4\pi\varepsilon_0 c^3}\left(1-\frac{v'^2}{c^2}\right)^{-3}
\tag{6.134}
$$

If we know $\mathbf{v}'(t')$, we can integrate this expression over $t'$ and obtain the total energy radiated during the acceleration or deceleration of the particle. This way we obtain a classical picture of *bremsstrahlung* (*braking radiation*, *free-free radiation*). Often, an atomistic treatment is required for obtaining an acceptable result.

### 6.5.4 Cyclotron and synchrotron radiation (magnetic bremsstrahlung)

Formula (6.105) and formula (6.106) on page 129 for the magnetic field and the radiation part of the electric field are general, valid for any kind of motion of the localised charge. A very important special case is circular motion, *i.e.* the case $\mathbf{v}' \perp \mathbf{a}'$.

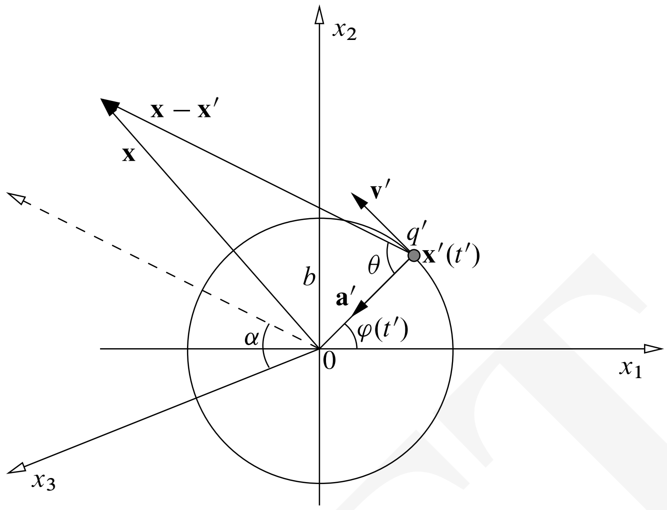


Figure 6.9: Coordinate system for the radiation from a charged particle at $\mathbf{x}'(t')$ in circular motion with velocity $\mathbf{v}'(t')$ along the tangent and constant acceleration $\mathbf{a}'(t')$ toward the origin. The $x_1x_2$ axes are chosen so that the relative field point vector $\mathbf{x}-\mathbf{x}'$ makes an angle $\alpha$ with the $x_3$ axis, which is normal to the plane of the orbital motion. The radius of the orbit is $b$.

With the charged particle orbiting in the $x_1x_2$ plane as in figure 6.9, an orbit radius $b$, and an angular frequency $\omega_0$, we obtain

$$\varphi(t') = \omega_0 t' \tag{6.135a}$$
$$\mathbf{x}'(t') = b[\hat{\mathbf{x}}_1 \cos\varphi(t') + \hat{\mathbf{x}}_2 \sin\varphi(t')] \tag{6.135b}$$
$$\mathbf{v}'(t') = \dot{\mathbf{x}}'(t') = b\omega_0[-\hat{\mathbf{x}}_1 \sin\varphi(t') + \hat{\mathbf{x}}_2 \cos\varphi(t')] \tag{6.135c}$$
$$v' = \left|\mathbf{v}'\right| = b\omega_0 \tag{6.135d}$$
$$\mathbf{a}' = \dot{\mathbf{v}}'(t') = \ddot{\mathbf{x}}'(t') = -b\omega_0^2[\hat{\mathbf{x}}_1 \cos\varphi(t') + \hat{\mathbf{x}}_2 \sin\varphi(t')] \tag{6.135e}$$
$$v' = \left|\mathbf{a}'\right| = b\omega_0^2 \tag{6.135f}$$

Because of the rotational symmetry we can, without loss of generality, rotate our coordinate system around the $x_3$ axis so the relative vector $\mathbf{x}-\mathbf{x}'$ from the source point to an arbitrary field point always lies in the $x_2x_3$ plane, *i.e.*

$$\mathbf{x}-\mathbf{x}' = \left|\mathbf{x}-\mathbf{x}'\right|(\hat{\mathbf{x}}_2 \sin\alpha + \hat{\mathbf{x}}_3 \cos\alpha) \tag{6.136}$$

where $\alpha$ is the angle between $\mathbf{x}-\mathbf{x}'$ and the normal to the plane of the particle orbit (see Figure 6.9). From the above expressions we obtain

$$(\mathbf{x}-\mathbf{x}') \cdot \mathbf{v}' = \left|\mathbf{x}-\mathbf{x}'\right| v' \sin\alpha \cos\varphi \tag{6.137a}$$
$$(\mathbf{x}-\mathbf{x}') \cdot \mathbf{a}' = -\left|\mathbf{x}-\mathbf{x}'\right| a' \sin\alpha \sin\varphi = \left|\mathbf{x}-\mathbf{x}'\right| a' \cos\theta \tag{6.137b}$$

where in the last step we simply used the definition of a scalar product and the fact that the angle between $\mathbf{a}'$ and $\mathbf{x}-\mathbf{x}'$ is $\theta$.

The energy flux is given by the Poynting vector, which, with the help of formula (6.105) on page 129, can be written

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{c\mu_0}|\mathbf{E}|^2 \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tag{6.138}$$

Inserting this into equation (6.128) on page 133, we obtain

$$\frac{dU^{\mathrm{rad}}(\alpha, \varphi)}{dt'} = \frac{|\mathbf{x} - \mathbf{x}'|\, s}{c\mu_0}|\mathbf{E}|^2 \tag{6.139}$$

where the retarded distance $s$ is given by expression (6.84) on page 124. With the radiation part of the electric field, expression (6.106) on page 129, inserted, and using (6.137a) and (6.137b) on the preceding page, one finds, after some algebra, that

$$\frac{dU^{\mathrm{rad}}(\alpha, \varphi)}{dt'} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c} \frac{\left(1 - \frac{v'}{c}\sin\alpha\cos\varphi\right)^2 - \left(1 - \frac{v'^2}{c^2}\right)\sin^2\alpha\sin^2\varphi}{\left(1 - \frac{v'}{c}\sin\alpha\cos\varphi\right)^5} \tag{6.140}$$

The angles $\theta$ and $\varphi$ vary in time during the rotation, so that $\theta$ refers to a *moving* coordinate system. But we can parametrise the solid angle $d\Omega$ in the angle $\varphi$ and the angle $\alpha$ so that $d\Omega = \sin\alpha\, d\alpha\, d\varphi$. Integration of equation (6.140) above over this $d\Omega$ gives, after some cumbersome algebra, the angular integrated expression

$$\frac{d\tilde{U}^{\mathrm{rad}}}{dt'} = \frac{\mu_0 q'^2 a'^2}{6\pi c} \frac{1}{\left(1 - \frac{v'^2}{c^2}\right)^2} \tag{6.141}$$

In equation (6.140) above, two limits are particularly interesting:

1. $v'/c \ll 1$ which corresponds to *cyclotron radiation*.
2. $v'/c \lesssim 1$ which corresponds to *synchrotron radiation*.

#### 6.5.4.1 Cyclotron radiation

For a non-relativistic speed $v' \ll c$, equation (6.140) above reduces to

$$\frac{dU^{\mathrm{rad}}(\alpha, \varphi)}{dt'} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c}(1 - \sin^2\alpha\sin^2\varphi) \tag{6.142}$$

But according to equation (6.137b) on the preceding page

$$\sin^2\alpha\sin^2\varphi = \cos^2\theta \tag{6.143}$$

where $\theta$ is defined in figure 6.9 on the facing page. This means that we can write

$$\frac{dU^{\mathrm{rad}}(\theta)}{dt'} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c}(1 - \cos^2\theta) = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c}\sin^2\theta \tag{6.144}$$

Consequently, a fixed observer near the orbit plane ($\alpha \approx \pi/2$) will observe cyclotron radiation twice per revolution in the form of two equally broad pulses of radiation with alternating polarisation.

### 6.5.4.2 Synchrotron radiation

When the particle is relativistic, $v' \lesssim c$, the denominator in equation (6.140) on the previous page becomes very small if $\sin\alpha\cos\varphi \approx 1$, which defines the forward direction of the particle motion ($\alpha \approx \pi/2,\ \varphi \approx 0$). The equation (6.140) on the preceding page becomes

$$\frac{\mathrm{d}U^{\text{rad}}(\pi/2,0)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c}\frac{1}{\left(1-\frac{v'}{c}\right)^3} \tag{6.145}$$

which means that an observer near the orbit plane sees a very strong pulse followed, half an orbit period later, by a much weaker pulse.

The two cases represented by equation (6.144) on the previous page and equation (6.145) above are very important results since they can be used to determine the characteristics of the particle motion both in particle accelerators and in astrophysical objects where a direct measurement of particle velocities are impossible.

In the orbit plane ($\alpha = \pi/2$), equation (6.140) on the previous page gives

$$\frac{\mathrm{d}U^{\text{rad}}(\pi/2,\varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2 a'^2}{16\pi^2 c}\frac{\left(1-\frac{v'}{c}\cos\varphi\right)^2 - \left(1-\frac{v'^2}{c^2}\right)\sin^2\varphi}{\left(1-\frac{v'}{c}\cos\varphi\right)^5} \tag{6.146}$$

This vanishes for angles $\varphi_0$ such that

$$\cos\varphi_0 = \frac{v'}{c} \tag{6.147a}$$

$$\sin\varphi_0 = \sqrt{1-\frac{v'^2}{c^2}} \tag{6.147b}$$

Hence, the angle $\varphi_0$ is a measure of the *synchrotron radiation lobe width* $\Delta\theta$; see figure 6.10 on the facing page. For ultra-relativistic particles, defined by

$$\gamma = \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}} \gg 1, \quad \sqrt{1-\frac{v'^2}{c^2}} \ll 1, \tag{6.148}$$

one can approximate

$$\varphi_0 \approx \sin\varphi_0 = \sqrt{1-\frac{v'^2}{c^2}} = \frac{1}{\gamma} \tag{6.149}$$

We see that synchrotron radiation from ultra-relativistic charges is characterized by a radiation lobe width which is approximately

$$\Delta\theta \approx \frac{1}{\gamma} \tag{6.150}$$

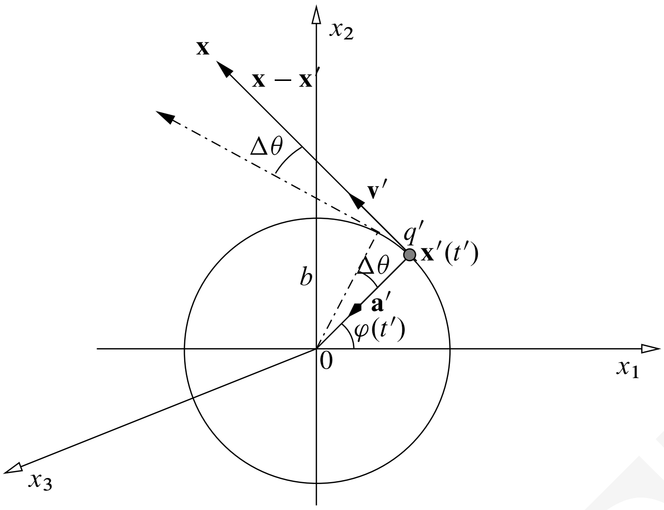


Figure 6.10: If the observation point $\mathbf{x}$ is in the plane of the particle orbit, *i.e.* if $\alpha = \pi/2$, the synchrotron radiation lobe width is given by $\Delta\theta$.

This angular interval is swept by the charge during the time interval

$$\Delta t' = \frac{\Delta\theta}{\omega_0} \tag{6.151}$$

during which the particle moves a length interval

$$\Delta l' = v'\Delta t' = v'\frac{\Delta\theta}{\omega_0} \tag{6.152}$$

in the direction toward the observer. The observer therefore measures a compressed pulse width of length

$$\begin{aligned}\Delta t &= \Delta t' - \frac{\Delta l'}{c} = \Delta t' - \frac{v'\Delta t'}{c} = \left(1-\frac{v'}{c}\right)\Delta t' \\ &= \left(1-\frac{v'}{c}\right)\frac{\Delta\theta}{\omega_0} \approx \left(1-\frac{v'}{c}\right)\frac{1}{\gamma\omega_0} = \frac{\left(1-\frac{v'}{c}\right)\left(1+\frac{v'}{c}\right)}{\underbrace{1+\frac{v'}{c}}_{\approx 2}}\frac{1}{\gamma\omega_0} \\ &\approx \underbrace{\left(1-\frac{v'^2}{c^2}\right)}_{1/\gamma^2}\frac{1}{2\gamma\omega_0} = \frac{1}{2\gamma^3}\frac{1}{\omega_0}\end{aligned} \tag{6.153}$$

Typically, the spectral width of a pulse of length $\Delta t$ is $\Delta\omega \lesssim 1/\Delta t$. In the ultrarelativistic synchrotron case one can therefore expect frequency components up to

$$\omega_{\max} \approx \frac{1}{\Delta t} = 2\gamma^3\omega_0 \tag{6.154}$$

A spectral analysis of the radiation pulse will therefore exhibit a (broadened) line spectrum of Fourier components $n\omega_0$ from $n = 1$ up to $n \approx 2\gamma^3$.

When many charged particles, $N$ say, contribute to the radiation, we can have three different situations depending on the relative phases of the radiation fields from the individual particles:

1. All $N$ radiating particles are spatially much closer to each other than a typical wavelength. Then the relative phase differences of the individual electric and magnetic fields radiated are negligible and the total radiated fields from all individual particles will add up to become $N$ times that from one particle. This means that the power radiated from the $N$ particles will be $N^2$ higher than for a single charged particle. This is called *coherent radiation*.

2. The charged particles are perfectly evenly distributed in the orbit. In this case the phases of the radiation fields cause a complete cancellation of the fields themselves. No radiation escapes.

3. The charged particles are somewhat unevenly distributed in the orbit. This happens for an open ring current, carried initially by evenly distributed charged particles, which is subject to thermal fluctuations. From statistical mechanics we know that this happens for all open systems and that the particle densities exhibit fluctuations of order $\sqrt{N}$. This means that out of the $N$ particles, $\sqrt{N}$ will exhibit deviation from perfect randomness — and thereby perfect radiation field cancellation — and give rise to net radiation fields which are proportional to $\sqrt{N}$. As a result, the radiated power will be proportional to $N$, and we speak about *incoherent radiation*. Examples of this can be found both in earthly laboratories and under cosmic conditions.

### 6.5.4.3 Radiation in the general case

We recall that the general expression for the radiation $\mathbf{E}$ field from a moving charge concentration is given by expression (6.106) on page 129. This expression in equation (6.139) on page 137 yields the general formula

$$\frac{\mathrm{d}U^{\mathrm{rad}}(\theta,\varphi)}{\mathrm{d}t'} = \frac{\mu_0 q'^2\,|\mathbf{x}-\mathbf{x}'|}{16\pi^2 c s^5}(\mathbf{x}-\mathbf{x}') \times \left[\left((\mathbf{x}-\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|\,\mathbf{v}'}{c}\right) \times \mathbf{a}'\right]^2 \tag{6.155}$$

Integration over the solid angle $\Omega$ gives the totally radiated power as

$$\frac{\mathrm{d}\tilde{U}^{\mathrm{rad}}}{\mathrm{d}t'} = \frac{\mu_0 q'^2 a'^2}{6\pi c}\,\frac{1 - \frac{v'^2}{c^2}\sin^2\psi}{\left(1 - \frac{v'^2}{c^2}\right)^3} \tag{6.156}$$

where $\psi$ is the angle between $\mathbf{v}'$ and $\mathbf{a}'$.

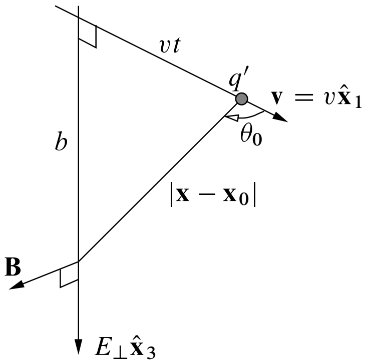


Figure 6.11: The perpendicular electric field of a charge $q'$ moving with velocity $\mathbf{v}' = v'\hat{\mathbf{x}}$ is $E_\perp \hat{\mathbf{z}}$.

If $\mathbf{v}'$ is collinear with $\mathbf{a}'$, so that $\sin\psi = 0$, we get *bremsstrahlung*. For $\mathbf{v}' \perp \mathbf{a}'$, $\sin\psi = 1$, which corresponds to *cyclotron radiation* or *synchrotron radiation*.

### 6.5.4.4 Virtual photons

Let us consider a charge $q'$ moving with constant, high velocity $\mathbf{v}'(t')$ along the $x_1$ axis. According to formula (6.198) on page 149 and figure 6.11, the perpendicular component along the $\mathbf{x}_3$ axis of the electric field from this moving charge is

$$E_\perp = E_3 = \frac{q'}{4\pi\varepsilon_0 s^3}\left(1 - \frac{v'^2}{c^2}\right)(\mathbf{x} - \mathbf{x}_0)\cdot\hat{\mathbf{x}}_3 \tag{6.157}$$

Utilising expression (6.113) on page 130 and simple geometrical relations, we can rewrite this as

$$E_\perp = \frac{q'}{4\pi\varepsilon_0} \frac{b}{\gamma^2\left[(v't')^2 + b^2/\gamma^2\right]^{3/2}} \tag{6.158}$$

This represents a contracted Coulomb field, approaching the field of a plane wave. The passage of this field 'pulse' corresponds to a frequency distribution of the field energy. Fourier transforming, we obtain

$$E_{\omega,\perp} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, E_\perp(t)\, \mathrm{e}^{\mathrm{i}\omega t} = \frac{q'}{4\pi^2\varepsilon_0 b v'}\left[\left(\frac{b\omega}{v'\gamma}\right) K_1\left(\frac{b\omega}{v'\gamma}\right)\right] \tag{6.159}$$

Here, $K_1$ is the *Kelvin function* (Bessel function of the second kind with imaginary argument) which behaves in such a way for small and large arguments that

$$E_{\omega,\perp} \sim \frac{q'}{4\pi^2\varepsilon_0 b v'}, \quad b\omega \ll v'\gamma \Leftrightarrow \frac{b}{v'\gamma}\omega \ll 1 \tag{6.160a}$$

$$E_{\omega,\perp} \sim 0, \quad b\omega \gg v'\gamma \Leftrightarrow \frac{b}{v'\gamma}\omega \gg 1 \tag{6.160b}$$

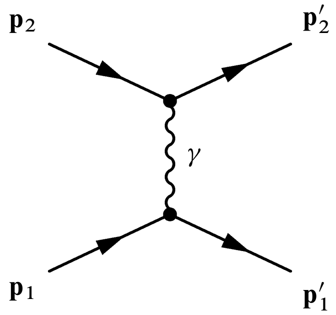


Figure 6.12: Diagrammatic representation of the semi-classical electron-electron interaction (Møller scattering).

showing that the 'pulse' length is of the order $b/(v'\gamma)$.

Due to the equipartitioning of the field energy into the electric and magnetic fields, the total field energy can be written

$$\tilde{U} = \varepsilon_0 \int_V \mathrm{d}^3x\, E_\perp^2 = \varepsilon_0 \int_{b_{\min}}^{b_{\max}} \mathrm{d}b\, 2\pi b \int_{-\infty}^{\infty} \mathrm{d}t\, v' E_\perp^2 \tag{6.161}$$

where the volume integration is over the plane perpendicular to $\mathbf{v}'$. With the use of *Parseval's identity* for Fourier transforms, formula (6.9) on page 106, we can rewrite this as

$$\begin{aligned}\tilde{U} &= \int_0^{\infty} \mathrm{d}\omega\, \tilde{U}_\omega = 4\pi\varepsilon_0 v' \int_{b_{\min}}^{b_{\max}} \mathrm{d}b\, 2\pi b \int_0^{\infty} \mathrm{d}\omega\, E_{\omega,\perp}^2 \\ &\approx \frac{q'^2}{2\pi^2\varepsilon_0 v'} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{b_{\min}}^{v'\gamma/\omega} \frac{\mathrm{d}b}{b}\end{aligned} \tag{6.162}$$

from which we conclude that

$$\tilde{U}_\omega \approx \frac{q'^2}{2\pi^2\varepsilon_0 v'} \ln\left(\frac{v'\gamma}{b_{\min}\omega}\right) \tag{6.163}$$

where an explicit value of $b_{\min}$ can be calculated in quantum theory only.

It is intriguing to quantise the energy into photons. We then find that

$$N_\omega\, \mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{c\gamma}{b_{\min}\omega}\right) \frac{\mathrm{d}\omega}{\omega} \tag{6.164}$$

where $\alpha = \mathrm{e}^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$ is the *fine structure constant*.

Let us consider the interaction of two (classical) electrons, 1 and 2. The result of this interaction is that they change their linear momenta from $\mathbf{p}_1$ to $\mathbf{p}'_1$ and $\mathbf{p}_2$ to $\mathbf{p}'_2$, respectively. Heisenberg's uncertainty principle gives $b_{\min} \sim \hbar/\left|\mathbf{p}_1 - \mathbf{p}'_1\right|$ so that the number of photons exchanged in the process is of the order

$$N_\omega\, \mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{c\gamma}{\hbar\omega}\left|\mathbf{p}_1 - \mathbf{p}'_1\right|\right) \frac{\mathrm{d}\omega}{\omega} \tag{6.165}$$

Since this change in momentum corresponds to a change in energy $\hbar\omega = E_1 - E'_1$ and $E_1 = m_0\gamma c^2$, we see that

$$N_\omega \, \mathrm{d}\omega \approx \frac{2\alpha}{\pi} \ln\left(\frac{E_1}{m_0c^2} \frac{\left|c\mathbf{p}_1 - c\mathbf{p}'_1\right|}{E_1 - E'_1}\right) \frac{\mathrm{d}\omega}{\omega} \tag{6.166}$$

a formula which gives a reasonable semi-classical account of a photon-induced electron-electron interaction process. In quantum theory, including only the lowest order contributions, this process is known as *Møller scattering*. A diagrammatic representation of (a semi-classical approximation of) this process is given in figure 6.12 on the preceding page.

## 6.6 Examples

▷Linear and angular momenta radiated from an electric dipole in vacuum —— EXAMPLE 6.1

The Fourier amplitudes of the fields generated by an electric dipole, $\mathbf{d}_\omega$, oscillating at the angular frequency $\omega$, are given by formulæ (6.45) on page 114. Inverse Fourier transforming to the time domain, and using a spherical coordinate system $(r, \theta, \varphi)$, the physically observable fields are readily found to be

$$\mathbf{B}(t,\mathbf{x}) = \frac{\omega\mu_0}{4\pi} d_\omega \sin\theta \left(\frac{1}{r^2}\sin(kr-\omega t) - \frac{k}{r}\cos(kr-\omega t)\right)\hat{\boldsymbol{\varphi}} \tag{6.167a}$$

$$\begin{aligned}\mathbf{E}(t,\mathbf{x}) = {} & \frac{1}{4\pi\varepsilon_0} d_\omega \sin\theta \left(\frac{1}{r^3}\cos(kr-\omega t) + \frac{k}{r^2}\sin(kr-\omega t) - \frac{k^2}{r}\cos(kr-\omega t)\right)\hat{\boldsymbol{\theta}} \\ & + \frac{1}{2\pi\varepsilon_0} d_\omega \cos\theta \left(\frac{1}{r^3}\cos(kr-\omega t) + \frac{k}{r^2}\sin(kr-\omega t)\right)\hat{\mathbf{r}}\end{aligned} \tag{6.167b}$$

Applying formula (4.45) on page 62 for the *electromagnetic linear momentum density* to the fields from a pure electric dipole, equations (6.167) above, one obtains

$$\begin{aligned}\mathbf{g}^{\mathrm{field}}(t,\mathbf{x}) = \varepsilon_0\mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) = & -\frac{\omega\mu_0}{8\pi^2} d_\omega^2 \sin\theta\cos\theta \bigg\{\frac{1}{r^5}\sin(kr-\omega t)\cos(kr-\omega t) \\ & + \frac{k}{r^4}\left[\sin^2(kr-\omega t) - \cos^2(kr-\omega t)\right] - \frac{k^2}{r^3}\sin(kr-\omega t)\cos(kr-\omega t)\bigg\}\hat{\boldsymbol{\theta}} \\ & + \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \sin^2\theta \bigg\{\frac{1}{r^5}\sin(kr-\omega t)\cos(kr-\omega t) \\ & + \frac{k}{r^4}\left[\sin^2(kr-\omega t) - \cos^2(kr-\omega t)\right] - 2\frac{k^2}{r^3}\sin(kr-\omega t)\cos(kr-\omega t) \\ & + \frac{k^3}{r^2}\cos^2(kr-\omega t)\bigg\}\hat{\mathbf{r}}\end{aligned} \tag{6.168}$$

Using the well-known double-angle trigonometric relations, this can be put in the form

$$
\begin{aligned}
\mathbf{g}^{\text{field}}(t,\mathbf{x}) = &-\frac{\omega\mu_0}{16\pi^2}d_\omega^2\sin\theta\cos\theta\bigg[\frac{1}{r^5}\sin[2(kr-\omega t)] - 2\frac{k}{r^4}\cos[2(kr-\omega t)] \\
&\qquad - \frac{k^2}{r^3}\sin[2(kr-\omega t)]\bigg]\hat{\boldsymbol{\theta}} \\
&+ \frac{\omega\mu_0}{32\pi^2}d_\omega^2\sin^2\theta\bigg[\frac{1}{r^5}\sin[2(kr-\omega t)] - 2\frac{k}{r^4}\cos[2(kr-\omega t)] \\
&\qquad - 2\frac{k^2}{r^3}\sin[2(kr-\omega t)] + \frac{k^3}{r^2}\big(1+\cos[2(kr-\omega t)]\big)\bigg]\hat{\mathbf{r}}
\end{aligned}
\tag{6.169}
$$

We see that the linear momentum density $\mathbf{g}^{\text{field}}$, and hence the Poynting vector $\mathbf{S} = \mathbf{g}^{\text{field}}c^2$ [recall identity (4.45) on page 62], has a perpendicular component (along $\hat{\boldsymbol{\theta}}$) and therefore performs, in general, a spiralling motion.

Consequently, at finite distances from the source the linear momentum has a perpendicular component. It is only at infinity that it is strictly radial (along $\hat{\mathbf{r}}$).

Applying formula (4.75) on page 67 for the *electromagnetic angular momentum density* around the momentum point $\mathbf{x}_0$, *i.e.*

$$
\mathbf{h}^{\text{field}}(t,\mathbf{x},\mathbf{x}_0) = (\mathbf{x}-\mathbf{x}_0)\times\mathbf{g}^{\text{field}}
\tag{6.170}
$$

and using equation (6.169) above, we find that for a pure electric dipole

$$
\begin{aligned}
\mathbf{h}^{\text{field}} = &-\frac{\omega\mu_0}{8\pi^2}d_\omega^2\sin\theta\cos\theta\bigg[\frac{1}{r^4}\sin(kr-\omega t)\cos(kr-\omega t) \\
&+ \frac{k}{r^3}\big[\sin^2(kr-\omega t) - \cos^2(kr-\omega t)\big] - \frac{k^2}{r^2}\sin(kr-\omega t)\cos(kr-\omega t)\bigg]\hat{\boldsymbol{\varphi}}
\end{aligned}
\tag{6.171}
$$

or

$$
\begin{aligned}
\mathbf{h}^{\text{field}} = &-\frac{\omega\mu_0}{16\pi^2}d_\omega^2\sin\theta\cos\theta\bigg[\frac{1}{r^4}\sin[2(kr-\omega t)] \\
&- 2\frac{k}{r^3}\cos[2(kr-\omega t)] - \frac{k^2}{r^2}\sin[2(kr-\omega t)]\bigg]\hat{\boldsymbol{\varphi}}
\end{aligned}
\tag{6.172}
$$

The *total electromagnetic linear momentum* is (*cf.* formula (4.51b) on page 63)

$$
\mathbf{p}^{\text{field}} = \int_{V'} \mathrm{d}^3x'\,\mathbf{g}^{\text{field}}(t',\mathbf{x}')
\tag{6.173}
$$

and the *total electromagnetic angular momentum* is (*cf.* formula (4.89) on page 69)

$$
\mathbf{J}^{\text{field}} = \int_{V'} \mathrm{d}^3x'\,\mathbf{h}^{\text{field}}(t',\mathbf{x}')
\tag{6.174}
$$

In order to get a total net $\mathbf{J}^{\text{field}}$, it is convenient to superimpose several individual dipoles of (possibly) different strengths and relative phases. Perhaps the most common configuration yielding a total net $\mathbf{J}^{\text{field}}$ is two orthogonal co-located dipoles with $\pi/2$ phase shift between them.

We note that in the far zone the linear and angular momentum densities tend to

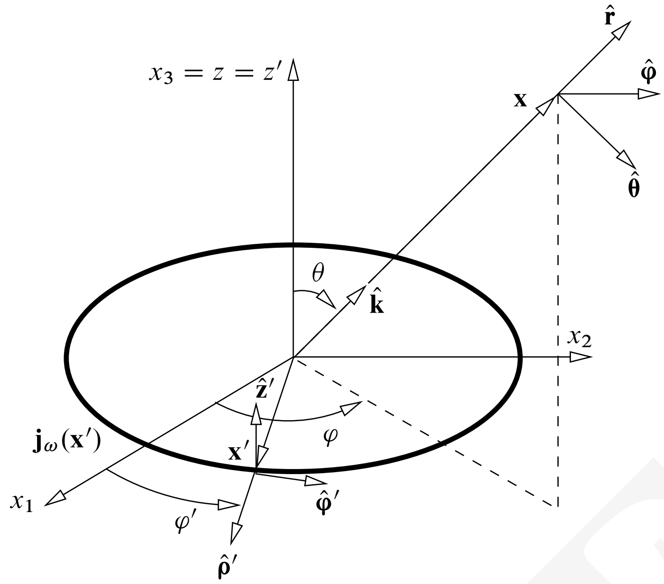


Figure 6.13: For the loop antenna the spherical coordinate system $(r, \theta, \varphi)$ describes the field point $\mathbf{x}$ (the radiation field) and the cylindrical coordinate system $(\rho', \varphi', z')$ describes the source point $\mathbf{x}'$ (the antenna current).

$$
\begin{aligned}
\mathbf{g}^{\text{field,far}}(t, \mathbf{x}) &\approx \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \frac{k^3}{r^2} \sin^2\theta \cos^2(kr - \omega t)\hat{\mathbf{r}} \\
&= \frac{\omega\mu_0}{32\pi^2} d_\omega^2 \frac{k^3}{r^2} \sin^2\theta \big(1 + \cos[2(kr - \omega t)]\big)\hat{\mathbf{r}}
\end{aligned} \tag{6.175}
$$

and

$$
\begin{aligned}
\mathbf{h}^{\text{field,far}}(t, \mathbf{x}) &\approx \frac{\omega\mu_0}{8\pi^2} d_\omega^2 \frac{k^2}{r^2} \sin\theta\cos\theta \sin(kr - \omega t)\cos(kr - \omega t)\hat{\boldsymbol{\varphi}} \\
&= \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \frac{k^2}{r^2} \sin\theta\cos\theta \sin[2(kr - \omega t)]\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{6.176}
$$

respectively. *I.e.* to leading order, both the linear momentum density $\mathbf{g}^{\text{field}}$ and the angular momentum density $\mathbf{h}^{\text{field}}$ fall off as $\sim 1/r^2$ far away from the source region. This means that when they are integrated over a spherical surface $\propto r^2$ located at a large distance from the source [*cf.* the last term in the LHS of formula (4.33) on page 60], there can be a net flux so that the integrated momenta do not fall off with distance and can therefore be transported all the way to infinity.

End of example 6.1 ◁

▷ Radiation from a two-dimensional current distribution — EXAMPLE 6.2

As an example of a two-dimensional current distribution we consider a circular *loop antenna* of radius $a$ and calculate the far-zone $\mathbf{E}^{\text{far}}$ and $\mathbf{B}^{\text{far}}$ fields from such an antenna. We choose the Cartesian coordinate system $x_1x_2x_3$ with its origin at the centre of the loop as in figure 6.13.

According to equation (5.33b) on page 95 the Fourier component of the radiation part of the magnetic field generated by an extended, monochromatic current source is

$$\mathbf{B}_{\omega}^{\text{far}} = \frac{-\mathrm{i}\mu_0 \mathrm{e}^{\mathrm{i}k|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \int_{V'} \mathrm{d}^3x'\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\, \mathbf{j}_{\omega} \times \mathbf{k} \tag{6.177}$$

In our case the generator produces a single frequency $\omega$ and we feed the antenna across a small gap where the loop crosses the positive $x_1$ axis. The circumference of the loop is chosen to be exactly one wavelength $\lambda = 2\pi c/\omega$. This means that the antenna current oscillates in the form of a sinusoidal standing current wave around the circular loop with a Fourier amplitude

$$\mathbf{j}_{\omega} = I_0 \cos\varphi' \delta(\rho' - a)\delta(z')\hat{\boldsymbol{\varphi}}' \tag{6.178}$$

For the spherical coordinate system of the field point, we recall from subsection F.1.2 on page 215 that the following relations between the base vectors hold:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\varphi\hat{\mathbf{x}}_1 + \sin\theta\sin\varphi\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3 \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\hat{\mathbf{x}}_1 + \cos\theta\sin\varphi\hat{\mathbf{x}}_2 - \sin\theta\hat{\mathbf{x}}_3 \\
\hat{\boldsymbol{\varphi}} &= -\sin\varphi\hat{\mathbf{x}}_1 + \cos\varphi\hat{\mathbf{x}}_2
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\
\hat{\mathbf{x}}_2 &= \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\
\hat{\mathbf{x}}_3 &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}
\end{aligned}$$

With the use of the above transformations and trigonometric identities, we obtain for the cylindrical coordinate system which describes the source:

$$\begin{aligned}
\hat{\boldsymbol{\rho}}' &= \cos\varphi'\hat{\mathbf{x}}_1 + \sin\varphi'\hat{\mathbf{x}}_2 \\
&= \sin\theta\cos(\varphi' - \varphi)\hat{\mathbf{r}} + \cos\theta\cos(\varphi' - \varphi)\hat{\boldsymbol{\theta}} + \sin(\varphi' - \varphi)\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{6.179}$$

$$\begin{aligned}
\hat{\boldsymbol{\varphi}}' &= -\sin\varphi'\hat{\mathbf{x}}_1 + \cos\varphi'\hat{\mathbf{x}}_2 \\
&= -\sin\theta\sin(\varphi' - \varphi)\hat{\mathbf{r}} - \cos\theta\sin(\varphi' - \varphi)\hat{\boldsymbol{\theta}} + \cos(\varphi' - \varphi)\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{6.180}$$

$$\hat{\mathbf{z}}' = \hat{\mathbf{x}}_3 = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \tag{6.181}$$

This choice of coordinate systems means that $\mathbf{k} = k\hat{\mathbf{r}}$ and $\mathbf{x}' = a\hat{\boldsymbol{\rho}}'$ so that

$$\mathbf{k}\cdot\mathbf{x}' = ka\sin\theta\cos(\varphi' - \varphi) \tag{6.182}$$

and

$$\hat{\boldsymbol{\varphi}}' \times \mathbf{k} = k[\cos(\varphi' - \varphi)\hat{\boldsymbol{\theta}} + \cos\theta\sin(\varphi' - \varphi)\hat{\boldsymbol{\varphi}}] \tag{6.183}$$

With these expressions inserted, recalling that in cylindrical coordinates $\mathrm{d}^3x' = \rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'$, the source integral becomes

$$\begin{aligned}
\int_{V'} \mathrm{d}^3x'\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\, \mathbf{j}_{\omega} \times \mathbf{k} &= a\int_0^{2\pi} \mathrm{d}\varphi'\, \mathrm{e}^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)}\, I_0\cos\varphi'\, \hat{\boldsymbol{\varphi}}' \times \mathbf{k} \\
&= I_0 ak \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)} \cos(\varphi' - \varphi)\cos\varphi'\,\mathrm{d}\varphi'\,\hat{\boldsymbol{\theta}} \\
&+ I_0 ak\,\cos\theta \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ka\sin\theta\cos(\varphi'-\varphi)} \sin(\varphi' - \varphi)\cos\varphi'\,\mathrm{d}\varphi'\,\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{6.184}$$

Utilising the periodicity of the integrands over the integration interval $[0, 2\pi]$, introducing the auxiliary integration variable $\varphi'' = \varphi' - \varphi$, and utilising standard trigonometric identities, the first integral in the RHS of (6.184) can be rewritten

$$\begin{aligned}
&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos\varphi''\cos(\varphi''+\varphi)\,d\varphi'' \\
&\quad= \cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos^2\varphi''\,d\varphi'' + \text{a vanishing integral} \\
&\quad= \cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\left(\frac{1}{2}+\frac{1}{2}\cos 2\varphi''\right)d\varphi'' \\
&\quad= \frac{1}{2}\cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,d\varphi'' \\
&\qquad+ \frac{1}{2}\cos\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos(2\varphi'')\,d\varphi''
\end{aligned} \tag{6.185}$$

Analogously, the second integral in the RHS of (6.184) can be rewritten

$$\begin{aligned}
&\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\sin\varphi''\cos(\varphi''+\varphi)\,d\varphi'' \\
&\quad= \frac{1}{2}\sin\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,d\varphi'' \\
&\qquad- \frac{1}{2}\sin\varphi\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos 2\varphi''\,d\varphi''
\end{aligned} \tag{6.186}$$

As is well-known from the theory of *Bessel functions*,

$$\begin{aligned}
J_n(-\xi) &= (-1)^n J_n(\xi) \\
J_n(-\xi) &= \frac{i^{-n}}{\pi}\int_0^{\pi} e^{-i\xi\cos\varphi}\cos n\varphi\,d\varphi = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{-i\xi\cos\varphi}\cos n\varphi\,d\varphi
\end{aligned} \tag{6.187}$$

which means that

$$\begin{aligned}
\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\,d\varphi'' &= 2\pi J_0(ka\sin\theta) \\
\int_0^{2\pi} e^{-ika\sin\theta\cos\varphi''}\cos 2\varphi''\,d\varphi'' &= -2\pi J_2(ka\sin\theta)
\end{aligned} \tag{6.188}$$

Putting everything together, we find that

$$\begin{aligned}
&\int_{V'} d^3x'\,e^{-i\mathbf{k}\cdot\mathbf{x}'}\,\mathbf{j}_\omega\times\mathbf{k} = \mathcal{I}_\theta\,\hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}} \\
&\quad= I_0 ak\pi\cos\varphi\,[J_0(ka\sin\theta) - J_2(ka\sin\theta)]\,\hat{\boldsymbol{\theta}} \\
&\qquad+ I_0 ak\pi\cos\theta\sin\varphi\,[J_0(ka\sin\theta) + J_2(ka\sin\theta)]\,\hat{\boldsymbol{\varphi}}
\end{aligned} \tag{6.189}$$

so that, in spherical coordinates where $|\mathbf{x}| = r$,

$$\mathbf{B}^{\text{far}}_\omega(\mathbf{x}) = \frac{-i\mu_0 e^{ikr}}{4\pi r}\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right) \tag{6.190}$$

To obtain the desired physical magnetic field in the radiation (far) zone we must Fourier transform back to $t$ space and take the real part:

$$
\begin{aligned}
\mathbf{B}^{\mathrm{far}}(t,\mathbf{x}) &= \mathrm{Re}\left\{\frac{-\mathrm{i}\mu_0 \mathrm{e}^{(\mathrm{i}kr-\omega t')}}{4\pi r}\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right)\right\} \\
&= \frac{\mu_0}{4\pi r}\sin(kr-\omega t')\left(\mathcal{I}_\theta\,\hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi\,\hat{\boldsymbol{\varphi}}\right) \\
&= \frac{I_0 a k \mu_0}{4r}\sin(kr-\omega t')\Big(\cos\varphi\,[J_0(ka\sin\theta) - J_2(ka\sin\theta)]\,\hat{\boldsymbol{\theta}} \\
&\quad + \cos\theta\sin\varphi\,[J_0(ka\sin\theta) + J_2(ka\sin\theta)]\,\hat{\boldsymbol{\varphi}}\Big)
\end{aligned}
\tag{6.191}
$$

From this expression for the radiated **B** field, we can obtain the radiated **E** field with the help of Maxwell's equations.

End of example 6.2 ◁

EXAMPLE 6.3 ▷ The fields from a uniformly moving charge

[4] This problem was first solved by OLIVER HEAVISIDE in 1888.

In the special case of uniform motion,[4] the localised charge moves in a field-free, isolated space and we know that it will not be affected by any external forces. It will therefore move uniformly in a straight line with the constant velocity $\mathbf{v}'$. This gives us the possibility to extrapolate its position at the observation time, $\mathbf{x}'(t)$, from its position at the retarded time, $\mathbf{x}'(t')$. Since the particle is not accelerated, $\dot{\mathbf{v}}' \equiv \mathbf{0}$, the virtual simultaneous coordinate $\mathbf{x}_0$ will be identical to the actual *simultaneous coordinate* of the particle at time $t$, *i.e.* $\mathbf{x}_0(t) = \mathbf{x}'(t)$. As depicted in figure 6.6 on page 124, the angle between $\mathbf{x} - \mathbf{x}_0$ and $\mathbf{v}'$ is $\theta_0$ while then angle between $\mathbf{x} - \mathbf{x}'$ and $\mathbf{v}'$ is $\theta'$.

In the case of uniform velocity $\mathbf{v}'$, *i.e.* a velocity that does not change with time, any physical observable $f(t,\mathbf{x})$ has the same value at time $t$ and position $\mathbf{x}$ as it has at time $t + \mathrm{d}t$ and position $\mathbf{x} + \mathbf{v}'\mathrm{d}t$. Hence,

$$
f(t,\mathbf{x}) = f(t + \mathrm{d}t, \mathbf{x} + \mathbf{v}'\mathrm{d}t) \tag{6.192}
$$

Taylor expanding $f(t + \mathrm{d}t, \mathbf{x} + \mathbf{v}'\mathrm{d}t)$, keeping only linear terms in the infinitesimally small $\mathrm{d}t$, we obtain

$$
f(t + \mathrm{d}t, \mathbf{x} + \mathbf{v}'\mathrm{d}t) = f(t,\mathbf{x}) + \frac{\partial f}{\partial t}\,\mathrm{d}t + \mathbf{v}'\cdot\boldsymbol{\nabla} f\,\mathrm{d}t + \mathcal{O}\big((\mathrm{d}t)^2\big) \tag{6.193}
$$

From this we conclude that for uniform motion

$$
\frac{\partial f}{\partial t} = -\mathbf{v}'\cdot\boldsymbol{\nabla} f \tag{6.194}
$$

Since $f$ is an arbitrary physical observable, the time and space derivatives must be related in the following way when they operate on any physical observable dependent on $\mathbf{x}(t)$ [*cf.* equation (1.34) on page 13]:

$$
\frac{\partial}{\partial t} = -\mathbf{v}'\cdot\boldsymbol{\nabla} \tag{6.195}
$$

Hence, the **E** and **B** fields can be obtained from formulæ (6.85) on page 124, with the potentials given by equations (6.83) on page 123 as follows:

$$
\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{1}{c^2}\frac{\partial \mathbf{v}'\phi}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\mathbf{v}'}{c^2}\frac{\partial \phi}{\partial t} \\
&= -\boldsymbol{\nabla}\phi + \frac{\mathbf{v}'}{c}\left(\frac{\mathbf{v}'}{c}\cdot\boldsymbol{\nabla}\phi\right) = -\left(1 - \frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2}\cdot\right)\boldsymbol{\nabla}\phi \qquad (6.196a) \\
&= \left(\frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2} - \mathbf{1}_3\right)\cdot\boldsymbol{\nabla}\phi \\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\left(\frac{\mathbf{v}'}{c^2}\phi\right) = \boldsymbol{\nabla}\phi\times\frac{\mathbf{v}'}{c^2} = -\frac{\mathbf{v}'}{c^2}\times\boldsymbol{\nabla}\phi \\
&= \frac{\mathbf{v}'}{c^2}\times\left[\left(\frac{\mathbf{v}'}{c}\cdot\boldsymbol{\nabla}\phi\right)\frac{\mathbf{v}'}{c} - \boldsymbol{\nabla}\phi\right] = \frac{\mathbf{v}'}{c^2}\times\left(\frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2} - \mathbf{1}_3\right)\cdot\boldsymbol{\nabla}\phi \qquad (6.196b) \\
&= \frac{\mathbf{v}'}{c^2}\times\mathbf{E}
\end{aligned}
$$

Here $\mathbf{1}_3 = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i$ is the unit dyad and we used the fact that $\mathbf{v}'\times\mathbf{v}'\equiv 0$. What remains is just to express $\boldsymbol{\nabla}\phi$ in quantities evaluated at $t$ and $\mathbf{x}$.

From equation (6.83a) on page 123 and equation (6.114) on page 131 we find that

$$
\begin{aligned}
\boldsymbol{\nabla}\phi &= \frac{q'}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{1}{s}\right) = -\frac{q'}{8\pi\varepsilon_0 s^3}\boldsymbol{\nabla}s^2 \\
&= -\frac{q'}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}'}{c}\times\left(\frac{\mathbf{v}'}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]
\end{aligned} \tag{6.197}
$$

When this expression for $\boldsymbol{\nabla}\phi$ is inserted into equation (6.196a) above, the following result

$$
\begin{aligned}
\mathbf{E}(t,\mathbf{x}) &= \left(\frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2} - \mathbf{1}_3\right)\cdot\boldsymbol{\nabla}\phi = -\frac{q'}{8\pi\varepsilon_0 s^3}\left(\frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2} - \mathbf{1}_3\right)\cdot\boldsymbol{\nabla}s^2 \\
&= \frac{q'}{4\pi\varepsilon_0 s^3}\left\{(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}'}{c}\times\left(\frac{\mathbf{v}'}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right. \\
&\qquad \left. - \frac{\mathbf{v}'}{c}\left(\frac{\mathbf{v}'}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right) - \frac{\mathbf{v}'\otimes\mathbf{v}'}{c^2}\cdot\left[\frac{\mathbf{v}'}{c}\times\left(\frac{\mathbf{v}'}{c}\times(\mathbf{x}-\mathbf{x}_0)\right)\right]\right\} \\
&= \frac{q'}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0) + \frac{\mathbf{v}'}{c}\left(\frac{\mathbf{v}'}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right) - (\mathbf{x}-\mathbf{x}_0)\frac{v'^2}{c^2}\right. \\
&\qquad \left. - \frac{\mathbf{v}'}{c}\left(\frac{\mathbf{v}'}{c}\cdot(\mathbf{x}-\mathbf{x}_0)\right)\right] \\
&= \frac{q'}{4\pi\varepsilon_0 s^3}(\mathbf{x}-\mathbf{x}_0)\left(1 - \frac{v'^2}{c^2}\right)
\end{aligned} \tag{6.198}
$$

obtains. Of course, the same result also follows from equation (6.102) on page 128 with $\dot{\mathbf{v}}'\equiv\mathbf{0}$ inserted.

From equation (6.198) above we conclude that $\mathbf{E}$ is directed along the vector from the simultaneous coordinate $\mathbf{x}_0(t)$ to the field (observation) coordinate $\mathbf{x}(t)$. In a similar way, the magnetic field can be calculated and one finds that

$$
\mathbf{B}(t,\mathbf{x}) = \frac{\mu_0 q'}{4\pi s^3}\left(1 - \frac{v'^2}{c^2}\right)\mathbf{v}'\times(\mathbf{x}-\mathbf{x}_0) = \frac{1}{c^2}\mathbf{v}'\times\mathbf{E} \tag{6.199}
$$

From these explicit formulae for the $\mathbf{E}$ and $\mathbf{B}$ fields and formula (6.113b) on page 130 for $s$, we can discern the following cases:

1. $v' \to 0 \Rightarrow \mathbf{E}$ goes over into the Coulomb field $\mathbf{E}^{\text{Coulomb}}$
2. $v' \to 0 \Rightarrow \mathbf{B}$ goes over into the Biot-Savart field
3. $v' \to c \Rightarrow \mathbf{E}$ becomes dependent on $\theta_0$
4. $v' \to c, \sin\theta_0 \approx 0 \Rightarrow \mathbf{E} \to (1 - v'^2/c^2)\mathbf{E}^{\text{Coulomb}}$
5. $v' \to c, \sin\theta_0 \approx 1 \Rightarrow \mathbf{E} \to (1 - v'^2/c^2)^{-1/2}\mathbf{E}^{\text{Coulomb}}$

End of example 6.3◁

EXAMPLE 6.4 ▷Bremsstrahlung for low speeds and short acceleration times

Calculate the bremsstrahlung when a charged particle, moving at a non-relativistic speed, is accelerated or decelerated during an infinitely short time interval.

We approximate the velocity change at time $t' = t_0$ by a delta function:

$$\dot{\mathbf{v}}'(t') = \Delta\mathbf{v}'\,\delta(t' - t_0) \tag{6.200}$$

which means that

$$\Delta\mathbf{v}'(t_0) = \int_{-\infty}^{\infty} \mathrm{d}t'\,\dot{\mathbf{v}}' \tag{6.201}$$

Also, we assume $v/c \ll 1$ so that, according to formula (6.84) on page 124,

$$s \approx |\mathbf{x} - \mathbf{x}'| \tag{6.202}$$

and, according to formula (6.101) on page 128,

$$\mathbf{x} - \mathbf{x}_0 \approx \mathbf{x} - \mathbf{x}' \tag{6.203}$$

From the general expression (6.105) on page 129 we conclude that $\mathbf{E} \perp \mathbf{B}$ and that it suffices to consider $E \equiv |\mathbf{E}^{\text{far}}|$. According to the 'bremsstrahlung expression' for $\mathbf{E}^{\text{far}}$, equation (6.123) on page 133,

$$E = \frac{q' \sin\theta'}{4\pi\varepsilon_0 c^2\,|\mathbf{x} - \mathbf{x}'|}\Delta v'\,\delta(t' - t_0) \tag{6.204}$$

In this simple case $B \equiv |\mathbf{B}^{\text{far}}|$ is given by

$$B = \frac{E}{c} \tag{6.205}$$

Because of the Dirac $\delta$ behaviour in time, Fourier transforming expression (6.204) above for $E$ is trivial, yielding

$$\begin{aligned} E_\omega &= \frac{q' \sin\theta'}{4\pi\varepsilon_0 c^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t'\,\frac{\Delta\mathbf{v}'\delta(t' - t_0)}{|\mathbf{x} - \mathbf{x}'(t')|}\,e^{\mathrm{i}\omega t'} \\ &= \frac{q' \sin\theta'}{8\pi^2\varepsilon_0 c^2\,|\mathbf{x} - \mathbf{x}'(t_0)|}\Delta v'(t_0)\,e^{\mathrm{i}\omega t_0} \end{aligned} \tag{6.206}$$

We note that the magnitude of this Fourier component is independent of $\omega$. This is a consequence of the infinitely short 'impulsive step' $\delta(t' - t_0)$ in the time domain which produces an infinite spectrum in the frequency domain.

The total radiation energy is given by the expression

$$
\begin{aligned}
\tilde{U}^{\text{rad}} &= \int_{-\infty}^{\infty} \mathrm{d}t' \, \frac{\mathrm{d}\tilde{U}^{\text{rad}}}{\mathrm{d}t'} = \int_{-\infty}^{\infty} \mathrm{d}t' \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \cdot \left( \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) \\
&= \frac{1}{\mu_0} \oint_{S'} \mathrm{d}^2x' \int_{-\infty}^{\infty} \mathrm{d}t' \, EB = \frac{1}{\mu_0 c} \oint_{S'} \mathrm{d}^2x' \int_{-\infty}^{\infty} \mathrm{d}t' \, E^2 \\
&= \varepsilon_0 c \oint_{S'} \mathrm{d}^2x' \int_{-\infty}^{\infty} \mathrm{d}t' \, E^2
\end{aligned}
\tag{6.207}
$$

According to *Parseval's identity* [*cf.* equation (6.9) on page 106] the following equality holds:

$$
\int_{-\infty}^{\infty} \mathrm{d}t' \, E^2 = 4\pi \int_{0}^{\infty} \mathrm{d}\omega \, |E_\omega|^2 \tag{6.208}
$$

which means that the radiated energy in the frequency interval $(\omega, \omega + \mathrm{d}\omega)$ is

$$
\tilde{U}^{\text{rad}}_\omega \mathrm{d}\omega = 4\pi\varepsilon_0 c \left( \oint_{S'} \mathrm{d}^2x' \, |E_\omega|^2 \right) \mathrm{d}\omega \tag{6.209}
$$

For our infinite spectrum, equation (6.206) on the facing page, we obtain

$$
\begin{aligned}
\tilde{U}^{\text{rad}}_\omega \mathrm{d}\omega &= \frac{q'^2 (\Delta v')^2}{16\pi^3 \varepsilon_0 c^3} \oint_{S'} \mathrm{d}^2x' \, \frac{\sin^2\theta'}{|\mathbf{x} - \mathbf{x}'|^2} \mathrm{d}\omega \\
&= \frac{q'^2 (\Delta v')^2}{16\pi^3 \varepsilon_0 c^3} \int_{0}^{2\pi} \mathrm{d}\varphi' \int_{0}^{\pi} \mathrm{d}\theta' \, \sin\theta' \, \sin^2\theta' \, \mathrm{d}\omega \\
&= \frac{q'^2}{3\pi\varepsilon_0 c} \left( \frac{\Delta v'}{c} \right)^2 \frac{\mathrm{d}\omega}{2\pi}
\end{aligned}
\tag{6.210}
$$

We see that the energy spectrum $\tilde{U}^{\text{rad}}_\omega$ is independent of frequency $\omega$. This means that if we would integrate it over all frequencies $\omega \in [0, \infty)$, a divergent integral would result.

In reality, all spectra have finite widths, with an upper *cutoff* limit set by the quantum condition

$$
\hbar\omega_{\max} = \frac{1}{2} m (v' + \Delta v')^2 - \frac{1}{2} m v'^2 \tag{6.211}
$$

which expresses that the highest possible frequency $\omega_{\max}$ in the spectrum is that for which all kinetic energy difference has gone into one single *field quantum* (*photon*) with energy $\hbar\omega_{\max}$. If we adopt the picture that the total energy is quantised in terms of $N_\omega$ photons radiated during the process, we find that

$$
\frac{\tilde{U}^{\text{rad}}_\omega \mathrm{d}\omega}{\hbar\omega} = \mathrm{d}N_\omega \tag{6.212}
$$

or, for an electron where $q' = -|e|$, where $e$ is the elementary charge,

$$
\mathrm{d}N_\omega = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \frac{2}{3\pi} \left( \frac{\Delta v'}{c} \right)^2 \frac{\mathrm{d}\omega}{\omega} \approx \frac{1}{137} \frac{2}{3\pi} \left( \frac{\Delta v'}{c} \right)^2 \frac{\mathrm{d}\omega}{\omega} \tag{6.213}
$$

where we used the value of the *fine structure constant* $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$.

Even if the number of photons becomes infinite when $\omega \to 0$, these photons have negligible energies so that the total radiated energy is still finite.

End of example 6.4◁

## 6.7 Bibliography

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# 7

# RELATIVISTIC ELECTRODYNAMICS

We saw in chapter chapter 3 how the introduction of electrodynamic potentials led, in a most natural way, to the existence of a characteristic, finite speed of propagation of electromagnetic fields (and related quantities) in free space (vacuum) that equals the speed of light $c = 1/\sqrt{\varepsilon_0 \mu_0}$ and which can be considered a constant of Nature. To take this finite speed of propagation of information into account, and to ensure that our laws of physics be independent of any specific coordinate frame, requires a treatment of electrodynamics in a relativistically covariant (coordinate independent) form. This is the objective of this chapter.[1]

The technique we shall use to study relativity is the mathematical apparatus developed for non-Euclidean spaces of arbitrary dimensions, here specialised to four dimensions. It turns out that this theory of Riemannian spaces, derived for more or less purely mathematical reasons only, is ideal for a formal description of relativistic physics. For the simple case of the *special theory of relativity*, the mathematics is quite simple, whereas for the *general theory of relativity* it becomes more complicated.

[1] *The Special Theory of Relativity*, by the physicist and philosopher DAVID JOSEPH BOHM (1917–1992), opens with the following paragraph:

> 'The theory of relativity is not merely a scientific development of great importance in its own right. It is even more significant as the first stage of a radical change in our basic concepts, which began in physics, and which is spreading into other fields of science, and indeed, even into a great deal of thinking outside of science. For as is well known, the modern trend is away from the notion of sure 'absolute' truth, (*i.e.* one which holds independently of all conditions, contexts, degrees, and types of approximation *etc.*) and toward the idea that a given concept has significance only in relation to suitable broader forms of reference, within which that concept can be given its full meaning.'

## 7.1 The special theory of relativity

An *inertial system*, or *inertial reference frame*, is a system of reference, or rigid coordinate system, in which the *law of inertia* (*Galileo's law*, *Newton's first law*) holds. In other words, an inertial system is a system in which free bodies move uniformly and do not experience any acceleration. The *special theory of relativity* describes how physical processes are interrelated when observed in different inertial systems in uniform, rectilinear motion relative to each other and is based on two postulates:

POSTULATE 7.1 (Relativity principle; POINCARÉ, 1905) *All laws of physics (except the laws of gravitation) are independent of the uniform translational motion of the system on which they operate.*

POSTULATE 7.2 (EINSTEIN, 1905) *The velocity of light in empty space is independent of the motion of the source that emits the light.*

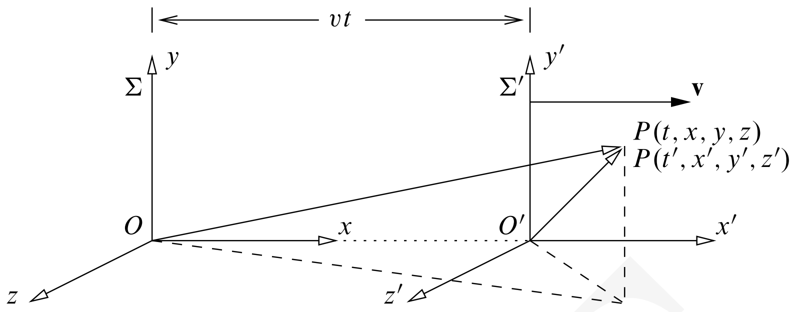


Figure 7.1: Two inertial systems $\Sigma$ and $\Sigma'$ in relative motion with velocity $\mathbf{v}$ along the $x = x'$ axis. At time $t = t' = 0$ the origin $O'$ of $\Sigma'$ coincided with the origin $O$ of $\Sigma$. At time $t$, the inertial system $\Sigma'$ has been translated a distance $vt$ along the $x$ axis in $\Sigma$. An event represented by $P(t, x, y, z)$ in $\Sigma$ is represented by $P(t', x', y', z')$ in $\Sigma'$.

A consequence of the first postulate is that all geometrical objects (vectors, tensors) in an equation describing a physical process must transform in a *covariant* manner, *i.e.* in the same way.

### 7.1.1 The Lorentz transformation

Let us consider two three-dimensional inertial systems $\Sigma$ and $\Sigma'$ in free space. They are in rectilinear motion relative to each other in such a way that $\Sigma'$ moves with constant velocity $\mathbf{v}$ along the $x$ axis of the $\Sigma$ system. The times and the spatial coordinates as measured in the two systems are $t$ and $(x, y, z)$, and $t'$ and $(x', y', z')$, respectively. At time $t = t' = 0$ the origins $O$ and $O'$ and the $x$ and $x'$ axes of the two inertial systems coincide and at a later time $t$ they have the relative location as depicted in figure 7.1, referred to as the *standard configuration*.

For convenience, let us introduce the two quantities

$$\beta = \frac{v}{c} \tag{7.1a}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{7.1b}$$

where $v = |\mathbf{v}|$. In the following, we shall make frequent use of these shorthand notations.

As shown by Einstein, the two postulates of special relativity require that the spatial coordinates and times as measured by an observer in $\Sigma$ and $\Sigma'$, respectively, are connected by the following transformation:

$$ct' = \gamma(ct - x\beta) \tag{7.2a}$$

$$x' = \gamma(x - vt) \tag{7.2b}$$

$$y' = y \tag{7.2c}$$

$$z' = z \tag{7.2d}$$

Taking the difference between the square of (7.2a) and the square of (7.2b) we find that

$$\begin{aligned} c^2 t'^2 - x'^2 &= \gamma^2 \left(c^2 t^2 - 2xc\beta t + x^2\beta^2 - x^2 + 2xvt - v^2 t^2\right) \\ &= \frac{1}{1 - \frac{v^2}{c^2}} \left[ c^2 t^2 \left(1 - \frac{v^2}{c^2}\right) - x^2 \left(1 - \frac{v^2}{c^2}\right)\right] \\ &= c^2 t^2 - x^2 \end{aligned} \tag{7.3}$$

From equations (7.2) on the facing page we see that the $y$ and $z$ coordinates are unaffected by the translational motion of the inertial system $\Sigma'$ along the $x$ axis of system $\Sigma$. Using this fact, we find that we can generalise the result in equation (7.3) above to

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2 \tag{7.4}$$

which means that if a light wave is transmitted from the coinciding origins $O$ and $O'$ at time $t = t' = 0$ it will arrive at an observer at $(x, y, z)$ at time $t$ in $\Sigma$ and an observer at $(x', y', z')$ at time $t'$ in $\Sigma'$ in such a way that both observers conclude that the speed (spatial distance divided by time) of light in vacuum is $c$. Hence, the speed of light in $\Sigma$ and $\Sigma'$ is the same. A linear coordinate transformation which has this property is called a (homogeneous) *Lorentz transformation*.

### 7.1.2 Lorentz space

Let us introduce an ordered quadruple of real numbers, enumerated with the help of upper indices $\mu = 0, 1, 2, 3$, where the zeroth component is $ct$ ($c$ is the speed of light and $t$ is time), and the remaining components are the components of the ordinary $\mathbb{R}^3$ position vector $\mathbf{x}$ defined in equation (M.1) on page 233:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \equiv (ct, \mathbf{x}) \tag{7.5}$$

In order that this quadruple $x^\mu$ represent a *physical observable*, it must transform as (the component form of) a *position four-vector* (*radius four-vector*) in a real, linear, *four-dimensional vector space*.[2] We require that this four-dimensional space be a *Riemannian space*, *i.e.* a metric space where a 'distance' and a scalar product are defined. In this space we therefore define a *metric tensor*, also known as the *fundamental tensor*, which we denote by $g_{\mu\nu}$.

[2] The British mathematician and philosopher ALFRED NORTH WHITEHEAD (1861–1947) writes in his book *The Concept of Nature*:

> 'I regret that it has been necessary for me in this lecture to administer a large dose of four-dimensional geometry. I do not apologise, because I am really not responsible for the fact that Nature in its most fundamental aspect is four-dimensional. Things are what they are. . . .'

#### 7.1.2.1 Radius four-vector in contravariant and covariant form

The position four-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x})$, as defined in equation (7.5) on the previous page, is, by definition, the prototype of a *contravariant vector* (or, more accurately, a vector in *contravariant component form*). To every such vector there exists a *dual vector*. The vector dual to $x^\mu$ is the *covariant vector* $x_\mu$, obtained as

$$x_\mu = g_{\mu\nu} x^\nu \tag{7.6}$$

where the upper index $\mu$ in $x^\mu$ is summed over and is therefore a *dummy index* and may be replaced by another dummy index $\nu$, say. This summation process is an example of *index contraction* and is often referred to as *index lowering*.

#### 7.1.2.2 Scalar product and norm

The scalar product of $x^\mu$ with itself in a Riemannian space is defined as

$$g_{\mu\nu} x^\nu x^\mu = x_\mu x^\mu \tag{7.7}$$

This scalar product acts as an invariant 'distance', or *norm*, in this space.

In order to put the physical property of Lorentz transformation invariance, described by equation (7.4) on the preceding page, into a convenient mathematical framework, we perceive this invariance as the manifestation of the conservation of the norm in a 4D Riemannian space.

#### 7.1.2.3 Metric tensor

In $\mathbb{L}^4$ one can choose the metric tensor $g_{\mu\nu}$ to take the simple form

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1,2,3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{7.8}$$

or, in matrix representation,

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.9}$$

*i.e.* a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+, -, -, -\}$. As we see, the index lowering operation in our flat 4D space $\mathbb{L}^4$ is nearly trivial.

In matrix representation the lowering of the indices of $x^\mu$ becomes

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \tag{7.10}$$

I four-tensor notation, this can be written

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -\mathbf{x}) \tag{7.11}$$

Hence, if the metric tensor is defined according to expression (7.8) on the preceding page, the covariant position four-vector $x_\mu$ is obtained from the contravariant position four-vector $x^\mu$ simply by changing the sign of the last three components. These components are referred to as the *space components*; the zeroth component is referred to as the *time component*.

As we see, for this particular choice of metric, the scalar product of $x^\mu$ with itself becomes

$$x_\mu x^\mu = (ct, \mathbf{x}) \cdot (ct, -\mathbf{x}) = c^2 t^2 - x^2 - y^2 - z^2 \tag{7.12}$$

which indeed is the desired Lorentz transformation invariance as required by equation (7.12) above. Without changing the physics, one can alternatively choose a signature $\{-,+,+,+\}$. The latter has the advantage that the transition from 3D to 4D becomes smooth, while it will introduce some annoying minus signs in the theory. In current physics literature, the signature $\{+,-,-,-\}$ seems to be the most commonly used one. Note that our space, regardless of signature chosen, will have an *indefinite norm*, *i.e.* a norm which can be positive definite, negative definite or even zero. This means that we deal with a *non-Euclidean space* and we call our four-dimensional space (or *space-time*) with this property *Lorentz space* and denote it $\mathbb{L}^4$. A corresponding real, linear 4D space with a *positive definite norm* which is conserved during ordinary rotations is a *Euclidean vector space*. We denote such a space $\mathbb{R}^4$.

The $\mathbb{L}^4$ metric tensor equation (7.8) on the preceding page has a number of interesting properties: firstly, we see that this tensor has a trace $\mathrm{Tr}\,\eta_{\mu\nu} = -2$ whereas in $\mathbb{R}^4$, as in any vector space with definite norm, the trace equals the space dimensionality. Secondly, we find, after trivial algebra, that the following relations between the contravariant, covariant and mixed forms of the metric tensor hold:

$$\eta_{\mu\nu} = \eta_{\nu\mu} \tag{7.13a}$$

$$\eta^{\mu\nu} = \eta_{\mu\nu} \tag{7.13b}$$

$$\eta_{\nu\kappa}\eta^{\kappa\mu} = \eta_\nu^\mu = \delta_\nu^\mu \tag{7.13c}$$

$$\eta^{\nu\kappa}\eta_{\kappa\mu} = \eta^\nu_\mu = \delta^\nu_\mu \tag{7.13d}$$

Here we have introduced the 4D version of the Kronecker delta $\delta^\mu_\nu$, a mixed four-tensor of rank 2 that fulfils

$$\delta^\mu_\nu = \delta^\nu_\mu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{7.14}$$

Clearly, the matrix representation of this tensor is

$$(\delta^\mu_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.15}$$

*i.e.* the $4 \times 4$ unit matrix.

#### 7.1.2.4 Invariant line element and proper time

The *differential distance* $\mathrm{d}s$ between the two points $x^\mu$ and $x^\mu + \mathrm{d}x^\mu$ in $\mathbb{L}^4$ can be calculated from the *Riemannian metric*, given by the *quadratic differential form*

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\nu \mathrm{d}x^\mu = \mathrm{d}x_\mu \mathrm{d}x^\mu = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{7.16}$$

where the metric tensor is as in equation (7.8) on page 156. As we see, this form is *indefinite* as expected for a non-Euclidean space. The square root of this expression is the *invariant line element*

$$\begin{aligned} \mathrm{d}s &= c\,\mathrm{d}t \sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^3}{\mathrm{d}t}\right)^2\right]} \\ &= c\,\mathrm{d}t \sqrt{1 - \frac{1}{c^2}\left[(v_x)^2 + (v_y)^2 + (v_z)^2\right]} = c\,\mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}} \\ &= c\,\mathrm{d}t \sqrt{1 - \beta^2} = c\frac{\mathrm{d}t}{\gamma} = c\,\mathrm{d}\tau \end{aligned} \tag{7.17}$$

where we introduced

$$\mathrm{d}\tau = \mathrm{d}t/\gamma \tag{7.18}$$

Since $\mathrm{d}\tau$ measures the time when no spatial changes are present, *i.e.* by a clock that is fixed relative the given frame of reference, it is called the *proper time*. As equation (7.18) above shows, the proper time of a moving object is always less than the corresponding interval in the rest system. One may say that moving clocks go slower than those at rest.

Expressing the property of the Lorentz transformation described by equations (7.4) on page 155 in terms of the differential interval $\mathrm{d}s$ and comparing with equation (7.16) on the preceding page, we find that

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{7.19}$$

is invariant, *i.e.* remains unchanged, during a Lorentz transformation. Conversely, we may say that every coordinate transformation which preserves this differential interval is a Lorentz transformation.

If in some inertial system

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 < c^2\mathrm{d}t^2 \tag{7.20}$$

$\mathrm{d}s$ is a *time-like interval*, but if

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 > c^2\mathrm{d}t^2 \tag{7.21}$$

$\mathrm{d}s$ is a *space-like interval*, whereas

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = c^2\mathrm{d}t^2 \tag{7.22}$$

is a *light-like interval*; we may also say that in this case we are on the *light cone*. A vector which has a light-like interval is called a *null vector*. The time-like, space-like or light-like aspects of an interval $\mathrm{d}s$ are invariant under a Lorentz transformation. *I.e.* it is not possible to change a time-like interval into a space-like one or *vice versa* via a Lorentz transformation.

#### 7.1.2.5 Four-vector fields

Any quantity which relative to any coordinate system has a quadruple of real numbers and transforms in the same way as the position four-vector $x^\mu$ does, is called a *four-vector*. In analogy with the notation for the position four-vector we introduce the notation $a^\mu = (a^0, \mathbf{a})$ for a general *contravariant four-vector field* in $\mathbb{L}^4$ and find that the 'lowering of index' rule, formula (7.6) on page 156, for such an arbitrary four-vector yields the dual *covariant four-vector field*

$$a_\mu(x^\kappa) = \eta_{\mu\nu}a^\nu(x^\kappa) = (a^0(x^\kappa), -\mathbf{a}(x^\kappa)) \tag{7.23}$$

The scalar product between this four-vector field and another one $b^\mu(x^\kappa)$ is

$$\eta_{\mu\nu}a^\nu(x^\kappa)b^\mu(x^\kappa) = (a^0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a^0b^0 - \mathbf{a}\cdot\mathbf{b} \tag{7.24}$$

which is a *scalar field*, *i.e.* an invariant scalar quantity $\alpha(x^\kappa)$ which depends on time and space, as described by $x^\kappa = (ct, x, y, z)$.

#### 7.1.2.6 The Lorentz transformation matrix

Introducing the transformation matrix

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.25}$$

the linear Lorentz transformation (7.2) on page 154, *i.e.* the coordinate transformation $x^{\mu} \to x'^{\mu} = x'^{\mu}(x^0, x^1, x^2, x^3)$, from one inertial system $\Sigma$ to another inertial system $\Sigma'$ in the standard configuration, can be written

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{7.26}$$

#### 7.1.2.7 The Lorentz group

It is easy to show, by means of direct algebra, that two successive Lorentz transformations of the type in equation (7.26) above, and defined by the speed parameters $\beta_1$ and $\beta_2$, respectively, correspond to a single transformation with speed parameter

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{7.27}$$

This means that the nonempty set of Lorentz transformations constitutes a *closed algebraic structure* with a binary operation (multiplication) that is *associative*. Furthermore, one can show that this set possesses at least one *identity element* and at least one *inverse element*. In other words, this set of Lorentz transformations constitutes a *mathematical group*. However tempting, we shall not make any further use of *group theory*.

### 7.1.3 Minkowski space

Specifying a point $x^{\mu} = (x^0, x^1, x^2, x^3)$ in 4D space-time is a way of saying that 'something takes place at a certain time $t = x^0/c$ and at a certain place $(x, y, z) = (x^1, x^2, x^3)$'. Such a point is therefore called an *event*. The trajectory for an event as a function of time and space is called a *world line*. For instance, the world line for a light ray that propagates in vacuum (free space) is the trajectory $x^0 = x^1$.

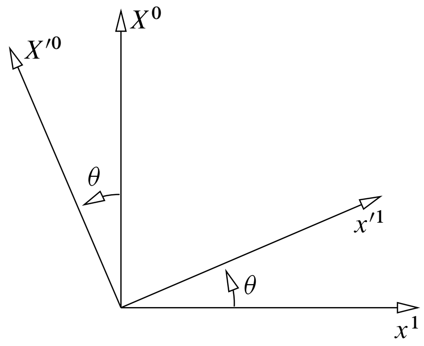


Figure 7.2: Minkowski space can be considered an ordinary Euclidean space where a Lorentz transformation from $(x^1, X^0 = \mathrm{i}ct)$ to $(x'^1, X'^0 = \mathrm{i}ct')$ corresponds to an ordinary rotation through an angle $\theta$. This rotation leaves the Euclidean distance $(x^1)^2 + (X^0)^2 = x^2 - c^2t^2$ invariant.

Introducing

$$X^0 = \mathrm{i}x^0 = \mathrm{i}ct \tag{7.28a}$$
$$X^1 = x^1 \tag{7.28b}$$
$$X^2 = x^2 \tag{7.28c}$$
$$X^3 = x^3 \tag{7.28d}$$
$$\mathrm{d}S = \mathrm{i}\mathrm{d}s \tag{7.28e}$$

where $\mathrm{i} = \sqrt{-1}$, we see that equation (7.16) on page 158 transforms into

$$\mathrm{d}S^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2 + (\mathrm{d}X^3)^2 \tag{7.29}$$

*i.e.* into a 4D differential form that is *positive definite* just as is ordinary 3D *Euclidean space* $\mathbb{R}^3$. We shall call the 4D Euclidean space constructed in this way the *Minkowski space* $\mathbb{M}^4$.[3]

As before, it suffices to consider the simplified case where the relative motion between $\Sigma$ and $\Sigma'$ is along the $x$ axes. Then

$$\mathrm{d}S^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 = (\mathrm{d}X^0)^2 + (\mathrm{d}x^1)^2 \tag{7.30}$$

and we consider the $X^0$ and $X^1 = x^1$ axes as orthogonal axes in a Euclidean space. As in all Euclidean spaces, every interval is invariant under a rotation of the $X^0x^1$ plane through an angle $\theta$ into $X'^0x'^1$:

$$X'^0 = -x^1 \sin\theta + X^0 \cos\theta \tag{7.31a}$$
$$x'^1 = x^1 \cos\theta + X^0 \sin\theta \tag{7.31b}$$

See figure 7.2.

If we introduce the angle $\varphi = -\mathrm{i}\theta$, often called the *rapidity* or the *Lorentz boost parameter*, and transform back to the original space and time variables by using equation (7.28) above backwards, we obtain

$$ct' = -x \sinh\varphi + ct \cosh\varphi \tag{7.32a}$$
$$x' = x \cosh\varphi - ct \sinh\varphi \tag{7.32b}$$

[3] The fact that our Riemannian space can be transformed in this way into a Euclidean one means that it is, strictly speaking, a *pseudo-Riemannian space*.

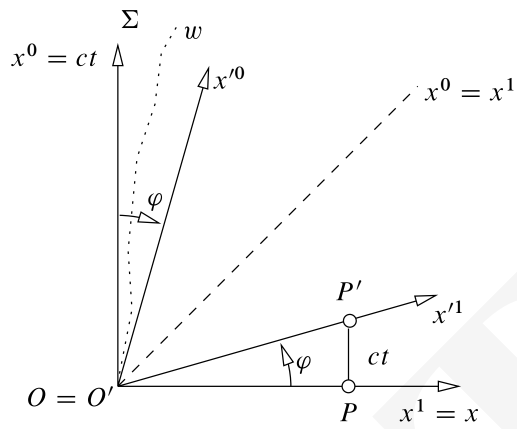


Figure 7.3: Minkowski diagram depicting geometrically the transformation (7.32) from the unprimed system to the primed system. Here $w$ denotes the world line for an event and the line $x^0 = x^1 \Leftrightarrow x = ct$ the world line for a light ray in vacuum. Note that the event $P$ is simultaneous with all points on the $x^1$ axis ($t = 0$), including the origin $O$. The event $P'$, which is simultaneous with all points on the $x'$ axis, including $O' = O$, to an observer at rest in the primed system, is not simultaneous with $O$ in the unprimed system but occurs there at time $|P - P'|/c$.

which are identical to the original transformation equations (7.2) on page 154 if we let

$$\sinh\varphi = \gamma\beta \tag{7.33a}$$

$$\cosh\varphi = \gamma \tag{7.33b}$$

$$\tanh\varphi = \beta \tag{7.33c}$$

It is therefore possible to envisage the Lorentz transformation as an 'ordinary' rotation in the 4D Euclidean space $\mathbb{M}^4$. Such a rotation in $\mathbb{M}^4$ corresponds to a coordinate change in $\mathbb{L}^4$ as depicted in figure 7.3. Equation (7.27) on page 160 for successive Lorentz transformation then corresponds to the tanh addition formula

$$\tanh(\varphi_1 + \varphi_2) = \frac{\tanh\varphi_1 + \tanh\varphi_2}{1 + \tanh\varphi_1\,\tanh\varphi_2} \tag{7.34}$$

The use of i$ct$ and $\mathbb{M}^4$, which leads to the interpretation of the Lorentz transformation as an 'ordinary' rotation, may, at best, be illustrative, but is not very physical. Besides, if we leave the flat $\mathbb{L}^4$ space and enter the curved space of general relativity, the 'i$ct$' trick will turn out to be an impasse. Let us therefore immediately return to $\mathbb{L}^4$ where all components are real valued.

## 7.2 Covariant classical mechanics

The invariance of the differential 'distance' $\mathrm{d}s$ in $\mathbb{L}^4$, and the associated differential proper time $\mathrm{d}\tau$ [see equation (7.17) on page 158] allows us to define the *four-velocity*

$$u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \gamma(c, \mathbf{v}) = \left( \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = (u^0, \mathbf{u}) \tag{7.35}$$

which, when multiplied with the scalar invariant $m_0$ yields the *four-momentum*

$$p^{\mu} = m_0 \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = m_0\gamma(c, \mathbf{v}) = \left( \frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = (p^0, \mathbf{p}) \tag{7.36}$$

From this we see that we can write

$$\mathbf{p} = m\mathbf{v} \tag{7.37}$$

where

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7.38}$$

We can interpret this such that the Lorentz covariance implies that the mass-like term in the ordinary 3D linear momentum is not invariant. A better way to look at this is that $\mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$ is the covariantly correct expression for the kinetic three-momentum.

Multiplying the zeroth (time) component of the four-momentum $p^{\mu}$ by the scalar invariant $c$, we obtain

$$cp^0 = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 \tag{7.39}$$

Since this component has the dimension of energy and is the result of a Lorentzcovariant description of the motion of a particle with its kinetic momentum described by the spatial components of the four-momentum, equation (7.36) above, we interpret $cp^0$ as the total energy $E$. Hence,

$$cp^{\mu} = (cp^0, c\mathbf{p}) = (E, c\mathbf{p}) \tag{7.40}$$

Scalar multiplying this four-vector with itself, we obtain

$$\begin{aligned} cp_{\mu}cp^{\mu} &= c^2\eta_{\mu\nu}p^{\nu}p^{\mu} = c^2[(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2] \\ &= (E, -c\mathbf{p}) \cdot (E, c\mathbf{p}) = E^2 - c^2\mathbf{p}^2 \\ &= \frac{(m_0c^2)^2}{1 - \frac{v^2}{c^2}} \left( 1 - \frac{v^2}{c^2} \right) = (m_0c^2)^2 \end{aligned} \tag{7.41}$$

Since this is an invariant, this equation holds in any inertial frame, particularly in the frame where $\mathbf{p} = \mathbf{0}$ and there we have

$$E = m_0 c^2 \tag{7.42}$$

This is probably the most famous formula in physics history.

## 7.3 Covariant classical electrodynamics

Let us consider a charge density which in its rest inertial system is denoted by $\rho_0$. The four-vector (in contravariant component form)

$$j^\mu = \rho_0 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \rho_0 u^\mu = \rho_0 \gamma(c, \mathbf{v}) = (\rho c, \rho \mathbf{v}) \tag{7.43}$$

with

$$\rho = \gamma \rho_0 \tag{7.44}$$

is the *four-current*.

The contravariant form of the four-del operator $\partial^\mu = \partial/\partial x_\mu$ is defined in equation (M.52) on page 242 and its covariant counterpart $\partial_\mu = \partial/\partial x^\mu$ in equation (M.53) on page 242, respectively. As is shown in example M.9 on page 258, the *d'Alembert operator* is the scalar product of the four-del with itself:

$$\Box^2 = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{7.45}$$

Since it has the characteristics of a four-scalar, the d'Alembert operator is invariant and, hence, the homogeneous wave equation $\Box^2 f(t, \mathbf{x}) = 0$ is Lorentz covariant.

### 7.3.1 The four-potential

If we introduce the *four-potential*

$$A^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right) \tag{7.46}$$

where $\Phi$ is the scalar potential and $\mathbf{A}$ the vector potential, defined in section 3.3 on page 35, we can write the uncoupled inhomogeneous wave equations, equations (3.17) on page 37, in the following compact (and covariant) way:

$$\Box^2 A^\mu = \mu_0 j^\mu \tag{7.47}$$

With the help of the above, we can formulate our electrodynamic equations covariantly. For instance, the covariant form of the *equation of continuity*, equation (4.20) on page 58 is

$$\partial_\mu j^\mu = 0 \tag{7.48}$$

and the *Lorenz-Lorentz gauge condition*, equation (3.16) on page 37, can be written

$$\partial_\mu A^\mu = 0 \tag{7.49}$$

The Lorenz-Lorentz gauge is sometimes called the *covariant gauge*. The gauge transformations (3.50) on page 43 in covariant form are

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu \Gamma(x^\nu) \tag{7.50}$$

If only one dimension Lorentz contracts (for instance, due to relative motion along the $x$ direction), a 3D spatial volume element transforms according to

$$\mathrm{d}V = \mathrm{d}^3x = \frac{1}{\gamma}\mathrm{d}V_0 = \mathrm{d}V_0\sqrt{1-\beta^2} = \mathrm{d}V_0\sqrt{1-\frac{v^2}{c^2}} \tag{7.51}$$

where $\mathrm{d}V_0$ denotes the volume element as measured in the rest system, then from equation (7.44) on the preceding page we see that

$$\rho \mathrm{d}V = \rho_0 \mathrm{d}V_0 \tag{7.52}$$

*i.e.* the charge in a given volume is conserved. We can therefore conclude that the elementary electric charge is a *universal constant*.

## 7.3.2 The Liénard-Wiechert potentials

Let us now solve the the inhomogeneous wave equations (3.17) on page 37 in vacuum for the case of a well-localised charge $q'$ at a source point defined by the position four-vector $x'^\mu \equiv (x'^0 = ct', x'^1, x'^2, x'^3)$. The field point (observation point) is denoted by the position four-vector $x^\mu = (x^0 = ct, x^1, x^2, x^3)$.

In the rest system we know that the solution is simply

$$(A^\mu)_0 = \left(\frac{\Phi}{c}, \mathbf{A}\right)_{\mathbf{v}=\mathbf{0}} = \left(\frac{q'}{4\pi\varepsilon_0}\frac{1}{c\,|\mathbf{x}-\mathbf{x}'|_0}, \mathbf{0}\right) \tag{7.53}$$

where $|\mathbf{x}-\mathbf{x}'|_0$ is the usual distance from the source point to the field point, evaluated in the rest system (signified by the index '0').

Let us introduce the relative position four-vector between the source point and the field point:

$$R^\mu = x^\mu - x'^\mu = \big(c(t-t'), \mathbf{x}-\mathbf{x}'\big) \tag{7.54}$$

Scalar multiplying this relative four-vector with itself, we obtain

$$R^\mu R_\mu = \big(c(t-t'), \mathbf{x}-\mathbf{x}'\big) \boldsymbol{\cdot} \big(c(t-t'), -(\mathbf{x}-\mathbf{x}')\big) = c^2(t-t')^2 - \left|\mathbf{x}-\mathbf{x}'\right|^2 \tag{7.55}$$

We know that in vacuum the signal (field) from the charge $q'$ at $x'^\mu$ propagates to $x^\mu$ with the speed of light $c$ so that

$$\left|\mathbf{x}-\mathbf{x}'\right| = c(t-t') \tag{7.56}$$

Inserting this into equation (7.55) above, we see that

$$R^\mu R_\mu = 0 \tag{7.57}$$

or that equation (7.54) on the preceding page can be written

$$R^\mu = (\left|\mathbf{x}-\mathbf{x}'\right|, \mathbf{x}-\mathbf{x}') \tag{7.58}$$

Now we want to find the correspondence to the rest system solution, equation (7.53) on the previous page, in an arbitrary inertial system. We note from equation (7.35) on page 163 that in the rest system

$$(u^\mu)_0 = \left( \frac{c}{\sqrt{1-\dfrac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1-\dfrac{v^2}{c^2}}} \right)_{\mathbf{v}=\mathbf{0}} = (c, \mathbf{0}) \tag{7.59}$$

and

$$(R^\mu)_0 = (\left|\mathbf{x}-\mathbf{x}'\right|, \mathbf{x}-\mathbf{x}')_0 = (\left|\mathbf{x}-\mathbf{x}'\right|_0, (\mathbf{x}-\mathbf{x}')_0) \tag{7.60}$$

As all scalar products, $u^\mu R_\mu$ is invariant, which means that we can evaluate it in any inertial system and it will have the same value in all other inertial systems. If we evaluate it in the rest system the result is:

$$\begin{aligned} u^\mu R_\mu &= \left(u^\mu R_\mu\right)_0 = (u^\mu)_0 (R_\mu)_0 \\ &= (c, \mathbf{0}) \boldsymbol{\cdot} (\left|\mathbf{x}-\mathbf{x}'\right|_0, -(\mathbf{x}-\mathbf{x}')_0) = c\left|\mathbf{x}-\mathbf{x}'\right|_0 \end{aligned} \tag{7.61}$$

We therefore see that the expression

$$A^\mu = \frac{q'}{4\pi\varepsilon_0} \frac{u^\mu}{c u^\nu R_\nu} \tag{7.62}$$

subject to the condition $R^\mu R_\mu = 0$ has the proper transformation properties (proper tensor form) and reduces, in the rest system, to the solution equation (7.53) on the previous page. It is therefore the correct solution, valid in any inertial system.

According to equation (7.35) on page 163 and equation (7.58) on the preceding page

$$u^\nu R_\nu = \gamma(c, \mathbf{v}) \cdot \left(\left|\mathbf{x}-\mathbf{x}'\right|, -(\mathbf{x}-\mathbf{x}')\right) = \gamma\left(c\left|\mathbf{x}-\mathbf{x}'\right| - \mathbf{v}\cdot(\mathbf{x}-\mathbf{x}')\right) \tag{7.63}$$

Generalising expression (7.1a) on page 154 to vector form:

$$\boldsymbol{\beta} = \beta\hat{\mathbf{v}} \stackrel{\text{def}}{\equiv} \frac{\mathbf{v}}{c} \tag{7.64}$$

and introducing

$$s \stackrel{\text{def}}{\equiv} \left|\mathbf{x}-\mathbf{x}'\right| - \frac{\mathbf{v}\cdot(\mathbf{x}-\mathbf{x}')}{c} \equiv \left|\mathbf{x}-\mathbf{x}'\right| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}') \tag{7.65}$$

we can write

$$u^\nu R_\nu = \gamma c s \tag{7.66}$$

and

$$\frac{u^\mu}{c u^\nu R_\nu} = \left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s}\right) \tag{7.67}$$

from which we see that the solution (7.62) can be written

$$A^\mu(x^\kappa) = \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s}\right) = \left(\frac{\Phi}{c}, \mathbf{A}\right) \tag{7.68}$$

where in the last step the definition of the four-potential, equation (7.46) on page 164, was used. Writing the solution in the ordinary 3D way, we conclude that for a very localised charge volume, moving relative an observer with a velocity $\mathbf{v}$, the scalar and vector potentials are given by the expressions

$$\Phi(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0}\frac{1}{s} = \frac{q'}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}')} \tag{7.69a}$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{s} = \frac{q'}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}}{|\mathbf{x}-\mathbf{x}'| - \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{x}')} \tag{7.69b}$$

These potentials are the *Liénard-Wiechert potentials* that we derived in a more complicated and restricted way in subsection 6.5.1 on page 123.

### 7.3.3 The electromagnetic field tensor

Consider a vectorial (cross) product $\mathbf{c}$ between two ordinary vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}\mathbf{c} = \mathbf{a} \times \mathbf{b} &= \epsilon_{ijk} a_i b_j \hat{\mathbf{x}}_k \\ &= (a_2 b_3 - a_3 b_2)\hat{\mathbf{x}}_1 + (a_3 b_1 - a_1 b_3)\hat{\mathbf{x}}_2 + (a_1 b_2 - a_2 b_1)\hat{\mathbf{x}}_3\end{aligned} \tag{7.70}$$

We notice that the $k$th component of the vector $\mathbf{c}$ can be represented as

$$c_k = a_i b_j - a_j b_i = c_{ij} = -c_{ji}, \quad i, j \neq k \tag{7.71}$$

In other words, the *pseudovector* $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ can be considered as an *antisymmetric tensor* of rank two. The same is true for the curl operator $\boldsymbol{\nabla}\times$ operating on a polar vector. For instance, the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.72}$$

can in this tensor notation be written

$$\frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j} = -\frac{\partial B_{ij}}{\partial t} \tag{7.73}$$

We know from chapter chapter 3 that the fields can be derived from the electromagnetic potentials in the following way:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{7.74a}$$

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{7.74b}$$

In component form, this can be written

$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = \partial_i A_j - \partial_j A_i \tag{7.75a}$$

$$E_i = -\frac{\partial \Phi}{\partial x^i} - \frac{\partial A_i}{\partial t} = -\partial_i \Phi - \partial_t A_i \tag{7.75b}$$

From this, we notice the clear difference between the *axial vector* (pseudovector) $\mathbf{B}$ and the *polar vector* ('ordinary vector') $\mathbf{E}$.

Our goal is to express the electric and magnetic fields in a tensor form where the components are functions of the covariant form of the four-potential, equation (7.46) on page 164:

$$A^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right) \tag{7.76}$$

Inspection of (7.76) and equation (7.75) above makes it natural to define the four-tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{7.77}$$

This anti-symmetric (*skew-symmetric*), four-tensor of rank 2 is called the *electromagnetic field tensor* or the *Faraday tensor*. In matrix representation, the *contravariant field tensor* can be written

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{7.78}$$

We note that the field tensor is a sort of four-dimensional curl of the four-potential vector $A^\mu$.

The *covariant field tensor* is obtained from the contravariant field tensor in the usual manner by index lowering

$$F_{\mu\nu} = \eta_{\mu\kappa}\eta_{\nu\lambda}F^{\kappa\lambda} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.79}$$

which in matrix representation becomes

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{7.80}$$

Comparing formula (7.80) above with formula (7.78) on the facing page we see that the covariant field tensor is obtained from the contravariant one by a transformation $\mathbf{E} \to -\mathbf{E}$.

That the two Maxwell source equations can be written

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{7.81}$$

is immediately observed by explicitly solving this covariant equation. Setting $\nu = 0$, corresponding to the first/leftmost column in the matrix representation of the covariant component form of the electromagnetic field tensor, $F^{\mu\nu}$, *i.e.* equation (7.78) on the preceding page, we see that

$$\begin{aligned}\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} &= 0 + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\ &= \frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E} = \mu_0 j^0 = \mu_0 c\rho\end{aligned} \tag{7.82}$$

or, equivalently (recalling that $\varepsilon_0\mu_0 = 1/c^2$),

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{7.83}$$

which we recognise as the Maxwell source equation for the electric field, equation (2.1a) on page 19.

For $\nu = 1$ [the second column in equation (7.78) on the preceding page], equation (7.81) above yields

$$\begin{aligned}\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} &= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + 0 + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ &= \mu_0 j^1 = \mu_0 \rho v_x\end{aligned} \tag{7.84}$$

This result can be rewritten as

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \varepsilon_0\mu_0\frac{\partial E_x}{\partial t} = \mu_0 j_x \tag{7.85}$$

or, equivalently, as

$$(\nabla \times \mathbf{B})_x = \mu_0 j_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} \tag{7.86}$$

and similarly for $\nu = 2, 3$. In summary, we can write the result in three-vector form as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}(t, \mathbf{x}) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{7.87}$$

which we recognise as the Maxwell source equation for the magnetic field, equation (2.1d) on page 19.

With the help of the fully antisymmetric pseudotensor of rank 4

$$\epsilon^{\mu\nu\kappa\lambda} = \begin{cases} 1 & \text{if } \mu, \nu, \kappa, \lambda \text{ is an } \textit{even} \text{ permutation of 0,1,2,3} \\ 0 & \text{if at least two of } \mu, \nu, \kappa, \lambda \text{ are equal} \\ -1 & \text{if } \mu, \nu, \kappa, \lambda \text{ is an } \textit{odd} \text{ permutation of 0,1,2,3} \end{cases} \tag{7.88}$$

which can be viewed as a 4D (rank 4) extension of the Levi-Civita rank 3 pseudotensor, formula (M.21) on page 237, we can introduce the *dual electromagnetic tensor*

$$^{\star}F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = -^{\star}F^{\nu\mu} \tag{7.89}$$

with the further property

$$^{\star}(^{\star}F^{\mu\nu}) = -F^{\mu\nu} \tag{7.90}$$

In matrix representation the dual field tensor is

$$(^{\star}F^{\mu\nu}) = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \tag{7.91}$$

*i.e.* the dual field tensor is obtained from the ordinary field tensor by the *duality transformation* $\mathbf{E} \to c\mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}/c$.

The covariant form of the two Maxwell field equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.92}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7.93}$$

can then be written

$$\partial_\mu {}^{\star}F^{\mu\nu} = 0 \tag{7.94}$$

Explicit evaluation shows that this corresponds to (no summation!)

$$\partial_\kappa F_{\mu\nu} + \partial_\mu F_{\nu\kappa} + \partial_\nu F_{\kappa\mu} = 0 \tag{7.95}$$

sometimes referred to as the *Jacobi identity*. Hence, equation (7.81) on page 169 and equation (7.95) above constitute Maxwell's equations in four-dimensional formalism.

It is interesting to note that equation (7.81) on page 169 and

$$\partial_\mu {}^\star F^{\mu\nu} = \mu_0 j_{\mathrm{m}}^\nu \tag{7.96}$$

where $j_{\mathrm{m}}$ is the *magnetic four-current*, represent the covariant form of Dirac's symmetrised Maxwell equations (2.2) on page 20.

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8

# ELECTROMAGNETIC FIELDS AND PARTICLES

In previous chapters, we calculated the electromagnetic fields and potentials from arbitrary, but prescribed distributions of charges and currents. In this chapter we first study the opposite situation, *viz.*, the dynamics of charged particles in arbitrary, but prescribed electromagnetic fields. Then we go on to consider the general problem of interaction between electric and magnetic fields and electrically charged particles. The analysis is based on Lagrangian and Hamiltonian methods, is fully covariant, and yields results which are relativistically correct.

## 8.1 Charged particles in an electromagnetic field

We first establish a relativistically correct theory describing the motion of charged particles in prescribed electric and magnetic fields. From these equations we may then calculate the charged particle dynamics in the most general case.

### 8.1.1 Covariant equations of motion

We will show that for our problem we can derive the correct equations of motion by using in four-dimensional $\mathbb{L}^4$ a function with similar properties as a Lagrange function in 3D and then apply a variational principle. We will also show that we can find a Hamiltonian-type function in 4D and solve the corresponding Hamilton-type equations to obtain the correct covariant formulation of classical electrodynamics.

#### 8.1.1.1 Lagrangian formalism

In analogy with particle dynamics in 3D Euclidean space, we introduce a generalised 4D action

$$S_4 = \int L_4(x^\mu, u^\mu)\, \mathrm{d}\tau \tag{8.1}$$

where $d\tau$ is the proper time defined via equation (7.17) on page 158, and $L_4$ acts as a kind of generalisation to the common 3D Lagrangian. As a result, the variational principle

$$\delta S_4 = \delta \int_{\tau_0}^{\tau_1} L_4(x^\mu, u^\mu)\, d\tau = 0 \tag{8.2}$$

with fixed endpoints $\tau_0, \tau_1$ must be fulfilled. We require that $L_4$ is a scalar invariant which does not contain higher than the second power of the four-velocity $u^\mu$ in order that the resulting equations of motion be linear.

According to formula (M.86) on page 249 the ordinary 3D Lagrangian is the difference between the kinetic and potential energies. A free particle has only kinetic energy. If the particle mass is $m_0$ then in 3D the kinetic energy is $m_0 v^2/2$. This suggests that in 4D the Lagrangian for a free particle should be

$$L_4^{\text{free}} = \frac{1}{2} m_0 u^\mu u_\mu \tag{8.3}$$

Again drawing inferences from analytical mechanics in 3D, we introduce a generalised interaction between the particles and the electromagnetic field with the help of the four-potential given by equation (7.76) on page 168 in the following way

$$L_4 = \frac{1}{2} m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu) \tag{8.4}$$

We call this the *four-Lagrangian* and shall now show how this function, together with the variation principle, formula (8.2), yields Lorentz covariant results which are physically correct.

The variation principle (8.2) with the 4D Lagrangian (8.4) inserted, leads to

$$\begin{aligned} \delta S_4 &= \delta \int_{\tau_0}^{\tau_1} \left( \frac{m_0}{2} u^\mu u_\mu + q u^\mu A_\mu(x^\nu) \right) d\tau \\ &= \int_{\tau_0}^{\tau_1} \left[ \frac{m_0}{2} \frac{\partial (u^\mu u_\mu)}{\partial u^\mu} \delta u^\mu + q \left( A_\mu \delta u^\mu + u^\mu \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu \right) \right] d\tau \\ &= \int_{\tau_0}^{\tau_1} \left[ m_0 u_\mu \delta u^\mu + q \left( A_\mu \delta u^\mu + u^\mu \partial_\nu A_\mu \delta x^\nu \right) \right] d\tau = 0 \end{aligned} \tag{8.5}$$

According to equation (7.35) on page 163, the four-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{8.6}$$

which means that we can write the variation of $u^\mu$ as a total derivative with respect to $\tau$:

$$\delta u^\mu = \delta \left( \frac{dx^\mu}{d\tau} \right) = \frac{d}{d\tau} (\delta x^\mu) \tag{8.7}$$

Inserting this into the first two terms in the last integral in equation (8.5) on the facing page, we obtain

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left( m_0 u_\mu \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) + qA_\mu \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x^\mu) + qu^\mu \partial_\nu A_\mu \delta x^\nu \right) \mathrm{d}\tau \tag{8.8}$$

Partial integration in the two first terms in the right hand member of (8.8) gives

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left( -m_0 \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} \delta x^\mu - q \frac{\mathrm{d}A_\mu}{\mathrm{d}\tau} \delta x^\mu + qu^\mu \partial_\nu A_\mu \delta x^\nu \right) \mathrm{d}\tau \tag{8.9}$$

where the integrated parts do not contribute since the variations at the endpoints vanish. A change of irrelevant summation index from $\mu$ to $\nu$ in the first two terms of the right hand member of (8.9) yields, after moving the ensuing common factor $\delta x^\nu$ outside the parenthesis, the following expression:

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left( -m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} - q \frac{\mathrm{d}A_\nu}{\mathrm{d}\tau} + qu^\mu \partial_\nu A_\mu \right) \delta x^\nu \, \mathrm{d}\tau \tag{8.10}$$

Applying well-known rules of differentiation and the expression (7.35) for the four-velocity, we can express $\mathrm{d}A_\nu/\mathrm{d}\tau$ as follows:

$$\frac{\mathrm{d}A_\nu}{\mathrm{d}\tau} = \frac{\partial A_\nu}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \partial_\mu A_\nu u^\mu \tag{8.11}$$

By inserting this expression (8.11) into the second term in right-hand member of equation (8.10) above, and noting the common factor $qu^\mu$ of the resulting term and the last term, we obtain the final variational principle expression

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left[ -m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} + qu^\mu \left( \partial_\nu A_\mu - \partial_\mu A_\nu \right) \right] \delta x^\nu \, \mathrm{d}\tau \tag{8.12}$$

Since, according to the variational principle, this expression shall vanish and $\delta x^\nu$ is arbitrary between the fixed end points $\tau_0$ and $\tau_1$, the expression inside $[\ ]$ in the integrand in the right hand member of equation (8.12) above must vanish. In other words, we have found an equation of motion for a charged particle in a prescribed electromagnetic field:

$$m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = qu^\mu \left( \partial_\nu A_\mu - \partial_\mu A_\nu \right) \tag{8.13}$$

With the help of formula (7.79) on page 169 for the covariant component form of the field tensor, we can express this equation in terms of the electromagnetic field tensor in the following way:

$$m_0 \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = qu^\mu F_{\nu\mu} \tag{8.14}$$

This is the sought-for covariant equation of motion for a particle in an electromagnetic field. It is often referred to as the *Minkowski equation*. As the reader may easily verify, the spatial part of this 4-vector equation is the covariant (relativistically correct) expression for the *Newton-Lorentz force equation*.

#### 8.1.1.2 Hamiltonian formalism

The usual *Hamilton equations* for a 3D space are given by equation (M.98) on page 250 in appendix M on page 231. These six first-order partial differential equations are

$$\frac{\partial H}{\partial p_i} = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{8.15a}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{8.15b}$$

where $H(p_i, q_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$ is the ordinary 3D Hamiltonian, $q_i$ is a *generalised coordinate* and $p_i$ is its *canonically conjugate momentum*.

We seek a similar set of equations in 4D space. To this end we introduce a *canonically conjugate four-momentum* $p^\mu$ in an analogous way as the ordinary 3D conjugate momentum

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} \tag{8.16}$$

and utilise the four-velocity $u^\mu$, as given by equation (7.35) on page 163, to define the *four-Hamiltonian*

$$H_4 = p^\mu u_\mu - L_4 \tag{8.17}$$

With the help of these, the position four-vector $x^\mu$, considered as the *generalised four-coordinate*, and the invariant line element $\mathrm{d}s$, defined in equation (7.17) on page 158, we introduce the following eight partial differential equations:

$$\frac{\partial H_4}{\partial p^\mu} = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} \tag{8.18a}$$

$$\frac{\partial H_4}{\partial x^\mu} = -\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} \tag{8.18b}$$

which form the *four-dimensional Hamilton equations*.

Our strategy now is to use equation (8.16) above and equations (8.18) above to derive an explicit algebraic expression for the canonically conjugate momentum four-vector. According to equation (7.40) on page 163, $c$ times a four-momentum has a zeroth (time) component which we can identify with the total energy. Hence we require that the component $p^0$ of the conjugate four-momentum vector defined according to equation (8.16) above be identical to the ordinary 3D Hamiltonian $H$ divided by $c$ and hence that this $cp^0$ solves the Hamilton equations, equations (8.15) above.[1] This latter consistency check is left as an exercise to the reader.

[1] Recall that in 3D, the Hamiltonian equals the total energy.

Using the definition of $H_4$, equation (8.17) on the preceding page, and the expression for $L_4$, equation (8.4) on page 174, we obtain

$$H_4 = p^\mu u_\mu - L_4 = p^\mu u_\mu - \frac{1}{2}m_0 u^\mu u_\mu - q u_\mu A^\mu(x^\nu) \tag{8.19}$$

Furthermore, from the definition (8.16) of the canonically conjugate four-momentum $p^\mu$, we see that

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} = \frac{\partial}{\partial u_\mu}\left(\frac{1}{2}m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu)\right) = m_0 u^\mu + q A^\mu \tag{8.20}$$

Inserting this into (8.19), we obtain

$$H_4 = m_0 u^\mu u_\mu + q A^\mu u_\mu - \frac{1}{2}m_0 u^\mu u_\mu - q u^\mu A_\mu(x^\nu) = \frac{1}{2}m_0 u^\mu u_\mu \tag{8.21}$$

Since the four-velocity scalar-multiplied by itself is $u^\mu u_\mu = c^2$, we clearly see from equation (8.21) above that $H_4$ is indeed a scalar invariant, whose value is simply

$$H_4 = \frac{m_0 c^2}{2} \tag{8.22}$$

However, at the same time (8.20) provides the algebraic relationship

$$u^\mu = \frac{1}{m_0}\left(p^\mu - q A^\mu\right) \tag{8.23}$$

and if this is used in (8.21) to eliminate $u^\mu$, one gets

$$\begin{aligned} H_4 &= \frac{m_0}{2}\left(\frac{1}{m_0}\left(p^\mu - qA^\mu\right)\frac{1}{m_0}\left(p_\mu - qA_\mu\right)\right) \\ &= \frac{1}{2m_0}\left(p^\mu - qA^\mu\right)\left(p_\mu - qA_\mu\right) \\ &= \frac{1}{2m_0}\left(p^\mu p_\mu - 2qA^\mu p_\mu + q^2 A^\mu A_\mu\right) \end{aligned} \tag{8.24}$$

That this four-Hamiltonian yields the correct covariant equation of motion can be seen by inserting it into the four-dimensional Hamilton equations (8.18) and using the relation (8.23):

$$\begin{aligned} \frac{\partial H_4}{\partial x^\mu} &= -\frac{q}{m_0}(p^\nu - qA^\nu)\frac{\partial A_\nu}{\partial x^\mu} \\ &= -\frac{q}{m_0} m_0 u^\nu \frac{\partial A_\nu}{\partial x^\mu} \\ &= -q u^\nu \frac{\partial A_\nu}{\partial x^\mu} \\ &= -\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = -m_0\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - q\frac{\partial A_\mu}{\partial x^\nu}u^\nu \end{aligned} \tag{8.25}$$

where in the last step equation (8.20) on the preceding page was used. Rearranging terms, and using equation (7.79) on page 169, we obtain

$$m_0 \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = qu^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = qu^\nu F_{\mu\nu} \tag{8.26}$$

which is identical to the covariant equation of motion equation (8.14) on page 175. We can therefore safely conclude that the Hamiltonian in question yields correct results.

Recalling expression (7.46) on page 164 for the four-potential, and representing the canonically conjugate four-momentum as $p^\mu = (p^0, \mathbf{p})$, we obtain the following scalar products:

$$p^\mu p_\mu = (p^0)^2 - (\mathbf{p})^2 \tag{8.27a}$$

$$A^\mu p_\mu = \frac{1}{c}\Phi p^0 - (\mathbf{p} \cdot \mathbf{A}) \tag{8.27b}$$

$$A^\mu A_\mu = \frac{1}{c^2}\Phi^2 - (\mathbf{A})^2 \tag{8.27c}$$

Inserting these explicit expressions into equation (8.24) on the preceding page, and using the fact that $H_4$ is equal to the scalar value $m_0c^2/2$, as derived in equation (8.22) on the previous page, we obtain the equation

$$\frac{m_0c^2}{2} = \frac{1}{2m_0}\left[(p^0)^2 - (\mathbf{p})^2 - \frac{2}{c}q\Phi p^0 + 2q(\mathbf{p} \cdot \mathbf{A}) + \frac{q^2}{c^2}\Phi^2 - q^2(\mathbf{A})^2\right] \tag{8.28}$$

which is a second-order algebraic equation in $p^0$:

$$(p^0)^2 - \frac{2q}{c}\Phi p^0 - \underbrace{\left[(\mathbf{p})^2 - 2q\mathbf{p} \cdot \mathbf{A} + q^2(\mathbf{A})^2\right]}_{(\mathbf{p}-q\mathbf{A})^2} + \frac{q^2}{c^2}\Phi^2 - m_0^2c^2 = 0 \tag{8.29}$$

with two possible solutions

$$p^0 = \frac{q}{c}\Phi \pm \sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2c^2} \tag{8.30}$$

Since the zeroth component (time component) $p^0$ of a four-momentum vector $p^\mu$ multiplied by $c$ represents the energy [*cf.* equation (7.40) on page 163], the positive solution in equation (8.30) above must be identified with the ordinary Hamilton function $H$ divided by $c$. Consequently,

$$H \equiv cp^0 = q\Phi + c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2c^2} \tag{8.31}$$

is the ordinary 3D Hamilton function for a charged particle moving in scalar and vector potentials associated with prescribed electric and magnetic fields.

The ordinary Lagrange and Hamilton functions $L$ and $H$ are related to each other by the 3D transformation [*cf.* the 4D transformation (8.17) between $L_4$ and $H_4$]

$$L = \mathbf{p} \cdot \mathbf{v} - H \tag{8.32}$$

Using the the explicit expressions given by equation (8.31) and equation (8.32), we obtain the explicit expression for the ordinary 3D Lagrange function

$$L = \mathbf{p} \cdot \mathbf{v} - q\Phi - c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2 c^2} \tag{8.33}$$

and if we make the identification

$$\mathbf{p} - q\mathbf{A} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\mathbf{v} \tag{8.34}$$

where the quantity $m\mathbf{v}$ is the usual *kinetic momentum*, we can rewrite this expression for the ordinary Lagrangian as follows:

$$\begin{aligned} L &= q\mathbf{A} \cdot \mathbf{v} + mv^2 - q\Phi - c\sqrt{m^2 v^2 + m_0^2 c^2} \\ &= mv^2 - q(\Phi - \mathbf{A} \cdot \mathbf{v}) - mc^2 = -q\Phi + q\mathbf{A} \cdot \mathbf{v} - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{8.35}$$

What we have obtained is the relativistically correct (covariant) expression for the Lagrangian describing the mechanical motion of a charged particle in scalar and vector potentials associated with prescribed electric and magnetic fields.

## 8.2 Covariant field theory

So far, we have considered two classes of problems. Either we have calculated the fields from given, prescribed distributions of charges and currents, or we have derived the equations of motion for charged particles in given, prescribed fields. Let us now put the fields and the particles on an equal footing and present a theoretical description which treats the fields, the particles, and their interactions in a unified way. This involves transition to a field picture with an infinite number of degrees of freedom. We shall first consider a simple mechanical problem whose solution is well known. Then, drawing inferences from this model problem, we apply a similar view on the electromagnetic problem.

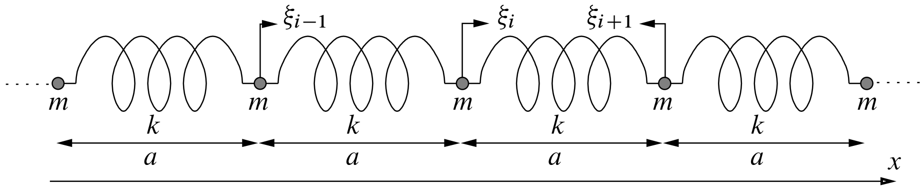


Figure 8.1: A one-dimensional chain consisting of $N$ discrete, identical mass points $m$, connected to their neighbours with identical, ideal springs with spring constants $k$. The equilibrium distance between the neighbouring mass points is $a$ and $\xi_{i-1}(t)$, $\xi_i(t)$, $\xi_{i+1}(t)$ are the instantaneous deviations, along the $x$ axis, of positions of the $(i-1)$th, $i$th, and $(i+1)$th mass point, respectively.

### 8.2.1 Lagrange-Hamilton formalism for fields and interactions

Consider the situation, illustrated in figure 8.1, with $N$ identical mass points, each with mass $m$ and connected to its neighbour along a one-dimensional straight line, which we choose to be the $x$ axis, by identical ideal springs with spring constants $k$ (*Hooke's law*). At equilibrium the mass points are at rest, distributed evenly with a distance $a$ to their two nearest neighbours so that the equilibrium coordinate for the $i$th particle is $\mathbf{x}_i = ia\hat{\mathbf{x}}$ . After perturbation, the motion of mass point $i$ will be a one-dimensional oscillatory motion along $\hat{\mathbf{x}}$. Let us denote the deviation for mass point $i$ from its equilibrium position by $\xi_i(t)\hat{\mathbf{x}}$.

As is well known, the solution to this mechanical problem can be obtained if we can find a *Lagrangian* (*Lagrange function*) $L$ which satisfies the variational equation

$$\delta \int L(\xi_i, \dot{\xi}_i, t)\,\mathrm{d}t = 0 \tag{8.36}$$

According to equation (M.86) on page 249, the Lagrangian is $L = T - V$ where $T$ denotes the *kinetic energy* and $V$ the *potential energy* of a classical mechanical system with *conservative forces*. In our case the Lagrangian is

$$L = \frac{1}{2}\sum_{i=1}^{N}\left[m\dot{\xi}_i^2 - k(\xi_{i+1} - \xi_i)^2\right] \tag{8.37}$$

Let us write the Lagrangian, as given by equation (8.37) above, in the following way:

$$L = \sum_{i=1}^{N} a\mathscr{L}_i \tag{8.38}$$

where

$$\mathscr{L}_i = \frac{1}{2}\left[\frac{m}{a}\dot{\xi}_i^2 - ka\left(\frac{\xi_{i+1} - \xi_i}{a}\right)^2\right] \tag{8.39}$$

is the so called linear *Lagrange density*, measured in $\mathrm{J\,m^{-1}}$. If we now let $N \to \infty$ and, at the same time, let the springs become infinitesimally short according to the following scheme:

$$a \to \mathrm{d}x \tag{8.40a}$$

$$\frac{m}{a} \to \frac{\mathrm{d}m}{\mathrm{d}x} = \mu \qquad \textit{linear mass density} \tag{8.40b}$$

$$ka \to Y \qquad \textit{Young's modulus} \tag{8.40c}$$

$$\frac{\xi_{i+1} - \xi_i}{a} \to \frac{\partial \xi}{\partial x} \tag{8.40d}$$

we obtain

$$L = \int \mathscr{L}\,\mathrm{d}x \tag{8.41}$$

where

$$\mathscr{L}\left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x}, t\right) = \frac{1}{2}\left[\mu\left(\frac{\partial \xi}{\partial t}\right)^2 - Y\left(\frac{\partial \xi}{\partial x}\right)^2\right] \tag{8.42}$$

Notice how we made a transition from a discrete description, in which the mass points were identified by a discrete integer variable $i = 1, 2, \ldots, N$, to a continuous description, where the infinitesimal mass points were instead identified by a continuous real parameter $x$, namely their position along $\hat{\mathbf{x}}$.

A consequence of this transition is that the number of degrees of freedom for the system went from the finite number $N$ to infinity! Another consequence is that $\mathscr{L}$ has now become dependent also on the partial derivative with respect to $x$ of the 'field coordinate' $\xi$. But, as we shall see, the transition is well worth the cost because it allows us to treat all fields, be it classical scalar or vectorial fields, or wave functions, spinors and other fields that appear in quantum physics, on an equal footing.

Under the assumption of time independence and fixed endpoints, the variation principle (8.36) on the facing page yields:

$$\begin{aligned}\delta \int L\,dt &= \delta \iint \mathscr{L}\left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t \\ &= \iint \left[\frac{\partial \mathscr{L}}{\partial \xi}\delta\xi + \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial t}\right)}\delta\left(\frac{\partial \xi}{\partial t}\right) + \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x}\right)}\delta\left(\frac{\partial \xi}{\partial x}\right)\right]\mathrm{d}x\,\mathrm{d}t = 0\end{aligned} \tag{8.43}$$

As before, the last integral can be integrated by parts. This results in the expression

$$\iint \left[\frac{\partial \mathscr{L}}{\partial \xi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial t}\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x}\right)}\right)\right]\delta\xi\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{8.44}$$

where the variation is arbitrary (and the endpoints fixed). This means that the integrand itself must vanish. If we introduce the *functional derivative*

$$\frac{\delta \mathscr{L}}{\delta \xi} = \frac{\partial \mathscr{L}}{\partial \xi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x}\right)}\right) \tag{8.45}$$

we can express this as

$$\frac{\delta \mathscr{L}}{\delta \xi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial t}\right)}\right) = 0 \tag{8.46}$$

which is the one-dimensional *Euler-Lagrange equation*.

Inserting the linear mass point chain Lagrangian density, equation (8.42) on the previous page, into equation (8.46) above, we obtain the equation of motion for our one-dimensional linear mechanical structure. It is:

$$\mu \frac{\partial^2 \xi}{\partial t^2} - Y \frac{\partial^2 \xi}{\partial x^2} = 0 \tag{8.47a}$$

or

$$\left(\frac{1}{v_\varphi^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\xi = 0 \tag{8.47b}$$

*i.e.* the one-dimensional wave equation for compression waves which propagate with phase speed $v_\varphi = \sqrt{Y/\mu}$ along the linear structure.

A generalisation of the above 1D results to a three-dimensional continuum is straightforward. For this 3D case we get the variational principle

$$\begin{aligned}\delta \int L \,\mathrm{d}t &= \delta \iint \mathscr{L}\,\mathrm{d}^3x\,\mathrm{d}t = \delta \int \mathscr{L}\left(\xi, \frac{\partial \xi}{\partial x^\mu}\right)\mathrm{d}^4x \\ &= \iint \left[\frac{\partial \mathscr{L}}{\partial \xi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x^\mu}\right)}\right)\right]\delta\xi\,\mathrm{d}^4x = 0\end{aligned} \tag{8.48}$$

where the variation $\delta\xi$ is arbitrary and the endpoints are fixed. This means that the integrand itself must vanish:

$$\frac{\partial \mathscr{L}}{\partial \xi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x^\mu}\right)}\right) = \frac{\partial \mathscr{L}}{\partial \xi} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \xi)}\right) = 0 \tag{8.49}$$

This constitutes the four-dimensional *Euler-Lagrange equations*.

Introducing the *three-dimensional functional derivative*

$$\frac{\delta \mathscr{L}}{\delta \xi} = \frac{\partial \mathscr{L}}{\partial \xi} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial x^i}\right)}\right) \tag{8.50}$$

we can express this as

$$\frac{\delta \mathscr{L}}{\delta \xi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial t}\right)}\right) = 0 \tag{8.51}$$

In analogy with particle mechanics (finite number of degrees of freedom), we may introduce the *canonically conjugate momentum density*

$$\pi(x^\mu) = \pi(t, \mathbf{x}) = \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \xi}{\partial t}\right)} \tag{8.52}$$

and define the *Hamilton density*

$$\mathscr{H}\left(\pi, \xi, \frac{\partial \xi}{\partial x^i}; t\right) = \pi \frac{\partial \xi}{\partial t} - \mathscr{L}\left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x^i}\right) \tag{8.53}$$

If, as usual, we differentiate this expression and identify terms, we obtain the following *Hamilton density equations*

$$\frac{\partial \mathscr{H}}{\partial \pi} = \frac{\partial \xi}{\partial t} \tag{8.54a}$$

$$\frac{\delta \mathscr{H}}{\delta \xi} = -\frac{\partial \pi}{\partial t} \tag{8.54b}$$

The Hamilton density functions are in many ways similar to the ordinary Hamilton functions for a system of a finite number of particles and lead to similar results. However, they describe the dynamics of a continuous system of infinitely many degrees of freedom.

#### 8.2.1.1 The electromagnetic field

Above, when we described the mechanical field, we used a scalar field $\xi(t, \mathbf{x})$. If we want to describe the electromagnetic field in terms of a Lagrange density $\mathscr{L}$ and Euler-Lagrange equations, it comes natural to express $\mathscr{L}$ in terms of the four-potential $A^\mu(x^\kappa)$.

The entire system of particles and fields consists of a mechanical part, a field part and an interaction part. We therefore assume that the total Lagrange density $\mathscr{L}^{\text{tot}}$ for this system can be expressed as

$$\mathscr{L}^{\text{tot}} = \mathscr{L}^{\text{mech}} + \mathscr{L}^{\text{interaction}} + \mathscr{L}^{\text{field}} \tag{8.55}$$

where the mechanical part has to do with the particle motion (kinetic energy). It is given by $L_4/V$ where $L_4$ is given by equation (8.3) on page 174 and $V$ is the volume. Expressed in the *rest mass density* $\varrho_0$, the *mechanical Lagrange density* can be written

$$\mathscr{L}^{\text{mech}} = \frac{1}{2}\varrho_0 u^\mu u_\mu \tag{8.56}$$

The $\mathscr{L}^{\text{interaction}}$ part describes the interaction between the charged particles and the external electromagnetic field. A convenient expression for this *interaction Lagrange density* is

$$\mathscr{L}^{\text{interaction}} = j^{\mu} A_{\mu} \tag{8.57}$$

For the field part $\mathscr{L}^{\text{field}}$ we choose the difference between magnetic and electric energy density (in analogy with the difference between kinetic and potential energy in a mechanical field). With the help of the field tensor, we express this *field Lagrange density* as

$$\mathscr{L}^{\text{field}} = \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{8.58}$$

so that the total Lagrangian density can be written

$$\mathscr{L}^{\text{tot}} = \frac{1}{2}\varrho_0 u^{\mu} u_{\mu} + j^{\mu} A_{\mu} + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{8.59}$$

From this we can calculate all physical quantities.

Using $\mathscr{L}^{\text{tot}}$ in the 3D Euler-Lagrange equations, equation (8.49) on page 182 (with $\xi$ replaced by $A_{\nu}$), we can derive the dynamics for the whole system. For instance, the electromagnetic part of the Lagrangian density

$$\mathscr{L}^{\text{EM}} = \mathscr{L}^{\text{interaction}} + \mathscr{L}^{\text{field}} = j^{\nu} A_{\nu} + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{8.60}$$

inserted into the Euler-Lagrange equations, expression (8.49) on page 182, yields two of Maxwell's equations. To see this, we note from equation (8.60) above and the results in Example 8.1 that

$$\frac{\partial \mathscr{L}^{\text{EM}}}{\partial A_{\nu}} = j^{\nu} \tag{8.61}$$

Furthermore,

$$\begin{aligned}
\partial_{\mu}\left[\frac{\partial \mathscr{L}^{\text{EM}}}{\partial(\partial_{\mu} A_{\nu})}\right] &= \frac{1}{4\mu_0}\partial_{\mu}\left[\frac{\partial}{\partial(\partial_{\mu} A_{\nu})}\left(F^{\kappa\lambda} F_{\kappa\lambda}\right)\right] \\
&= \frac{1}{4\mu_0}\partial_{\mu}\left\{\frac{\partial}{\partial(\partial_{\mu} A_{\nu})}\left[(\partial^{\kappa} A^{\lambda} - \partial^{\lambda} A^{\kappa})(\partial_{\kappa} A_{\lambda} - \partial_{\lambda} A_{\kappa})\right]\right\} \\
&= \frac{1}{4\mu_0}\partial_{\mu}\left\{\frac{\partial}{\partial(\partial_{\mu} A_{\nu})}\left[\partial^{\kappa} A^{\lambda}\partial_{\kappa} A_{\lambda} - \partial^{\kappa} A^{\lambda}\partial_{\lambda} A_{\kappa}\right.\right. \\
&\qquad \left.\left. - \partial^{\lambda} A^{\kappa}\partial_{\kappa} A_{\lambda} + \partial^{\lambda} A^{\kappa}\partial_{\lambda} A_{\kappa}\right]\right\} \\
&= \frac{1}{2\mu_0}\partial_{\mu}\left[\frac{\partial}{\partial(\partial_{\mu} A_{\nu})}\left(\partial^{\kappa} A^{\lambda}\partial_{\kappa} A_{\lambda} - \partial^{\kappa} A^{\lambda}\partial_{\lambda} A_{\kappa}\right)\right]
\end{aligned} \tag{8.62}$$

But

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial^\kappa A^\lambda \partial_\kappa A_\lambda\right) &= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial^\kappa A^\lambda \\
&= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\eta^{\kappa\alpha}\partial_\alpha \eta^{\lambda\beta} A_\beta \\
&= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \eta^{\kappa\alpha}\eta^{\lambda\beta}\partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\alpha A_\beta \\
&= \partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\kappa A_\lambda + \partial^\alpha A^\beta \frac{\partial}{\partial(\partial_\mu A_\nu)}\partial_\alpha A_\beta \\
&= 2\partial^\mu A^\nu
\end{aligned} \tag{8.63}$$

Similarly,

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial^\kappa A^\lambda \partial_\lambda A_\kappa\right) = 2\partial^\nu A^\mu \tag{8.64}$$

so that

$$\partial_\mu \left[\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial(\partial_\mu A_\nu)}\right] = \frac{1}{\mu_0}\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{1}{\mu_0}\partial_\mu F^{\mu\nu} \tag{8.65}$$

This means that the Euler-Lagrange equations, expression (8.49) on page 182, for the Lagrangian density $\mathscr{L}^{\mathrm{EM}}$ and with $A_\nu$ as the field quantity become

$$\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial A_\nu} - \partial_\mu \left[\frac{\partial \mathscr{L}^{\mathrm{EM}}}{\partial(\partial_\mu A_\nu)}\right] = j^\nu - \frac{1}{\mu_0}\partial_\mu F^{\mu\nu} = 0 \tag{8.66}$$

or

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{8.67}$$

which, according to equation (7.81) on page 169, is a Lorentz covariant formulation of Maxwell's source equations.

▷Field energy difference expressed in the field tensor —— EXAMPLE 8.1

Show, by explicit calculation, that

$$\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2\right) \tag{8.68}$$

*i.e.* the difference between the magnetic and electric field energy densities.

From formula (7.78) on page 168 we recall that

$$\left(F^{\mu\nu}\right) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{8.69}$$

and from formula (7.80) on page 169 that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{8.70}$$

where $\mu$ denotes the row number and $\nu$ the column number. Then, Einstein summation and direct substitution yields

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{00}F_{00} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} \\ &\quad + F^{10}F_{10} + F^{11}F_{11} + F^{12}F_{12} + F^{13}F_{13} \\ &\quad + F^{20}F_{20} + F^{21}F_{21} + F^{22}F_{22} + F^{23}F_{23} \\ &\quad + F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} + F^{33}F_{33} \\ &= 0 - E_x^2/c^2 - E_y^2/c^2 - E_z^2/c^2 \\ &\quad - E_x^2/c^2 + 0 + B_z^2 + B_y^2 \\ &\quad - E_y^2/c^2 + B_z^2 + 0 + B_x^2 \\ &\quad - E_z^2/c^2 + B_y^2 + B_x^2 + 0 \\ &= -2E_x^2/c^2 - 2E_y^2/c^2 - 2E_z^2/c^2 + 2B_x^2 + 2B_y^2 + 2B_z^2 \\ &= -2E^2/c^2 + 2B^2 = 2(B^2 - E^2/c^2) \end{aligned} \tag{8.71}$$

or [*cf.* equation (2.44a) on page 30]

$$\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \frac{1}{c^2\mu_0}E^2\right) = \frac{1}{2}\left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2\right) = -\frac{\varepsilon_0}{2}(E^2 - c^2B^2) \tag{8.72}$$

where, in the last step, the identity $\varepsilon_0\mu_0 = 1/c^2$ was used. QED ■

End of example 8.1 ◁

### 8.2.1.2 Other fields

In general, the dynamic equations for most fields, and not only electromagnetic ones, can be derived from a Lagrangian density together with a variational principle (the Euler-Lagrange equations). Both linear and non-linear fields are studied with this technique. As a simple example, consider a real, scalar field $\xi$ which has the following Lagrange density:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\xi\partial^\mu\xi - m^2\xi^2\right) \tag{8.73}$$

Insertion into the 1D Euler-Lagrange equation, equation (8.46) on page 182, yields the dynamic equation

$$(\Box^2 - m^2)\xi = 0 \tag{8.74}$$

with the solution

$$\xi = \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\frac{\mathrm{e}^{-m|\mathbf{x}|}}{|\mathbf{x}|} \tag{8.75}$$

which describes the *Yukawa meson field* for a scalar meson with mass $m$. With

$$\pi = \frac{1}{c^2}\frac{\partial \xi}{\partial t} \tag{8.76}$$

we obtain the Hamilton density

$$\mathscr{H} = \frac{1}{2}\left[c^2\pi^2 + (\boldsymbol{\nabla}\xi)^2 + m^2\xi^2\right] \tag{8.77}$$

which is positive definite.

Another Lagrangian density which has attracted quite some interest is the *Proca Lagrangian*

$$\mathscr{L}^{\mathrm{EM}} = \mathscr{L}^{\mathrm{interaction}} + \mathscr{L}^{\mathrm{field}} = j^{\nu}A_{\nu} + \frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^{\mu}A_{\mu} \tag{8.78}$$

where $m$ is a mass term. This leads to the dynamic equation

$$\partial_{\mu}F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^{\nu} = \mu_0 j^{\nu} \tag{8.79}$$

Clearly, this equation describes an electromagnetic field with mass $m$, or, in other words, *massive photons*. If massive photons do exist, large-scale magnetic fields, including those of the earth and galactic spiral arms, should be significantly modified from what they are to yield measurable discrepancies from their usual form. Space experiments of this kind on board satellites have led to stringent upper bounds on the photon mass. If the photon really has a mass, it will have an impact on electrodynamics as well as on cosmology and astrophysics.

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# 9

# ELECTROMAGNETIC FIELDS AND MATTER

The microscopic Maxwell equations derived in chapter 1, which in chapter 2 were chosen as the axiomatic basis for the treatment in the remainder of the book, are valid on all scales where a classical description is good. They provide a correct physical picture for arbitrary field and source distributions, on macroscopic and, under certain assumptions, microscopic scales. A more complete and accurate theory, valid also when quantum effects are significant, is provided by *quantum electrodynamics*. *QED* gives a consistent description of how electromagnetic fields are quantised into *photons* and describes their intrinsic and extrinsic properties. However, this theory is beyond the scope of the current book.

In a material medium, be it in a solid, fluid or gaseous state or a combination thereof, it is sometimes convenient to replace the Maxwell-Lorentz equations (2.1) on page 19 by the corresponding macroscopic Maxwell equations in which auxiliary, derived fields are introduced. These auxiliary fields, *viz.*, the *electric displacement vector* $\mathbf{D}$ (measured in $\mathrm{C\,m^{-2}}$) and the *magnetising field* $\mathbf{H}$ (measured in $\mathrm{A\,m^{-1}}$), incorporate intrinsic electromagnetic properties of macroscopic matter, or properties that appear when the medium is immersed fully or partially in an electromagnetic field. Consequently, they represent, respectively, electric and magnetic field quantities in which, in an average sense, the material properties of the substances are already included. In the most general case, these derived fields are complicated, possibly non-local and nonlinear, functions of the primary fields $\mathbf{E}$ and $\mathbf{B}$ :

$$\mathbf{D} = \mathbf{D}(t, \mathbf{x}; \mathbf{E}, \mathbf{B}) \tag{9.1a}$$

$$\mathbf{H} = \mathbf{H}(t, \mathbf{x}; \mathbf{E}, \mathbf{B}) \tag{9.1b}$$

An example of this are *chiral media*.

A general treatment of these fields will not be included here. Only simplified, but important and illuminating examples will be given.

## 9.1 Maxwell's macroscopic theory

Under certain conditions, for instance for small magnitudes of the primary field strengths $\mathbf{E}$ and $\mathbf{B}$, we may assume that the response of a substance to the fields can be approximated by a linear one so that

$$\mathbf{D} \approx \varepsilon\mathbf{E} \tag{9.2}$$

$$\mathbf{H} \approx \mu^{-1}\mathbf{B} \tag{9.3}$$

*i.e.* that the *electric displacement vector* $\mathbf{D}(t,\mathbf{x})$ is only linearly dependent on the *electric field* $\mathbf{E}(t,\mathbf{x})$, and the *magnetising field* $\mathbf{H}(t,\mathbf{x})$ is only linearly dependent on the *magnetic field* $\mathbf{B}(t,\mathbf{x})$. In this chapter we derive these linearised forms, and then consider a simple, explicit linear model for a medium from which we derive the expression for the *dielectric permittivity* $\varepsilon(t,\mathbf{x})$, the *magnetic susceptibility* $\mu(t,\mathbf{x})$, and the *refractive index* or *index of refraction* $n(t,\mathbf{x})$ of this medium. Using this simple model, we study certain interesting aspects of the propagation of electromagnetic particles and waves in the medium.

### 9.1.1 Polarisation and electric displacement

By writing the first microscopic Maxwell-Lorentz equation (2.1a) on page 19 as in equation (6.26) on page 110, *i.e.* in a form where the total charge density $\rho(t,\mathbf{x})$ is split into the charge density for free, 'true' charges, $\rho^{\text{true}}$, and the charge density, $\rho^{\text{pol}}$, for bound *polarisation charges* induced by the applied field $\mathbf{E}$, as

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho^{\text{total}}(t,\mathbf{x})}{\varepsilon_0} = \frac{\rho^{\text{true}}(t,\mathbf{x}) + \rho^{\text{pol}}(t,\mathbf{x})}{\varepsilon_0} = \frac{\rho^{\text{true}}(t,\mathbf{x}) - \boldsymbol{\nabla}\cdot\mathbf{P}(t,\mathbf{x})}{\varepsilon_0} \tag{9.4}$$

and at the same time introducing the *electric displacement vector* ($\mathrm{C\,m^{-2}}$)

$$\mathbf{D}(t,\mathbf{x}) = \varepsilon_0\mathbf{E}(t,\mathbf{x}) + \mathbf{P}(t,\mathbf{x}) \tag{9.5}$$

one can reshuffle expression (9.4) above to obtain

$$\boldsymbol{\nabla}\cdot[\varepsilon_0\mathbf{E}(t,\mathbf{x}) + \mathbf{P}(t,\mathbf{x})] = \boldsymbol{\nabla}\cdot\mathbf{D}(t,\mathbf{x}) = \rho^{\text{true}}(t,\mathbf{x}) \tag{9.6}$$

This is one of the original macroscopic Maxwell equations. It is important to remember that only the induced electric dipole moment of matter, subject to the field $\mathbf{E}$, was included in the above separation into true and induced charge densities. Contributions to $\mathbf{D}$ from higher-order electric moments were neglected. This is one of the approximations assumed.

Another approximation is the assumption that there exists a simple linear relationship between **P** and **E** in the material medium under consideration

$$\mathbf{P}(t,\mathbf{x}) = \varepsilon_0 \chi_{\mathrm{e}}(t,\mathbf{x})\mathbf{E}(t,\mathbf{x}) \tag{9.7}$$

This approximation is often valid for regular media if the field strength $|\mathbf{E}|$ is low enough. Here the variations in time and space of the the material dependent *electric susceptibility*, $\chi_{\mathrm{e}}$, are usually on much slower and longer scales than for **E** itself.[1] Inserting the approximation (9.7) into equation (9.5) on the facing page, we can write the latter

$$\mathbf{D}(t,\mathbf{x}) = \varepsilon(t,\mathbf{x})\mathbf{E}(t,\mathbf{x}) \tag{9.8}$$

where, approximately,

$$\varepsilon(t,\mathbf{x}) = \varepsilon_0[1 + \chi_{\mathrm{e}}(t,\mathbf{x})] = \varepsilon_0 \kappa_{\mathrm{e}}(t,\mathbf{x}) \tag{9.9}$$

For an electromagnetically *anisotropic medium* such as a *magnetised plasma* or a *birefringent crystal*, the susceptibility $\chi_{\mathrm{e}}$ or, equivalently the *relative dielectric permittivity*

$$\kappa_{\mathrm{e}}(t,\mathbf{x}) = \frac{\varepsilon(t,\mathbf{x})}{\varepsilon_0} = 1 + \chi_{\mathrm{e}}(t,\mathbf{x}) \tag{9.10}$$

will have to be replaced by a tensor. This would still describe a linear relationship between **E** and **P** but one where the linear proportionality factor, or, as we shall call it, the *dispersive property* of the medium, is dependent on the direction in space.

In general, however, the relationship is not of a simple linear form as in equation (9.7) above but non-linear terms are important. In such a situation the principle of superposition is no longer valid and non-linear effects such as frequency conversion and mixing can be expected.[2]

[1] The fact that the relation between the dipole moment per unit volume **P** and the applied electric field **E** is local in time and space is yet another approximation assumed in macroscopic Maxwell theory.

[2] The nonlinearity of semiconductor diodes is used, *e.g.* in radio receivers to convert high radio frequencies into lower ones, or into the audible spectrum. These techniques are called *heterodyning* and *demodulation*, respectively. Another example of the nonlinear response of a medium is the *Kerr effect*.

## 9.1.2 Magnetisation and the magnetising field

An analysis of the properties of magnetic media and the associated currents shows that three such types of currents exist:

1. In analogy with true charges for the electric case, we may have true currents $\mathbf{j}^{\mathrm{true}}$, *i.e.* a physical transport of true (free) charges.

2. In analogy with the electric polarisation **P** there may be a form of charge transport associated with the changes of the polarisation with time. Such currents, induced by an external field, are called *polarisation currents* and are identified with $\partial\mathbf{P}/\partial t$.

3. There may also be intrinsic currents of a microscopic, often atomistic, nature that are inaccessible to direct observation, but which may produce net effects at discontinuities and boundaries. These *magnetisation currents* are denoted $\mathbf{j}^{\mathrm{M}}$.

Free magnetic monopoles have not yet been unambiguously identified in experiments. So there is no correspondence in the magnetic case to the electric monopole moment, formula (6.17a) on page 108. The lowest order magnetic moment, corresponding to the electric dipole moment, formula (6.17b) on page 108, is the *magnetic dipole moment* [*cf.* the Fourier component expression (6.55) on page 116]

$$\mathbf{m}(t) = \frac{1}{2}\int_{V'} \mathrm{d}^3x'\,(\mathbf{x}' - \mathbf{x}_0) \times \mathbf{j}(t', \mathbf{x}') \tag{9.11}$$

Analogously to the electric case, one may, for a distribution of magnetic dipole moments in a volume, describe this volume in terms of its *magnetisation*, or *magnetic dipole moment per unit volume*, $\mathbf{M}$. Via the definition of the vector potential $\mathbf{A}$ one can show that the magnetisation current and the magnetisation is simply related:

$$\mathbf{j}^{\mathrm{M}} = \boldsymbol{\nabla} \times \mathbf{M} \tag{9.12}$$

In a stationary medium we therefore have a total current which is (approximately) the sum of the three currents enumerated above:

$$\mathbf{j}^{\mathrm{total}} = \mathbf{j}^{\mathrm{true}} + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M} \tag{9.13}$$

One might then be led to think that the right-hand side (RHS) of the $\boldsymbol{\nabla} \times \mathbf{B}$ Maxwell equation (2.1d) on page 19 should be

$$\mathrm{RHS} = \mu_0 \left( \mathbf{j}^{\mathrm{true}} + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M} \right)$$

However, moving the term $\boldsymbol{\nabla} \times \mathbf{M}$ from the right hand side (RHS) to the left hand side (LHS) and introducing the *magnetising field* (*magnetic field intensity*, *Ampère-turn density*) as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{9.14}$$

and using the definition for $\mathbf{D}$, equation (9.5) on page 190, we find that

$$\mathrm{LHS} = \boldsymbol{\nabla} \times \mathbf{H}$$
$$\mathrm{RHS} = \mathbf{j}^{\mathrm{true}} + \frac{\partial \mathbf{P}}{\partial t} = \mathbf{j}^{\mathrm{true}} + \frac{\partial \mathbf{D}}{\partial t} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Hence, in this simplistic view, we would pick up a term $-\varepsilon_0 \partial \mathbf{E}/\partial t$ which makes the equation inconsistent: the divergence of the left hand side vanishes while the divergence of the right hand side does not! Maxwell realised this and to overcome this inconsistency he was forced to add his famous displacement current term which precisely compensates for the last term the RHS expression.[3] In chapter 1, we discussed an alternative way, based on the postulate of conservation of electric charge, to introduce the displacement current.

We may, in analogy with the electric case, introduce a *magnetic susceptibility* for the medium. Denoting it $\chi_\mathrm{m}$, we can write

$$\mathbf{H}(t, \mathbf{x}) = \mu^{-1}(t, \mathbf{x})\mathbf{B}(t, \mathbf{x}) \tag{9.15}$$

where, approximately,

$$\mu(t, \mathbf{x}) = \mu_0[1 + \chi_\mathrm{m}(t, \mathbf{x})] = \mu_0 \kappa_\mathrm{m}(t, \mathbf{x}) \tag{9.16}$$

and

$$\kappa_\mathrm{m}(t, \mathbf{x}) = \frac{\mu(t, \mathbf{x})}{\mu_0} = 1 + \chi_\mathrm{m}(t, \mathbf{x}) \tag{9.17}$$

is the *relative permeability*. In the case of anisotropy, $\kappa_\mathrm{m}$ will be a tensor, but it is still only a linear approximation.[4]

[3] This term, which ensures that electric charge is conserved also in non-stationary problems, is the one that makes it possible to turn the Maxwell equations into wave equations (see chapter 2) and, hence, the term that, in a way, is the basis for radio communications and other engineering applications of the theory.

[4] This is the case for the *Hall effect* which produces a potential difference across an electric conduction current channel, perpendicular to this current, in the presence of an external magnetic field that is likewise perpendicular to the current. This effect was discovered 1879 by the US physicist EDWIN HERBERT HALL (1855–1938).

### 9.1.3 Macroscopic Maxwell equations

Field equations, expressed in terms of the derived, and therefore in principle superfluous, field quantities **D** and **H** are obtained from the Maxwell-Lorentz microscopic equations (2.1) on page 19, by replacing the **E** and **B** in the two *source equations* by using the approximate relations formula (9.8) on page 191 and formula (9.15) above, respectively:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho^{\mathrm{true}} \tag{9.18a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.18b}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{9.18c}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}^{\mathrm{true}} + \frac{\partial \mathbf{D}}{\partial t} \tag{9.18d}$$

This set of differential equations, originally derived by Maxwell himself, are called *Maxwell's macroscopic equations*. Together with the boundary conditions and the constitutive relations, they describe uniquely (but only approximately) the properties of the electric and magnetic fields in matter and are convenient to use in certain simple cases, particularly in engineering applications. However, the structure of these equations rely on certain linear approximations and there are many situations where they are not useful or even applicable. Therefore, these equations, which are the original Maxwell equations (albeit expressed in their modern vector form as introduced by OLIVER HEAVISIDE), should be used with some care.[5]

[5] It should be recalled that Maxwell formulated these macroscopic equations before it was known that matter has an atomistic structure and that there exist electrically charged particles such as electrons and protons, which possess a quantum mechanical property called spin that gives rise to magnetism!

## 9.2 Phase velocity, group velocity and dispersion

If we introduce the *phase velocity* in the medium as

$$v_\varphi = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\kappa_\mathrm{e}\varepsilon_0\kappa_\mathrm{m}\mu_0}} = \frac{c}{\sqrt{\kappa_\mathrm{e}\kappa_\mathrm{m}}} \tag{9.19}$$

where, according to equation (1.12) on page 6, $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light, *i.e.* the phase speed of electromagnetic waves, in vacuum. Associated with the phase speed of a medium for a wave of a given frequency $\omega$ we have a *wave vector*, defined as

$$\mathbf{k} \stackrel{\text{def}}{\equiv} k\hat{\mathbf{k}} = k\hat{\mathbf{v}}_\varphi = \frac{\omega}{v_\varphi}\frac{\mathbf{v}_\varphi}{v_\varphi} \tag{9.20}$$

The ratio of the phase speed in vacuum and in the medium

$$\frac{c}{v_\varphi} = \sqrt{\kappa_\mathrm{e}\kappa_\mathrm{m}} = c\sqrt{\varepsilon\mu} \stackrel{\text{def}}{\equiv} n \tag{9.21}$$

where the material dependent quantity

$$n(t,\mathbf{x}) \stackrel{\text{def}}{\equiv} \frac{c}{v_\varphi} = \sqrt{\kappa_\mathrm{e}(t,\mathbf{x})\kappa_\mathrm{m}(t,\mathbf{x})} \tag{9.22}$$

is called the *refractive index* of the medium and describes its refractive and reflective properties.[6] In general $n$ is a function of frequency. If the medium is *anisotropic* or *birefringent*, the refractive index is a rank-two tensor field. Under our simplifying assumptions, in the material medium that we consider $n = Const$ for each frequency component of the fields. In certain materials, the refractive index is larger than unity (*e.g.* glass and water at optical frequencies), in others, it can be smaller than unity (*e.g.* plasma and metals at radio and optical frequencies).

[6] In fact, there exist *metamaterials* where $\kappa_\mathrm{e}$ and $\kappa_\mathrm{m}$ are negative. For such materials, the refractive index becomes negative:

$$\begin{aligned} n &= \mathrm{i}\sqrt{|\kappa_\mathrm{e}|}\,\mathrm{i}\sqrt{|\kappa_\mathrm{m}|} \\ &= -|\kappa_\mathrm{e}\kappa_\mathrm{m}|^{1/2} \end{aligned}$$

Such *negative refractive index* materials, have quite remarkable electromagnetic properties.

It is important to notice that depending on the electric and magnetic properties of a medium, and, hence, on the value of the refractive index $n$, the phase speed in the medium can be smaller or larger than the speed of light:

$$v_\varphi = \frac{c}{n} = \frac{\omega}{k} \tag{9.23}$$

where, in the last step, we used equation (9.20) on the preceding page.

If the medium has a refractive index which, as is usually the case, dependent on frequency $\omega$, we say that the medium is *dispersive*. Because in this case also $\mathbf{k}(\omega)$ and $\omega(\mathbf{k})$, so that the *group velocity*

$$v_{\mathrm{g}} = \frac{\partial \omega}{\partial k} \tag{9.24}$$

has a unique value for each frequency component, and is different from $v_\varphi$. Except in regions of *anomalous dispersion*, $v_{\mathrm{g}}$ is always smaller than $c$. In a gas of free charges, such as a *plasma*, the refractive index is given by the expression

$$n^2(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{9.25}$$

where

$$\omega_{\mathrm{p}}^2 = \sum_\sigma \frac{N_\sigma q_\sigma^2}{\varepsilon_0 m_\sigma} \tag{9.26}$$

is the square of the *plasma frequency* $\omega_{\mathrm{p}}$. Here $m_\sigma$ and $N_\sigma$ denote the mass and number density, respectively, of charged particle species $\sigma$. In an inhomogeneous plasma, $N_\sigma = N_\sigma(\mathbf{x})$ so that the refractive index and also the phase and group velocities are space dependent. As can be easily seen, for each given frequency, the phase and group velocities in a plasma are different from each other. If the frequency $\omega$ is such that it coincides with $\omega_{\mathrm{p}}$ at some point in the medium, then at that point $v_\varphi \to \infty$ while $v_{\mathrm{g}} \to 0$ and the wave Fourier component at $\omega$ is reflected there.

## 9.3 Radiation from charges in a material medium

When electromagnetic radiation is propagating through matter, new phenomena may appear which are (at least classically) not present in vacuum. As mentioned earlier, one can under certain simplifying assumptions include, to some extent, the influence from matter on the electromagnetic fields by introducing new, derived field quantities $\mathbf{D}$ and $\mathbf{H}$ according to

$$\mathbf{D} = \varepsilon(t, \mathbf{x})\mathbf{E} = \kappa_{\mathrm{e}}(t, \mathbf{x})\varepsilon_0 \mathbf{E} \tag{9.27}$$

$$\mathbf{B} = \mu(t, \mathbf{x})\mathbf{H} = \kappa_{\mathrm{m}}(t, \mathbf{x})\mu_0 \mathbf{H} \tag{9.28}$$

### 9.3.1 Vavilov-Čerenkov radiation

As we saw in example 6.3 on page 148, a charge in uniform, rectilinear motion *in vacuum* does not give rise to any radiation; see in particular equation (6.196a) on page 149. Let us now consider a charge in uniform, rectilinear motion *in a medium* with electric properties which are different from those of a (classical) vacuum. Specifically, consider a medium where

$$\varepsilon = Const > \varepsilon_0 \tag{9.29a}$$

$$\mu = \mu_0 \tag{9.29b}$$

This implies that in this medium the phase speed is

$$v_\varphi = \frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu_0}} < c \tag{9.30}$$

Hence, in this particular medium, the speed of propagation of (the phase planes of) electromagnetic waves is less than the speed of light in vacuum, which we know is an absolute limit for the motion of anything, including particles. A medium of this kind has the interesting property that particles, entering into the medium at high speeds $|\mathbf{v}'|$, which, of course, are below the phase speed *in vacuum*, can experience that the particle speeds are *higher* than the phase speed *in the medium*. This is the basis for the *Vavilov-Čerenkov radiation*, more commonly known in the western literature as *Cherenkov radiation*, that we shall now study.

If we recall the general derivation, in the vacuum case, of the retarded (and advanced) potentials in chapter 3 and the Liénard-Wiechert potentials, equations (6.83) on page 123, we realise that we obtain the latter in the medium by a simple formal replacement $c \mapsto c/n$ in the expression (6.84) on page 124 for $s$. Hence, the Liénard-Wiechert potentials in a medium characterized by a refractive index $n$, are

$$\Phi(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q'}{\left||\mathbf{x}-\mathbf{x}'| - n\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c}\right|} = \frac{1}{4\pi\varepsilon_0}\frac{q'}{s} \tag{9.31a}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\frac{q'\mathbf{v}'}{\left||\mathbf{x}-\mathbf{x}'| - n\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c}\right|} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{q'\mathbf{v}'}{s} \tag{9.31b}$$

where now

$$s = \left||\mathbf{x}-\mathbf{x}'| - n\frac{(\mathbf{x}-\mathbf{x}')\cdot\mathbf{v}'}{c}\right| \tag{9.32}$$

The need for the absolute value of the expression for $s$ is obvious in the case when $v'/c \geq 1/n$ because then the second term can be larger than the first term; if $v'/c \ll 1/n$ we recover the well-known vacuum case but with modified phase speed. We also note that the retarded and advanced times in the medium are [*cf.*

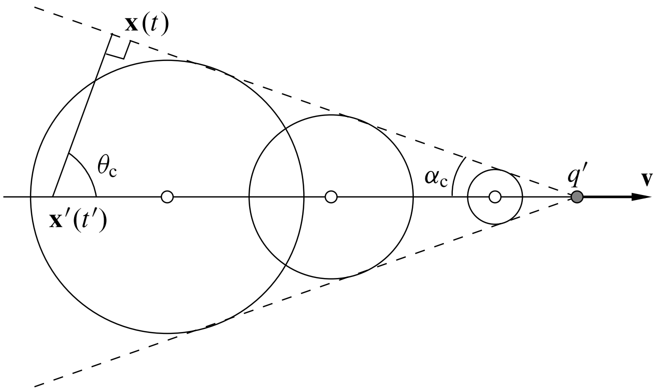


Figure 9.1: Instantaneous picture of the expanding field spheres from a point charge moving with constant speed $v'/c > 1/n$ in a medium where $n > 1$. This generates a Vavilov-Čerenkov shock wave in the form of a cone.

equation (3.33) on page 40]

$$t'_{\text{ret}} = t'_{\text{ret}}(t, |\mathbf{x} - \mathbf{x}'|) = t - \frac{k\,|\mathbf{x} - \mathbf{x}'|}{\omega} = t - \frac{|\mathbf{x} - \mathbf{x}'|\,n}{c} \tag{9.33a}$$

$$t'_{\text{adv}} = t'_{\text{adv}}(t, |\mathbf{x} - \mathbf{x}'|) = t + \frac{k\,|\mathbf{x} - \mathbf{x}'|}{\omega} = t + \frac{|\mathbf{x} - \mathbf{x}'|\,n}{c} \tag{9.33b}$$

so that the usual time interval $t - t'$ between the time measured at the point of observation and the retarded time *in a medium* becomes

$$t - t' = \frac{|\mathbf{x} - \mathbf{x}'|\,n}{c} \tag{9.34}$$

For $v'/c \geq 1/n$, the retarded distance $s$, and therefore the denominators in equations (9.31) on the preceding page, vanish when

$$n(\mathbf{x} - \mathbf{x}') \cdot \frac{\mathbf{v}'}{c} = |\mathbf{x} - \mathbf{x}'| \frac{nv'}{c} \cos\theta_c = |\mathbf{x} - \mathbf{x}'| \tag{9.35}$$

or, equivalently, when

$$\cos\theta_c = \frac{c}{nv'} \tag{9.36}$$

In the direction defined by this angle $\theta_c$, the potentials become singular. During the time interval $t-t'$ given by expression (9.34) above, the field exists within a sphere of radius $|\mathbf{x} - \mathbf{x}'|$ around the particle while the particle moves a distance

$$l' = (t - t')v' \tag{9.37}$$

along the direction of $\mathbf{v}'$.

In the direction $\theta_c$ where the potentials are singular, all field spheres are tangent to a straight cone with its apex at the instantaneous position of the particle and with the apex half angle $\alpha_c$ defined according to

$$\sin\alpha_c = \cos\theta_c = \frac{c}{nv'} \tag{9.38}$$

This is illustrated in figure 9.1 on the previous page.

The cone of potential singularities and field sphere circumferences propagates with speed $c/n$ in the form of a *shock front*. The first observation of this type of radiation was reported by MARIE SKLODOWSKA CURIE in 1910, but she never pursued the exploration of it. This radiation in question is therefore called *Vavilov-Čerenkov radiation*.[7]

In order to make some quantitative estimates of this radiation, we note that we can describe the motion of each charged particle $q'$ as a current density:

$$\mathbf{j} = q'\mathbf{v}'\,\delta(\mathbf{x}' - \mathbf{v}'t') = q'v'\,\delta(x' - v't')\delta(y')\delta(z')\hat{\mathbf{x}}_1 \tag{9.39}$$

which has the trivial Fourier transform

$$\mathbf{j}_\omega = \frac{q'}{2\pi}\,\mathrm{e}^{\mathrm{i}\omega x'/v'}\,\delta(y')\delta(z')\hat{\mathbf{x}}_1 \tag{9.40}$$

This Fourier component can be used in the formulæ derived for a linear current in subsection 6.4.1 on page 117 if only we make the replacements

$$\varepsilon_0 \mapsto \varepsilon = n^2\varepsilon_0 \tag{9.41a}$$

$$k \mapsto \frac{n\omega}{c} \tag{9.41b}$$

In this manner, using $\mathbf{j}_\omega$ from equation (9.40) above, the resulting Fourier transforms of the Vavilov-Čerenkov magnetic and electric radiation fields can be calculated from the expressions (5.23) on page 92 and (5.18) on page 90, respectively.

The total energy content is then obtained from equation (6.9) on page 106 (integrated over a closed sphere at large distances). For a Fourier component one obtains [*cf.* equation (6.12) on page 107]

$$\begin{aligned} U_\omega^{\mathrm{rad}}\mathrm{d}\Omega &\approx \frac{1}{4\pi\varepsilon_0 n c}\left|\int_{V'}\mathrm{d}^3x'\,(\mathbf{j}_\omega \times \mathbf{k})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\right|^2\mathrm{d}\Omega \\ &= \frac{q'^2 n\omega^2}{16\pi^3\varepsilon_0 c^3}\left|\int_{-\infty}^{\infty}\exp\left[\mathrm{i}x'\left(\frac{\omega}{v'} - k\cos\theta\right)\right]\mathrm{d}x'\right|^2\sin^2\theta\,\mathrm{d}\Omega \end{aligned} \tag{9.42}$$

where $\theta$ is the angle between the direction of motion, $\hat{\mathbf{x}}'_1$, and the direction to the observer, $\hat{\mathbf{k}}$. The integral in (9.42) is singular of a 'Dirac delta type'. If we limit the spatial extent of the motion of the particle to the closed interval $[-X, X]$ on the $x'$ axis we can evaluate the integral to obtain

$$U_\omega^{\mathrm{rad}}\mathrm{d}\Omega = \frac{q'^2 n\omega^2\sin^2\theta}{4\pi^3\varepsilon_0 c^3}\,\frac{\sin^2\left[\left(1 - \frac{nv'}{c}\cos\theta\right)\frac{X\omega}{v'}\right]}{\left[\left(1 - \frac{nv'}{c}\cos\theta\right)\frac{\omega}{v'}\right]^2}\mathrm{d}\Omega \tag{9.43}$$

which has a maximum in the direction $\theta_c$ as expected. The magnitude of this maximum grows and its width narrows as $X \to \infty$. The integration of (9.43)

[7] The first systematic exploration of this radiation was made in 1934 by PAVEL ALEKSEEVICH ČERENKOV(1904–1990), who was then a doctoral student in SERGEY IVANOVICH VAVILOV's (1891–1951) research group at the Lebedev Physical Institute in Moscow. Vavilov wrote a manuscript with the experimental findings, put Čerenkov as the author, and submitted it to *Nature*. In the manuscript, Vavilov explained the results in terms of radioactive particles creating Compton electrons which gave rise to the radiation. This was indeed the correct interpretation, but the paper was rejected. The paper was then sent to *Physical Review* and was, after some controversy with the American editors, who claimed the results to be wrong, eventually published in 1937. In the same year, IGOR' EVGEN'EVICH TAMM (1895–1975) and ILYA MIKHAILOVICH FRANK (1908–1990) published the theory for the effect ('the singing electron').

In fact, predictions of a similar effect had been made as early as 1888 by OLIVER HEAVISIDE (1850–1925), and by ARNOLD JOHANNES WILHELM SOMMERFELD (1868–1951) in his 1904 paper 'Radiating body moving with velocity of light'. On 8 May, 1937, Sommerfeld sent a letter to Tamm via Austria, saying that he was surprised that his old 1904 ideas were now becoming interesting. Tamm, Frank and Čerenkov received the Nobel Prize in 1958 'for the discovery and the interpretation of the Čerenkov effect' [VITALIY LAZAREVICH GINZBURG (1916–2009), *private communication*]. The *Vavilov-Čerenkov cone* is similar in nature to the *Mach cone* in acoustics.

over $\Omega$ therefore picks up the main contributions from $\theta \approx \theta_c$. Consequently, we can set $\sin^2\theta \approx \sin^2\theta_c$ and the result of the integration is

$$\begin{aligned}\tilde{U}^{\rm rad}_\omega &= 2\pi\int_0^\pi U^{\rm rad}_\omega(\theta)\sin\theta\,{\rm d}\theta = \lceil\cos\theta = -\xi\rfloor = 2\pi\int_{-1}^1 U^{\rm rad}_\omega(\xi)\,{\rm d}\xi \\ &\approx \frac{q'^2 n\omega^2\sin^2\theta_c}{2\pi^2\varepsilon_0 c^3}\int_{-1}^1 \frac{\sin^2\left[\left(1+\frac{nv'\xi}{c}\right)\frac{X\omega}{v'}\right]}{\left[\left(1+\frac{nv'\xi}{c}\right)\frac{\omega}{v'}\right]^2}\,{\rm d}\xi\end{aligned} \tag{9.44}$$

The integrand in (9.44) is strongly peaked near $\xi = -c/(nv')$, or, equivalently, near $\cos\theta_c = c/(nv')$. This means that the integrand function is practically zero outside the integration interval $\xi \in [-1, 1]$. Consequently, one may extend the $\xi$ integration interval to $(-\infty, \infty)$ without introducing too large an error. Via yet another variable substitution we can therefore approximate

$$\begin{aligned}\sin^2\theta_c\int_{-1}^1 \frac{\sin^2\left[\left(1+\frac{nv'\xi}{c}\right)\frac{X\omega}{v'}\right]}{\left[\left(1+\frac{nv'\xi}{c}\right)\frac{\omega}{v'}\right]^2}\,{\rm d}\xi &\approx \left(1-\frac{c^2}{n^2v'^2}\right)\frac{cX}{\omega n}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,{\rm d}x \\ &= \frac{cX\pi}{\omega n}\left(1-\frac{c^2}{n^2v'^2}\right)\end{aligned} \tag{9.45}$$

leading to the final approximate result for the total energy loss in the frequency interval $(\omega, \omega+{\rm d}\omega)$

$$\tilde{U}^{\rm rad}_\omega{\rm d}\omega = \frac{q'^2X}{2\pi\varepsilon_0c^2}\left(1-\frac{c^2}{n^2v'^2}\right)\omega\,{\rm d}\omega \tag{9.46}$$

As mentioned earlier, the refractive index is usually frequency dependent. Realising this, we find that the radiation energy per frequency unit and *per unit length* is

$$\frac{\tilde{U}^{\rm rad}_\omega{\rm d}\omega}{2X} = \frac{q'^2\omega}{4\pi\varepsilon_0c^2}\left(1-\frac{c^2}{n^2(\omega)v'^2}\right){\rm d}\omega \tag{9.47}$$

This result was derived under the assumption that $v'/c > 1/n(\omega)$, *i.e.* under the condition that the expression inside the parentheses in the right hand side is positive. For all media it is true that $n(\omega) \to 1$ when $\omega \to \infty$, so there exist always a highest frequency for which we can obtain Vavilov-Čerenkov radiation from a fast charge in a medium. Our derivation above for a fixed value of $n$ is valid for each individual Fourier component.

## 9.4 Electromagnetic waves in a medium

In section 2.3 on page 23 in chapter 2 we derived the wave equations for the electric and magnetic fields, **E** and **B**, respectively,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\frac{\boldsymbol{\nabla}\rho}{\varepsilon_0} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \tag{9.48a}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mu_0 \boldsymbol{\nabla} \times \mathbf{j} \tag{9.48b}$$

where the charge density $\rho$ and the current density $\mathbf{j}$ were viewed as the sources of the wave fields. As we recall, these wave equations were derived from the Maxwell-Lorentz equations (2.1) on page 19, taken as an axiomatic foundation, or postulates, of electromagnetic theory. As such, these equations just state what relations exist between (the second order derivatives of) the fields, *i.e.* essentially dynamic generalisations of the Coulomb and Ampère forces, and the dynamics of the charges (charge and current densities) in the region under study.

Even if the $\rho$ and $\mathbf{j}$ terms in the Maxwell-Lorentz equations are often referred to as the source terms, they can equally well be viewed as terms that describe the impact on matter in a particular region upon which an electromagnetic wave, produced in another region with its own charges and currents, impinges. In order to do so, one needs to find the *constitutive relations* that describe how charge and current densities are induced by the impinging fields. Then one can solve the wave equations (9.48) above. In general, this is a formidable task, and one must often resort to numerical methods.

Let us, for simplicity, assume that the linear relations, as given by formula (9.8) on page 191 and formula (9.15) on page 193, hold, and that there is also a linear relation between the electric field **E** and the current density, known as *Ohm's law*:

$$\mathbf{j}(t,\mathbf{x}) = \sigma(t,\mathbf{x})\mathbf{E}(t,\mathbf{x}) \tag{9.49}$$

where $\sigma$ is the *conductivity* of the medium. Let us make the further assumption that $\varepsilon = \varepsilon(\mathbf{x})$, $\mu = \mu(\mathbf{x})$, and $\sigma(\mathbf{x})$ are not explicitly dependent on time and are local in space. Then we obtain the coupled wave equations

$$\begin{aligned}\nabla^2 \mathbf{E} - \mu(\mathbf{x})\sigma(\mathbf{x})\frac{\partial \mathbf{E}}{\partial t} - \varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2 \mathbf{E}}{\partial t^2} &= (\boldsymbol{\nabla} \times \mathbf{E}) \times \boldsymbol{\nabla} \ln \mu(\mathbf{x}) \\ &\quad - \boldsymbol{\nabla}[\boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \cdot \mathbf{E}] + \boldsymbol{\nabla}\frac{\rho}{\varepsilon}\end{aligned} \tag{9.50a}$$

$$\begin{aligned}\nabla^2 \mathbf{H} - \mu(\mathbf{x})\sigma(\mathbf{x})\frac{\partial \mathbf{H}}{\partial t} - \varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2 \mathbf{H}}{\partial t^2} &= (\boldsymbol{\nabla} \times \mathbf{H}) \times \boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \\ - \boldsymbol{\nabla}[\boldsymbol{\nabla} \ln \mu(\mathbf{x}) \cdot \mathbf{H}] + \boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \times (\sigma(\mathbf{x})\mathbf{E}) &- [\boldsymbol{\nabla}\sigma(\mathbf{x})] \times \mathbf{E}\end{aligned} \tag{9.50b}$$

For the case $\mu = \mu_0$ (no magnetisation) and $\rho = Const$ in the medium, equations (9.50) on the preceding page simplify to

$$\nabla^2 \mathbf{E} - \mu_0 \sigma(\mathbf{x}) \frac{\partial \mathbf{E}}{\partial t} - \varepsilon(\mathbf{x}) \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\boldsymbol{\nabla}[\boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \cdot \mathbf{E}] \tag{9.51a}$$

$$\begin{aligned} \nabla^2 \mathbf{B} - \mu_0 \sigma(\mathbf{x}) \frac{\partial \mathbf{B}}{\partial t} - \varepsilon(\mathbf{x}) \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} &= (\boldsymbol{\nabla} \times \mathbf{B}) \times \boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \\ &+ \mu_0 \boldsymbol{\nabla} \ln \varepsilon(\mathbf{x}) \times (\sigma \mathbf{E}) - \mu_0 [\boldsymbol{\nabla} \sigma(\mathbf{x})] \times \mathbf{E} \end{aligned} \tag{9.51b}$$

Making the further assumption that the medium is not conductive, *i.e.* that $\sigma = 0$, the uncoupled wave equations

$$\nabla^2 \mathbf{E} - \frac{\varepsilon(\mathbf{x})}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\boldsymbol{\nabla}\{[\boldsymbol{\nabla} \ln \varepsilon(\mathbf{x})] \cdot \mathbf{E}\} \tag{9.52a}$$

$$\nabla^2 \mathbf{B} - \frac{\varepsilon(\mathbf{x})}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = (\boldsymbol{\nabla} \times \mathbf{B}) \times [\boldsymbol{\nabla} \ln \varepsilon(\mathbf{x})] \tag{9.52b}$$

are obtained.

### 9.4.1 Constitutive relations

In a solid, fluid or gaseous medium the source terms in the microscopic Maxwell equations (2.1) on page 19 must include all charges and currents in the medium, *i.e.* also the intrinsic ones (*e.g.* the polarisation charges in electrets, and atomistic magnetisation currents in magnets) and the self-consistently imposed ones (*e.g.* polarisation currents). This is of course also true for the inhomogeneous wave equations derived from Maxwell's equations.

From now one we assume that $\rho$ and $\mathbf{j}$ represent only the charge and current densities (*i.e.* polarisation and conduction charges and currents, respectively) that are induces by the $\mathbf{E}$ and $\mathbf{B}$ fields of the waves impinging upon the medium of interest.[8]

Let us for simplicity consider a medium containing free electrons only and which is not penetrated by a magnetic field, *i.e.* the medium is assumed to be isotropic with no preferred direction(s) in space.[9]

Each of these electrons are assumed to be accelerated by the Lorentz force, formula (4.53) on page 63. However, if the fields are those of an electromagnetic wave one can, for reasonably high oscillation frequencies, neglect the force from the magnetic field. Of course, this is also true if there is no magnetic field present. So the equation of motion for each electron in the medium can be written

$$m \frac{\mathrm{d}^2 \mathbf{x}}{\mathrm{d}t^2} + m\nu \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = q\mathbf{E} \tag{9.53}$$

[8] If one includes also the effect of the charges on $\mathbf{E}$ and $\mathbf{B}$, *i.e.* treat $\rho$ and $\mathbf{j}$ as sources for fields, singularities will appear in the theory. Such so called *self-force effects* will not be treated here.

[9] To this category belongs *unmagnetised plasma*. So do also, to a good approximation, fluid or solid metals.

where $m$ and $q$ are the mass and charge of the electron, respectively, $\nu$ the effective *collision frequency* representing the frictional dissipative force from the surrounding medium, and $\mathbf{E}$ the effective applied electric field sensed by the electron. For a Fourier component of the electric field $\mathbf{E} = \mathbf{E}_0 \exp(-\mathrm{i}\omega t)$, the equation of motion becomes

$$\omega^2 q\mathbf{x}(t) - \mathrm{i}\omega\nu q\mathbf{x}(t) = \frac{q^2}{m}\mathbf{E} \tag{9.54}$$

If the electron is at equilibrium $\mathbf{x} = \mathbf{0}$ when $\mathbf{E} = \mathbf{0}$, then its dipole moment is $\mathbf{d}(t) = q\mathbf{x}(t)$. Inserting this in equation (9.54) above, we obtain

$$\mathbf{d} = -\frac{q^2}{m(\omega^2 + \mathrm{i}\omega\nu)}\mathbf{E} \tag{9.55}$$

This is the the lowest order contribution to the dipole moment of the medium from each electron under the influence of the assumed electric field. If $N_{\mathbf{d}}(\mathbf{x})$ electrons per unit volume can be assumed to give rise to the electric polarisation $\mathbf{P}$, this becomes

$$\mathbf{P} = N_{\mathbf{d}}\mathbf{d} = -\frac{N_{\mathbf{d}}q^2}{m(\omega^2 + \mathrm{i}\omega\nu)}\mathbf{E} \tag{9.56}$$

Using this in formula (9.5) on page 190, one finds that

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon(\mathbf{x})\mathbf{E}(t, \mathbf{x}) \tag{9.57}$$

where

$$\varepsilon(\mathbf{x}) = \varepsilon_0 - \frac{N_{\mathbf{d}}(\mathbf{x})q^2}{m(\omega^2 + \mathrm{i}\omega\nu)} = \varepsilon_0\left(1 - \frac{N_{\mathbf{d}}(\mathbf{x})q^2}{\varepsilon_0 m}\frac{1}{\omega^2 + \mathrm{i}\omega\nu}\right) \tag{9.58}$$

The quantity

$$\omega_{\mathrm{p}}(\mathbf{x}) = \sqrt{\frac{N_{\mathbf{d}}(\mathbf{x})q^2}{\varepsilon_0 m}} \tag{9.59}$$

is the *plasma frequency* and

$$n(\mathbf{x}) = \sqrt{\frac{\varepsilon(\mathbf{x})}{\varepsilon_0}} = \sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + \mathrm{i}\omega\nu}} \tag{9.60}$$

is the *refractive index*. At points in the medium where the wave frequency $\omega$ equals this plasma frequency and the collision frequency $\nu$ vanishes, the refractive index $n = 0$, and the wave is totally reflected. In the ionised outer part of the atmosphere called the *ionosphere* this happens for radio waves of frequencies up to about 5–10 MHz dependent on the solar radiation which causes most of the ionisation. This is the basis for over-the-horizon radio communications and is also the reason why low-frequency radio signals from space do not reach the surface of the Earth.

## 9.4.2 Electromagnetic waves in a conducting medium

We shall now restrict ourselves to the wave equations for the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$ in a electrically conductive and neutral medium, *i.e.* a volume where there exist no net electric charge, $\rho = 0$, no dielectric effects, $\varepsilon = \varepsilon_0$, and no electromotive force, $\mathbf{E}^{\mathrm{emf}} = \mathbf{0}$. A highly conductive metal is a good example of such a medium.

### 9.4.2.1 The wave equations for E and B

To a good approximation, metals and other conductors in free space have a conductivity $\sigma$ that is not dependent on $t$ or $\mathbf{x}$. The wave equations (9.51) on page 201 are then simplified to

$$\nabla^2\mathbf{E} - \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0} \tag{9.61}$$

$$\nabla^2\mathbf{B} - \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \mathbf{0} \tag{9.62}$$

which are the *homogeneous vector wave equations* for $\mathbf{E}$ and $\mathbf{B}$ in a conducting medium without EMF.

We notice that for the simple propagation media considered here, the wave equation for the magnetic field $\mathbf{B}$ has exactly the same mathematical form as the wave equation for the electric field $\mathbf{E}$, equation (9.61) above. Therefore, in this case it suffices to consider only the $\mathbf{E}$ field, since the results for the $\mathbf{B}$ field follow trivially. For EM waves propagating in more complicated media, containing, *e.g.* inhomogeneities, the wave equations for $\mathbf{E}$ and for $\mathbf{B}$ do not have the same mathematical form.

Following the spectral component prescription leading to equation (2.23) on page 26, we obtain, in the special case under consideration, the following time-independent wave equation

$$\nabla^2\mathbf{E}_0 + \frac{\omega^2}{c^2}\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right)\mathbf{E}_0 = \mathbf{0} \tag{9.63}$$

Multiplying by $\mathrm{e}^{-\mathrm{i}\omega t}$ and introducing the *relaxation time* $\tau = \varepsilon_0/\sigma$ of the medium in question, we see that the differential equation for each spectral component can be written

$$\nabla^2\mathbf{E}(t,\mathbf{x}) + \frac{\omega^2}{c^2}\left(1 + \frac{\mathrm{i}}{\tau\omega}\right)\mathbf{E}(t,\mathbf{x}) = \mathbf{0} \tag{9.64}$$

In the limit of long $\tau$ (low conductivity $\sigma$), (9.64) tends to

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = \mathbf{0} \tag{9.65}$$

which is a *time-independent wave equation* for $\mathbf{E}$, representing undamped propagating waves. In the short $\tau$ (high conductivity $\sigma$) limit we have instead

$$\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\sigma\mathbf{E} = \mathbf{0} \tag{9.66}$$

which is a *time-independent diffusion equation* for $\mathbf{E}$.

For most metals $\tau \sim 10^{-14}$ s, which means that the diffusion picture is good for all frequencies lower than optical frequencies. Hence, in metallic conductors, the propagation term $\partial^2\mathbf{E}/c^2\partial t^2$ is negligible even for VHF, UHF, and SHF signals. Alternatively, we may say that the displacement current $\varepsilon_0\partial\mathbf{E}/\partial t$ is negligible relative to the conduction current $\mathbf{j} = \sigma\mathbf{E}$.

If we introduce the *vacuum wave number*

$$k = \frac{\omega}{c} \tag{9.67}$$

we can write, using the fact that $c = 1/\sqrt{\varepsilon_0\mu_0}$ according to equation (1.12) on page 6,

$$\frac{1}{\tau\omega} = \frac{\sigma}{\varepsilon_0\omega} = \frac{\sigma}{\varepsilon_0}\frac{1}{ck} = \frac{\sigma}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\sigma}{k}R_0 \tag{9.68}$$

where in the last step we used the *characteristic impedance of vacuum* defined according to formula (6.4) on page 105.

### 9.4.2.2 Plane waves

Consider now the case where all fields depend only on the distance $\zeta$ to a given plane with unit normal $\hat{\mathbf{n}}$. Then the *del* operator becomes

$$\boldsymbol{\nabla} = \hat{\mathbf{n}}\frac{\partial}{\partial\zeta} = \hat{\mathbf{n}}\nabla \tag{9.69}$$

and the microscopic Maxwell equations attain the form

$$\hat{\mathbf{n}}\cdot\frac{\partial\mathbf{E}}{\partial\zeta} = 0 \tag{9.70a}$$

$$\hat{\mathbf{n}}\times\frac{\partial\mathbf{E}}{\partial\zeta} = -\frac{\partial\mathbf{B}}{\partial t} \tag{9.70b}$$

$$\hat{\mathbf{n}}\cdot\frac{\partial\mathbf{B}}{\partial\zeta} = 0 \tag{9.70c}$$

$$\hat{\mathbf{n}}\times\frac{\partial\mathbf{B}}{\partial\zeta} = \mu_0\mathbf{j}(t,\mathbf{x}) + \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\sigma\mathbf{E} + \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t} \tag{9.70d}$$

Scalar multiplying (9.70d) by $\hat{\mathbf{n}}$, we find that

$$0 = \hat{\mathbf{n}}\cdot\left(\hat{\mathbf{n}}\times\frac{\partial\mathbf{B}}{\partial\zeta}\right) = \hat{\mathbf{n}}\cdot\left(\mu_0\sigma + \varepsilon_0\mu_0\frac{\partial}{\partial t}\right)\mathbf{E} \tag{9.71}$$

which simplifies to the first-order ordinary differential equation for the normal component $E_n$ of the electric field

$$\frac{\mathrm{d}E_n}{\mathrm{d}t} + \frac{\sigma}{\varepsilon_0} E_n = 0 \tag{9.72}$$

with the solution

$$E_n = E_{n_0} \mathrm{e}^{-\sigma t/\varepsilon_0} = E_{n_0} \mathrm{e}^{-t/\tau} \tag{9.73}$$

This, together with (9.70a), shows that the *longitudinal component* of $\mathbf{E}$, *i.e.* the component which is perpendicular to the plane surface is independent of $\zeta$ and has a time dependence which exhibits an exponential decay, with a decrement given by the relaxation time $\tau$ in the medium.

Scalar multiplying (9.70b) by $\hat{\mathbf{n}}$, we similarly find that

$$0 = \hat{\mathbf{n}} \cdot \left( \hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} \right) = -\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \tag{9.74}$$

or

$$\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{9.75}$$

From this, and (9.70c), we conclude that the only longitudinal component of $\mathbf{B}$ must be constant in both time and space. In other words, the only non-static solution must consist of *transverse components*.

### 9.4.2.3 Telegrapher's equation

In analogy with equation (9.61) on page 203, we can easily derive a wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{9.76}$$

describing the propagation of plane waves along $\zeta$ in a conducting medium. This equation is is called the *telegrapher's equation*. If the medium is an insulator so that $\sigma = 0$, then the equation takes the form of the *one-dimensional wave equation*

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \tag{9.77}$$

As is well known, each component of this equation has a solution which can be written

$$E_i = f(\zeta - ct) + g(\zeta + ct), \quad i = 1, 2, 3 \tag{9.78}$$

where $f$ and $g$ are arbitrary (non-pathological) functions of their respective arguments. This general solution represents perturbations which propagate along $\zeta$, where the $f$ perturbation propagates in the positive $\zeta$ direction and the $g$ perturbation propagates in the negative $\zeta$ direction. In a medium, the general solution to each component of equation (9.99) on page 208 is given by

$$E_i = f(\zeta - v_\varphi t) + g(\zeta + v_\varphi t), \quad i = 1, 2, 3 \tag{9.79}$$

If we assume that our electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are represented by a Fourier component proportional to $\exp(-\mathrm{i}\omega t)$, the solution of equation (9.77) on the previous page becomes

$$\mathbf{E} = \mathbf{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t \pm k\zeta)} = \mathbf{E}_0 \mathrm{e}^{\mathrm{i}(\mp k\zeta - \omega t)} \tag{9.80}$$

By introducing the *wave vector*

$$\mathbf{k} = k\hat{\mathbf{n}} = \frac{\omega}{c}\hat{\mathbf{n}} = \frac{\omega}{c}\hat{\mathbf{k}} \tag{9.81}$$

this solution can be written as

$$\mathbf{E} = \mathbf{E}_0 \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{9.82}$$

Let us consider the lower sign in front of $k\zeta$ in the exponent in (9.80). This corresponds to a wave which propagates in the direction of increasing $\zeta$. Inserting this solution into equation (9.70b) on page 204, gives

$$\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} = \mathrm{i}\omega\mathbf{B} = \mathrm{i}k\hat{\mathbf{n}} \times \mathbf{E} \tag{9.83}$$

or, solving for $\mathbf{B}$,

$$\mathbf{B} = \frac{k}{\omega}\hat{\mathbf{n}} \times \mathbf{E} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E} = \sqrt{\varepsilon_0\mu_0}\,\hat{\mathbf{n}} \times \mathbf{E} \tag{9.84}$$

Hence, to each transverse component of $\mathbf{E}$, there exists an associated magnetic field given by equation (9.84) above. If $\mathbf{E}$ and/or $\mathbf{B}$ has a direction in space which is constant in time, we have a *plane wave*.

Allowing now for a finite conductivity $\sigma$ in our medium, and making the spectral component Ansatz in equation (9.76) on the preceding page, we find that the *time-independent telegrapher's equation* can be written

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} + \varepsilon_0\mu_0\omega^2\mathbf{E} + \mathrm{i}\mu_0\sigma\omega\mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial \zeta^2} + K^2\mathbf{E} = \mathbf{0} \tag{9.85}$$

where

$$K^2 = \varepsilon_0\mu_0\omega^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = \frac{\omega^2}{c^2}\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = k^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) \tag{9.86}$$

where, in the last step, equation (9.67) on page 204 was used to introduce the wave number $k$. Taking the square root of this expression, we obtain

$$K = k\sqrt{1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}} = \alpha + \mathrm{i}\beta \tag{9.87}$$

Squaring, one finds that

$$k^2\left(1 + \mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right) = (\alpha^2 - \beta^2) + 2\mathrm{i}\alpha\beta \tag{9.88}$$

or

$$\beta^2 = \alpha^2 - k^2 \tag{9.89}$$

$$\alpha\beta = \frac{k^2\sigma}{2\varepsilon_0\omega} \tag{9.90}$$

Squaring the latter and combining with the former, one obtains the second order algebraic equation (in $\alpha^2$)

$$\alpha^2(\alpha^2 - k^2) = \frac{k^4\sigma^2}{4\varepsilon_0^2\omega^2} \tag{9.91}$$

which can be easily solved and one finds that

$$\alpha = k\sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0\omega}\right)^2} + 1}{2}} \tag{9.92a}$$

$$\beta = k\sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0\omega}\right)^2} - 1}{2}} \tag{9.92b}$$

As a consequence, the solution of the time-independent telegrapher's equation, equation (9.85) on the facing page, can be written

$$\mathbf{E} = \mathbf{E}_0\mathrm{e}^{-\beta\zeta}\mathrm{e}^{\mathrm{i}(\alpha\zeta - \omega t)} \tag{9.93}$$

With the aid of equation (9.84) on the preceding page we can calculate the associated magnetic field, and find that it is given by

$$\mathbf{B} = \frac{1}{\omega}K\hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\omega}(\hat{\mathbf{k}} \times \mathbf{E})(\alpha + \mathrm{i}\beta) = \frac{1}{\omega}(\hat{\mathbf{k}} \times \mathbf{E})\,|A|\,\mathrm{e}^{\mathrm{i}\gamma} \tag{9.94}$$

where we have, in the last step, rewritten $\alpha + \mathrm{i}\beta$ in the amplitude-phase form $|A|\exp(\mathrm{i}\gamma)$. From the above, we immediately see that $\mathbf{E}$, and consequently also $\mathbf{B}$, is damped, and that $\mathbf{E}$ and $\mathbf{B}$ in the wave are out of phase.

In the limit $\varepsilon_0\omega \ll \sigma$, we can approximate $K$ as follows:

$$\begin{aligned}K &= k\left(1+\mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\right)^{\frac{1}{2}} = k\left[\mathrm{i}\frac{\sigma}{\varepsilon_0\omega}\left(1-\mathrm{i}\frac{\varepsilon_0\omega}{\sigma}\right)\right]^{\frac{1}{2}} \approx k(1+\mathrm{i})\sqrt{\frac{\sigma}{2\varepsilon_0\omega}} \\ &= \sqrt{\varepsilon_0\mu_0}\,\omega(1+\mathrm{i})\sqrt{\frac{\sigma}{2\varepsilon_0\omega}} = (1+\mathrm{i})\sqrt{\frac{\mu_0\sigma\omega}{2}}\end{aligned} \tag{9.95}$$

In this limit we find that when the wave impinges perpendicularly upon the medium, the fields are given, *inside* the medium, by

$$\mathbf{E}' = \mathbf{E}_0 \exp\left\{-\sqrt{\frac{\mu_0\sigma\omega}{2}}\zeta\right\}\exp\left\{\mathrm{i}\left(\sqrt{\frac{\mu_0\sigma\omega}{2}}\zeta-\omega t\right)\right\} \tag{9.96a}$$

$$\mathbf{B}' = (1+\mathrm{i})\sqrt{\frac{\mu_0\sigma}{2\omega}}\,(\hat{\mathbf{n}}\times\mathbf{E}') \tag{9.96b}$$

Hence, both fields fall off by a factor $1/e$ at a distance

$$\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}} \tag{9.97}$$

This distance $\delta$ is called the *skin depth*.

Assuming for simplicity that the *electric permittivity* $\varepsilon$ and the *magnetic permeability* $\mu$, and hence the *relative permittivity* $\kappa_\mathrm{e}$ and the *relative permeability* $\kappa_\mathrm{m}$ all have fixed values, independent on time and space, for each type of material we consider, we can derive the general *telegrapher's equation* [*cf.* equation (9.76) on page 205]

$$\frac{\partial^2\mathbf{E}}{\partial\zeta^2} - \sigma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0} \tag{9.98}$$

describing (1D) wave propagation in a material medium.

In chapter 2 we concluded that the existence of a finite conductivity, manifesting itself in a *collisional interaction* between the charge carriers, causes the waves to decay exponentially with time and space. Let us therefore assume that in our medium $\sigma = 0$ so that the wave equation simplifies to

$$\frac{\partial^2\mathbf{E}}{\partial\zeta^2} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0} \tag{9.99}$$

As in the vacuum case discussed in chapter 2, assuming that $\mathbf{E}$ is time-harmonic, *i.e.* can be represented by a Fourier component proportional to $\exp(-\mathrm{i}\omega t)$, the solution of equation (9.99) can be written

$$\mathbf{E} = \mathbf{E}_0 \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{9.100}$$

where now $\mathbf{k}$ is the wave vector *in the medium* given by equation (9.20) on page 194. With these definitions, the vacuum formula for the associated magnetic field, equation (9.84) on page 206,

$$\mathbf{B} = \sqrt{\varepsilon\mu}\,\hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{v_\varphi}\hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E} \tag{9.101}$$

is valid also in a material medium (assuming, as mentioned, that $n$ has a fixed constant scalar value). A consequence of a $\kappa_\mathrm{e} \neq 1$ is that the electric field will, in general, have a longitudinal component.

▷Electromagnetic waves in an electrically and magnetically conducting medium—— EXAMPLE 9.1

Derive the wave equation for the $\mathbf{E}$ field described by the electromagnetodynamic equations (Dirac's symmetrised Maxwell equations) [*cf.* equations (2.2) on page 20]

$$\nabla \cdot \mathbf{E} = \frac{\rho^\mathrm{e}}{\varepsilon_0} \tag{9.102a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{j}^\mathrm{m} \tag{9.102b}$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^\mathrm{m} \tag{9.102c}$$

$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}^\mathrm{e} \tag{9.102d}$$

under the assumption of vanishing net electric and magnetic charge densities and in the absence of electromotive and magnetomotive forces. Interpret this equation physically.

Assume, for symmetry reasons, that there exists a linear relation between the *magnetic* current density $\mathbf{j}^\mathrm{m}$ and the magnetic field $\mathbf{B}$ (the magnetic dual of Ohm's law for *electric* currents, $\mathbf{j}^\mathrm{e} = \sigma^\mathrm{e}\mathbf{E}$)

$$\mathbf{j}^\mathrm{m} = \sigma^\mathrm{m}\mathbf{B} \tag{9.103}$$

Taking the curl of (2.2c) and using (2.2d), one finds, noting that $\varepsilon_0\mu_0 = 1/c^2$, that

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= -\mu_0 \nabla \times \mathbf{j}^\mathrm{m} - \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= -\mu_0\sigma^\mathrm{m}\nabla \times \mathbf{B} - \frac{\partial}{\partial t}\left(\mu_0 \mathbf{j}^\mathrm{e} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \\ &= -\mu_0\sigma^\mathrm{m}\left(\mu_0\sigma^\mathrm{e}\mathbf{E} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) - \mu_0\sigma^\mathrm{e}\frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\end{aligned} \tag{9.104}$$

Using the vector operator identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$, and the fact that $\nabla \cdot \mathbf{E} = 0$ for a vanishing net electric charge, we can rewrite the wave equation as

$$\nabla^2\mathbf{E} - \mu_0\left(\sigma^\mathrm{e} + \frac{\sigma^\mathrm{m}}{c^2}\right)\frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0^2\sigma^\mathrm{m}\sigma^\mathrm{e}\mathbf{E} = \mathbf{0} \tag{9.105}$$

This is the homogeneous electromagnetodynamic wave equation for $\mathbf{E}$ that we were after.

Compared to the ordinary electrodynamic wave equation for $\mathbf{E}$, equation (9.61) on page 203, we see that we pick up extra terms. In order to understand what these extra terms mean physically, we analyse the time-independent wave equation for a single Fourier component. Then our wave equation becomes

$$\begin{aligned}&\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} - \mu_0^2\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\mathbf{E}\\ &= \nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left[\left(1 - \frac{1}{\omega^2}\frac{\mu_0}{\varepsilon_0}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right) + \mathrm{i}\frac{\sigma^{\mathrm{e}} + \sigma^{\mathrm{m}}/c^2}{\varepsilon_0\omega}\right]\mathbf{E} = \mathbf{0}\end{aligned} \tag{9.106}$$

Realising that, according to formula (6.4) on page 105, $\mu_0/\varepsilon_0$ is the square of the vacuum radiation resistance $R_0$, and rearranging a bit, we obtain the time-independent wave equation in Dirac's symmetrised electrodynamics

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\left(1 - \frac{R_0^2}{\omega^2}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right)\left(1 + \mathrm{i}\frac{\sigma^{\mathrm{e}} + \sigma^{\mathrm{m}}/c^2}{\varepsilon_0\omega\left(1 - \frac{R_0^2}{\omega^2}\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}\right)}\right)\mathbf{E} = \mathbf{0}, \qquad \omega \neq R_0\sqrt{\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}} \tag{9.107}$$

From this equation we conclude that the existence of magnetic charges (magnetic monopoles), and non-vanishing electric and magnetic conductivities would lead to a shift in the effective wave number of the wave. Furthermore, even if the electric conductivity $\sigma^{\mathrm{e}}$ vanishes, the imaginary term does not necessarily vanish and the wave therefore experiences damping or growth according as $\sigma^{\mathrm{m}}$ is positive or negative, respectively. This would happen in a hypothetical medium which is a perfect insulator for electric currents but which can carry magnetic currents.

Finally, we note that in the particular case $\omega = R_0\sqrt{\sigma^{\mathrm{m}}\sigma^{\mathrm{e}}} \stackrel{\mathrm{def}}{\equiv} \omega_m$, the time-independent wave equation equation (9.106) above becomes a time-independent diffusion equation

$$\nabla^2\mathbf{E} + \mathrm{i}\omega\mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)\mathbf{E} = \mathbf{0} \tag{9.108}$$

which in time domain corresponds to the *time-dependent diffusion equation*

$$\frac{\partial\mathbf{E}}{\partial t} - D\nabla^2\mathbf{E} - = \mathbf{0} \tag{9.109}$$

with a *diffusion coefficient* given by

$$D = \frac{1}{\mu_0\left(\sigma^{\mathrm{e}} + \frac{\sigma^{\mathrm{m}}}{c^2}\right)} \tag{9.110}$$

Hence, electromagnetic waves with this particular frequency do not propagate. This means that if magnetic charges (monopoles) exist in a region in the Universe, electromagnetic waves propagating through this region would, in this simplistic model, exhibit a lower cutoff at $\omega = \omega_m$. This would in fact impose a lower limit on the mass of the *photon*, the quantum of the electromagnetic field that we shall come across later.

End of example 9.1 ◁

## 9.5 Bibliography

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# F

# FORMULÆ

This appendix contains an comprehensive collection of vector and tensor algebra and calculus formulæ and identities, including perhaps a few that are not included in every electrodynamics textbook. It also lists the most important electrodynamics formulæ from the various chapters.

## F.1 Vector and tensor fields in 3D Euclidean space

Let $\mathbf{x}$ be the *position vector* (*radius vector*, *coordinate vector*) from the origin of the Euclidean space $\mathbb{R}^3$ coordinate system to the coordinate point $(x_1, x_2, x_3)$ in the same system and let $|\mathbf{x}|$ denote the magnitude ('length') of $\mathbf{x}$. Let further $\alpha(\mathbf{x}), \beta(\mathbf{x}), \ldots$, be arbitrary scalar fields, $\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}), \ldots$, arbitrary vector fields, and $\mathsf{A}(\mathbf{x}), \mathsf{B}(\mathbf{x}), \ldots$, arbitrary rank two tensor fields in this space. Let $^*$ denote *complex conjugate* and $^\dagger$ denote *Hermitian conjugate* (transposition and, where applicable, complex conjugation).

The differential vector operator $\boldsymbol{\nabla}$ is in Cartesian coordinates given by

$$\boldsymbol{\nabla} \equiv \sum_{i=1}^{3} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \boldsymbol{\partial} \tag{F.1}$$

where $\hat{\mathbf{x}}_i$, $i = 1, 2, 3$ is the $i$th unit vector and $\hat{\mathbf{x}}_1 \equiv \hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2 \equiv \hat{\mathbf{y}}$, and $\hat{\mathbf{x}}_3 \equiv \hat{\mathbf{z}}$. In component (tensor) notation $\boldsymbol{\nabla}$ can be written

$$\nabla_i = \partial_i = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{F.2}$$

The differential vector operator $\boldsymbol{\nabla}'$ is defined as

$$\boldsymbol{\nabla}' \equiv \sum_{i=1}^{3} \hat{\mathbf{x}}_i \frac{\partial}{\partial x'_i} \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i \frac{\partial}{\partial x'_i} \stackrel{\text{def}}{\equiv} \boldsymbol{\partial}' \tag{F.3}$$

or

$$\nabla'_i = \partial'_i = \left( \frac{\partial}{\partial x'_1}, \frac{\partial}{\partial x'_2}, \frac{\partial}{\partial x'_3} \right) = \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) \tag{F.4}$$

## F.1.1 Cylindrical circular coordinates

### F.1.1.1 Base vectors

CARTESIAN TO CYLINDRICAL CIRCULAR

$$\hat{\boldsymbol{\rho}} = \cos\theta\,\hat{\mathbf{x}}_1 + \sin\theta\,\hat{\mathbf{x}}_2 \tag{F.5a}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\theta\,\hat{\mathbf{x}}_1 + \cos\theta\,\hat{\mathbf{x}}_2 \tag{F.5b}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{x}}_3 \tag{F.5c}$$

CYLINDRICAL CIRCULAR TO CARTESIAN

$$\hat{\mathbf{x}}_1 = \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\boldsymbol{\varphi}} \tag{F.6a}$$

$$\hat{\mathbf{x}}_2 = \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\boldsymbol{\varphi}} \tag{F.6b}$$

$$\hat{\mathbf{x}}_3 = \hat{\mathbf{z}} \tag{F.6c}$$

### F.1.1.2 Directed line element

$$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \rho\,\mathrm{d}\varphi\;\hat{\boldsymbol{\varphi}} + \mathrm{d}z\hat{\mathbf{z}} \tag{F.7}$$

### F.1.1.3 Directed area element

$$\mathrm{d}\mathbf{S} = \rho\,\mathrm{d}\varphi\,\mathrm{d}z\,\hat{\boldsymbol{\rho}} + \mathrm{d}\rho\,\mathrm{d}z\,\hat{\boldsymbol{\varphi}} + \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\hat{\mathbf{z}} \tag{F.8}$$

### F.1.1.4 Volume element

$$\mathrm{d}V = \mathrm{d}^3x = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z \tag{F.9}$$

### F.1.1.5 Spatial differential operators

In the following we assume that the scalar field $\alpha = \alpha(\rho, \varphi, z)$ and that the vector field $\mathbf{a} = \mathbf{a}(\rho, \varphi, z)$.

THE GRADIENT

$$\boldsymbol{\nabla}\alpha = \hat{\boldsymbol{\rho}}\,\frac{\partial\alpha}{\partial\rho} + \hat{\boldsymbol{\theta}}\,\frac{1}{\rho}\frac{\partial\alpha}{\partial\varphi} + \hat{\mathbf{z}}\,\frac{\partial\alpha}{\partial z} \tag{F.10}$$

THE DIVERGENCE

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_z}{\partial z} \tag{F.11}$$

#### The curl

$$\nabla \times \mathbf{a} = \hat{\boldsymbol{\rho}} \left( \frac{1}{\rho} \frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z} \right) + \hat{\boldsymbol{\varphi}} \left( \frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial \rho} \right) + \hat{\mathbf{z}} \frac{1}{\rho} \left( \frac{\partial (\rho a_\varphi)}{\partial \rho} - \frac{\partial a_\rho}{\partial \varphi} \right) \tag{F.12}$$

#### The Laplacian

$$\nabla^2 \alpha = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \alpha}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \alpha}{\partial \varphi^2} + \frac{\partial^2 \alpha}{\partial z^2} \tag{F.13}$$

## F.1.2 Spherical polar coordinates

### F.1.2.1 Base vectors

#### Cartesian to spherical polar

$$\hat{\mathbf{r}} = \sin\theta \cos\varphi \, \hat{\mathbf{x}}_1 + \sin\theta \sin\varphi \, \hat{\mathbf{x}}_2 + \cos\theta \, \hat{\mathbf{x}}_3 \tag{F.14a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta \cos\varphi \, \hat{\mathbf{x}}_1 + \cos\theta \sin\varphi \, \hat{\mathbf{x}}_2 - \sin\theta \, \hat{\mathbf{x}}_3 \tag{F.14b}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi \, \hat{\mathbf{x}}_1 + \cos\varphi \, \hat{\mathbf{x}}_2 \tag{F.14c}$$

#### Spherical polar to Cartesian

$$\hat{\mathbf{x}}_1 = \sin\theta \cos\varphi \, \hat{\mathbf{r}} + \cos\theta \cos\varphi \, \hat{\boldsymbol{\theta}} - \sin\varphi \, \hat{\boldsymbol{\varphi}} \tag{F.15a}$$

$$\hat{\mathbf{x}}_2 = \sin\theta \sin\varphi \, \hat{\mathbf{r}} + \cos\theta \sin\varphi \, \hat{\boldsymbol{\theta}} + \cos\varphi \, \hat{\boldsymbol{\varphi}} \tag{F.15b}$$

$$\hat{\mathbf{x}}_3 = \cos\theta \, \hat{\mathbf{r}} - \sin\theta \, \hat{\boldsymbol{\theta}} \tag{F.15c}$$

### F.1.2.2 Directed line element

$$\mathrm{d}\mathbf{l} = \mathrm{d}x \, \hat{\mathbf{x}} = \mathrm{d}r \, \hat{\mathbf{r}} + r \, \mathrm{d}\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, \mathrm{d}\varphi \, \hat{\boldsymbol{\varphi}} \tag{F.16}$$

### F.1.2.3 Solid angle element

$$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \tag{F.17}$$

### F.1.2.4 Directed area element

$$\mathrm{d}\mathbf{S} = r^2 \mathrm{d}\Omega \, \hat{\mathbf{r}} + r \sin\theta \, \mathrm{d}r \, \mathrm{d}\varphi \, \hat{\boldsymbol{\theta}} + r \, \mathrm{d}r \, \mathrm{d}\theta \, \hat{\boldsymbol{\varphi}} \tag{F.18}$$

#### F.1.2.5 Volume element

$$\mathrm{d}V = \mathrm{d}^3x = \mathrm{d}r\, r^2 \mathrm{d}\Omega \tag{F.19}$$

#### F.1.2.6 Spatial differential operators

In the following we assume that the scalar field $\alpha = \alpha(r, \theta, \varphi)$ and that the vector field $\mathbf{a} = \mathbf{a}(r, \theta, \varphi)$.

THE GRADIENT

$$\boldsymbol{\nabla}\alpha = \hat{\mathbf{r}}\,\frac{\partial\alpha}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial\alpha}{\partial\theta} + \hat{\boldsymbol{\varphi}}\,\frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\varphi} \tag{F.20}$$

THE DIVERGENCE

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi} \tag{F.21}$$

THE CURL

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{a} = \hat{\mathbf{r}}\,&\frac{1}{r\sin\theta}\left(\frac{\partial(a_\varphi\sin\theta)}{\partial\theta} - \frac{\partial a_\theta}{\partial\varphi}\right)\\ &+ \hat{\boldsymbol{\theta}}\,\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{\partial(r a_\varphi)}{\partial r}\right)\\ &+ \hat{\boldsymbol{\varphi}}\,\frac{1}{r}\left(\frac{\partial(r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)\end{aligned} \tag{F.22}$$

THE LAPLACIAN

$$\nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\alpha}{\partial\varphi^2} \tag{F.23}$$

### F.1.3 Vector and tensor field formulæ

#### F.1.3.1 The three-dimensional unit tensor of rank two

$$\mathbf{1}_3 = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i \tag{F.24}$$

with matrix representation, denoted by $(\dots)$,

$$(\mathbf{1}_3) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{F.25}$$

### F.1.3.2 The 3D Kronecker delta tensor

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{F.26}$$

has the same matrix representation as $\mathbf{1}_3$:

$$(\delta_{ij}) = (\mathbf{1}_3) \tag{F.27}$$

### F.1.3.3 The fully antisymmetric Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of 1,2,3} \\ 0 & \text{if at least two of } i, j, k \text{ are equal} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of 1,2,3} \end{cases} \tag{F.28}$$

has the properties

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \tag{F.29}$$

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} \tag{F.30}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{F.31}$$

### F.1.3.4 Rotational matrices

The rotational matrix vector

$$\mathbf{S} = \mathsf{S}_i\hat{\mathbf{x}}_i = -\mathrm{i}(\epsilon_{ijk})\hat{\mathbf{x}}_i \tag{F.32}$$

has the matrices $\mathsf{S}_i$ as components, where

$$\mathsf{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} = -\mathrm{i}(\epsilon_{1jk}) \tag{F.33a}$$

$$\mathsf{S}_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} = -\mathrm{i}(\epsilon_{2jk}) \tag{F.33b}$$

$$\mathsf{S}_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathrm{i}(\epsilon_{3jk}) \tag{F.33c}$$

#### F.1.3.5 General vector and tensor algebra identities

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \delta_{ij}a_i b_j = ab\cos\theta \tag{F.34}$$

$$\mathbf{a}\cdot\mathbf{a} = a^2 \tag{F.35}$$

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \epsilon_{ijk}\hat{\mathbf{x}}_i a_j b_k = \epsilon_{jki}\hat{\mathbf{x}}_j a_k b_i = \epsilon_{kij}\hat{\mathbf{x}}_k a_i b_j \tag{F.36}$$

$$(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\times\mathbf{b})^2 = a^2b^2 \tag{F.37}$$

$$\mathsf{A} = A_{ij}\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \tag{F.38}$$

$$\mathsf{A}^\dagger = A^*_{ij}\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \tag{F.39}$$

$$A_{ij} = \hat{\mathbf{x}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{x}}_j \tag{F.40}$$

$$\mathrm{Tr}(\mathsf{A}) = A_{ii} \tag{F.41}$$

$$\mathbf{a}\otimes\mathbf{b} \equiv \mathbf{ab} = \hat{\mathbf{x}}_i a_i b_j \hat{\mathbf{x}}_j \tag{F.42}$$

$$\mathrm{Tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} \tag{F.43}$$

$$\mathbf{a}\otimes(\mathbf{b}+\mathbf{c}) = \mathbf{a}\otimes\mathbf{b} + \mathbf{a}\otimes\mathbf{c} \tag{F.44}$$

$$(\mathbf{a}+\mathbf{b})\otimes\mathbf{c} = \mathbf{a}\otimes\mathbf{c} + \mathbf{b}\otimes\mathbf{c} \tag{F.45}$$

$$\mathbf{c}\cdot\mathbf{a}\otimes\mathbf{b} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} \tag{F.46}$$

$$\mathbf{a}\otimes\mathbf{b}\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{F.47}$$

$$(\mathbf{a}\times\mathbf{b}) = -\mathrm{i}\mathbf{a}\cdot\mathsf{S}\otimes\mathbf{b} = -\mathrm{i}\mathbf{a}\otimes\mathsf{S}\cdot\mathbf{b} \tag{F.48}$$

$$\mathbf{c}\times\mathbf{a}\otimes\mathbf{b} = (\mathbf{c}\times\mathbf{a})\otimes\mathbf{b} \tag{F.49}$$

$$\mathbf{a}\otimes\mathbf{b}\times\mathbf{c} = \mathbf{a}\otimes(\mathbf{b}\times\mathbf{c}) \tag{F.50}$$

$$\mathbf{a}\otimes\mathbf{b}\cdot\mathbf{c}\otimes\mathbf{d} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\otimes\mathbf{d} \tag{F.51}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{F.52}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\otimes\mathbf{a}\cdot\mathbf{c} - \mathbf{c}\otimes\mathbf{a}\cdot\mathbf{b} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \tag{F.53}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}\otimes\mathbf{a}\cdot\mathbf{c} - \mathbf{a}\otimes\mathbf{b}\cdot\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{F.54}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0} \tag{F.55}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{F.56}$$

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\times\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})\mathbf{d} \tag{F.57}$$

#### F.1.3.6 Special vector and tensor algebra identities

$$\mathsf{1}_3\cdot\mathbf{a} = \mathbf{a}\cdot\mathsf{1}_3 = \mathbf{a} \tag{F.58}$$

$$\mathsf{1}_3\times\mathbf{a} = \mathbf{a}\times\mathsf{1}_3 \tag{F.59}$$

$$(\mathsf{1}_3\times\mathbf{a}) = (\mathbf{a}\times\mathsf{1}_3) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = -\mathrm{i}\mathsf{S}\cdot\mathbf{a} \tag{F.60}$$

$$\mathbf{a}\cdot(\mathsf{1}_3\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} \tag{F.61}$$

$$\mathbf{a} \cdot (\mathbf{1}_3 \times \mathbf{b}) = \mathbf{a} \times \mathbf{b} \tag{F.62}$$
$$\mathbf{1}_3 \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \otimes \mathbf{a} - \mathbf{a} \otimes \mathbf{b} \tag{F.63}$$

### F.1.3.7 General vector and tensor calculus identities

$$\boldsymbol{\nabla}\alpha = \hat{\mathbf{x}}_i \partial_i \alpha \tag{F.64}$$
$$\boldsymbol{\nabla} \cdot \mathbf{a} = \partial_i a_i \tag{F.65}$$
$$\mathbf{a} \cdot \boldsymbol{\nabla} = a_i \partial_i \tag{F.66}$$
$$\boldsymbol{\nabla} \times \mathbf{a} = \epsilon_{ijk} \hat{\mathbf{x}}_i \partial_j a_k \tag{F.67}$$
$$\mathbf{a} \times \boldsymbol{\nabla} = \epsilon_{ijk} \hat{\mathbf{x}}_i a_j \partial_k \tag{F.68}$$
$$\boldsymbol{\nabla} \otimes \mathbf{a} = \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \partial_i a_j \tag{F.69}$$
$$\mathbf{a} \otimes \boldsymbol{\nabla} = \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j a_i \partial_j \tag{F.70}$$
$$\mathrm{Tr}(\boldsymbol{\nabla} \otimes \mathbf{a}) = \boldsymbol{\nabla} \cdot \mathbf{a} \tag{F.71}$$
$$\boldsymbol{\nabla} \cdot \mathsf{A} = \hat{\mathbf{x}}_j \partial_i A_{ij} \tag{F.72}$$
$$\mathbf{a} \otimes \boldsymbol{\nabla} \cdot \mathbf{b} = \mathbf{a} \partial_i b_i \tag{F.73}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\alpha = \nabla^2 \alpha \tag{F.74}$$
$$\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \cdot \mathbf{a} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) = \hat{\mathbf{x}}_i \partial_i \partial_j a_j \tag{F.75}$$
$$\mathbf{a} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} = (\mathbf{a} \cdot \boldsymbol{\nabla})\boldsymbol{\nabla} = \hat{\mathbf{x}}_i a_j \partial_j \partial_i \tag{F.76}$$
$$\boldsymbol{\nabla}(\alpha\beta) = \alpha \boldsymbol{\nabla}\beta + \beta \boldsymbol{\nabla}\alpha \tag{F.77}$$
$$\boldsymbol{\nabla} \otimes (\alpha \mathbf{a}) = (\boldsymbol{\nabla}\alpha) \otimes \mathbf{a} + \alpha \boldsymbol{\nabla} \otimes \mathbf{a} \tag{F.78}$$
$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) + \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{a}) + \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{b} + \mathbf{b} \cdot \boldsymbol{\nabla} \otimes \mathbf{a} \tag{F.79}$$
$$\boldsymbol{\nabla}(\mathbf{a} \times \mathbf{b}) = (\boldsymbol{\nabla}\mathbf{a}) \times \mathbf{b} - (\boldsymbol{\nabla}\mathbf{b}) \times \mathbf{a} \tag{F.80}$$
$$\boldsymbol{\nabla} \cdot (\alpha \mathbf{a}) = \mathbf{a} \cdot \boldsymbol{\nabla}\alpha + \alpha \boldsymbol{\nabla} \cdot \mathbf{a} \tag{F.81}$$
$$(\boldsymbol{\nabla}\alpha) \cdot \boldsymbol{\nabla}\beta = \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla}\beta) - \alpha \nabla^2 \beta \tag{F.82}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) \tag{F.83}$$
$$(\boldsymbol{\nabla}\alpha) \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = -\boldsymbol{\nabla} \cdot (\mathbf{a} \times \boldsymbol{\nabla}\alpha) \tag{F.84}$$
$$(\boldsymbol{\nabla} \times \mathbf{a}) \cdot (\boldsymbol{\nabla} \times \mathbf{b}) = \mathbf{b} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})] - \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \times \mathbf{a}) \times \mathbf{b}] \tag{F.85}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{b} \tag{F.86}$$
$$\boldsymbol{\nabla} \times (\alpha \mathbf{a}) = \alpha \boldsymbol{\nabla} \times \mathbf{a} - \mathbf{a} \times \boldsymbol{\nabla}\alpha \tag{F.87}$$
$$\boldsymbol{\nabla} \times (\alpha \boldsymbol{\nabla}\beta) = (\boldsymbol{\nabla}\alpha) \times \boldsymbol{\nabla}\beta \tag{F.88}$$
$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \otimes \boldsymbol{\nabla} \cdot \mathbf{b} - \mathbf{b} \otimes \boldsymbol{\nabla} \cdot \mathbf{a} + \mathbf{b} \cdot \boldsymbol{\nabla} \otimes \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{b} \tag{F.89}$$
$$\boldsymbol{\nabla} \times (\mathbf{a} \otimes \mathbf{b}) = (\boldsymbol{\nabla} \times \mathbf{a}) \otimes \mathbf{b} - \mathbf{a} \times \boldsymbol{\nabla} \otimes \mathbf{b} \tag{F.90}$$
$$\mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) = (\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{b} \tag{F.91}$$
$$\mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) = (\boldsymbol{\nabla} \otimes \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{b} \tag{F.92}$$

$$\mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \tfrac{1}{2}\boldsymbol{\nabla}(a^2) - \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{a} \tag{F.93}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \times \mathbf{b} = (\boldsymbol{\nabla} \otimes \mathbf{b}) \cdot \mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} \tag{F.94}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \times \boldsymbol{\nabla} = \mathbf{a} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} - \mathbf{a}\nabla^2 \tag{F.95}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \cdot \mathbf{a} - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\mathbf{a} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2\mathbf{a} \tag{F.96}$$

$$\mathbf{a} \cdot \boldsymbol{\nabla} \otimes (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{b}) \times \mathbf{c} + \mathbf{b} \times (\mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{c}) \tag{F.97}$$

$$\mathbf{a} \cdot \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{a} \cdot \boldsymbol{\nabla} \otimes \mathbf{c}) - \mathbf{a} \cdot (\mathbf{b} \cdot \boldsymbol{\nabla} \otimes \mathbf{c}) \tag{F.98}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = 0 \tag{F.99}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\alpha = \mathbf{0} \tag{F.100}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{\nabla}) \cdot \mathbf{a} = 0 \tag{F.101}$$

### F.1.3.8 Special vector and tensor calculus identities

In the following $\mathbf{S}$ is the matrix vector defined in formula (F.32) and $\mathbf{k}$ is an arbitrary *constant* vector.

$$\boldsymbol{\nabla} \cdot (\mathbf{1}_3\alpha) = \boldsymbol{\nabla}\alpha \tag{F.102}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{1}_3 \cdot \mathbf{a}) = \boldsymbol{\nabla} \cdot \mathbf{a} \tag{F.103}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{1}_3 \times \mathbf{a}) = \boldsymbol{\nabla} \times \mathbf{a} \tag{F.104}$$

$$(\boldsymbol{\nabla} \times \mathbf{a}) = -\mathrm{i}\boldsymbol{\nabla} \cdot \mathbf{S} \otimes \mathbf{a} = -\mathrm{i}\boldsymbol{\nabla} \otimes \mathbf{S} \cdot \mathbf{a} \tag{F.105}$$

$$\boldsymbol{\nabla} \cdot \mathbf{x} = 3 \tag{F.106}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \mathbf{0} \tag{F.107}$$

$$\boldsymbol{\nabla} \otimes \mathbf{x} = \mathbf{1}_3 \tag{F.108}$$

$$\boldsymbol{\nabla}(\mathbf{x} \cdot \mathbf{k}) = \mathbf{k} \tag{F.109}$$

$$\boldsymbol{\nabla}(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{a}) + (\mathbf{x} \times \boldsymbol{\nabla}) \times \mathbf{a} \tag{F.110}$$

$$\boldsymbol{\nabla}|\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \tag{F.111}$$

$$\boldsymbol{\nabla}\left(|\mathbf{x} - \mathbf{x}'|\right) = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} = -\boldsymbol{\nabla}'\left(|\mathbf{x} - \mathbf{x}'|\right) \tag{F.112}$$

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{F.113}$$

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) \tag{F.114}$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) = -\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = 4\pi\delta(\mathbf{x}) \tag{F.115}$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\right) = -\nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = 4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{F.116}$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{k}}{|\mathbf{x}|}\right) = \mathbf{k} \cdot \left[\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}|}\right)\right] = -\frac{\mathbf{k} \cdot \mathbf{x}}{|\mathbf{x}|^3} \tag{F.117}$$

$$\boldsymbol{\nabla} \times \left[\mathbf{k} \times \left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\right)\right] = -\boldsymbol{\nabla}\left(\frac{\mathbf{k} \cdot \mathbf{x}}{|\mathbf{x}|^3}\right) \text{ if } |\mathbf{x}| \neq 0 \tag{F.118}$$

$$\nabla^2\left(\frac{\mathbf{k}}{|\mathbf{x}|}\right) = \mathbf{k}\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\mathbf{k}\delta(\mathbf{x}) \tag{F.119}$$

$$\boldsymbol{\nabla} \times (\mathbf{k} \times \mathbf{a}) = \mathbf{k}(\boldsymbol{\nabla} \cdot \mathbf{a}) + \mathbf{k} \times (\boldsymbol{\nabla} \times \mathbf{a}) - \boldsymbol{\nabla}(\mathbf{k} \cdot \mathbf{a}) \tag{F.120}$$

#### F.1.3.9 Integral identities

##### Divergence theorem and related theorems

Let $V(S)$ be the volume bounded by the closed surface $S(V)$. Denote the 3D volume element by $\mathrm{d}^3x (\equiv \mathrm{d}V)$ and the surface element, directed along the outward pointing surface normal unit vector $\hat{\mathbf{n}}$, by $\mathrm{d}\mathbf{S}(\equiv \mathrm{d}^2x\, \hat{\mathbf{n}})$. Then

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla}\alpha = \oint_S \mathrm{d}\mathbf{S}\,\alpha \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}}\alpha \tag{F.121a}$$

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \cdot \mathbf{a} = \oint_S \mathrm{d}\mathbf{S} \cdot \mathbf{a} \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathbf{a} \tag{F.121b}$$

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \times \mathbf{a} = \oint_S \mathrm{d}\mathbf{S} \times \mathbf{a} \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \times \mathbf{a} \tag{F.121c}$$

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \otimes \mathbf{a} = \oint_S \mathrm{d}\mathbf{S} \otimes \mathbf{a} \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \otimes \mathbf{a} \tag{F.121d}$$

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \cdot \mathsf{A} = \oint_S \mathrm{d}\mathbf{S} \cdot \mathsf{A} \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \mathsf{A} \tag{F.121e}$$

$$\int_V \mathrm{d}^3x\, \boldsymbol{\nabla} \times \mathsf{A} = \oint_S \mathrm{d}\mathbf{S} \times \mathsf{A} \equiv \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \times \mathsf{A} \tag{F.121f}$$

If $S(C)$ is an open surface bounded by the contour $C(S)$, whose line element is $\mathrm{d}\mathbf{l}$, then

$$\oint_C \mathrm{d}\mathbf{l}\,\alpha = \int_S \mathrm{d}\mathbf{S} \times \boldsymbol{\nabla}\alpha \tag{F.122}$$

$$\oint_C \mathrm{d}\mathbf{l} \cdot \mathbf{a} = \int_S \mathrm{d}\mathbf{S} \cdot \boldsymbol{\nabla} \times \mathbf{a} \tag{F.123}$$

##### Green's first identity

$$\int_V \mathrm{d}^3x\, [(\boldsymbol{\nabla}\alpha) \cdot \boldsymbol{\nabla}\beta + \alpha\nabla^2\beta] = \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \cdot \alpha\boldsymbol{\nabla}\beta \tag{F.124}$$

follows from formula (F.82).

##### Generic partial integration identity

$$\boldsymbol{\nabla} \circ \int_{V'} \mathrm{d}^3x'\, \frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \circ \mathcal{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \oint_{S'} \mathrm{d}^2x'\, \hat{\mathbf{n}}' \circ \frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{F.125}$$

where $\circ$ is either (i) nothing (juxtaposition) and $\mathcal{A} = \alpha$, or (ii) $\circ = \boldsymbol{\cdot}$ or $\circ = \times$ and $\mathcal{A} = \mathbf{a}$ or $\mathcal{A} = \mathbf{A}$.

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\nabla} \int_{V'} \mathrm{d}^3x' \, \frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \mathcal{A}(\mathbf{x}') \nabla^2 \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi \mathcal{A}(\mathbf{x}) \quad \text{(F.126)}$$

where $\mathcal{A} = \alpha$ or $\mathcal{A} = \mathbf{a}$.

## Specific integral identities

$$\begin{aligned} \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \boldsymbol{\cdot} \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} &= - \int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \boldsymbol{\cdot} \mathbf{a}(\mathbf{x}')] \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &\quad + \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \boldsymbol{\cdot} \left( \frac{\mathbf{a}(\mathbf{x}') \otimes (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right) \end{aligned} \quad \text{(F.127)}$$

$$\begin{aligned} \boldsymbol{\nabla} \times \left( \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) &= \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \boldsymbol{\cdot} \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &\quad - \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\nabla} \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= 4\pi \mathbf{a}(\mathbf{x}) - \int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \boldsymbol{\cdot} \mathbf{a}(\mathbf{x}')] \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &\quad + \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \boldsymbol{\cdot} \left( \frac{\mathbf{a}(\mathbf{x}') \otimes (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right) \end{aligned} \quad \text{(F.128)}$$

## Helmholtz decomposition

Any regular, differentiable vector field $\mathbf{u}$ that falls off sufficiently fast asymptotically can be decomposed into two components, one irrotational and one rotational, such that

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) + \mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \quad \text{(F.129a)}$$

where

$$\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) = -\boldsymbol{\nabla} \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \boldsymbol{\cdot} \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \quad \text{(F.129b)}$$

$$\mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) = \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \times \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \quad \text{(F.129c)}$$

If $\mathbf{v}$ is a vector field with the same properties as $\mathbf{u}$, or $\mathbf{u}$ itself, then

$$\int_V \mathrm{d}^3x \, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \boldsymbol{\cdot} \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = 0 \quad \text{(F.130a)}$$

$$\int_V \mathrm{d}^3x \, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) = \mathbf{0} \quad \text{(F.130b)}$$

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) = \int_V \mathrm{d}^3x\, [\boldsymbol{\nabla} \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x})] \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \cdot \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \tag{F.130c}$$

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \int_V \mathrm{d}^3x\, [\boldsymbol{\nabla} \times \mathbf{u}^{\mathrm{rotat}}(\mathbf{x})] \cdot \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \tag{F.130d}$$

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \int_V \mathrm{d}^3x\, [\boldsymbol{\nabla} \times \mathbf{u}^{\mathrm{rotat}}(\mathbf{x})] \times \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \tag{F.130e}$$

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \int_V \mathrm{d}^3x\, [\boldsymbol{\nabla} \cdot \mathbf{u}^{\mathrm{irrot}}(\mathbf{x})] \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \tag{F.130f}$$

## F.2 The electromagnetic field

### F.2.1 Microscopic Maxwell-Lorentz equations in Dirac's symmetrised form

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho^{\mathrm{e}}}{\varepsilon_0} \tag{F.131}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu_0 \rho^{\mathrm{m}} \tag{F.132}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{j}^{\mathrm{m}} \tag{F.133}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}^{\mathrm{e}} \tag{F.134}$$

#### F.2.1.1 Constitutive relations

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad (= 299\,792\,458\ \mathrm{m\,s^{-1}}) \tag{F.135}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = R_0 \quad (= 119.9169832\pi\ \Omega \approx 376.7\ \Omega) \tag{F.136}$$

$$\mathbf{D} \approx \varepsilon \mathbf{E} \tag{F.137}$$

$$\mathbf{H} \approx \mu^{-1} \mathbf{B} \tag{F.138}$$

$$\mathbf{j} \approx \sigma \mathbf{E} \tag{F.139}$$

$$\mathbf{P} \approx \varepsilon_0 \chi \mathbf{E} \tag{F.140}$$

## F.2.2 Fields and potentials

### F.2.2.1 Vector and scalar potentials

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{F.141a}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{F.141b}$$

### F.2.2.2 The velocity gauge condition in free space

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{\alpha}{c^2}\frac{\partial \Phi}{\partial t} = 0, \quad \alpha = \frac{c^2}{v^2}, \quad \begin{cases} \alpha = 1 \Rightarrow \text{ Lorenz-Lorentz gauge} \\ \alpha = 0 \Rightarrow \text{Coulomb gauge} \\ \alpha = -1 \Rightarrow \text{Kirchhoff gauge} \end{cases} \tag{F.142}$$

### F.2.2.3 Gauge transformation

$$\Phi(t, \mathbf{x}) \mapsto \Phi'(t, \mathbf{x}) = \Phi(t, \mathbf{x}) - \frac{\partial \Gamma(t, \mathbf{x})}{\partial t} \tag{F.143a}$$

$$\mathbf{A}(t, \mathbf{x}) \mapsto \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) + \boldsymbol{\nabla}\Gamma(t, \mathbf{x}) \tag{F.143b}$$

## F.2.3 Energy and momentum

### F.2.3.1 Electromagnetic field energy density in free space

$$u^{\text{field}} = \frac{1}{2}\varepsilon_0(\mathbf{E} \cdot \mathbf{E} + c^2\mathbf{B} \cdot \mathbf{B}) \tag{F.144}$$

### F.2.3.2 Poynting vector in free space

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{F.145}$$

### F.2.3.3 Linear momentum density in free space

$$\mathbf{g} = \varepsilon_0\mathbf{E} \times \mathbf{B} \tag{F.146}$$

#### F.2.3.4 Linear momentum flux tensor in free space

$$\mathsf{T} = u^{\text{field}}\mathbf{1}_3 - \varepsilon_0\big(\mathbf{E}\otimes\mathbf{E} + c^2\mathbf{B}\otimes\mathbf{B}\big) \tag{F.147}$$

$$T_{ij} = u^{\text{field}}\delta_{ij} - \varepsilon_0 E_i E_j - \varepsilon_0 c^2 B_i B_j \tag{F.148}$$

#### F.2.3.5 Angular momentum density around $\mathbf{x}_0$ in free space

$$\mathbf{h} = (\mathbf{x}-\mathbf{x}_0)\times\mathbf{g} = (\mathbf{x}-\mathbf{x}_0)\times\varepsilon_0\mathbf{E}\times\mathbf{B} \tag{F.149}$$

#### F.2.3.6 Angular momentum flux tensor around $\mathbf{x}_0$ in free space

$$\mathsf{K}(\mathbf{x}_0) = (\mathbf{x}-\mathbf{x}_0)\times\mathsf{T} \tag{F.150}$$

### F.2.4 Electromagnetic radiation

#### F.2.4.1 The far fields from an extended source distribution

$$\mathbf{E}^{\text{far}}_\omega(\mathbf{x}) \approx -\mathrm{i}\frac{k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\Big[\Big(\mathbf{j}_\omega\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}\Big)\times\hat{\mathbf{k}}\Big] \tag{F.151}$$

$$\approx -\mathrm{i}\frac{k}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}(\boldsymbol{\mathcal{I}}_\omega(\mathbf{x}_0)\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} \tag{F.152}$$

$$\mathbf{B}^{\text{far}}_\omega(\mathbf{x}) \approx -\mathrm{i}\frac{k}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\int_{V'}\mathrm{d}^3x'\,\Big(\mathbf{j}_\omega\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}_0)}\times\hat{\mathbf{k}}\Big) \tag{F.153}$$

$$\approx -\mathrm{i}\frac{k}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\boldsymbol{\mathcal{I}}_\omega(\mathbf{x}_0)\times\hat{\mathbf{n}} \tag{F.154}$$

$$\boldsymbol{\mathcal{I}}_\omega(\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,\mathbf{j}_\omega(\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{n}}\cdot(\mathbf{x}'-\mathbf{x}_0)} \tag{F.155}$$

#### F.2.4.2 The far fields from an electric dipole

$$\mathbf{E}^{\text{far}}_\omega(\mathbf{x}) = -\frac{k^2}{4\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}(\mathbf{d}_\omega(\mathbf{x}_0)\times\hat{\mathbf{k}})\times\hat{\mathbf{k}} \tag{F.156}$$

$$\mathbf{B}^{\text{far}}_\omega(\mathbf{x}) = -\frac{k^2}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\mathbf{d}_\omega(\mathbf{x}_0)\times\hat{\mathbf{k}} \tag{F.157}$$

$$\mathbf{d}(t,\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\,\rho(t'_{\text{ret}},\mathbf{x}') \tag{F.158}$$

#### F.2.4.3 The far fields from a magnetic dipole

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{k^2}{4\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\,\mathbf{m}_\omega(\mathbf{x}_0)\times\hat{\mathbf{k}} \tag{F.159}$$

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = -\frac{k^2}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\left(\mathbf{m}_\omega(\mathbf{x}_0)\times\hat{\mathbf{k}}\right)\times\hat{\mathbf{k}} \tag{F.160}$$

$$\mathbf{m}(t,\mathbf{x}_0) = \frac{1}{2}\int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\times\mathbf{j}(t'_{\text{ret}},\mathbf{x}') \tag{F.161}$$

#### F.2.4.4 The far fields from an electric quadrupole

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \mathrm{i}\frac{k^3}{8\pi\varepsilon_0}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\left[\left(\hat{\mathbf{k}}\cdot\mathbf{Q}_\omega(\mathbf{x}_0)\right)\times\hat{\mathbf{k}}\right]\times\hat{\mathbf{k}} \tag{F.162}$$

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = \mathrm{i}\frac{k^3}{8\pi\varepsilon_0 c}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\left(\hat{\mathbf{k}}\cdot\mathbf{Q}_\omega(\mathbf{x}_0)\right)\times\hat{\mathbf{k}} \tag{F.163}$$

$$\mathbf{Q}(t,\mathbf{x}_0) = \int_{V'}\mathrm{d}^3x'\,(\mathbf{x}'-\mathbf{x}_0)\otimes(\mathbf{x}'-\mathbf{x}_0)\,\rho(t'_{\text{ret}},\mathbf{x}') \tag{F.164}$$

#### F.2.4.5 Relationship between the field vectors in a plane wave

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{E} \tag{F.165}$$

#### F.2.4.6 The fields from a point charge in arbitrary motion

$$\mathbf{E}(t,\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 s^3}\left[(\mathbf{x}-\mathbf{x}_0)\left(1-\frac{v'^2}{c^2}\right)+(\mathbf{x}-\mathbf{x}')\times\frac{(\mathbf{x}-\mathbf{x}_0)\times\mathbf{a}'}{c^2}\right] \tag{F.166}$$

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{c}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{E}(t,\mathbf{x}) \tag{F.167}$$

$$\mathbf{v}'(t') = \frac{\mathrm{d}\mathbf{x}'(t')}{\mathrm{d}t'} \tag{F.168}$$

$$\mathbf{a}'(t') = \frac{\mathrm{d}\mathbf{v}'(t')}{\mathrm{d}t'} \tag{F.169}$$

$$s = \left|\mathbf{x}-\mathbf{x}'\right| - (\mathbf{x}-\mathbf{x}')\cdot\frac{\mathbf{v}'}{c} \tag{F.170}$$

$$\mathbf{x}-\mathbf{x}_0 = (\mathbf{x}-\mathbf{x}') - \left|\mathbf{x}-\mathbf{x}'\right|\frac{\mathbf{v}'}{c} \tag{F.171}$$

$$\left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} = \frac{|\mathbf{x}-\mathbf{x}'|}{s} \tag{F.172}$$

# F.3 Special relativity

## F.3.1 Metric tensor for flat 4D space

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{F.173}$$

## F.3.2 Lorentz transformation of a four-vector

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{F.174}$$

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{F.175}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{F.176}$$

$$\beta = \frac{v}{c} \tag{F.177}$$

## F.3.3 Covariant and contravariant four-vectors

### F.3.3.1 Position four-vector (radius four-vector)

CONTRAVARIANT REPRESENTATION

$$x^{\mu} = (x^0, x^2, x^2, x^3) = (ct, x, y, z) \stackrel{\text{def}}{\equiv} (ct, \mathbf{x}) \tag{F.178}$$

COVARIANT REPRESENTATION

$$\begin{aligned} x_{\mu} &= (x_0, x_2, x_2, x_3) = (x^0, -x^2, -x^2, -x^3) \\ &= (ct, -x, -y, -z) \stackrel{\text{def}}{\equiv} (ct, -\mathbf{x}) \end{aligned} \tag{F.179}$$

### F.3.3.2 Arbitrary four-vector field

$$a_{\mu}(x^{\kappa}) = \eta_{\mu\nu} a^{\nu}(x^{\kappa}) \tag{F.180}$$

#### F.3.3.3 Four-del operator

$$\partial^\mu = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) \stackrel{\text{def}}{\equiv} \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{F.181a}$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \stackrel{\text{def}}{\equiv} \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{F.181b}$$

D'ALEMBERT OPERATOR

$$\Box^2 = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \boldsymbol{\cdot} \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{F.182}$$

#### F.3.3.4 Invariant line element

$$\mathrm{d}s = c\,\frac{\mathrm{d}t}{\gamma} = c\,\mathrm{d}\tau \tag{F.183}$$

#### F.3.3.5 Four-velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(c, \mathbf{v}) \tag{F.184}$$

#### F.3.3.6 Four-momentum

$$p^\mu = m_0 u^\mu = \left(\frac{E}{c}, \mathbf{p}\right) \tag{F.185}$$

#### F.3.3.7 Four-current density

$$j^\mu = \rho_0 u^\mu = (\rho c, \rho \mathbf{v}) \tag{F.186}$$

#### F.3.3.8 Four-potential

$$A^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right) \tag{F.187}$$

### F.3.4 Field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{F.188}$$

$$
(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{F.189}
$$

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M

# MATHEMATICAL METHODS

Physics is the academic discipline that systematically studies and describes the physical world, postulates new fundamental laws of Nature or generalises existing ones that govern Nature's behaviour under various conditions, and — perhaps most important of all — makes new predictions about Nature, based on these new postulates. Then these predictions are put to systematic tests in independent, carefully designed and performed, repeatable experiments that produce objective emprical data. Merely describing Nature and explaining physical experiments in terms of already existing laws is not physics in the true sense of the word.[1] Had this non-creative, static view been adopted by all physicists since the days of Newton, we would still be doing essentially Newtonian physics.

[1] As STEVEN WEINBERG puts it in the Preface To Volume I of *The Quantum Theory of Fields*:

> '...after all, our purpose in theoretical physics is not just to describe the world as we find it, but to explain — in terms of a few fundamental principles — why the world is the way it is.'

Even if such a scientific giant as MICHAEL FARADAY, who had very little mathematical training, was able to make truly remarkable contributions to physics (and chemistry) using practically no formal mathematics whatsoever, it is for us mere mortals most convenient to use the shorthand language of mathematics, together with the formal methods of logic (*inter alia propositional calculus*), in physics. After all, mathematics was once introduced by us human beings to make it easier to quantitatively and systematically describe, understand and predict the physical world around us. Examples of this from ancient times are arithmetics and geometry. A less archaic example is differential calculus, needed by SIR ISAAC NEWTON to formulate, in a compact and unambiguous manner, the physical laws that bear his name. Another more modern example is the delta 'function' introduced by PAUL ADRIEN MAURICE DIRAC. But the opposite is also very common: the expansion and generalisation of mathematics has more than once provided excellent tools for creating new physical ideas and to better analyse observational data. Examples of the latter include non-Euclidean geometry and group theory.

Unlike mathematics *per se*, where the criterion of logical consistency is both necessary and sufficient, a physical theory that is supposed to describe the physical reality has to fulfil the additional criterion that its predictions be empirically testable. Ultimately, as GALILEO GALILEI taught us, physical reality is defined by the outcome of experiments and observations, and not by mere Aristotelean logical reasoning, however mathematically correct and logically consistent this may be. Common sense is not enough and logic and reasoning can never 'outsmart' Nature. Should the outcome of repeated, carefully performed, independent experiments produce results that systematically contradict predictions of a theory, the only conclusion one can draw is that the theory in question, however logically stringent and mathematically correct it may be, is wrong.[2]

On the other hand, extending existing physical theories by mathematical and logical generalisations is a very powerful way of making hypotheses that predict the existence of possilbe new physical phenomena. History shows that if one has several alternative ways of generalising a theory, the theory with most generality, simplicity, elegance and beauty often is the best one. But it is not until these hypotheses and predictions have withstood tough empirical tests in well-designed, systematic physical experiments that they can be said to extend physics and our knowledge about Nature.

This appendix describes briefly some of the more common mathematical methods and tools that are used in Classical Electrodynamics.

[2] The theoretical physicist Sir Rudolf Peierls (1907–1995) described an ideal physical theory in the following way:

> 'It must firstly leave undisturbed the successes of earlier work and not upset the explanations of observations that had been used in support of earlier ideas. Secondly it must explain in a reasonable manner the new evidence which brought the previous ideas into doubt and which suggested the new hypothesis. And thirdly it must predict new phenomena or new relationships between different phenomena, which were not known or not clearly understood at the time when it was invented.'

## M.1 Scalars, vectors and tensors

Every physical observable can be represented by a mathematical object. We have chosen to describe the observables in classical electrodynamics in terms of scalars, pseudoscalars, vectors, pseudovectors, tensors and pseudotensors, all of which obey certain canonical rules of transformation under a change of coordinate systems and are completely defined by these rules. Despite certain advantages (and some shortcomings), differential forms will not be exploited to any significant degree in our mathematical description of physical observables.

A *scalar*, which may or may not be constant in time and/or space, describes the scaling of a physical quantity. A *vector* describes some kind of physical motion along a curve in space due to vection.[3] A *tensor* describes the local motion or deformation of a surface or a volume due to some form of tension and is therefore a relation between a set of vectors. However, generalisations to more abstract notions of these quantities have proved useful and are therefore commonplace. The difference between a scalar, vector and tensor and a *pseudoscalar*, *pseudovector* and a *pseudotensor* is that the latter behave differently under those coordinate transformations that cannot be reduced to pure rotations.

[3] The Latin word 'vector' means 'carrier'.

For computational convenience, it is often useful to allow mathematical objects representing physical observables to be complex valued, *i.e.* to let them be analytically continued into (a domain of) the complex plane. However, since by definition our physical world is real, care must be exercised when comparing mathematical results with physical observables, *i.e.* real-valued numbers obtained from physical measurements.

Throughout we adopt the convention that Latin indices $i, j, k, l, \ldots$ run over the range $\{1, 2, 3\}$ to denote vector or tensor components in the real Euclidean three-dimensional (3D) configuration space, and Greek indices $\mu, \nu, \kappa, \lambda, \ldots$, which are used in four-dimensional (4D) spacetime, run over the range $\{0, 1, 2, 3\}$.

## M.1.1 Vectors

Mathematically, a vector can be represented in a number of different ways. One suitable representation in a vector space of dimension $N$ is in terms of an ordered $N$-tuple of real or complex[4] numbers $(a_1, a_2, \ldots, a_N)$ of the components along $N$ orthogonal coordinate axes that span the vector space under consideration. Note, however, that there are many ordered $N$-tuples of numbers that do not comprise a vector, *i.e.* do not have the necessary vector transformation properties!

[4] It is often very convenient to use *complex notation* in physics. This notation can simplify the mathematical treatment considerably. But since all physical observables are real, we must in the final step of our mathematical analysis of a physical problem always ensure that the results to be compared with experimental values are real-valued. In classical physics this is achieved by taking the real (or imaginary) part of the mathematical result, whereas in quantum physics one takes the absolute value.

### M.1.1.1 Position vector

The most basic vector, and the prototype against which all other vectors are benchmarked, is the *position vector* (*radius vector*, *coordinate vector*) which is the vector from the origin of the chosen coordinate system to the actual point of interest. Its $N$-tuple representation simply enumerates the coordinates of the position of this point. In this sense, the vector from the origin to a point is synonymous with the coordinates of the point itself.

In the 3D Euclidean space $\mathbb{R}^3$, we have $N = 3$ and the position vector $\mathbf{x}$ can be represented by the triplet $(x_1, x_2, x_3)$ of its coordinates $x_i \in \mathbb{R}$, $i = 1, 2, 3$. The coordinates $x_i$ are scalar quantities which describe the position along the unit base vectors $\hat{\mathbf{x}}_i$ which span $\mathbb{R}^3$. Therefore one convenient representation of the position vector in $\mathbb{R}^3$ is[5]

$$\mathbf{x} = \sum_{i=1}^{3} x_i \hat{\mathbf{x}}_i \stackrel{\text{def}}{\equiv} x_i \hat{\mathbf{x}}_i \tag{M.1}$$

[5] We introduce the symbol $\stackrel{\text{def}}{\equiv}$ which may be read 'is, by definition, to equal in meaning', or 'equals by definition', or, formally, *definiendum* $\stackrel{\text{def}}{\equiv}$ *definiens*. Another symbol sometimes used is $:=$.

where we have introduced *Einstein's summation convention* (E$\Sigma$) that states that a repeated index in a term implies summation over the range of the index in question. Whenever possible and convenient we shall in the following always assume E$\Sigma$ and suppress explicit summation in our formulæ. Typographically, we represent vectors as well as prefix and infix vector operators in 3D Euclidean space by a boldface letter or symbol in a Roman font, for instance $\mathbf{a}$, $\boldsymbol{\nabla}$, $\boldsymbol{\cdot}$, $\boldsymbol{\times}$, and $\boldsymbol{\otimes}$.

Alternatively, we can describe the position vector $\mathbf{x}$ in *component notation* as $x_i$ where

$$x_i \stackrel{\text{def}}{\equiv} (x_1, x_2, x_3) \tag{M.2}$$

In Cartesian coordinates

$$(x_1, x_2, x_3) = (x, y, z) \tag{M.3}$$

This component notation is particularly useful in a 4-dimensional *Riemannian space* where we can represent the (one and the same) position vector either in its *contravariant component form*, (superscript index form) as the quartet

$$x^\mu \stackrel{\text{def}}{\equiv} (x^0, x^1, x^2, x^3) \tag{M.4}$$

or its *covariant component form* (subscript index form)

$$x_\mu \stackrel{\text{def}}{\equiv} (x_0, x_1, x_2, x_3) \tag{M.5}$$

The contravariant and covariant forms represent the same vector but the numerical values of the components of the two may have different numerical values. The relation between them is determined by the *metric tensor* (also known as the *fundamental tensor*) whose actual form is dictated by the properties of the vector space in question. The dual representation of vectors in contravariant and covariant forms is most convenient when we work in a vector space with an indefinite *metric*. An example of 4D Riemannian space is *Lorentz space* $\mathbb{L}^4$ which is a frequently employed to formulate the special theory of relativity.

## M.1.2 Fields

A *field* is a physical entity that depends on one or more continuous parameters. Such a parameter can be viewed as a 'continuous index' that enumerates the infinitely many 'coordinates' of the field. In particular, in a field that depends on the usual position vector $\mathbf{x}$ of $\mathbb{R}^3$, each point in this space can be considered as one degree of freedom so that a field is a representation of a physical entity with an infinite number of degrees of freedom.

### M.1.2.1 Scalar fields

We denote an arbitrary *scalar field* in $\mathbb{R}^3$ by

$$\alpha(\mathbf{x}) = \alpha(x_1, x_2, x_3) \stackrel{\text{def}}{\equiv} \alpha(x_i) \tag{M.6}$$

This field describes how the scalar quantity $\alpha$ varies continuously in 3D $\mathbb{R}^3$ space.

In 4D, a *four-scalar* field is denoted

$$\alpha(x^0, x^1, x^2, x^3) \stackrel{\text{def}}{\equiv} \alpha(x^\mu) \tag{M.7}$$

which indicates that the four-scalar $\alpha$ depends on all four coordinates spanning this space. Since a four-scalar has the same value at a given point regardless of coordinate system, it is also called an *invariant*.

### M.1.2.2 Vector fields

We can represent an arbitrary 3D *real vector field* $\mathbf{a}(\mathbf{x})$ as follows:

$$\mathbf{a}(\mathbf{x}) = a_i(\mathbf{x})\hat{\mathbf{x}}_i \in \mathbb{R}^3 \tag{M.8}$$

In component notation this same vector can be represented as

$$a_i(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x})) = a_i(x_j) \tag{M.9}$$

A 3D *complex vector field* $\mathbf{c}(\mathbf{x})$ is a vector in $\mathbb{C}^3$ (or, if we like, in $\mathbb{R}^6$), expressed in terms of two real vectors $c_\mathrm{R}$ and $c_\mathrm{I}$ in $\mathbb{R}^3$ in the following way

$$\mathbf{c}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \mathbf{c}_\mathrm{R}(\mathbf{x}) + \mathrm{i}\mathbf{c}_\mathrm{I}(\mathbf{x}) = c_\mathrm{R}(\mathbf{x})\hat{\mathbf{c}}_\mathrm{R} + \mathrm{i}c_\mathrm{I}(\mathbf{x})\hat{\mathbf{c}}_\mathrm{I} \stackrel{\text{def}}{\equiv} c(\mathbf{x})\hat{\mathbf{c}} \in \mathbb{C}^3 \tag{M.10}$$

which means that

$$\mathrm{Re}\,\{\mathbf{c}\} = \mathbf{c}_\mathrm{R} = c_\mathrm{R}\hat{\mathbf{c}}_\mathrm{R} \in \mathbb{R}^3 \tag{M.11a}$$
$$\mathrm{Im}\,\{\mathbf{c}\} = \mathbf{c}_\mathrm{I} = c_\mathrm{I}\hat{\mathbf{c}}_\mathrm{I} \in \mathbb{R}^3 \tag{M.11b}$$

The use of complex vectors is in many situations a very convenient and powerful technique but requires extra care since physical observables must be represented by real vectors ($\in \mathbb{R}^3$).

In 4D, an arbitrary *four-vector* field in contravariant component form can be represented as

$$a^\mu(x^\nu) = (a^0(x^\nu), a^1(x^\nu), a^2(x^\nu), a^3(x^\nu)) \tag{M.12}$$

or, in *covariant* component form, as

$$a_\mu(x^\nu) = (a_0(x^\nu), a_1(x^\nu), a_2(x^\nu), a_3(x^\nu)) \tag{M.13}$$

where $x^\nu$ is the *position four-vector* (*radius four-vector*, *coordinate four-vector*). Again, the relation between $a^\mu$ and $a_\mu$ is determined by the metric of the physical 4D system under consideration.

### M.1.2.3 Coordinate transformations

We note that for a change of coordinates $x^\mu \mapsto x'^\mu = x'^\mu(x^0, x^1, x^2, x^3)$, due to a transformation from one coordinate system $\Sigma$ to another coordinate system $\Sigma'$, the differential position vector $\mathrm{d}x^\mu$ transforms as

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu \tag{M.14}$$

This follows trivially from the rules of differentiation of $x'^{\mu}$ considered as a function of four variables $x^{\nu}$, *i.e.* $x'^{\mu} = x'^{\mu}(x^{\nu})$. Analogous to the transformation rule for the differential $\mathrm{d}x^{\mu}$, equation (M.14) above, the transformation rule for the differential operator $\partial/\partial x^{\mu}$ under a transformation $x^{\mu} \mapsto x'^{\mu}$ becomes

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} \tag{M.15}$$

which, again, follows trivially from the rules of differentiation.

Whether an arbitrary $N$-tuple fulfils the requirement of being an ($N$-dimensional) contravariant vector or not, depends on its transformation properties during a change of coordinates. For instance, in 4D an assemblage $y^{\mu} = (y^0, y^1, y^2, y^3)$ constitutes a *contravariant four-vector* (or the contravariant components of a four-vector) if and only if, during a transformation from a system $\Sigma$ with coordinates $x^{\mu}$ to a system $\Sigma'$ with coordinates $x'^{\mu}$, it transforms to the new system according to the rule

$$y'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} y^{\nu} \tag{M.16}$$

*i.e.* in the same way as the differential coordinate element $\mathrm{d}x^{\mu}$ transforms according to equation (M.14) on the preceding page.

The analogous requirement for a *covariant four-vector* is that it transforms, during the change from $\Sigma$ to $\Sigma'$, according to the rule

$$y'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} y_{\nu} \tag{M.17}$$

*i.e.* in the same way as the differential operator $\partial/\partial x^{\mu}$ transforms according to equation (M.15) above.

### M.1.2.4 Tensor fields

We denote an arbitrary *tensor field* in $\mathbb{R}^3$ by $\mathbf{A}(\mathbf{x})$. This tensor field can be represented in a number of ways, for instance in the following *matrix representation*:[6]

$$(\mathbf{A}(\mathbf{x})) \stackrel{\text{def}}{\equiv} (A_{ij}(x_k)) \stackrel{\text{def}}{\equiv} \begin{pmatrix} A_{11}(\mathbf{x}) & A_{12}(\mathbf{x}) & A_{13}(\mathbf{x}) \\ A_{21}(\mathbf{x}) & A_{22}(\mathbf{x}) & A_{23}(\mathbf{x}) \\ A_{31}(\mathbf{x}) & A_{32}(\mathbf{x}) & A_{33}(\mathbf{x}) \end{pmatrix} \tag{M.18}$$

Strictly speaking, the tensor field described here is a tensor of *rank* two.

A particularly simple rank two tensor in $\mathbb{R}^3$ is the 3D *Kronecker delta tensor* $\delta_{ij}$, with the following properties:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{M.19}$$

[6] When a mathematical object representing a physical observable is given in matrix representation, we indicate this by enclosing the mathematical object in question in parentheses, *i.e.* $(\ldots)$.

The 3D Kronecker delta tensor has the following matrix representation

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{M.20}$$

Another common and useful tensor is the *fully antisymmetric tensor* of rank three, also known as the *Levi-Civita tensor*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of 1,2,3} \\ 0 & \text{if at least two of } i, j, k \text{ are equal} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of 1,2,3} \end{cases} \tag{M.21}$$

Clearly, this tensor fulfils the relations

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \tag{M.22}$$

and

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} \tag{M.23}$$

and has the following further property

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{M.24}$$

In fact, tensors may have any rank $n$. In this picture, a scalar is considered to be a tensor of rank $n = 0$ and a vector to be a tensor of rank $n = 1$. Consequently, the notation where a vector (tensor) is represented in its component form is called the *tensor notation*. A tensor of rank $n = 2$ may be represented by a two-dimensional array or matrix, and a tensor of rank $n = 3$ may be represented as a vector of tensors of rank $n = 2$. Assuming that one of the indices of the Levi-Civita tensor $\epsilon_{ijk}$, *e.g.* the first index $i = 1, 2, 3$, denotes the component of such a vector of tensors, these components have the matrix representations (the second and third indices, $j, k = 1, 2, 3$, are the matrix indices)

$$\big((\epsilon_{ijk})_{i=1}\big) = \begin{pmatrix} \epsilon_{111} & \epsilon_{112} & \epsilon_{113} \\ \epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\ \epsilon_{131} & \epsilon_{132} & \epsilon_{133} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathrm{i}\mathsf{S}_1 \tag{M.25a}$$

$$\big((\epsilon_{ijk})_{i=2}\big) = \begin{pmatrix} \epsilon_{211} & \epsilon_{212} & \epsilon_{213} \\ \epsilon_{221} & \epsilon_{222} & \epsilon_{223} \\ \epsilon_{231} & \epsilon_{232} & \epsilon_{233} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathrm{i}\mathsf{S}_2 \tag{M.25b}$$

$$\big((\epsilon_{ijk})_{i=3}\big) = \begin{pmatrix} \epsilon_{311} & \epsilon_{312} & \epsilon_{313} \\ \epsilon_{321} & \epsilon_{322} & \epsilon_{323} \\ \epsilon_{331} & \epsilon_{332} & \epsilon_{333} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{i}\mathsf{S}_3 \tag{M.25c}$$

Here we have introduced the matrix vector

$$\mathsf{S} = \mathsf{S}_i \hat{\mathbf{x}}_i \tag{M.26}$$

where the vector components $\mathsf{S}_i$ are the matrices

$$\mathsf{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad \mathsf{S}_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \quad \mathsf{S}_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{M.27}$$

which satisfy the *angular momentum commutation rule*

$$[\mathsf{S}_i, \mathsf{S}_j] = -\mathrm{i}\epsilon_{ijk}\mathsf{S}_k \tag{M.28}$$

Tensors of rank higher than 3 are best represented in their tensor notation (component form). It is important to remember that a tensor of any rank is fully and totally characterized by its transformation properties under the change of coordinates. This is a very strict constraint.

In 4D, we have three forms of *four-tensor fields* of rank $n$. We speak of

- a *contravariant four-tensor field*, denoted $A^{\mu_1\mu_2\ldots\mu_n}(x^\nu)$,
- a *covariant four-tensor field*, denoted $A_{\mu_1\mu_2\ldots\mu_n}(x^\nu)$,
- a *mixed four-tensor field*, denoted $A^{\mu_1\mu_2\ldots\mu_k}_{\mu_{k+1}\ldots\mu_n}(x^\nu)$.

The 4D *metric tensor* (*fundamental tensor*) mentioned above is a particularly important four-tensor of rank two. In covariant component form we shall denote it $g_{\mu\nu}$. This metric tensor determines the relation between an arbitrary contravariant four-vector $a^\mu$ and its covariant counterpart $a_\mu$ according to the following rule:

$$a_\mu(x^\kappa) \stackrel{\text{def}}{\equiv} g_{\mu\nu}a^\nu(x^\kappa) \tag{M.29}$$

This rule is often called *lowering of index*. The *raising of index* analogue of the index lowering rule is:

$$a^\mu(x^\kappa) \stackrel{\text{def}}{\equiv} g^{\mu\nu}a_\nu(x^\kappa) \tag{M.30}$$

More generally, the following lowering and raising rules hold for arbitrary rank $n$ mixed tensor fields:

$$g_{\mu_k\nu_k} A^{\nu_1\nu_2\ldots\nu_{k-1}\nu_k}_{\nu_{k+1}\nu_{k+2}\ldots\nu_n}(x^\kappa) = A^{\nu_1\nu_2\ldots\nu_{k-1}}_{\mu_k\nu_{k+1}\ldots\nu_n}(x^\kappa) \tag{M.31}$$

$$g^{\mu_k\nu_k} A^{\nu_1\nu_2\ldots\nu_{k-1}}_{\nu_k\nu_{k+1}\ldots\nu_n}(x^\kappa) = A^{\nu_1\nu_2\ldots\nu_{k-1}\mu_k}_{\nu_{k+1}\nu_{k+2}\ldots\nu_n}(x^\kappa) \tag{M.32}$$

Successive lowering and raising of more than one index is achieved by a repeated application of this rule. For example, a dual application of the lowering operation on a rank two tensor in its contravariant form yields

$$A_{\mu\nu} = g_{\mu\kappa}g_{\lambda\nu}A^{\kappa\lambda} \tag{M.33}$$

*i.e.* the same rank two tensor in its covariant form.

## M.2 Vector algebra

### M.2.1 Scalar product

The *scalar product* (*dot product*, *inner product*) of two arbitrary 3D vectors **a** and **b** in Euclidean $\mathbb{R}^3$ space is the scalar number

$$\mathbf{a} \cdot \mathbf{b} = a_i \hat{\mathbf{x}}_i \cdot b_j \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j a_i b_j = \delta_{ij} a_i b_j = a_i b_i \tag{M.34}$$

where we used the fact that the scalar product $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j$ is a representation of the Kronecker delta $\delta_{ij}$ defined in equation (M.19) on page 236.[7] The scalar product of a vector **a** in $\mathbb{R}^3$ with itself is

[7] In the Russian literature, the 3D scalar product is often denoted **(ab)**.

$$\mathbf{a} \cdot \mathbf{a} \stackrel{\text{def}}{\equiv} (\mathbf{a})^2 = |\mathbf{a}|^2 = (a_i)^2 = a^2 \tag{M.35}$$

and similarly for **b**. This allows us to write

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \tag{M.36}$$

where $\theta$ is the angle between **a** and **b**.

In 4D space we define the scalar product of two arbitrary four-vectors $a^\mu$ and $b^\mu$ in the following way

$$a_\mu b^\mu = g_{\nu\mu} a^\nu b^\mu = a^\nu b_\nu = g^{\mu\nu} a_\mu b_\nu \tag{M.37}$$

where we made use of the index lowering and raising rules (M.29) and (M.30). The result is a four-scalar, *i.e.* an invariant which is independent of in which 4D coordinate system it is measured.

The *quadratic differential form*

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\nu \mathrm{d}x^\mu = \mathrm{d}x_\mu \mathrm{d}x^\mu \tag{M.38}$$

*i.e.* the scalar product of the differential position four-vector with itself, is an invariant called the *metric*. It is also the square of the *line element* $\mathrm{d}s$ which is the distance between neighbouring points with coordinates $x^\mu$ and $x^\mu + \mathrm{d}x^\mu$.

### M.2.2 Vector product

The *vector product* or *cross product* of two arbitrary 3D vectors **a** and **b** in ordinary $\mathbb{R}^3$ space is the vector[8]

[8] Sometimes the 3D vector product of **a** and **b** is denoted $\mathbf{a} \wedge \mathbf{b}$ or, particularly in the Russian literature, **[ab]**.

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} \hat{\mathbf{x}}_i a_j b_k = \epsilon_{ijk} a_j b_k \hat{\mathbf{x}}_i \tag{M.39}$$

Here $\epsilon_{ijk}$ is the Levi-Civita tensor defined in equation (M.21) on page 237. Alternatively,

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta \, \hat{\mathbf{e}} \tag{M.40}$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.

A spatial reversal of the coordinate system, $(x'_1, x'_2, x'_3) = (-x_1, -x_2, -x_3)$, known as a *parity transformation*, changes sign of the components of the vectors $\mathbf{a}$ and $\mathbf{b}$ so that in the new coordinate system $\mathbf{a}' = -\mathbf{a}$ and $\mathbf{b}' = -\mathbf{b}$, which is to say that the direction of an ordinary vector is not dependent on the choice of the directions of the coordinate axes. On the other hand, as is seen from equation (M.39) on the preceding page, the cross product vector $\mathbf{c}$ does *not* change sign. Therefore $\mathbf{a}$ (or $\mathbf{b}$) is an example of a 'true' vector, or *polar vector*, whereas $\mathbf{c}$ is an example of an *pseudovector* or *axial vector*.

A prototype for a pseudovector is the angular momentum vector $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ and hence the attribute 'axial'. Pseudovectors transform as ordinary vectors under translations and proper rotations, but reverse their sign relative to ordinary vectors for any coordinate change involving reflection. Tensors (of any rank) that transform analogously to pseudovectors are called *pseudotensors*. Scalars are tensors of rank zero, and zero-rank pseudotensors are therefore also called *pseudoscalars*, an example being the pseudoscalar $\hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_k) = \hat{\mathbf{x}}_i \cdot (\epsilon_{ijk} \hat{\mathbf{x}}_i)$. This triple product is a representation of the $ijk$ component of the rank three Levi-Civita pseudotensor $\epsilon_{ijk}$.

## M.2.3 Dyadic product

The *dyadic product* $\mathbf{A}(\mathbf{x}) \equiv \mathbf{a}(\mathbf{x}) \otimes \mathbf{b}(\mathbf{x})$ of two vector fields $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ is the *outer product* of $\mathbf{a}$ and $\mathbf{b}$ known as a *dyad*. Here $\mathbf{a}$ is called the *antecedent* and $\mathbf{b}$ the *consequent*.[9]

[9] In electrodynamics, it is very common that the dyadic product is represented by a juxtaposition of the antecedent and the consequent, *i.e.* $\mathbf{A} = \mathbf{ab}$.

The dyadic product between vectors (rank one tensors) is a special instance of the *direct product* $\otimes$ between arbitrary rank tensors. This associative but non-commuting operation is also called the *tensor product* or, when applied to matrices (matrix representations of tensors), the *Kronecker product*.

Written out in explicit form, the dyadic product $\mathbf{A}$ becomes

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{b} = a_1\hat{\mathbf{x}}_1 \otimes b_1\hat{\mathbf{x}}_1 + a_1\hat{\mathbf{x}}_1 \otimes b_2\hat{\mathbf{x}}_2 + \cdots + a_3\hat{\mathbf{x}}_3 \otimes b_3\hat{\mathbf{x}}_3 \tag{M.41a}$$

$$= a_i\hat{\mathbf{x}}_i \otimes b_j\hat{\mathbf{x}}_j = a_i b_j \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i a_i \otimes b_j \hat{\mathbf{x}}_j \tag{M.41b}$$

$$= \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \end{pmatrix} \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \tag{M.41c}$$

In matrix representation

$$(\mathbf{A}) = (\mathbf{a} \otimes \mathbf{b}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \tag{M.42}$$

which we identify with expression (M.18) on page 236, *viz.* a tensor in matrix representation. Hence, a dyadic of two vectors is intimately related to a rank two tensor, emphasising its vectorial characteristics.

Scalar multiplication from the right or from the left of the dyad $\mathsf{A} = \mathbf{a} \otimes \mathbf{b}$ by a vector $\mathbf{c}$, produces other vectors according to the scheme

$$\mathsf{A} \cdot \mathbf{c} = \mathbf{a} \otimes \mathbf{b} \cdot \mathbf{c} \stackrel{\text{def}}{\equiv} \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = a_j b_i c_i \hat{\mathbf{x}}_j \tag{M.43a}$$

$$\mathbf{c} \cdot \mathsf{A} = \mathbf{c} \cdot \mathbf{a} \otimes \mathbf{b} \stackrel{\text{def}}{\equiv} (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = a_i b_j c_i \hat{\mathbf{x}}_j \tag{M.43b}$$

respectively. These two vectors, proportional to $\mathbf{a}$ and $\mathbf{b}$, respectively, are in general *not* identical to each other. In the first case, $\mathbf{c}$ is known as the *postfactor*, in the second case as the *prefactor*.

Specifically, if $\mathbf{c} = \hat{\mathbf{x}}_j$, then

$$\mathsf{A} \cdot \hat{\mathbf{x}}_j = \mathbf{a} b_j = a_i b_j \hat{\mathbf{x}}_i \tag{M.44a}$$

$$\hat{\mathbf{x}}_j \cdot \mathsf{A} = a_j \mathbf{b} = a_j b_i \hat{\mathbf{x}}_i \tag{M.44b}$$

which means that

$$\hat{\mathbf{x}}_i \cdot \mathsf{A} \cdot \hat{\mathbf{x}}_j = a_i b_j = A_{ij} \tag{M.44c}$$

The vector product can be represented in matrix form as follows:

$$(\mathbf{c}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = -\mathrm{i}\mathbf{a} \cdot \mathsf{S} \otimes \mathbf{b} = -\mathrm{i}\mathbf{a} \otimes \mathsf{S} \cdot \mathbf{b} \tag{M.45}$$

where $\mathsf{S} \otimes \mathbf{b}$ is the dyadic product of the matrix vector $\mathsf{S}$, given by formula (M.26) on page 238, and the vector $\mathbf{b}$, and $\mathbf{a} \otimes \mathsf{S}$ is the dyadic product of the vector $\mathbf{a}$ and the matrix vector $\mathsf{S}$.

Vector multiplication from the right and from the left of the dyad $\mathsf{A}$ by a vector is another dyad according to the scheme

$$\mathsf{A} \times \mathbf{c} = \mathbf{a} \otimes \mathbf{b} \times \mathbf{c} \stackrel{\text{def}}{\equiv} \mathbf{a} \otimes (\mathbf{b} \times \mathbf{c}) = \epsilon_{jkl} a_i b_k c_l \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \tag{M.46a}$$

$$\mathbf{c} \times \mathsf{A} = \mathbf{c} \times \mathbf{a} \otimes \mathbf{b} \stackrel{\text{def}}{\equiv} (\mathbf{c} \times \mathbf{a}) \otimes \mathbf{b} = \epsilon_{jkl} a_l b_i c_k \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i = \epsilon_{ikl} a_l b_j c_k \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \tag{M.46b}$$

respectively. In general, the two new dyads thus created are *not* identical to each other.

Specifically, if $\mathsf{A} = \mathsf{1}_3 = \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i$, *i.e.* the *unit dyad* or the second-rank *unit tensor*, then

$$\begin{aligned} \mathsf{1}_3 \times \mathbf{c} = & -c_3 \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + c_2 \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 \\ & + c_3 \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1 - c_1 \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 \\ & - c_2 \hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + c_1 \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2 = \mathbf{c} \times \mathsf{1}_3 \end{aligned} \tag{M.47}$$

or, in matrix representation,

$$(\mathbf{1}_3 \times \mathbf{c}) = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} = (\mathbf{c} \times \mathbf{1}_3) \tag{M.48}$$

Using the matrix vector formula (M.26) on page 238, we can write this as

$$(\mathbf{1}_3 \times \mathbf{c}) = (\mathbf{c} \times \mathbf{1}_3) = -\mathrm{i}\mathbf{S} \cdot \mathbf{c} \tag{M.49}$$

One can extend the dyadic scheme and introduce **abc**, called a *tryad*, and so on. In this vein, a vector **a** is sometimes called a *monad*.

## M.3 Vector calculus

### M.3.1 The *del* operator

In $\mathbb{R}^3$ the *del operator* is a *differential vector operator*, denoted in *Gibbs' notation* by the boldface nabla symbol $\boldsymbol{\nabla}$ and defined as[10]

$$\boldsymbol{\nabla} = \hat{\mathbf{x}}_i \nabla_i \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{\equiv} \frac{\partial}{\partial \mathbf{x}} \stackrel{\text{def}}{\equiv} \boldsymbol{\partial} \tag{M.50}$$

where $\hat{\mathbf{x}}_i$ is the $i$th unit vector in a Cartesian coordinate system. Since the operator in itself has vectorial properties, we denote it with a boldface nabla ($\boldsymbol{\nabla}$).

In 'component' (tensor) notation the *del* operator can be written

$$\partial_i = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{M.51}$$

In 4D, the contravariant component representation of the *four-del operator* is defined by

$$\partial^\mu = \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{M.52}$$

whereas the covariant four-del operator is

$$\partial_\mu = \left( \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \tag{M.53}$$

We can use this four-del operator to express the transformation properties (M.16) and (M.17) on page 236 as

$$y'^\mu = \left( \partial_\nu x'^\mu \right) y^\nu \tag{M.54}$$

and

$$y'_\mu = \left( \partial'_\mu x^\nu \right) y_\nu \tag{M.55}$$

respectively.

[10] This operator was introduced by WILLIAM ROWEN HAMILTON (1805–1865) who, however, used the symbol $\triangleright$ for it. It is therefore sometimes called the *Hamilton operator*.

## M.3.2 The gradient of a scalar field

The *gradient* of an $\mathbb{R}$ scalar field $\alpha(\mathbf{x})$, is an $\mathbb{R}^3$ vector field

$$\operatorname{grad}\alpha(\mathbf{x}) \stackrel{\text{def}}{\equiv} \boldsymbol{\nabla}\alpha(\mathbf{x}) \stackrel{\text{def}}{\equiv} \boldsymbol{\partial}\alpha(\mathbf{x}) \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i\partial_i\alpha(\mathbf{x}) \stackrel{\text{def}}{\equiv} \hat{\mathbf{x}}_i\partial_i\alpha(x_j) \tag{M.56}$$

If the scalar field depends only on one coordinate, $\zeta$ say, then

$$\boldsymbol{\nabla}\alpha(\mathbf{x}) = \boldsymbol{\nabla}\alpha(\zeta) = \hat{\boldsymbol{\zeta}}\frac{\partial\alpha(\zeta)}{\partial\zeta} = \hat{\boldsymbol{\zeta}}\nabla\alpha(\zeta) \tag{M.57}$$

and, therefore, $\boldsymbol{\nabla} = \hat{\boldsymbol{\zeta}}\nabla$. From this we see that the boldface notation for the gradient ($\boldsymbol{\nabla}$) is very handy as it elucidates its 3D vectorial property and separates it from the nabla operator ($\nabla$) which has a scalar property.

In 4D, the *four-gradient* of a four-scalar is a covariant vector, formed as a derivative of a four-scalar field $\alpha(x^\mu)$, with the following component form:

$$\partial_\mu\alpha(x^\nu) = \frac{\partial\alpha(x^\nu)}{\partial x^\mu} \tag{M.58a}$$

with the contravariant form

$$\partial^\mu\alpha(x^\nu) = \frac{\partial\alpha(x^\nu)}{\partial x_\mu} \tag{M.58b}$$

## M.3.3 The divergence of a vector field

We define the 3D *divergence* of a vector field $\mathbf{a}$ in $\mathbb{R}^3$ as

$$\operatorname{div}\mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \boldsymbol{\nabla}\cdot\mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \frac{\partial a_i(\mathbf{x})}{\partial x_i} \stackrel{\text{def}}{\equiv} \partial_i a_i(\mathbf{x}) \stackrel{\text{def}}{\equiv} \partial_i a_i(x_j) \tag{M.59}$$

which, as indicated by the notation $\alpha(\mathbf{x})$, is a *scalar* field in $\mathbb{R}^3$.

The *four-divergence* of a four-vector $a^\mu$ is the four-scalar

$$\partial_\mu a^\mu(x^\nu) = \frac{\partial a^\mu(x^\nu)}{\partial x^\mu} \tag{M.60}$$

## M.3.4 The curl of a vector field

In $\mathbb{R}^3$ the *curl* of a vector field $\mathbf{a}(\mathbf{x})$ is another $\mathbb{R}^3$ vector field defined in the following way:

$$\operatorname{curl}\mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{\equiv} \epsilon_{ijk}\hat{\mathbf{x}}_i\frac{\partial a_k(\mathbf{x})}{\partial x_j} \stackrel{\text{def}}{\equiv} \epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j a_k(\mathbf{x}) \stackrel{\text{def}}{\equiv} \epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j a_k(x_l) \tag{M.61}$$

where use was made of the Levi-Civita tensor, introduced in equation (M.21) on page 237. If $\mathbf{a}$ is an ordinary vector (polar vector), then $\boldsymbol{\nabla}\times\mathbf{a}$ is a pseudovector (axial vector) and vice versa.

Similarly to formula (M.45) on page 241, we can write the matrix representation of the curl in $\mathbb{R}^3$ as

$$(\boldsymbol{\nabla}\times\mathbf{a}) = \begin{pmatrix} \partial_2 a_3 - \partial_3 a_2 \\ \partial_3 a_1 - \partial_1 a_3 \\ \partial_1 a_2 - \partial_2 a_1 \end{pmatrix} = -\mathrm{i}\boldsymbol{\nabla}\cdot\mathbf{S}\otimes\mathbf{a} = -\mathrm{i}\boldsymbol{\nabla}\otimes\mathbf{S}\cdot\mathbf{a} \tag{M.62}$$

where $\mathbf{S}$ is the matrix vector given by formula (M.26) on page 238.

The covariant 4D generalisation of the curl of a four-vector field $a^\mu(x^\nu)$ is the antisymmetric four-tensor field

$$A_{\mu\nu}(x^\kappa) = \partial_\mu a_\nu(x^\kappa) - \partial_\nu a_\mu(x^\kappa) = -A_{\nu\mu}(x^\kappa) \tag{M.63}$$

A vector with vanishing curl is said to be *irrotational*.

## M.3.5 The Laplacian

The 3D *Laplace operator* or *Laplacian* can be described as the divergence of the *del* operator:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \nabla^2 \stackrel{\text{def}}{\equiv} \Delta \stackrel{\text{def}}{\equiv} \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} \stackrel{\text{def}}{\equiv} \frac{\partial^2}{\partial x_i^2} \stackrel{\text{def}}{\equiv} \partial_i^2 \tag{M.64}$$

The symbol $\nabla^2$ is sometimes read *del squared*. If, for a scalar field $\alpha(\mathbf{x})$, $\nabla^2\alpha < 0$ at some point in 3D space, $\alpha$ has a *concentration* at that point.

Numerous vector algebra and vector calculus formulæ are given in appendix F on page 213. Those which are not found there can often be easily derived by using the component forms of the vectors and tensors, together with the Kronecker and Levi-Civita tensors and their generalisations to higher ranks and higher dimensions.

## M.3.6 Vector and tensor integrals

In this subsection we derive some (Riemann) integral identities involving vectors and/or tensors, Of particular interest, importance and usefulness are identities where the integrand contains the scalar $1/\left|\mathbf{x}-\mathbf{x}'\right|$.

### M.3.6.1 First order derivatives

Let us start with the gradient of a volume integral of a scalar field divided by $|\mathbf{x}-\mathbf{x}'|$. It can be written

$$\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x' \, \frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \alpha(\mathbf{x}')\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{M.65}$$

where we used formula (F.77) on page 219 with $\beta(\mathbf{x}) = 1/|\mathbf{x}-\mathbf{x}'|$, noticing that $\alpha$ is a function of $\mathbf{x}'$ only and therefore behaves as a constant under differentiation with respect to $\mathbf{x}$. The results obtained in example M.10 on page 258 allow us to make the replacement $\boldsymbol{\nabla} \mapsto -\boldsymbol{\nabla}'$ in the integrand, leading to

$$\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x' \, \frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = -\int_{V'} \mathrm{d}^3x' \, \alpha(\mathbf{x}')\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{M.66}$$

We can now integrate the RHS by part by invoking formula (F.77) once more, but this time with $\boldsymbol{\nabla}'$ instead of $\boldsymbol{\nabla}$. The result is

$$\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x' \, \frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}'\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \int_{V'} \mathrm{d}^3x' \, \boldsymbol{\nabla}'\left(\frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{M.67}$$

Formula (F.121a) on page 221 enables us to replace the last volume integral with a surface integral, yielding the final result

$$\boldsymbol{\nabla}\int_{V'} \mathrm{d}^3x' \, \frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}'\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}} \frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{M.68}$$

An analogous approach for the divergence of a volume integral of a regular vector field $\mathbf{a}(\mathbf{x}')$ divided by $|\mathbf{x}-\mathbf{x}'|$ yields, with the use of formula (F.81) on page 219 and the results in example M.10 on page 258,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} &= \int_{V'} \mathrm{d}^3x' \, \mathbf{a}(\mathbf{x}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\ &= -\int_{V'} \mathrm{d}^3x' \, \mathbf{a}(\mathbf{x}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{M.69}$$

We integrate this by part, again employing identity (F.81) on page 219 and formula (F.121b) on page 221, to obtain

$$\boldsymbol{\nabla}\cdot\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}'\cdot\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}'\cdot\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{M.70}$$

For the curl of the same vector integral, we do not repeat the steps but only quote the final result

$$\boldsymbol{\nabla}\times\int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}'\times\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}'\times\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{M.71}$$

The above results can be summarised in the general partial integration formula

$$\boldsymbol{\nabla}\circ\int_{V'}\mathrm{d}^3x'\,\frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}=\int_{V'}\mathrm{d}^3x'\,\frac{\boldsymbol{\nabla}'\circ\mathcal{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\oint_{S'}\mathrm{d}^2x'\,\hat{\mathbf{n}}'\circ\frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{M.72}$$

where $\circ$ is either (i) nothing (juxtaposition) and $\mathcal{A}=\alpha$, or (ii) $\circ=\boldsymbol{\cdot}$ or $\circ=\boldsymbol{\times}$ and $\mathcal{A}=\mathbf{a}$. In the surface integrals in the formulæ above, the surface element $\mathrm{d}^2x'=\mathrm{d}\mathbf{S}'\boldsymbol{\cdot}\hat{\mathbf{n}}'$ is proportional to $r^2=|\mathbf{x}-\mathbf{x}'|^2$ [*cf.* formula (F.18) on page 215]. Hence, if $\mathcal{A}(\mathbf{x}')$ falls off faster than $1/r$, this surface integral vanishes when we let the radius $r$ of the sphere $S'$, over which the surface integral is to be evaluated, tend to infinity. If $\mathcal{A}(\mathbf{x}')$ falls off exactly as $1/r$ for large $r$, the surface integral tends to a constant, which, in the special cases $\circ=\boldsymbol{\cdot}$, $\mathcal{A}(\mathbf{x}')=\mathbf{a}\perp\hat{\mathbf{n}}'$ and $\circ=\boldsymbol{\times}$, $\mathcal{A}(\mathbf{x}')=\mathbf{a}\parallel\hat{\mathbf{n}}'$ is zero. If $\mathcal{A}(\mathbf{x}')$ falls off slower than $1/r$ at infinity, the surface integral is singular and, consequently, the above formulæ are inapplicable. When the surface integral vanishes the following simple and very useful formula obtains:

$$\boldsymbol{\nabla}\circ\int_{V'}\mathrm{d}^3x'\,\frac{\mathcal{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}=\int_{V'}\mathrm{d}^3x'\,\frac{\boldsymbol{\nabla}'\circ\mathcal{A}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{M.73}$$

### M.3.6.2 Second order derivatives

Let us also derive two identities involving second derivatives of an integral over $V'$ where the integrand is either a regular, differentiable scalar $\alpha(\mathbf{x}')$, or a regular vector field $\mathbf{a}(\mathbf{x}')$ divided by $|\mathbf{x}-\mathbf{x}'|$.

The divergence of the gradient of an integral where the integrand is of the first kind can be written

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}&=-\int_{V'}\mathrm{d}^3x'\,\alpha(\mathbf{x}')\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\&=-\int_{V'}\mathrm{d}^3x'\,\alpha(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{M.74}$$

The representation of the Dirac delta function given by formula (F.116) on page 220 immediately yields the simple result that

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\alpha(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}=-4\pi\alpha(\mathbf{x}) \tag{M.75}$$

The curl of the curl of an integral with an integrand of the second kind is

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\times}\left(\boldsymbol{\nabla}\boldsymbol{\times}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)&=\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\boldsymbol{\cdot}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\&\quad-\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\int_{V'}\mathrm{d}^3x'\,\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{M.76}$$

where we used the identity (F.96) on page 220. The second integral in the RHS is the vector form of formula (M.75) on the preceding page :

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \mathbf{a}(\mathbf{x}) \tag{M.77}$$

Using the fact that $\boldsymbol{\nabla}$ operates on unprimed coordinates whereas $\mathbf{a}$ depends only on $\mathbf{x}'$, we can rewrite the first integral in the RHS as

$$\int_{V'} \mathrm{d}^3x' \, \mathbf{a}(\mathbf{x}') \cdot \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \int_{V'} \mathrm{d}^3x' \, \mathbf{a}(\mathbf{x}') \cdot \boldsymbol{\nabla}' \otimes \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \tag{M.78}$$

where we twice used the fact that $\boldsymbol{\nabla} = -\boldsymbol{\nabla}'$ when they operate on $1/|\mathbf{x} - \mathbf{x}'|$ and its gradient. Now we can utilise identity (F.86) on page 219 with $\mathbf{b} = \boldsymbol{\nabla}'(1/|\mathbf{x} - \mathbf{x}'|)$ to integrate the RHS by parts as follows:

$$\begin{aligned}
&\int_{V'} \mathrm{d}^3x' \, \mathbf{a}(\mathbf{x}') \cdot \boldsymbol{\nabla}' \otimes \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
&= -\int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \cdot \mathbf{a}(\mathbf{x}')] \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) + \int_{V'} \mathrm{d}^3x' \, \boldsymbol{\nabla}' \cdot \left[ \mathbf{a}(\mathbf{x}') \otimes \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \\
&= -\int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \cdot \mathbf{a}(\mathbf{x}')] \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) + \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \cdot \left[ \mathbf{a}(\mathbf{x}') \otimes \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right]
\end{aligned} \tag{M.79}$$

where, in the last step, we used the divergence theorem for tensors/dyadics, formula (F.121e) on page 221. Putting it all together, we finally obtain the integral identity

$$\begin{aligned}
\boldsymbol{\nabla} \times \left( \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) &= 4\pi \mathbf{a}(\mathbf{x}) - \int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \cdot \mathbf{a}(\mathbf{x}')] \boldsymbol{\nabla}' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
&\quad + \oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \cdot \left( \frac{\mathbf{a}(\mathbf{x}') \otimes (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right)
\end{aligned} \tag{M.80}$$

where the surface integral vanishes for any well-behaved $\mathbf{a}$.

## M.3.7 Helmholtz's theorem

Let us consider an unspecified but well-behaved vector field $\mathbf{u}(\mathbf{x})$ that is continuously differentiable. From equation (M.76) and equation (M.77) above we see that we can always represent such a vector field in the following way:

$$\mathbf{u}(\mathbf{x}) = -\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} \cdot \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} + \boldsymbol{\nabla} \times \left( \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \right) \tag{M.81}$$

If $\mathbf{u}(\mathbf{x})$ is regular and falls off rapidly enough with distance $r = |\mathbf{x} - \mathbf{x}'|$ (typically faster than $1/r$ when $r \to \infty$), we can use formula (M.73) on page 246 to rewrite this as

$$\mathbf{u}(\mathbf{x}) = -\boldsymbol{\nabla}\alpha(\mathbf{x}) + \boldsymbol{\nabla} \times \mathbf{a}(\mathbf{x}) \tag{M.82a}$$

where

$$\alpha(\mathbf{x}) = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \cdot \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.82b}$$

$$\mathbf{a}(\mathbf{x}) = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \times \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.82c}$$

Since, according to example M.13 on page 259, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\alpha) = \mathbf{0}$, *i.e.* $\boldsymbol{\nabla}\alpha$ is *irrotational* (also called *rotation-less* or *lamellar*), and, according to example M.14 on page 260, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) = 0$, *i.e.* $\boldsymbol{\nabla} \times \mathbf{a}$ is *rotational* (also called *divergence-less* or *solenoidal*), $\mathbf{u}$ can always be decomposed into one irrotational and one rotational component

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\text{irrot}}(\mathbf{x}) + \mathbf{u}^{\text{rotat}}(\mathbf{x}) \tag{M.83a}$$

where

$$\mathbf{u}^{\text{irrot}}(\mathbf{x}) = -\boldsymbol{\nabla}\alpha(\mathbf{x}) = -\boldsymbol{\nabla} \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \cdot \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.83b}$$

$$\mathbf{u}^{\text{rotat}}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{a}(\mathbf{x}) = \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \times \mathbf{u}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.83c}$$

This is called *Helmholtz decomposition*.

Furthermore, since

$$\boldsymbol{\nabla} \times \mathbf{u}^{\text{irrot}} = \mathbf{0} \tag{M.84a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}^{\text{rotat}} = 0 \tag{M.84b}$$

we notice, by invoking also equations (M.83) above, that

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{u}^{\text{irrot}} = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla}\alpha) = -\nabla^2\alpha \tag{M.85a}$$

$$\boldsymbol{\nabla} \times \mathbf{u} = \boldsymbol{\nabla} \times \mathbf{u}^{\text{rotat}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) \tag{M.85b}$$

from which we see that a vector field that is well-behaved at large distances is completely and uniquely determined if we know its divergence and curl at all points $\mathbf{x}$ in 3D space (and at any given fixed time $t$ if the vector field is time dependent). This is the (first) *Helmholtz's theorem*, also called the *fundamental theorem of vector calculus*.

# M.4 Analytical mechanics

## M.4.1 Lagrange's equations

As is well known from elementary analytical mechanics, the *Lagrange function* or *Lagrangian* $L$ is given by

$$L(q_i, \dot{q}_i, t) = L\left(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t\right) = T - V \tag{M.86}$$

where $q_i$ is the *generalised coordinate*,

$$\dot{q}_i \stackrel{\text{def}}{\equiv} \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{M.87}$$

the *generalised velocity*, $T$ the *kinetic energy*, and $V$ the *potential energy* of a mechanical system, If we use the action

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_i, \dot{q}_i, t) \tag{M.88}$$

and the *variational principle* with fixed endpoints $t_1$ and $t_2$,

$$\delta S = 0 \tag{M.89}$$

we find that the Lagrangian $L$ satisfies the *Euler-Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{M.90}$$

To the generalised coordinate $q_i$ one defines a *canonically conjugate momentum* $p_i$ according to

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{M.91}$$

and note from equation (M.90) above that

$$\frac{\partial L}{\partial q_i} = \dot{p}_i \tag{M.92}$$

If we introduce an arbitrary, continuously differentiable function $\alpha = \alpha(q_i, t)$ and a new Lagrangian $L'$ related to $L$ in the following way

$$L' = L + \frac{\mathrm{d}\alpha}{\mathrm{d}t} = L + \dot{q}_i \frac{\partial \alpha}{\partial q_i} + \frac{\partial \alpha}{\partial t} \tag{M.93}$$

then

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial \alpha}{\partial q} \tag{M.94a}$$

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \alpha}{\partial q} \tag{M.94b}$$

Or, in other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{M.95}$$

where

$$p'_i = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial \alpha}{\partial q_i} = p_i + \frac{\partial \alpha}{\partial q_i} \tag{M.96a}$$

and

$$q'_i = -\frac{\partial L'}{\partial \dot{p}_i} = \frac{\partial L}{\partial \dot{p}} = q_i \tag{M.96b}$$

### M.4.2 Hamilton's equations

From $L$, the *Hamiltonian* (*Hamilton function*) $H$ can be defined via the *Legendre transformation*

$$H(p_i, q_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{M.97}$$

After differentiating the left and right hand sides of this definition and setting them equal we obtain

$$\frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i + \frac{\partial H}{\partial t}\mathrm{d}t = \dot{q}_i \mathrm{d}p_i + p_i \mathrm{d}\dot{q}_i - \frac{\partial L}{\partial q_i}\mathrm{d}q_i - \frac{\partial L}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i - \frac{\partial L}{\partial t}\mathrm{d}t \tag{M.98}$$

According to the definition of $p_i$, equation (M.91) on the preceding page, the second and fourth terms on the right hand side cancel. Furthermore, noting that according to equation (M.92) on the previous page the third term on the right hand side of equation (M.98) above is equal to $-\dot{p}_i \mathrm{d}q_i$ and identifying terms, we obtain the *Hamilton equations*:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{M.99a}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -\frac{\mathrm{d}p_i}{\mathrm{d}t} \tag{M.99b}$$

## M.5 Examples

EXAMPLE M.1 ▷The physical interpretation of a complex vector

Study the physical meaning of vectors in complex notation where both the magnitudes and the base vectors are complex.

Let us choose the complex magnitude as

$$c = |c|e^{i\phi} = \sqrt{2}C(\cos\phi + i\sin\phi)\,, C \in \mathbb{R} \tag{M.100}$$

and, without lack of generality, choose as base vectors the two complex unit base vectors, expressed in the two orthogonal base vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$

$$\hat{\mathbf{h}}_\pm = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \tag{M.101}$$

that fulfil the conditions

$$\hat{\mathbf{h}}_+ \cdot \hat{\mathbf{h}}_+^* = \hat{\mathbf{h}}_- \cdot \hat{\mathbf{h}}_-^* = 1 \tag{M.102}$$

$$\hat{\mathbf{h}}_+ \cdot \hat{\mathbf{h}}_-^* = \hat{\mathbf{h}}_- \cdot \hat{\mathbf{h}}_+^* = 0 \tag{M.103}$$

With these choices, our two vectors can be written

$$\begin{aligned} \mathbf{c}_\pm &= |c|e^{i\phi}\hat{\mathbf{h}}\pm = C(\cos\phi + i\sin\phi)(\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \\ &= C(\cos\phi\,\hat{\mathbf{x}}_1 \mp \sin\phi\,\hat{\mathbf{x}}_2) + iC(\sin\phi\,\hat{\mathbf{x}}_1 \pm \cos\phi\,\hat{\mathbf{x}}_2) \end{aligned} \tag{M.104}$$

In order to interpret this expression correctly in physical terms, we must take the real part

$$\operatorname{Re}\{\mathbf{c}_\pm\} = C(\cos\phi\,\hat{\mathbf{x}}_1 \mp \sin\phi\,\hat{\mathbf{x}}_2) \tag{M.105}$$

If $\phi = \omega t$, as is be the case when $\phi$ measures the angle of rotation with angular frequency $\omega$, then

$$\operatorname{Re}\{\mathbf{c}_\pm\} = C\big(\cos(\omega t)\,\hat{\mathbf{x}}_1 \mp \sin(\omega t)\,\hat{\mathbf{x}}_2\big) \tag{M.106}$$

We see that the physically meaningful real part describes a rotation in positive or negative sense, depending on the choice of sign. and therefore $\hat{\mathbf{h}}_\pm$ are called *helical base vectors*.

End of example M.1 ◁

▷ Tensors in 3D space — EXAMPLE M.2

Consider a tetrahedron-like volume element $V$ of a solid, fluid, or gaseous body, whose atomistic structure is irrelevant for the present analysis; figure M.1 on the next page indicates how this volume may look like. Let $d\mathbf{S} = d^2x\,\hat{\mathbf{n}}$ be the directed surface element of this volume element and let the vector $\mathbf{T}_{\hat{\mathbf{n}}}\,d^2x$ be the force that matter, lying on the side of $d^2x$ toward which the unit normal vector $\hat{\mathbf{n}}$ points, acts on matter which lies on the opposite side of $d^2x$. This force concept is meaningful only if the forces are short-range enough that they can be assumed to act only in the surface proper. According to Newton's third law, this surface force fulfils

$$\mathbf{T}_{-\hat{\mathbf{n}}} = -\mathbf{T}_{\hat{\mathbf{n}}} \tag{M.107}$$

Using (M.107) and Newton's second law, we find that the matter of mass $m$, which at a given instant is located in $V$ obeys the equation of motion

$$\mathbf{T}_{\hat{\mathbf{n}}}\,d^2x - \cos\theta_1\mathbf{T}_{\hat{\mathbf{x}}_1}\,d^2x - \cos\theta_2\mathbf{T}_{\hat{\mathbf{x}}_2}\,d^2x - \cos\theta_3\mathbf{T}_{\hat{\mathbf{x}}_3}\,d^2x + \mathbf{F}_{\text{ext}} = m\mathbf{a} \tag{M.108}$$

where $\mathbf{F}_{\text{ext}}$ is the external force and $\mathbf{a}$ is the acceleration of the volume element. In other words

$$\mathbf{T}_{\hat{\mathbf{n}}} = n_1\mathbf{T}_{\hat{\mathbf{x}}_1} + n_2\mathbf{T}_{\hat{\mathbf{x}}_2} + n_3\mathbf{T}_{\hat{\mathbf{x}}_3} + \frac{m}{d^2x}\left(\mathbf{a} - \frac{\mathbf{F}_{\text{ext}}}{m}\right) \tag{M.109}$$

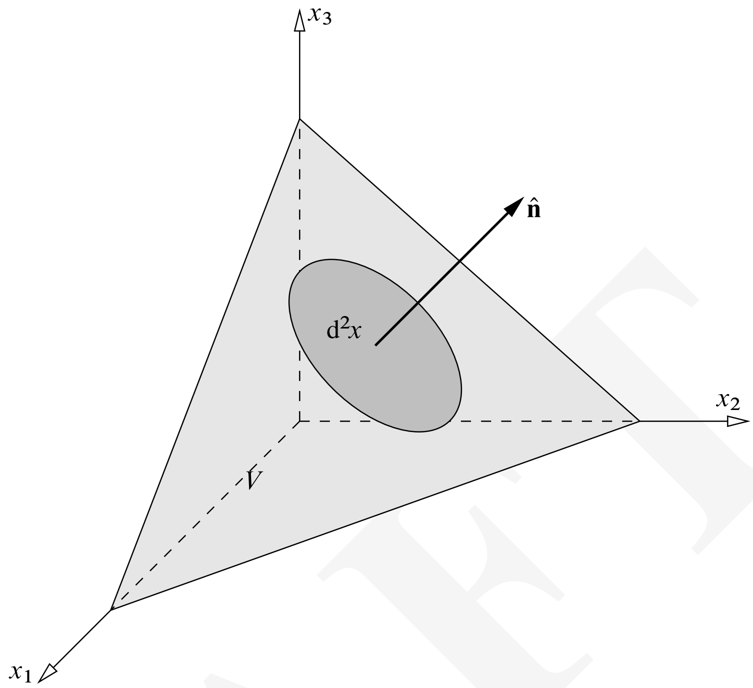


Figure M.1: Tetrahedron-like volume element $V$ containing matter.

Since both $\mathbf{a}$ and $\mathbf{F}_{\mathrm{ext}}/m$ remain finite whereas $m/\mathrm{d}^2x \to 0$ as $V \to 0$, one finds that in this limit

$$\mathbf{T}_{\hat{\mathbf{n}}} = \sum_{i=1}^{3} n_i \mathbf{T}_{\hat{\mathbf{x}}_i} \equiv n_i \mathbf{T}_{\hat{\mathbf{x}}_i} \tag{M.110}$$

From the above derivation it is clear that equation (M.110) above is valid not only in equilibrium but also when the matter in $V$ is in motion.

Introducing the notation

$$T_{ij} = \left(\mathbf{T}_{\hat{\mathbf{x}}_i}\right)_j \tag{M.111}$$

for the $j$th component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, we can write equation (M.110) above in component form as follows

$$T_{\hat{\mathbf{n}}j} = (\mathbf{T}_{\hat{\mathbf{n}}})_j = \sum_{i=1}^{3} n_i T_{ij} \equiv n_i T_{ij} \tag{M.112}$$

Using equation (M.112) above, we find that the component of the vector $\mathbf{T}_{\hat{\mathbf{n}}}$ in the direction of an arbitrary unit vector $\hat{\mathbf{m}}$ is

$$\begin{aligned} T_{\hat{\mathbf{n}}\hat{\mathbf{m}}} &= \mathbf{T}_{\hat{\mathbf{n}}} \cdot \hat{\mathbf{m}} \\ &= \sum_{j=1}^{3} T_{\hat{\mathbf{n}}j} m_j = \sum_{j=1}^{3} \left( \sum_{i=1}^{3} n_i T_{ij} \right) m_j \equiv n_i T_{ij} m_j = \hat{\mathbf{n}} \cdot \mathsf{T} \cdot \hat{\mathbf{m}} \end{aligned} \tag{M.113}$$

Hence, the $j$th component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, here denoted $T_{ij}$, can be interpreted as the $ij$th component of a tensor $\mathsf{T}$. Note that $T_{\hat{\mathbf{n}}\hat{\mathbf{m}}}$ is independent of the particular coordinate system used in the derivation.

We shall now show how one can use the momentum law (force equation) to derive the equation of motion for an arbitrary element of mass in the body. To this end we consider a part $V$ of the body. If the external force density (force per unit volume) is denoted by $\mathbf{f}$ and the velocity for a mass element d$m$ is denoted by $\mathbf{v}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{v}\,\mathrm{d}m = \int_V \mathbf{f}\,\mathrm{d}^3x + \int_S \mathbf{T}_{\hat{\mathbf{n}}}\,\mathrm{d}^2x \tag{M.114}$$

The $j$th component of this equation can be written

$$\int_V \frac{\mathrm{d}}{\mathrm{d}t}v_j\,\mathrm{d}m = \int_V f_j\,\mathrm{d}^3x + \int_S T_{\hat{\mathbf{n}}j}\,\mathrm{d}^2x = \int_V f_j\,\mathrm{d}^3x + \int_S n_i T_{ij}\,\mathrm{d}^2x \tag{M.115}$$

where, in the last step, equation (M.112) on the preceding page was used. Setting $\mathrm{d}m = \rho\,\mathrm{d}^3x$ and using the divergence theorem on the last term, we can rewrite the result as

$$\int_V \rho\frac{\mathrm{d}}{\mathrm{d}t}v_j\,\mathrm{d}^3x = \int_V f_j\,\mathrm{d}^3x + \int_V \frac{\partial T_{ij}}{\partial x_i}\,\mathrm{d}^3x \tag{M.116}$$

Since this formula is valid for any arbitrary volume, we must require that

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \tag{M.117}$$

or, equivalently

$$\rho\frac{\partial v_j}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \tag{M.118}$$

Note that $\partial v_j/\partial t$ is the rate of change with time of the velocity component $v_j$ at a *fixed* point $\mathbf{x} = (x_1, x_1, x_3)$.

End of example M.2◁

▷Contravariant and covariant vectors in flat Lorentz space — EXAMPLE M.3

The 4D Lorentz space $\mathbb{L}^4$ has a simple metric which can be described by the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1,2,3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{M.119}$$

which in matrix notation becomes

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{M.120}$$

*i.e.* a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+,-,-,-\}$. Alternatively, one can define the metric tensor in $\mathbb{L}^4$ as

$$\eta_{\mu\nu} = \begin{cases} -1 & \text{if } \mu = \nu = 0 \\ 1 & \text{if } \mu = \nu = i = j = 1,2,3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{M.121}$$

which in matrix representation becomes

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{M.122}$$

*i.e.* a matrix with signature $\{-,+,+,+\}$. Of course, the physics is unaffected by the choice of metric tensor.

Consider an arbitrary contravariant four-vector $a^\nu$ in this space. In component form it can be written:

$$a^\nu \stackrel{\text{def}}{\equiv} (a^0, a^1, a^2, a^3) = (a^0, \mathbf{a}) \tag{M.123}$$

According to the index lowering rule, equation (M.29) on page 238, we obtain the covariant version of this vector as

$$a_\mu \stackrel{\text{def}}{\equiv} (a_0, a_1, a_2, a_3) = \eta_{\mu\nu} a^\nu \tag{M.124}$$

In the $\{+,-,-,-\}$ metric we obtain

$$\mu = 0: \quad a_0 = 1 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = a^0 \tag{M.125}$$
$$\mu = 1: \quad a_1 = 0 \cdot a^0 - 1 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = -a^1 \tag{M.126}$$
$$\mu = 2: \quad a_2 = 0 \cdot a^0 + 0 \cdot a^1 - 1 \cdot a^2 + 0 \cdot a^3 = -a^2 \tag{M.127}$$
$$\mu = 3: \quad a_3 = 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 - 1 \cdot a^3 = -a^3 \tag{M.128}$$

or

$$a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) = (a^0, -\mathbf{a}) \tag{M.129}$$

The radius 4-vector itself in $\mathbb{L}^4$ and in this metric is given by

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{x}) \\ x_\mu &= (x_0, x_1, x_2, x_3) = (ct, -x^1, -x^2, -x^3) = (ct, -\mathbf{x}) \end{aligned} \tag{M.130}$$

Analogously, using the $\{-,+,+,+\}$ metric we obtain

$$a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3) = (-a^0, \mathbf{a}) \tag{M.131}$$

End of example M.3 ◁

▷Scalar products in complex vector space —— EXAMPLE M.4

Let $\mathbf{c}$ be a complex vector defined as in expression (M.10) on page 235.

The inner product of $\mathbf{c}$ with itself may be defined as

$$\begin{aligned}\mathbf{c}^2 &\stackrel{\text{def}}{\equiv} \mathbf{c}\cdot\mathbf{c} = (\mathbf{c}_\mathrm{R} + \mathrm{i}\mathbf{c}_\mathrm{I})\cdot(\mathbf{c}_\mathrm{R} + \mathrm{i}\mathbf{c}_\mathrm{I}) = \mathbf{c}_\mathrm{R}^2 - \mathbf{c}_\mathrm{I}^2 + 2\mathrm{i}\mathbf{c}_\mathrm{R}\cdot\mathbf{c}_\mathrm{I} \\ &= c_\mathrm{R}^2 - c_\mathrm{I}^2 + 2\mathrm{i}\mathbf{c}_\mathrm{R}\cdot\mathbf{c}_\mathrm{I} \stackrel{\text{def}}{\equiv} c^2 \in \mathbb{C}\end{aligned} \tag{M.132}$$

from which we find that

$$c = \sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 + 2\mathrm{i}\mathbf{c}_\mathrm{R}\cdot\mathbf{c}_\mathrm{I}} = \sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 + 2\mathrm{i}c_\mathrm{R}c_\mathrm{I}\cos\theta} \in \mathbb{C} \tag{M.133}$$

where $\theta$ is the (real-valued) angle between $\mathbf{c}_\mathrm{R}$ and $\mathbf{c}_\mathrm{I}$.

Using this in equation (M.10) on page 235, we see that we can define the complex unit vector as being

$$\begin{aligned}\hat{\mathbf{c}} = \frac{\mathbf{c}}{c} &= \frac{c_\mathrm{R}}{\sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 + 2\mathrm{i}c_\mathrm{R}c_\mathrm{I}\cos\theta}}\hat{\mathbf{c}}_\mathrm{R} + \mathrm{i}\frac{c_\mathrm{I}}{\sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 + 2\mathrm{i}c_\mathrm{R}c_\mathrm{I}\cos\theta}}\hat{\mathbf{c}}_\mathrm{I} \\ &= \frac{c_\mathrm{R}\sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 - 2\mathrm{i}c_\mathrm{R}c_\mathrm{I}\cos\theta}}{(c_\mathrm{R}^2 + c_\mathrm{I}^2)\sqrt{1 - \dfrac{4c_\mathrm{R}^2c_\mathrm{I}^2\sin^2\theta}{(c_\mathrm{R}^2 + c_\mathrm{I}^2)^2}}}\hat{\mathbf{c}}_\mathrm{R} + \mathrm{i}\frac{c_\mathrm{I}\sqrt{c_\mathrm{R}^2 - c_\mathrm{I}^2 - 2\mathrm{i}c_\mathrm{R}c_\mathrm{I}\cos\theta}}{(c_\mathrm{R}^2 + c_\mathrm{I}^2)\sqrt{1 - \dfrac{4c_\mathrm{R}^2c_\mathrm{I}^2\sin^2\theta}{(c_\mathrm{R}^2 + c_\mathrm{I}^2)^2}}}\hat{\mathbf{c}}_\mathrm{I} \in \mathbb{C}^3\end{aligned} \tag{M.134}$$

On the other hand, the definition of the scalar product in terms of the inner product of a complex vector with its own complex conjugate yields

$$|\mathbf{c}|^2 \stackrel{\text{def}}{\equiv} \mathbf{c}\cdot\mathbf{c}^* = (\mathbf{c}_\mathrm{R} + \mathrm{i}\mathbf{c}_\mathrm{I})\cdot(\mathbf{c}_\mathrm{R} + \mathrm{i}\mathbf{c}_\mathrm{I})^* = \mathbf{c}_\mathrm{R}^2 + \mathbf{c}_\mathrm{I}^2 = c_\mathrm{R}^2 + c_\mathrm{I}^2 = |c|^2 \tag{M.135}$$

with the help of which we can define the unit vector as

$$\begin{aligned}\hat{\mathbf{c}} = \frac{\mathbf{c}}{|c|} &= \frac{c_\mathrm{R}}{\sqrt{c_\mathrm{R}^2 + c_\mathrm{I}^2}}\hat{\mathbf{c}}_\mathrm{R} + \mathrm{i}\frac{c_\mathrm{I}}{\sqrt{c_\mathrm{R}^2 + c_\mathrm{I}^2}}\hat{\mathbf{c}}_\mathrm{I} \\ &= \frac{c_\mathrm{R}\sqrt{c_\mathrm{R}^2 + c_\mathrm{I}^2}}{c_\mathrm{R}^2 + c_\mathrm{I}^2}\hat{\mathbf{c}}_\mathrm{R} + \mathrm{i}\frac{c_\mathrm{I}\sqrt{c_\mathrm{R}^2 + c_\mathrm{I}^2}}{c_\mathrm{R}^2 + c_\mathrm{I}^2}\hat{\mathbf{c}}_\mathrm{I} \in \mathbb{C}^3\end{aligned} \tag{M.136}$$

End of example M.4◁

EXAMPLE M.5 ▷Scalar product, norm and metric in Lorentz space

In $\mathbb{L}^4$ the metric tensor attains a simple form [see example M.3 on page 253] and, hence, the scalar product in equation (M.37) on page 239 can be evaluated almost trivially. For the $\{+,-,-,-\}$ signature it becomes

$$a_\mu b^\mu = (a_0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a_0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{M.137}$$

In $\mathbb{L}^4$ the important scalar product of the position four-vector with itself therefore takes the simple form

$$\begin{aligned} x_\mu x^\mu &= (x_0, -\mathbf{x}) \cdot (x^0, \mathbf{x}) = (ct, -\mathbf{x}) \cdot (ct, \mathbf{x}) \\ &= (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = s^2 \end{aligned} \tag{M.138}$$

which is the indefinite, real *norm* of $\mathbb{L}^4$. The $\mathbb{L}^4$ metric is the quadratic differential form

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = c^2(\mathrm{d}t)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \tag{M.139}$$

End of example M.5◁

EXAMPLE M.6 ▷The vector triple product

The vector triple product is the vector product of a vector $\mathbf{a}$ with a vector product $\mathbf{b} \times \mathbf{c}$ and can, with the help of formula (M.24) on page 237, be evaluated as

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \epsilon_{ijk} \hat{\mathbf{x}}_i a_j (\mathbf{b} \times \mathbf{c})_k = \epsilon_{ijk} \hat{\mathbf{x}}_i a_j (\epsilon_{lmn} \hat{\mathbf{x}}_l b_m c_n)_k \\ &= \epsilon_{ijk} \hat{\mathbf{x}}_i a_j \epsilon_{kmn} b_m c_n = \epsilon_{ijk} \epsilon_{kmn} \hat{\mathbf{x}}_i a_j b_m c_n \\ &= \epsilon_{kij} \epsilon_{kmn} \hat{\mathbf{x}}_i a_j b_m c_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \hat{\mathbf{x}}_i a_j b_m c_n \\ &= \delta_{im}\delta_{jn} \hat{\mathbf{x}}_i a_j b_m c_n - \delta_{in}\delta_{jm} \hat{\mathbf{x}}_i a_j b_m c_n = a_j c_j b_i \hat{\mathbf{x}}_i - a_j b_j c_i \hat{\mathbf{x}}_i \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \equiv \mathbf{b} \otimes \mathbf{a} \cdot \mathbf{c} - \mathbf{c} \otimes \mathbf{a} \cdot \mathbf{b} \end{aligned} \tag{M.140}$$

which is formula (F.53) on page 218. This is sometimes called *Lagrange's formula*, but is more often referred to as the *bac-cab rule*.

End of example M.6◁

EXAMPLE M.7 ▷Matrix representation of the vector product in $\mathbb{R}^3$

Prove that the matrix representation of the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is given by formula (M.45) on page 241.

According to formula (M.43) on page 241, the scalar multiplication of a dyadic product of two vectors (in our case $\mathsf{S}$ and $\mathbf{b}$) from the left by a vector (in our case $\mathbf{a}$) is interpreted as $\mathbf{a} \cdot \mathsf{S} \otimes \mathbf{b} = (\mathbf{a} \cdot \mathsf{S})\mathbf{b}$ where

$$\mathbf{a} \cdot \mathsf{S} = a_i \mathsf{S}_i \tag{M.141}$$

and the components $\mathsf{S}_i$ are given by formula (M.27) on page 238. Hence

$$\begin{aligned} \mathbf{a} \cdot \mathsf{S} &= a_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{i} \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_1 \\ 0 & a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & 0 \\ -a_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -a_3 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ &= \mathrm{i} \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \end{aligned} \tag{M.142}$$

from which we find that

$$(\mathbf{a}\cdot\mathbf{S})\mathbf{b} = \mathrm{i}\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \mathrm{i}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \tag{M.143}$$

In other words,

$$-\mathrm{i}(\mathbf{a}\cdot\mathbf{S})\mathbf{b} = -\mathrm{i}\mathbf{a}\cdot\mathbf{S}\otimes\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = (\mathbf{a}\times\mathbf{b}) \tag{M.144}$$

Likewise, $\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{b} = \mathbf{a}(\mathbf{S}\cdot\mathbf{b})$ where

$$\begin{aligned}\mathbf{S}\cdot\mathbf{b} = \mathsf{S}_i b_i &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} b_1 + \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} b_2 + \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} b_3 \\ &= \mathrm{i}\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}\end{aligned} \tag{M.145}$$

and

$$\mathbf{a}(\mathbf{S}\cdot\mathbf{b}) = \mathrm{i}\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \tag{M.146}$$

from which follows

$$-\mathrm{i}\mathbf{a}(\mathbf{S}\cdot\mathbf{b}) = -\mathrm{i}\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = (\mathbf{a}\times\mathbf{b}) \tag{M.147}$$

Hence, $(\mathbf{a}\times\mathbf{b}) = -\mathrm{i}\mathbf{a}\cdot\mathbf{S}\otimes\mathbf{b} = -\mathrm{i}\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{b}$ QED ■

End of example M.7◁

▷Gradient of a scalar product of two vector fields — EXAMPLE M.8

The gradient of the scalar product of two vector fields **a** and **b** can be calculated in the following way:

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) &= (\hat{\mathbf{x}}_i\partial_i)(a_j\hat{\mathbf{x}}_j\cdot b_k\hat{\mathbf{x}}_k) \\ &= [(\hat{\mathbf{x}}_i\partial_i)(a_j\hat{\mathbf{x}}_j)]\cdot(b_k\hat{\mathbf{x}}_k) + (a_j\hat{\mathbf{x}}_j)\cdot[(\hat{\mathbf{x}}_i\partial_i)(b_k\hat{\mathbf{x}}_k)] \\ &= (\hat{\mathbf{x}}_i\partial_i a_j\hat{\mathbf{x}}_j)\cdot\mathbf{b} + (\hat{\mathbf{x}}_i\partial_i b_k\hat{\mathbf{x}}_k)\cdot\mathbf{a} = (\boldsymbol{\nabla}\otimes\mathbf{a})\cdot\mathbf{b} + (\boldsymbol{\nabla}\otimes\mathbf{b})\cdot\mathbf{a}\end{aligned} \tag{M.148}$$

This is the first version of formula (F.79) on page 219.

End of example M.8◁

EXAMPLE M.9 ▷The four-del operator in Lorentz space

In $\mathbb{L}^4$ the contravariant form of the four-*del* operator can be represented as

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\partial}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{M.149}$$

and the covariant form as

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\partial}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{M.150}$$

Taking the scalar product of these two, one obtains

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2 \tag{M.151}$$

which is the *d'Alembert operator*, sometimes denoted $\Box$, and sometimes defined with an opposite sign convention.

End of example M.9◁

EXAMPLE M.10 ▷Gradients of scalar functions of relative distances in 3D

Very often electrodynamic quantities are dependent on the relative distance in $\mathbb{R}^3$ between two vectors $\mathbf{x}$ and $\mathbf{x}'$, *i.e.* on $\left|\mathbf{x} - \mathbf{x}'\right|$. In analogy with equation (M.50) on page 242, we can define the primed *del* operator in the following way:

$$\boldsymbol{\nabla}' = \hat{\mathbf{x}}_i \frac{\partial}{\partial x'_i} = \boldsymbol{\partial}' \tag{M.152}$$

Using this, the corresponding unprimed version, *viz.*, equation (M.50) on page 242, and elementary rules of differentiation, we obtain the following very useful result:

$$\begin{aligned}\boldsymbol{\nabla}\left(\left|\mathbf{x}-\mathbf{x}'\right|\right) &= \hat{\mathbf{x}}_i \frac{\partial\left|\mathbf{x}-\mathbf{x}'\right|}{\partial x_i} = \hat{\mathbf{x}}_i \frac{\partial\sqrt{(x_1-x'_1)^2+(x_2-x'_2)^2+(x_3-x'_3)^2}}{\partial x_i} \\ &= \hat{\mathbf{x}}_i \frac{(x_i-x'_i)}{\left|\mathbf{x}-\mathbf{x}'\right|} = \frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|} = -\hat{\mathbf{x}}_i \frac{\partial\left|\mathbf{x}-\mathbf{x}'\right|}{\partial x'_i} = -\boldsymbol{\nabla}'\left(\left|\mathbf{x}-\mathbf{x}'\right|\right)\end{aligned} \tag{M.153}$$

Likewise

$$\boldsymbol{\nabla}\left(\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|^3} = -\boldsymbol{\nabla}'\left(\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) \tag{M.154}$$

End of example M.10◁

EXAMPLE M.11 ▷Divergence and curl of a vector field divided by the relative distance

For an arbitrary $\mathbb{R}^3$ vector field $\mathbf{a}(\mathbf{x}')$, the following relations hold:

$$\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{a}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) = \frac{\boldsymbol{\nabla}'\cdot\mathbf{a}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} + \mathbf{a}(\mathbf{x}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) \tag{M.155a}$$

$$\boldsymbol{\nabla}'\times\left(\frac{\mathbf{a}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) = \frac{\boldsymbol{\nabla}'\times\mathbf{a}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|} + \mathbf{a}(\mathbf{x}')\times\boldsymbol{\nabla}'\left(\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) \tag{M.155b}$$

This demonstrates how the primed divergence and curl, defined in terms of the primed *del* operator in equation (M.152) above, work.

End of example M.11◁

▷The Laplacian and the Dirac delta 'function' — EXAMPLE M.12

A very useful formula in $\mathbb{R}^3$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=-4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{M.156}$$

where $\delta(\mathbf{x}-\mathbf{x}')$ is the 3D *Dirac delta* 'function'. This formula follows directly from the fact that

$$\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\int_{V'}\mathrm{d}^3x'\,\boldsymbol{\nabla}\cdot\left(-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)=\oint_{S'}\mathrm{d}^2x'\,\hat{\mathbf{n}}'\cdot\left(-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) \tag{M.157}$$

equals $-4\pi$ if the integration volume $V'(S')$, enclosed by the surface $S'(V')$, includes $\mathbf{x}'=\mathbf{x}$, and equals 0 otherwise.

End of example M.12◁

▷The curl of a gradient — EXAMPLE M.13

Using the definition of the $\mathbb{R}^3$ curl, equation (M.61) on page 243, and the gradient, equation (M.56) on page 243, we see that

$$\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\alpha(\mathbf{x})]=\epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j[\boldsymbol{\nabla}\alpha(\mathbf{x})]_k=\epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j\partial_k\alpha(\mathbf{x}) \tag{M.158}$$

which, due to the assumed well-behavedness of $\alpha(\mathbf{x})$, vanishes:

$$\begin{aligned}\epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j\partial_k\alpha(\mathbf{x})&=\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\alpha(\mathbf{x})\hat{\mathbf{x}}_i\\&=\left(\frac{\partial^2}{\partial x_2\partial x_3}-\frac{\partial^2}{\partial x_3\partial x_2}\right)\alpha(\mathbf{x})\hat{\mathbf{x}}_1\\&\quad+\left(\frac{\partial^2}{\partial x_3\partial x_1}-\frac{\partial^2}{\partial x_1\partial x_3}\right)\alpha(\mathbf{x})\hat{\mathbf{x}}_2\\&\quad+\left(\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial x_2\partial x_1}\right)\alpha(\mathbf{x})\hat{\mathbf{x}}_3\\&\equiv\mathbf{0}\end{aligned} \tag{M.159}$$

We thus find that

$$\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\alpha(\mathbf{x})]\equiv\mathbf{0} \tag{M.160}$$

for any arbitrary, well-behaved $\mathbb{R}^3$ scalar field $\alpha(\mathbf{x})$.

In 4D we note that for any well-behaved four-scalar field $\alpha(x^\kappa)$

$$(\partial_\mu\partial_\nu-\partial_\nu\partial_\mu)\alpha(x^\kappa)\equiv 0 \tag{M.161}$$

so that the four-curl of a four-gradient vanishes just as does a curl of a gradient in $\mathbb{R}^3$.

Hence, a gradient is always *irrotational*.

End of example M.13◁

EXAMPLE M.14 ▷The divergence of a curl

With the use of the definitions of the divergence (M.59) and the curl, equation (M.61) on page 243, we find that

$$\boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x})] = \partial_i[\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x})]_i = \epsilon_{ijk}\partial_i\partial_j a_k(\mathbf{x}) \tag{M.162}$$

Using the definition for the Levi-Civita symbol, defined by equation (M.21) on page 237, we find that, due to the assumed well-behavedness of $\mathbf{a}(\mathbf{x})$,

$$\begin{aligned}\partial_i\epsilon_{ijk}\partial_j a_k(\mathbf{x}) &= \frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}a_k\\ &= \left(\frac{\partial^2}{\partial x_2\partial x_3}-\frac{\partial^2}{\partial x_3\partial x_2}\right)a_1(\mathbf{x})\\ &\quad+\left(\frac{\partial^2}{\partial x_3\partial x_1}-\frac{\partial^2}{\partial x_1\partial x_3}\right)a_2(\mathbf{x})\\ &\quad+\left(\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial x_2\partial x_1}\right)a_3(\mathbf{x})\\ &\equiv 0\end{aligned} \tag{M.163}$$

*i.e.* that

$$\boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x})] \equiv 0 \tag{M.164}$$

for any arbitrary, well-behaved $\mathbb{R}^3$ vector field $\mathbf{a}(\mathbf{x})$. The 3D curl is therefore *solenoidal* (has a vanishing divergence).

In 4D, the four-divergence of the four-curl is *not* zero, for

$$\partial^\nu A^\mu_\nu = \partial^\mu\partial_\nu a^\nu(x^\kappa) - \Box^2 a^\mu(x^\kappa) \neq 0 \tag{M.165}$$

End of example M.14◁

EXAMPLE M.15 ▷The curl of the curl of a vector field — the 'bac-cab' rule for the *del* operator

The curl of the curl of a vector field can be viewed as the *del* operator version of the bac-cab that was derived in example M.6 on page 256. By using formula (M.24) on page 237 it can be evaluated as

$$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{a}) &= \epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j(\boldsymbol{\nabla}\times\mathbf{a})_k = \epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j(\epsilon_{lmn}\hat{\mathbf{x}}_l\partial_m a_n)_k\\ &= \epsilon_{ijk}\hat{\mathbf{x}}_i\partial_j\epsilon_{kmn}\partial_m a_n = \epsilon_{ijk}\epsilon_{kmn}\hat{\mathbf{x}}_i\partial_j\partial_m a_n\\ &= \epsilon_{kij}\epsilon_{kmn}\hat{\mathbf{x}}_i\partial_j\partial_m a_n = (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\hat{\mathbf{x}}_i\partial_j\partial_m a_n\\ &= \delta_{im}\delta_{jn}\hat{\mathbf{x}}_i\partial_j\partial_m a_n - \delta_{in}\delta_{jm}\hat{\mathbf{x}}_i\partial_j\partial_m a_n\\ &= \hat{\mathbf{x}}_i\partial_j\partial_i a_j - \hat{\mathbf{x}}_i\partial_j\partial_j a_i = \hat{\mathbf{x}}_i\partial_i\partial_j a_j - \hat{\mathbf{x}}_i\partial_j^2 a_i\\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{a}) - \nabla^2\mathbf{a} \equiv \boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\cdot\mathbf{a} - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{a}\end{aligned} \tag{M.166}$$

which is formula (F.96) on page 220.

End of example M.15◁

▷Products of rotational and irrotational components of vectors — EXAMPLE M.16

Derive the scalar and vector products of the rotational and irrotational components of two vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ that have been Helmholtz decomposed into $\mathbf{u} = \mathbf{u}^{\mathrm{rotat}} + \mathbf{u}^{\mathrm{irrot}}$ and $\mathbf{v} = \mathbf{v}^{\mathrm{rotat}} + \mathbf{v}^{\mathrm{irrot}}$, respectively.

Let us, in addition to expressions (M.82), introduce the definitions

$$\beta(\mathbf{x}) = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \cdot \mathbf{v}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.167a}$$

$$\mathbf{b}(\mathbf{x}) = \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \times \mathbf{v}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \tag{M.167b}$$

in order to simplify our notations and calculations.

## SCALAR PRODUCTS

The scalar product of the two irrotational components $\mathbf{u}^{\mathrm{irrot}}$ and $\mathbf{v}^{\mathrm{irrot}}$ is given by the formula

$$\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) = [\boldsymbol{\nabla}\alpha(\mathbf{x})] \cdot [\boldsymbol{\nabla}\beta(\mathbf{x})] \tag{M.168}$$

where we can rewrite the RHS by using formula (F.82) on page 219 so that

$$[\boldsymbol{\nabla}\alpha(\mathbf{x})] \cdot [\boldsymbol{\nabla}\beta(\mathbf{x})] = \boldsymbol{\nabla} \cdot [\alpha(\mathbf{x})\boldsymbol{\nabla}\beta(\mathbf{x})] - \alpha(\mathbf{x})\nabla^2\beta(\mathbf{x}) \tag{M.169}$$

If we insert the expression (M.82b) on page 248 for $\alpha(\mathbf{x})$ and expression (M.167a) for $\beta(\mathbf{x})$ and then integrate over $V$, the first term in the RHS can, with help of the divergence theorem, be written as a surface integral where, at large distances $r = |\mathbf{x} - \mathbf{x}'|$, the integrand tends to zero as $1/r^3$, ensuring that this integral vanishes. Then, also using the identity (M.77) on page 247 and equations (M.85) on page 248, we obtain the following non-local scalar product expression:

$$\begin{aligned}\int_V \mathrm{d}^3x \, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) &= \int_V \mathrm{d}^3x \int_{V'} \mathrm{d}^3x' \, \frac{[\boldsymbol{\nabla}' \cdot \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}')][\boldsymbol{\nabla} \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x})]}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \\ &= \int_V \mathrm{d}^3x \, [\boldsymbol{\nabla} \cdot \mathbf{v}^{\mathrm{irrot}}(\mathbf{x})] \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \cdot \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|}\end{aligned} \tag{M.170}$$

This is identity (F.130c) on page 223.

We now evaluate the scalar product of the two rotational (solenoidal) components $\mathbf{u}^{\mathrm{rotat}}$ and $\mathbf{v}^{\mathrm{rotat}}$

$$\mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = [\boldsymbol{\nabla} \times \mathbf{a}(\mathbf{x})] \cdot [\boldsymbol{\nabla} \times \mathbf{b}(\mathbf{x})] \tag{M.171}$$

Formula (F.85) on page 219 allows us to write

$$\mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \cdot \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \mathbf{b} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})] - \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \times \mathbf{a}) \times \mathbf{b}] \tag{M.172}$$

As the last term in the RHS will vanish when we integrate over $V$ (divergence theorem), we focus our attention on the first term. According to formula (F.96) on page 220

$$\mathbf{b} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})] = \mathbf{b} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2\mathbf{a}] \tag{M.173}$$

and, according to equation (M.82c) on page 248 and formula (F.81) on page 219, we obtain

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot \int_{V'} \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}' \times \mathbf{v}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} = \int_{V'} \mathrm{d}^3x' \, [\boldsymbol{\nabla}' \times \mathbf{v}(\mathbf{x}')] \cdot \boldsymbol{\nabla}\left(\frac{1}{4\pi \, |\mathbf{x} - \mathbf{x}'|}\right) \tag{M.174}$$

With the help of formula (F.114) on page 220, formula (M.155) on page 258, and formula (F.99) on page 220, we can rewrite this as

$$\nabla \cdot \mathbf{a} = -\int_{V'} \mathrm{d}^3x' \, \nabla \cdot \left( \frac{\nabla' \times \mathbf{v}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \right) = -\oint_{S'} \mathrm{d}^2x' \, \hat{\mathbf{n}}' \cdot \left( \frac{\nabla' \times \mathbf{v}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \right) \tag{M.175}$$

which tends to zero for a surface at large distances. Hence, we have found that

$$\mathbf{u}^{\text{rotat}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) = -\mathbf{b} \cdot \nabla^2 \mathbf{a} - \nabla \cdot [(\nabla \times \mathbf{a}) \times \mathbf{b}] \tag{M.176}$$

Integration over $V$ and the use of the divergence theorem, where the resulting surface integral vanishes, and using equations (M.85) on page 248 gives the result

$$\begin{aligned} \int_V \mathrm{d}^3x \, \mathbf{u}^{\text{rotat}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) &= \int_V \mathrm{d}^3x \int_{V'} \mathrm{d}^3x' \, \frac{[\nabla \times \mathbf{u}^{\text{rotat}}(\mathbf{x})] \cdot [\nabla' \times \mathbf{v}^{\text{rotat}}(\mathbf{x}')]}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \\ &= \int_V \mathrm{d}^3x \, [\nabla \times \mathbf{u}^{\text{rotat}}(\mathbf{x})] \cdot \int_{V'} \mathrm{d}^3x' \, \frac{\nabla' \times \mathbf{v}^{\text{rotat}}(\mathbf{x}')}{4\pi \, |\mathbf{x} - \mathbf{x}'|} \end{aligned} \tag{M.177}$$

This is identity (F.130d) on page 223.

The scalar product of $\mathbf{u}^{\text{irrot}}$ and $\mathbf{v}^{\text{rotat}}$ becomes

$$\mathbf{u}^{\text{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) = -[\nabla \alpha(\mathbf{x})] \cdot [\nabla \times \mathbf{b}(\mathbf{x})] \tag{M.178}$$

With the use of a standard vector analytic identity (F.84) on page 219, we can rewrite this as

$$\mathbf{u}^{\text{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) = -\nabla \cdot \big(\mathbf{b}(\mathbf{x}) \times [\nabla \alpha(\mathbf{x})]\big) \tag{M.179}$$

If we integrate this over $V$ and use the divergence theorem, we obtain

$$\int_V \mathrm{d}^3x \, \mathbf{u}^{\text{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) = -\oint_S \mathrm{d}^2x \, \hat{\mathbf{n}} \cdot \big(\mathbf{b}(\mathbf{x}) \times [\nabla \alpha(\mathbf{x})]\big) \tag{M.180}$$

Again, the surface integral vanishes and we find that the non-local *orthogonality condition*

$$\int_V \mathrm{d}^3x \, \mathbf{u}^{\text{irrot}}(\mathbf{x}) \cdot \mathbf{v}^{\text{rotat}}(\mathbf{x}) = 0 \tag{M.181}$$

holds between the irrotational and rotational components of any two arbitrary, continuously differentiable vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$. This is identity (F.130a) on page 222. If, in particular, $\mathbf{v} \equiv \mathbf{u}$, then

$$\int_V \mathrm{d}^3x \, \mathbf{u}^{\text{irrot}}(\mathbf{x}) \cdot \mathbf{u}^{\text{rotat}}(\mathbf{x}) = 0 \tag{M.182}$$

### Vector products

According to formulæ (M.83) on page 248, the vector product of the two irrotational components $\mathbf{u}^{\text{irrot}}$ and $\mathbf{v}^{\text{irrot}}$ is

$$\mathbf{u}^{\text{irrot}}(\mathbf{x}) \times \mathbf{v}^{\text{irrot}}(\mathbf{x}) = [\nabla \alpha(\mathbf{x})] \times [\nabla \beta(\mathbf{x})] \tag{M.183}$$

which can be evaluated in a similar manner as the scalar product. Using formula (F.88) on page 219 we immediately see that the result is

$$\mathbf{u}^{\text{irrot}}(\mathbf{x}) \times \mathbf{v}^{\text{irrot}}(\mathbf{x}) = \nabla \times [\alpha(x) \nabla \beta(\mathbf{x})] \tag{M.184}$$

Integration over $V$ yields, with the use of formula (F.121c) on page 221, the result

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) = \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}} \times \alpha(x) \boldsymbol{\nabla}\beta(\mathbf{x}) \tag{M.185}$$

where the RHS goes to zero asymptotically, resulting in the non-local *parallelity condition*

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{irrot}}(\mathbf{x}) = \mathbf{0} \tag{M.186}$$

This is identity (F.130b) on page 222.

In order to evaluate the vector product of two rotational components $\mathbf{u}^{\mathrm{rotat}}$ and $\mathbf{v}^{\mathrm{rotat}}$

$$\mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = (\boldsymbol{\nabla} \times \mathbf{a}) \times (\boldsymbol{\nabla} \times \mathbf{b}) \tag{M.187}$$

[see formulæ (M.83)], we use the vector identity (F.79) on page 219 that gives (with $\mathbf{c}$ replacing $\mathbf{a}$)

$$\mathbf{c} \times (\boldsymbol{\nabla} \times \mathbf{b}) = \boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{c}) - \mathbf{c} \cdot \boldsymbol{\nabla} \otimes \mathbf{b} - \mathbf{b} \cdot \boldsymbol{\nabla} \otimes \mathbf{c} \tag{M.188}$$

Next, we use the identity (F.86) on page 219 to replace $\mathbf{c} \cdot \boldsymbol{\nabla} \otimes \mathbf{b}$ and find that we can write

$$\mathbf{c} \times (\boldsymbol{\nabla} \times \mathbf{b}) = -\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{c}) - \mathbf{b} \cdot \boldsymbol{\nabla} \otimes \mathbf{c} + (\boldsymbol{\nabla} \cdot \mathbf{c})\mathbf{b} + \boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{b}) + \boldsymbol{\nabla} \cdot (\mathbf{c} \otimes \mathbf{b}) \tag{M.189}$$

Finally we make the change $\mathbf{c} \mapsto \boldsymbol{\nabla} \times \mathbf{a}$ with the result that we obtain the identity

$$\begin{aligned}(\boldsymbol{\nabla} \times \mathbf{a}) \times (\boldsymbol{\nabla} \times \mathbf{b}) = {} & -\mathbf{b} \times [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a})] - \mathbf{b} \cdot [\boldsymbol{\nabla} \otimes (\boldsymbol{\nabla} \times \mathbf{a})] + [\underbrace{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{a})}_{=0}]\mathbf{b} \\ & + \boldsymbol{\nabla}[(\boldsymbol{\nabla} \times \mathbf{a}) \cdot \mathbf{b}] + \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \times \mathbf{a}) \otimes \mathbf{b}]\end{aligned} \tag{M.190}$$

Recalling that $\mathbf{a}$ is given by formula (M.82c) on page 248, we see that the expression within square brackets in the first term of the RHS of equation (M.190) is

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) &= \boldsymbol{\nabla} \times \left( \boldsymbol{\nabla} \times \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}' \times \mathbf{u}(\mathbf{x}')}{4\pi\, |\mathbf{x} - \mathbf{x}'|} \right) \\ &= \boldsymbol{\nabla} \times \mathbf{u}(\mathbf{x}) - \int_{V'} \mathrm{d}^3x'\, \underbrace{\{\boldsymbol{\nabla}' \cdot [\boldsymbol{\nabla}' \times \mathbf{u}(\mathbf{x}')]\}}_{0} \boldsymbol{\nabla}' \left( \frac{1}{4\pi\, |\mathbf{x} - \mathbf{x}'|} \right) \\ &= \boldsymbol{\nabla} \times \mathbf{u}(\mathbf{x})\end{aligned} \tag{M.191}$$

where we used formula (M.80) on page 247, not including the vanishing surface integral (last term in the RHS of that formula). Consequently,

$$\begin{aligned}\mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) &= (\boldsymbol{\nabla} \times \mathbf{a}) \times (\boldsymbol{\nabla} \times \mathbf{b}) \\ &= -\mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{b} \cdot \boldsymbol{\nabla} \otimes (\boldsymbol{\nabla} \times \mathbf{a}) \\ &\quad + \boldsymbol{\nabla}[(\boldsymbol{\nabla} \times \mathbf{a}) \cdot \mathbf{b}] + \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla} \times \mathbf{a}) \otimes \mathbf{b}]\end{aligned} \tag{M.192}$$

Integrating the RHS of this expression over $V$, using the divergence theorem and formula (F.121a) on page 221, and assuming that the ensuing surface integrals vanish, we therefore obtain

$$\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = -\int_V \mathrm{d}^3x\, \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{u}) - \int_V \mathrm{d}^3x\, \mathbf{b} \cdot \boldsymbol{\nabla} \otimes (\boldsymbol{\nabla} \times \mathbf{a}) \tag{M.193}$$

Integration by parts of the second term in the RHS of this expression over $V$, making use of the identity (F.86) on page 219 with $\mathbf{a} = \mathbf{b}$ and $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$, gives the result

$$
\begin{aligned}
-\int_V \mathrm{d}^3x\, \mathbf{b}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\times\mathbf{a}) &= \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla}\cdot\mathbf{b})(\boldsymbol{\nabla}\times\mathbf{a}) - \int_V \mathrm{d}^3x\, \boldsymbol{\nabla}\cdot[\mathbf{b}\otimes(\boldsymbol{\nabla}\times\mathbf{a})] \\
&= \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla}\cdot\mathbf{b})(\boldsymbol{\nabla}\times\mathbf{a}) - \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}}\cdot\mathbf{b}\otimes(\boldsymbol{\nabla}\times\mathbf{a})
\end{aligned}
\tag{M.194}
$$

where we used formula (F.121e) on page 221. We have thereby shown that

$$
\begin{aligned}
\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = &-\int_V \mathrm{d}^3x\, \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{u}) + \int_V \mathrm{d}^3x\, (\boldsymbol{\nabla}\cdot\mathbf{b})(\boldsymbol{\nabla}\times\mathbf{a}) \\
&- \oint_S \mathrm{d}^2x\, \hat{\mathbf{n}}\cdot\mathbf{b}\otimes(\boldsymbol{\nabla}\times\mathbf{a})
\end{aligned}
\tag{M.195}
$$

When we insert $\mathbf{b}$ as given by expression (M.167b) on page 261, the surface integral vanishes and we get

$$
\begin{aligned}
\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = &-\int_V \mathrm{d}^3x \int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\times(\boldsymbol{\nabla}\times\mathbf{u}) \\
&+ \int_V \mathrm{d}^3x \left(\boldsymbol{\nabla}\cdot\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\right)(\boldsymbol{\nabla}\times\mathbf{a})
\end{aligned}
\tag{M.196}
$$

However, for the case at hand with sufficiently rapid fall-off of the integrands in the surface integrals, we can, according to formula (M.73) on page 246 with $\mathcal{A} = \boldsymbol{\nabla}'\cdot\mathbf{v}(\mathbf{x}')$, rewrite the expression within the large parentheses in the last term as

$$
\boldsymbol{\nabla}\cdot\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int_{V'} \mathrm{d}^3x'\, \frac{\overbrace{\boldsymbol{\nabla}'\cdot[\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')]}^{0}}{|\mathbf{x}-\mathbf{x}'|} = 0
\tag{M.197}
$$

This means that the second term itself vanishes and we are left with the final result

$$
\begin{aligned}
\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{rotat}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) &= \int_V \mathrm{d}^3x \int_{V'} \mathrm{d}^3x'\, \frac{[\boldsymbol{\nabla}\times\mathbf{u}(\mathbf{x})]\times[\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')]}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \\
&= \int_V \mathrm{d}^3x\, [\boldsymbol{\nabla}\times\mathbf{u}(\mathbf{x})]\times\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}'\times\mathbf{v}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}
\end{aligned}
\tag{M.198}
$$

This is identity (F.130e) on page 223.

The vector product of an irrotational and a rotational component is

$$
\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = -[\boldsymbol{\nabla}\alpha(\mathbf{x})]\times[\boldsymbol{\nabla}\times\mathbf{b}(\mathbf{x})]
\tag{M.199}
$$

Using identity (F.79), identity (F.89) with $\mathbf{a} = \boldsymbol{\nabla}\alpha$, and identity (F.100) on page 220, together with identity (F.74) on page 219, this can be written

$$
\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \mathbf{b}\cdot\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}\alpha - \mathbf{b}\nabla^2\alpha - \boldsymbol{\nabla}[(\boldsymbol{\nabla}\alpha)\cdot\mathbf{b}] + \boldsymbol{\nabla}\cdot[(\boldsymbol{\nabla}\alpha)\otimes\mathbf{b}]
\tag{M.200}
$$

When we insert the expression (M.83b) on page 248 and integrate over $V$, employing identity (F.121a) and identity (F.121b) on page 221 and neglecting the resulting surface integrals, we get

$$
\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = -\mathbf{b}\cdot\boldsymbol{\nabla}\otimes\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) - \mathbf{b}\nabla^2\int_{V'} \mathrm{d}^3x'\, \frac{\boldsymbol{\nabla}'\cdot\mathbf{u}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}
\tag{M.201}
$$

With the help of the generic formula (F.126) on page 222 we can simplify the last integral and obtain

$$
\int_V \mathrm{d}^3x\, \mathbf{u}^{\mathrm{irrot}}(\mathbf{x})\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \int_V \mathrm{d}^3x\, \mathbf{b}(\mathbf{x})[\boldsymbol{\nabla}\cdot\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})] - \int_V \mathrm{d}^3x\, \mathbf{b}(\mathbf{x})\cdot\boldsymbol{\nabla}\otimes\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})
\tag{M.202}
$$

After integrating the last integral on the RHS by parts, making use of formula (F.86) on page 219 and discarding the surface integral which results after applying the divergence theorem, we can write

$$\int_V \mathrm{d}^3x\,\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) = \int_V \mathrm{d}^3x\,\mathbf{b}(\mathbf{x})[\boldsymbol{\nabla}\cdot\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})] + \int_V \mathrm{d}^3x\,[\boldsymbol{\nabla}\cdot\mathbf{b}(\mathbf{x})]\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \tag{M.203}$$

To obtain the final expression, we use formula (M.167b) on page 261 and recall that $\boldsymbol{\nabla}\cdot\mathbf{b} = 0$. The result is

$$\begin{aligned}\int_V \mathrm{d}^3x\,\mathbf{u}^{\mathrm{irrot}}(\mathbf{x}) \times \mathbf{v}^{\mathrm{rotat}}(\mathbf{x}) &= \int_V \mathrm{d}^3x \int_{V'} \mathrm{d}^3x'\,\frac{[\boldsymbol{\nabla}\cdot\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})][\boldsymbol{\nabla}'\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}')]}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\\ &= \int_V \mathrm{d}^3x\,[\boldsymbol{\nabla}\cdot\mathbf{u}^{\mathrm{irrot}}(\mathbf{x})]\int_{V'} \mathrm{d}^3x'\,\frac{\boldsymbol{\nabla}'\times\mathbf{v}^{\mathrm{rotat}}(\mathbf{x}')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{M.204}$$

This is identity (F.130f) on page 223.

End of example M.16◁

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