Chapter 5

Testing General Linear Hypothesis (II)

Review

- We have discussed the following in Part (1)
- Full model and reduced model
- Linear hypothesis : $C\underline{\beta} = \underline{0}$
- Sum of Squares due to hypothesis $C\underline{\beta} = \underline{0}$ is given by $SSE_H - SSE$
- Test statistics: $F = \dfrac{(SSE_H - SSE)/q}{SSE/[n-(p+1)]}$
- Reject H_0 at sig. level α if $F_{\text{obs}} > F_\alpha(q, n-(p+1))$.
- 2 examples

Example 4

- Let $u_1, u_2, \cdots, u_{n_1}$ be independent observations from $N(\mu_1, \sigma^2)$ and

- $v_1, v_2, \cdots, v_{n_2}$ be independent observations from $N(\mu_2, \sigma^2)$.

- Derive a test for testing

  H_0: $\mu_1 = \mu_2$ against H_1: $\mu_1 \neq \mu_2$.

Example 4 (Continued)

We can write

$$u_i = \mu_1 + \epsilon_i, \qquad i = 1, \cdots, n_1,$$

and

$$v_j = \mu_2 + \epsilon_j, \qquad j = 1, \cdots, n_2$$

where $\epsilon_k \sim N(0, \sigma^2)$, $k = 1, 2, \cdots, n_1 + n_2$.

Example 4 (Continued)

- The above model can then be expressed in matrix form as follows

$$E(\underline{y}) = X\underline{\beta},$$

where $\underline{y} = \begin{pmatrix} u_1 \\ \vdots \\ u_{n_1} \\ v_1 \\ \vdots \\ v_{n_2} \end{pmatrix}, X = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$ and $\underline{\beta} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

Example 4 (Continued)

- Hence

$$X'X = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} \qquad X'\underline{y} = \begin{pmatrix} \sum_{i=1}^{n_1} u_i \\ \sum_{j=1}^{n_2} v_j \end{pmatrix}$$

- Therefore

$$\underline{\hat{\beta}} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} = (X'X)^{-1}X'\underline{y}$$

$$= \begin{pmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n_2} \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n_1} u_i \\ \sum_{j=1}^{n_2} v_j \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

Example 4 (Continued)

$$SSE = \underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y}$$

$$= \left(\sum_{i=1}^{n_1} u_i^2 + \sum_{j=1}^{n_2} v_j^2\right) - \left(n_1\bar{u}^2 + n_2\bar{v}^2\right)$$

$$= \left(\sum_{i=1}^{n_1} u_i^2 - n_1\bar{u}^2\right) + \left(\sum_{j=1}^{n_2} v_j^2 - n_2\bar{v}^2\right)$$

$$= \sum_{i=1}^{n_1}(u_i - \bar{u})^2 + \sum_{j=1}^{n_2}(v_i - \bar{v})^2$$

with $n_1 + n_2 - 2$ d.f.

Note:

$SSE/(n_1 + n_2 - 2)$ in this case is known as the pooled variance in the 2-sample t-test.

Example 4 (Continued)

- Under the hypothesis H_0: $\mu_1 = \mu_2 \ (= \mu)$, the model is reduced to

$$E(\underline{y}) = \underline{1}_n \mu, \qquad \text{where } n = n_1 + n_2.$$

- LSE of μ is given by

$$\hat{\mu} = \underbrace{\boxed{\left(\underline{1}_n{}'\underline{1}_n\right)^{-1}}}_{1/n} \underbrace{\boxed{\underline{1}_n{}'\underline{y}}}_{\Sigma y_i} = \bar{y} = \frac{\sum_{i=1}^{n_1} u_i + \sum_{j=1}^{n_2} v_j}{n_1 + n_2}$$

and

$$SSE_H = \underline{y}'\underline{y} - \hat{\mu}\underline{1}_n{}'\underline{y} = \left(\sum_{i=1}^{n_1} u_i^2 + \sum_{j=1}^{n_2} v_j^2\right) - n\hat{\mu}^2$$

with $n_1 + n_2 - 1$ d.f.

Example 4 (Continued)

- Let

$$F = \frac{(SSE_H - SSE)/1}{SSE/(n_1 + n_2 - 2)}$$

- Hence under H_0

$$F \sim F(1, n_1 + n_2 - 2)$$

- Reject H_0 at the level of significance α if

$$F_{\text{obs}} > F_\alpha(1, n_1 + n_2 - 2).$$

Example 4 (Continued)

- It can be shown as follows that

$$SSE_H - SSE = \frac{1}{1/n_1 + 1/n_2}(\bar{u} - \bar{v})^2$$

- Note: $SSE_H - SSE = n_1\bar{u}^2 + n_2\bar{v}^2 - n\hat{\mu}^2$ (See Slide 5.29 and 5.30)

$$= n_1\bar{u}^2 + n_2\bar{v}^2 - \frac{(n_1\bar{u} + n_2\bar{v})^2}{n_1 + n_2}$$

$$= \left(n_1 - \frac{n_1^2}{n_1+n_2}\right)\bar{u}^2 - \frac{2n_1n_2}{n_1+n_2}\bar{u}\bar{v} + \left(n_2 - \frac{n_2^2}{n_1+n_2}\right)\bar{v}^2$$

$$= \frac{n_1n_2}{n_1 + n_2}(\bar{u} - \bar{v})^2 = \frac{1}{1/n_1 + 1/n_2}(\bar{u} - \bar{v})^2$$

Example 4 (Continued)

- Therefore

$$F = \frac{\boxed{\dfrac{(\bar{u}-\bar{v})^2}{(1/n_1+1/n_2)}}}{s^2} \quad \boxed{(SSE_H - SSE)/1}$$

where $s^2 = \frac{SSE}{n_1+n_2-2}$

- Note that the F statistic obtained here is the square of t statistic in the two sample t-test.

Example 5

- Consider the model $E(\underline{y}) = X\underline{\beta}$,

where

$$\underline{y} = \begin{pmatrix} 1 \\ 4 \\ 8 \\ 9 \\ 3 \\ 8 \\ 9 \end{pmatrix}, \ \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \end{pmatrix} \text{ and } X = \begin{array}{c} \begin{matrix} 1 & x_1 & x_2 & x_1^2 \end{matrix} \\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} \end{array}$$

Example 5 (Continued)

Test the hypothesis that H_0: $C\underline{\beta} = \underline{0}$, where

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix}, \text{ and } \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \end{pmatrix}$$

That is, H_0: $\beta_{11} = 0$,

$\beta_1 - \beta_2 = 0$,

$\beta_1 - \beta_2 + \beta_{11} = 0$ and

$2\beta_1 - 2\beta_2 + 3\beta_{11} = 0$

Example 5 (Continued)

- When the original full model

$$E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2$$

is fitted, we have

$$X'X = \begin{pmatrix} 7 & 0 & 3 & 4 \\ 0 & 4 & 0 & 0 \\ 3 & 0 & 9 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} 1/2 & 0 & -1/6 & -1/2 \\ 0 & 1/4 & 0 & 0 \\ -1/6 & 0 & 1/6 & 1/6 \\ -1/2 & 0 & 1/6 & 3/4 \end{pmatrix}$$

Example 5 (Continued)

$$X'\underline{y} = \begin{pmatrix} 42 \\ 4 \\ 38 \\ 22 \end{pmatrix} \text{ and } \underline{\hat{\beta}} = (X'X)^{-1}X'\underline{y} = \begin{pmatrix} 11/3 \\ 1 \\ 3 \\ 11/6 \end{pmatrix}$$

$$SSE = \underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y} = 316 - 312.3333 = 3.6667$$

with 7 – 4 = 3 d.f.

Example 5 (Continued)

- The equations under H_0: $C\underline{\beta} = \underline{0}$ are

$$0 = \beta_{11} \qquad (1)$$
$$0 = \beta_1 - \beta_2 \qquad (2)$$
$$0 = \beta_1 - \beta_2 + \beta_{11} \qquad (3)$$
$$0 = 2\beta_1 - 2\beta_2 + 3\beta_{11} \qquad (4)$$

- Note the third and fourth equations are linear combinations of the first and second equations.
  - Eq (3) is equivalent to Eq (1) + Eq(2)
  - Eq (4) is equivalent to 2 Eq(2) + 3 Eq(1)
  - Hence there are only 2 linearly independent equations specified in H_0

Example 5 (Continued)

- Therefore, the null hypothesis can be expressed as H_0: $\beta_{11} = 0$ and $\beta_2 = \beta_1$,

- Substituting these conditions in the model gives the following reduced model

$$\begin{aligned} E(y) &= \beta_0 + \beta_1 x_1 + \beta_1 x_2 + 0x_1^2 \\ &= \beta_0 + \beta_1(x_1 + x_2) \end{aligned}$$

or

$$E(y) = \alpha_0 + \alpha_1 z,$$

where $\alpha_0 = \beta_0, \alpha_1 = \beta_1$ and $z = (x_1 + x_2)$.

Example 5 (Continued)

- Reduced model: $E\left(\underline{y}\right) = Z\underline{\alpha}$

where

$$Z = \begin{pmatrix} 1 & -1-1 \\ 1 & 1-1 \\ 1 & -1+1 \\ 1 & 1+1 \\ 1 & 0+0 \\ 1 & 0+1 \\ 1 & 0+2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 0 \\ 1 & 2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \underline{\alpha} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

Example 5 (Continued)

- Therefore

$$(Z'Z)^{-1} = \begin{pmatrix} 7 & 3 \\ 3 & 13 \end{pmatrix}^{-1} = \frac{1}{82}\begin{pmatrix} 13 & -3 \\ -3 & 7 \end{pmatrix} \text{ and } Z'\underline{y} = \begin{pmatrix} 42 \\ 42 \end{pmatrix}$$

- Hence $\underline{\hat{\alpha}} = (Z'Z)^{-1}Z'\underline{y} = \frac{21}{41}\begin{pmatrix} 10 \\ 4 \end{pmatrix}$

and $SSE_H = \underline{y}'\underline{y} - \underline{\hat{\alpha}}'Z'\underline{y} = 316 - 301.1707$
$= 14.8293$ with 7 – 2 = 5 d.f.

Therefore
$$SSE_H - SSE = 14.8293 - 3.6667 = 11.1626$$
with 2 d.f.

Example 5 (Continued)

- With n = 7, no. of parameters in the full model = 4 and q = 2, we have

$$F = \frac{(SSE_H - SSE)/2}{SSE/3} = \frac{11.1626/2}{3.6667/3} = 4.567$$

- Since $F_{\text{obs}} = 4.567 < F_{0.05}(2, 3) = 9.55$, we do not reject H_0.

Example 5 (Continued)

- Hence we do not have strong evidence to support the alternate model $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2$.

- The non-rejection of the null hypothesis implies that $\beta_{11} = 0$ and $\beta_2 = \beta_1$, hence a more plausible model would be

$$E(y) = \beta_0 + \beta_1 (x_1 + x_2).$$

Example 6 (Partial F test)

- Consider the model $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

- We want to know if the contribution of x_1 given that x_2 has been included in the model

- Hence we want to test H_0: $\beta_1 = 0$ against H_1:$\beta_1 \neq 0$

- The Partial F test in Chapter 3 is used to test the above hypothesis (see Slide 3.19 and 3.23)

Example 6 (Continued)

- The F test statistic in the partial F-test is given by

$$F = \frac{SSR(x_1|x_2)}{MSE}$$

Or

$$F = \frac{\big(SSR(x_1, x_2) - SSR(x_2)\big)/1}{SSE(x_1, x_2)/(n-3)}$$

- Question: Can we derive the test statistic using the testing general linear hypothesis approach?

Example 6 (Continued)

- In the testing general linear hypothesis approach, we need to identify the full model and the reduced model under H_0

- Full model: $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

- The SSE for the full model is given by

$$SSE = SSE(x_1, x_2) = SST - SSR(x_1, x_2)$$

with n – 3 d.f.

Example 6 (Continued)

- Under $H_0: \beta_1 = 0$, the reduced model is given by $E(y) = \beta_0 + \beta_2 x_2$

- The SSE for the reduced model is given by

$$SSE_H = SSE(x_2) = SST - SSR(x_2)$$

with n – 2 d.f.

Example 6 (Continued)

- The F test using the testing for general linear hypothesis is given by

$$F = \frac{(SSE_H - SSE)/1}{SSE/(n-3)}$$

- We have $SE_H = SST - SSR(x_2)$ and $SSE = SST - SSR(x_1, x_2)$
- Hence

$$\begin{aligned} SSE_H - \ SSE \\ &= \big(SST - SSR(x_2)\big) - \big(SST - SSR(x_1, x_2)\big) \\ &= SSR(x_1, x_2) - SSR(x_2) \end{aligned}$$

Example 6 (Continued)

- Therefore the F test using the testing for general linear hypothesis approach is given by

$$F = \frac{(SSE_H - SSE)/1}{SSE/(n-3)}$$

Or

$$F = \frac{\big(SSR(x_1, x_2) - SSR(x_2)\big)/1}{SSE(x_1, x_2)/(n-3)}$$

- which is the same as the partial F-test.

Recap

- Linear hypothesis: $C\,\underline{\beta} = \underline{0}$
- Full model: $E\left(\underline{y}\right) = X\,\underline{\beta}$
- Reduced model: $E\left(\underline{y}\right) = Z\underline{\alpha}$
- Test H_0:$C\,\underline{\beta} = \underline{0}$ against H_1: $C\,\underline{\beta} \neq \underline{0}$
- Test statistic:$F = \dfrac{(SSE_H - SSE)/q}{SSE/(n-(p+1))}$
  - where q = no. of linearly independent equations in H_0
  - n = no. of observations
  - $p + 1$ = no. of parameters in the full model
- Reject H_0 at the sig. level α if $F_{\text{obs}}(q, n-(p+1))$