Ch 4, part I: Moving average and autoregressive processes

- We just finished Ch.2 and learned about stochastic processes, their mean/autocovariance/autocorrelation functions and stationarity.
- We now move onto discussing moving average and autoregressive processes, which are very important building blocks for time series modeling.
- Both processes will be defined, and we will discuss examples and properties (in particular, their autocorrelation functions).
- Note: these slides contain material from Ch 4.2 and 4.3. We skipped Ch.3 and Ch 4.1 but will return to that material later.

Moving average processes

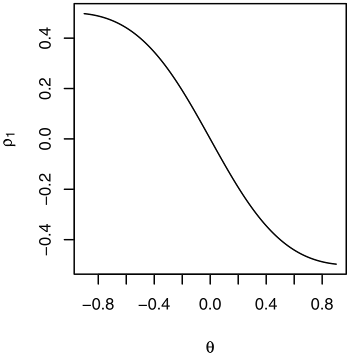
- A moving average process of order q, denoted by MA(q), is defined as:
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q},$$
where the θ's are unknown parameters (weights) and $e_t \sim WN(0, \sigma_e^2)$.
- Example MA(1): $Y_t = e_t - \theta e_{t-1}$, where $\theta = \theta_1$.
- What are the mean and autocovariance function of the MA(1) process?
- $\mu_t = E(Y_t) = E(e_t - \theta e_{t-1}) = 0$ for all t.

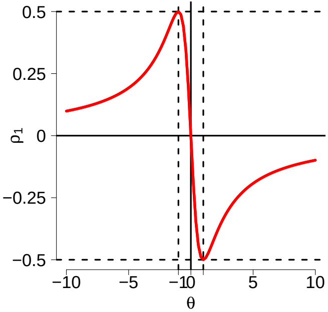
Autocorrelation function for MA(1)

- The autocorrelation function for the MA(1) $Y_t = e_t - \theta e_{t-1}$ is given by (*verify!*):

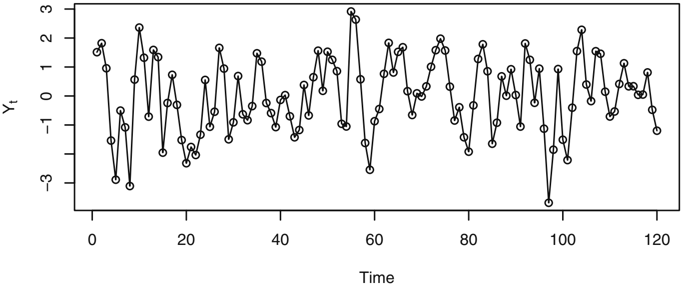
$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-\theta}{1+\theta^2} & \text{for } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

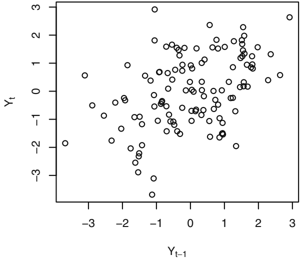
where ρ_k is short-hand notation for correlation at lag k (e.g. $\rho_{t,t-k}$).

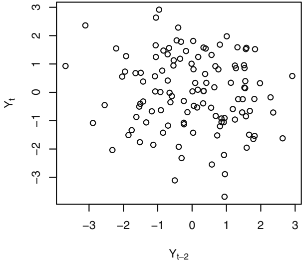




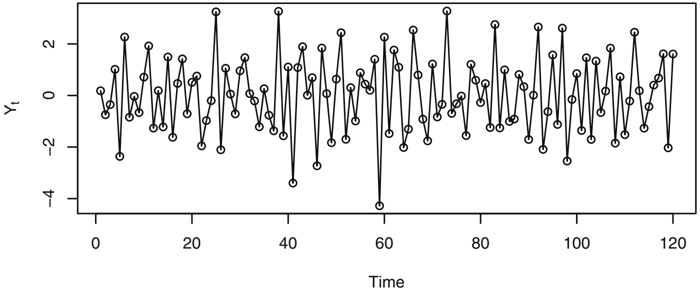
Simulations of the MA(1) with $\theta = -0.9$

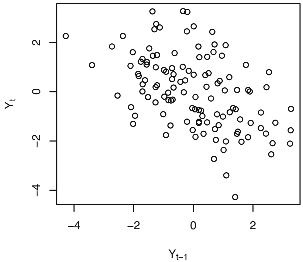


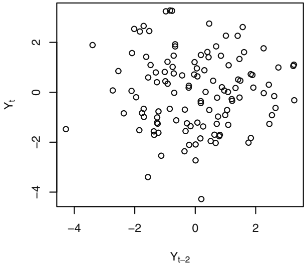




Simulations of the MA(1) with $\theta = 0.9$







Autocorrelation function for the MA(q) process

- A moving average process of order q, denoted by MA(q), is defined as:
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q},$$
where the θ's are unknown parameters (weights) and $e_t \sim WN(0, \sigma_e^2)$.
- MA(q) processes are stationary, with $E(Y_t) = 0$ and autocorrelation function:

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

- Autocorrelation is always zero after lag q:
Moving average processes can be used to model stationary time series with autocorrelation functions that cut off at a certain lag.
- What if the autocorrelation function extends beyond a finite lag?
Use autoregressive processes!

Autoregressive processes

- An autoregressive process of order p, denoted by AR(p), is defined as:

  $$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_q Y_{t-p} + e_t,$$

  where the ϕ's are unknown parameters and $e_t \sim WN(0, \sigma_e^2)$.
- Example AR(1): $Y_t = \phi Y_{t-1} + e_t$, where $\phi = \phi_1$.
- What are the mean and autocovariance function of the AR(1) process?
  - We will assume that
    - $-1 < \phi < 1$ with $\phi \neq 0$,
    - e_t is independent of Y_{t-k} for all $k \geq 0$.

    The resulting AR(1) process is stationary with $E(Y_t) = 0$.
  - These assumptions and stationarity for the AR(1) process will be discussed in more detail later.

Autocovariance function for the AR(1) process

- Assumptions: $-1 < \phi < 1$ with $\phi \neq 0$, e_t is independent of Y_{t-k} for all $k \geq 0$, process is stationary.
- To obtain variance of AR(1):

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t, \\ Var(Y_t) &= Var(\phi Y_{t-1} + e_t), \\ \gamma_0 &= \phi^2 \gamma_0 + \sigma_e^2, \end{aligned}$$

resulting in $\gamma_0 = \sigma_e^2/(1-\phi^2)$.

Autocovariance function for the AR(1) process (ctd)

- Assumptions: $-1 < \phi < 1$ with $\phi \neq 0$, e_t is independent of Y_{t-k} for all $k \geq 0$, process is stationary with $E(Y_t) = 0$.
- Multiply by Y_{t-k} to obtain the autocovariance of AR(1) at lag k:

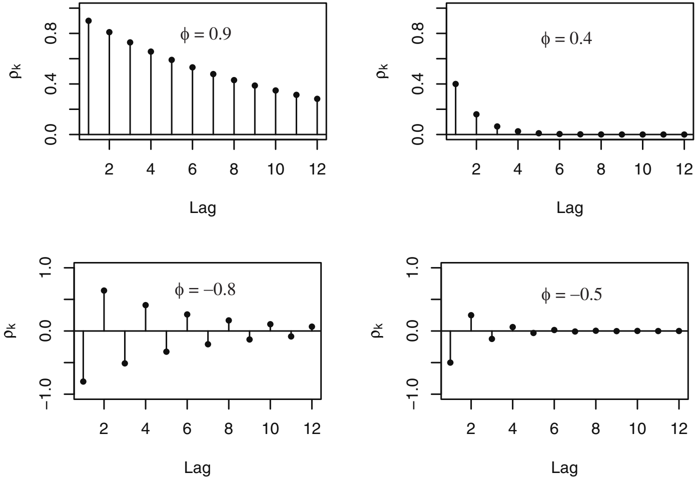
$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t, \\
Y_{t-k} \cdot Y_t &= \phi Y_{t-k} Y_{t-1} + Y_{t-k} e_t, \\
E(Y_{t-k} Y_t) &= E(\phi Y_{t-k} Y_{t-1}) + E(Y_{t-k} e_t), \\
\gamma_k &= \phi \gamma_{k-1},
\end{aligned}
$$

  since $Cov(Y_{t-k}, Y_t) = E(Y_{t-k} Y_t) - E(Y_{t-k}) E(Y_t) = E(Y_{t-k} Y_t)$ and $E(Y_{t-k} e_t) = E(Y_{t-k}) E(e_t) = 0$.
- The autocorrelation function is

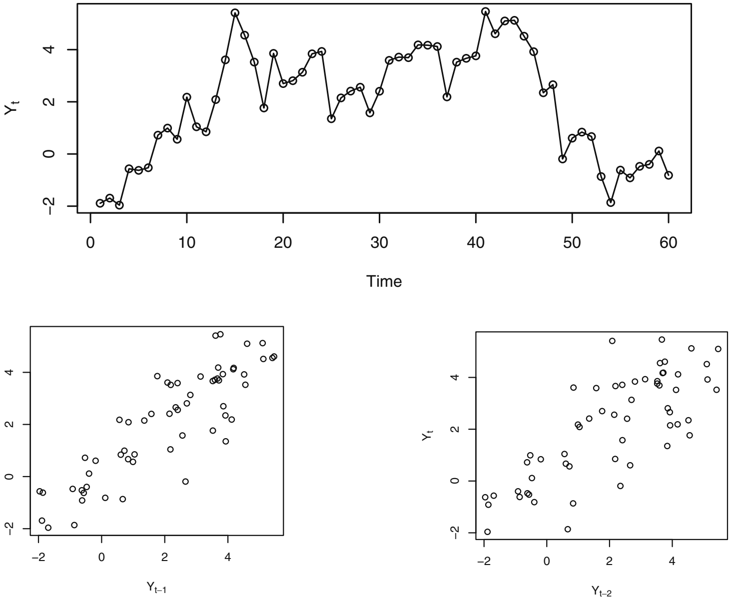
$$
\begin{aligned}
\gamma_k / \gamma_0 &= \phi \gamma_{k-1} / \gamma_0, \\
\rho_k &= \phi \rho_{k-1},
\end{aligned}
$$

  and because $\rho_0 = 1$, we find that $\rho_k = \phi^k$.

Examples of the AR(1) autocorrelation functions $\rho_k = \phi^k$



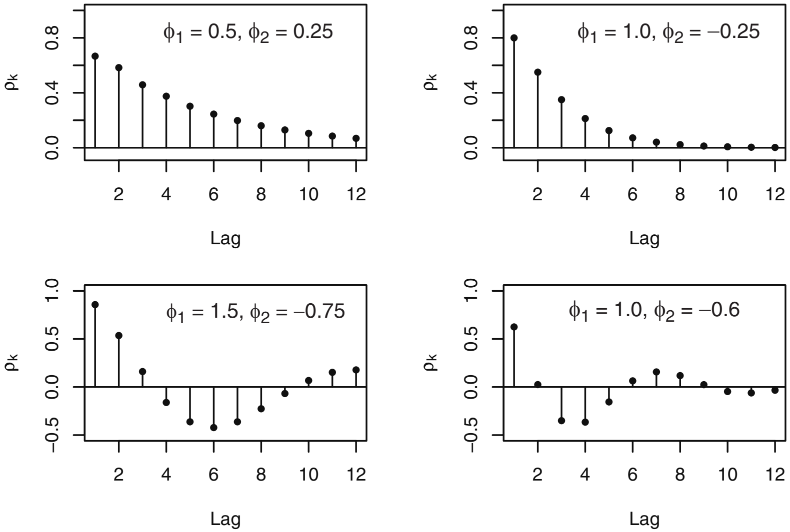
Simulations of the AR(1) with $\phi = 0.9$



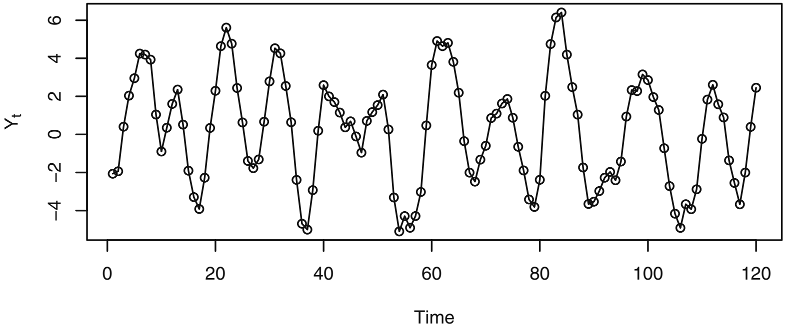
Autocorrelation function for the AR(p) process

- An autoregressive process of order p, denoted by AR(p), is defined as:
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_q Y_{t-p} + e_t,$$
where the ϕ's are unknown parameters and $e_t \sim WN(0, \sigma_e^2)$.
- As for the AR(1), if we assume stationarity and zero mean, we can multiply by Y_{t-k}, take expectations and divide by γ_0 to obtain
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \phi_3 \rho_{k-3} + \ldots + \phi_p \rho_{k-p}. \qquad (1)$$
- We can use Eq.(1) to obtain $\rho_1, \ldots, \rho_p$ by solving the following set of (Yule-Walker) equations (based on plugging in $k = 1, \ldots, p$, and noting that $\rho_k = \rho_{-k}$):
$$\left.\begin{aligned} \rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1} \\ \rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2} \\ &\vdots \\ \rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p \end{aligned}\right\}$$
- This can results in lots of interesting autocorrelation functions...

Examples AR(2) autocorrelation functions



Example of AR(2) simulation with $\phi_1 = 1.5$ and $\phi_2 = -0.75$



Summary

- We discussed moving average and autoregressive processes, which are very important building blocks for time series modeling.
- We found that the autocorrelation function for the MA(q) process cuts off at lag q while the autocorrelation function of the AR(p) process can take on various forms.
- Next time we will continue with Ch.4 to discuss both processes in more detail and combine them into an ARMA(p,q) process!