Ch 4: Models for stationary time series

- So far, we learned about stochastic processes and their mean/autocovariance/autocorrelation functions, and discussed stationary MA and AR processes.
- Now, we will get into a bit more detail on the AR and MA processes, to then combine them into ARMA processes and discuss the properties of such processes.
- Big picture:
  - Recap of our goal in time series analysis: Given an observed time series, that is typically not deterministic but contains a random component, find out what time series process(es) could have resulted in the observed time series; use a stochastic process as a model for the time series of interest, to capture its autocovariance structure.
  - If the random component is stationary, then we can use autoregressive moving-average (ARMA) processes as discussed in this chapter to represent that component. This is a powerful approach because for a large class of autocovariance functions (ACVFs), it is possible to find an ARMA process with an ACVF that closely approximates the ACVF of the stochastic process being modeled.

Ch 4: Models for stationary time series

- In this chapter, we assume that the process of interest has mean zero (no deterministic trend) and is stationary. We will discuss how to model deterministic trends and non-stationary processes in later chapters.
- Topics in Ch 4-Part II:
  - Stationarity of an AR(p) process
  - General linear process
  - Causality and invertibility
  - ARMA process
- Relevant material in Ch 4: all except for calculations related to stationarity of AR(2) process (mid p.71 - 75)

Stationarity of AR(p) process

- Let's find out when an AR(p) process is stationary.
- We will define the
  - stationarity condition for an AR(p) process, based on
  - AR(p) characteristic function and equation

  and examine what that condition implies for the coefficient of an AR(1) process.

Some definitions

- An autoregressive process of order p, denoted by AR(p), is defined as:
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t,$$
where the ϕ's are unknown parameters and $e_t \sim WN(0, \sigma_e^2)$.
- The corresponding AR characteristic polynomial (function) is defined as
$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p,$$
and the AR characteristic equation is defined as
$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0.$$
    - Q: What is the characteristic equation for an AR(1) process with $Y_t = \phi Y_{t-1} + e_t$?
- z is called a root for the AR characteristic equation if $\phi(z) = 0$.
    - Q: What is the root for the characteristic equation for an AR(1) process?

When is an AR(p) process stationary?

- The AR(p) process, with e_t be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$, is stationary if and only if the roots of characteristic equation exceed 1 in absolute value (modulus).
- In other words, the AR(p) process is stationary if the roots z_i (for $i = 1$ up to p) of AR characteristic equation

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

  satisfy $|z_i| > 1$.
- We'll focus on understanding the condition and show that if the stationarity condition is satisfied for an AR(1) process, the process is indeed stationary.
- The proof is outside the class material (see reference material).
- Assume that e_t does not depend on $Y_{t-1}, Y_{t-2}, \ldots$ unless otherwise stated.

Stationarity for the AR(1) process

- AR(1): $Y_t = \phi Y_{t-1} + e_t$.
- AR(1) characteristic equation is defined as

$$\phi(x) = 1 - \phi x = 0.$$

- The root is given by $z_1 = 1/\phi$.
- Q: When is an AR(1) process stationary?
- We can show that the AR(1) process with $|\phi| < 1$ is indeed stationary, by writing this process as a general linear process.

General linear processes

- Definition: A general linear process $\{Y_t\}$ is a process that can be represented as a weighted linear combination of present and past white noise terms:

$$Y_t = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \ldots,$$

  with $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ and $\psi_0 = 1$.
- We can write any AR(p) process, that satisfies the stationary condition, as a general linear process.
- Example: AR(1) with $|\phi| < 1$:
  - The general linear process $Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$ with $\psi_j = \phi^j$ satisfies the AR(1) equation $Y_t = \phi Y_{t-1} + e_t$ (show this!).
    - Note that $\sum_{j=1}^{\infty} \psi_j^2 = \sum_{j=1}^{\infty} (\phi^j)^2 < \infty$ (remember that $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$ if $|a| < 1$).
  - Given the general linear process representation for the AR(1) with $|\phi| < 1$, we can derive the mean and autocovariance function and find that they do not depend on t (we get the same expressions as discussed in Ch4-partI), thus that the process is stationary (see book, p.55-56).

Summary: Stationarity of an AR(p) process

- An autoregressive process of order p, denoted by AR(p), with

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t,$$

  and e_t be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$, is stationary if and only if the roots z_i (for $i = 1$ up to p) of AR characteristic equation

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

  satisfy $|z_i| > 1$.
- A stationary AR(p) process can be written as a general linear process.

Causality

- For the AR(1) process with $Y_t = \phi Y_{t-1} + e_t$, what happens when $|\phi| = 1$ or $|\phi| > 1$?
- When $\phi = \pm 1$, ...
- When $|\phi| > 1$, e.g. $\phi = 3$ (exercise 4.16):
  - An equivalent process is given by $Y_t = -\sum_{j=1}^{\infty} (1/\phi)^j e_{t+j}$ (show by substitution).
  - This process is stationary: the mean and ACVF do not depend on time t.
- However, we do not consider AR(1) processes with $|\phi| > 1$ because Y_t depends on future e_t's or equivalently, e_t depends on past Y_t's.
- Instead, we focus on *causal* processes, where e_t is independent of $Y_{t-1}, Y_{t-2}, \ldots$.
- General summary for AR(p) processes with root z_i:
  - if $|z_i| = 1$, the AR(p) process will NOT be stationary,
  - if $|z_i| > 1$, the AR(p) process will be stationary and causal,
  - if $|z_i| < 1$, the AR(p) process will be stationary but NOT causal.

  We focus on stationary causal processes.

A similar yet different issue for MA processes

- The autocorrelation function for the MA(1) process $Y_t = e_t - \theta e_{t-1}$ is given by:

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-\theta}{1+\theta^2} & \text{for } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

- What is ρ_1 for $\theta = 3$; what is ρ_1 for $\theta = 1/3$?
- This could lead to problems when fitting an MA(1) process to data: if the true but unknown process is MA(1) with $\theta = 1/3$, we can end up with an estimate for θ around $1/3$ OR around 3.
  $\Rightarrow$ Without restrictions on the MA-parameters, we do not have a 1-to-1 correspondence between an MA-model specification and an MA-autocorrelation function.
- To avoid problems, we will impose some restrictions on the MA-parameters and focus on *invertible* processes only.

Invertibility

- An MA(q) process

  $$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q},$$

  is called invertible if we can write as an "AR(∞) process":

  $$Y_t = e_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \ldots$$

- An MA(q) process is invertible if and only if the roots of the MA characteristic equation exceed one in modulus.
  - (unsurprisingly) the MA characteristic equation is given by:

    $$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q.$$

- When is the MA(1) process invertible? (relate back to example on previous slide!)
- We will only consider the "more sensible class" of invertible MA(q) models.

ARMA(p,q) processes

- Let's combine the MA(q) and AR(p) process!
- A mixed autoregressive moving average process $\{Y_t\}$ of orders p and q, denoted by ARMA(p, q), is defined as:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \end{aligned}$$

  with $e_t \sim WN(0, \sigma_e^2)$.
- Example ARMA(1,1): $Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$.
- What are the mean and autocovariance function of the ARMA(1,1) process if it is stationary, with $E(Y_t) = 0$ and e_t independent of $Y_{t-1}, Y_{t-2}, \ldots$?

ARMA(1,1) autocovariance function

- As before, multiply by Y_{t-k} and take expectations at both sides to find that

$$\left.\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k &= \phi\gamma_{k-1} \quad \text{for } k \geq 2 \end{aligned}\right\}$$

  - How did γ_k depend on γ_{k-1} for the AR(1) process for $k = 1, 2, \ldots$?

- To find the ACFV and ACF, use equations for γ_0 and γ_1 to obtain γ_0, and then γ_k and ρ_k:

$$\boxed{\gamma_0 = \frac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_e^2}$$

$$\boxed{\rho_k = \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2}\phi^{k-1} \quad \text{for } k \geq 1}$$

Finding the autocovariance function for a stationary ARMA(p, q) process

The approach is not new (multiply by Y_t and take expectations) but solving this can become a bit of work and needs the general linear process representation.... Use software!

Thus the autocovariance must satisfy

$$\begin{aligned}\gamma_k &= E(Y_{t+k}Y_t) = E\left[\left(\sum_{j=1}^{p}\phi_j Y_{t+k-j} - \sum_{j=0}^{q}\theta_j e_{t+k-j}\right)Y_t\right] \\ &= \sum_{j=1}^{p}\phi_j\gamma_{k-j} - \sigma_e^2\sum_{j=k}^{q}\theta_j\psi_{j-k}\end{aligned} \qquad (4.C.3)$$

where $\theta_0 = -1$ and the last sum is absent if $k > q$. Setting $k = 0, 1, \ldots, p$ and using $\gamma_{-k} = \gamma_k$ leads to $p + 1$ linear equations in $\gamma_0, \gamma_1, \ldots, \gamma_p$.

$$\left.\begin{aligned}\gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p - \sigma_e^2(\theta_0 + \theta_1\psi_1 + \cdots + \theta_q\psi_q) \\ \gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 + \cdots + \phi_p\gamma_{p-1} - \sigma_e^2(\theta_1 + \theta_2\psi_1 + \cdots + \theta_q\psi_{q-1}) \\ &\vdots \\ \gamma_p &= \phi_1\gamma_{p-1} + \phi_2\gamma_{p-2} + \cdots + \phi_p\gamma_0 - \sigma_e^2(\theta_p + \theta_{p+1}\psi_1 + \cdots + \theta_q\psi_{q-p})\end{aligned}\right\} \qquad (4.C.4)$$

where $\theta_j = 0$ if $j > q$.

Stationarity and the general linear process

- The ARMA(p, q) process, with e_t be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$, is (causal and) stationary if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus).
  - same as for AR(p)!
- If the ARMA process is stationary, we can write it as a linear process $Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$, and find the ψ's by equating the coefficients of the e_{t-j}'s, e.g. $\psi_0 = 1$, $\psi_1 = -\theta_1 + \phi_1$, etc, because

$$
\begin{aligned}
Y_t &= \sum_{j=0}^{\infty} \psi_j e_{t-j} = \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \ldots, \\
&= e_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} \\
&\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \\
&= e_t + \phi_1(e_{t-1} + \phi_1 Y_{t-2} + \phi_2 Y_{t-3} + \ldots + \phi_p Y_{t-p-1} \\
&\quad -\theta_1 e_{t-2} - \theta_2 e_{t-3} - \ldots - \theta_q e_{t-q-1}) + \phi_2 Y_{t-2} + \ldots \\
&\quad +\phi_p Y_{t-p} - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \\
&=
\end{aligned}
$$

Invertibility

- The ARMA(p, q) process is called invertible if we can write as an "AR(∞) process":

$$Y_t = e_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \dots$$

- An ARMA(p, q) process is invertible if and only if the roots of the MA characteristic equation exceed one in modulus, with the MA characteristic equation is given by:

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \dots - \theta_q x^q.$$

- Same as for MA(q)!
- We will only consider invertible ARMA(p, q) models.

Summary

- In Ch 4, we learned about a family of time series models, which are ARMA(p, q) processes.
- From now on, when referring to an ARMA(p,q) process, we will assume that it is causal, stationary and invertible unless otherwise stated.
- Next: non-stationary time series models!