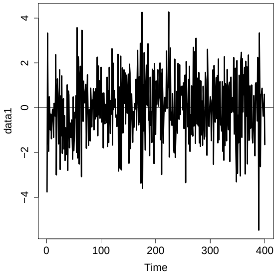
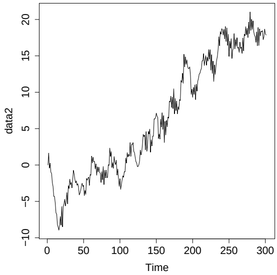
Time series analysis: review Ch 1- 6

- Motivating data series below:
  How to forecast future outcomes?





Step 1 in time series analysis

- Use the sample to investigate/explore/examine/analyze the properties of the underlying time series process.
- What is a time series process? What properties are we interested in?

Time series process

- The sequence of random variables $\{Y_t : t = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ is called a stochastic process, also referred to as a time series process.
- We are interested in:
  - mean function $\mu_t = E(Y_t)$
  - autocovariance function $\gamma_{t,s} = Cov(Y_t, Y_s)$
  - (second-order) stationarity:
    - Is μ_t constant with time?
    - Is $\gamma_{t,t-k} = \gamma_{0,k} = \gamma_k$ (free of t) for all time t and lags k?

Back to step 1

- Use the sample to investigate/explore/examine/analyze the properties of the underlying time series process.
- How to find out about the mean function, autocovariance function (or equivalently variance and autocorrelation function) and stationarity?
- Use exploratory tools: sample autocorrelation functions (ch. 6)!

(Sample) autocorrelation function

- For a stationary time series process, the autocorrelation function (ACF) is given by:

$$\rho_k \quad = \quad Corr(Y_t, Y_{t+k}) = \frac{Cov(Y_t, Y_{t+k})}{Var(Y_t)} = \frac{\gamma_k}{\gamma_0}.$$

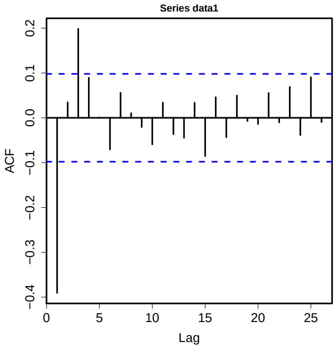
- Based on an observed time series $Y_1, \ldots, Y_n$, the sample autocorrelation function r_k (sample ACF) for time lag $k = 1, 2, \ldots$ is defined as

$$r_k = \frac{\sum_{t=k+1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2}.$$

- For a stationary time series process $\{Y_t\}$, r_k is an estimator for the true autocorrelation function $\rho_k = Corr(Y_t, Y_{t-k})$.

Example of sample ACF for data series 1

- In R: acf(data1) (using TSA library).
- Note that r_k is an estimate for ρ_k based on one observed series and thus subject to sampling error.
- Question: Are the sample autocorrelations in data series 1 higher than expected if the observations would be uncorrelated?
  - For an independent series of observations (white noise), approximately $r_k \sim N(0, 1/n)$ for large n.



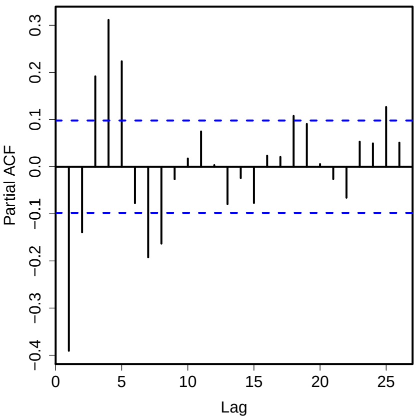
(Sample) partial autocorrelation function

- For a stationary time series process, the partial autocorrelation between Y_t and Y_{t-k}, which is denoted by ϕ_{kk}, is the autocorrelation between Y_t and Y_{t-k}
  - "that is not explained by" $Y_{t-1}, \ldots, Y_{t-k+1}$,
  - after "controlling for" $Y_{t-1}, \ldots, Y_{t-k+1}$.
- ϕ_{kk} is the coefficient of Y_{t-k} in an AR(k) model, and can be found by solving the following set of (YW) equations:

$$\left.\begin{array}{rrrcl}
\phi_{k1} + & \rho_1\phi_{k2} + & \rho_2\phi_{k3} & + \cdots + \rho_{k-1}\phi_{kk} = & \rho_1 \\
\rho_1\phi_{k1} + & \phi_{k2} + & \rho_1\phi_{k3} & + \cdots + \rho_{k-2}\phi_{kk} = & \rho_2 \\
\vdots & & & & \\
\rho_{k-1}\phi_{k1} + & \rho_{k-2}\phi_{k2} + & \rho_{k-3}\phi_{k3} & + \cdots + \qquad \phi_{kk} = & \rho_k
\end{array}\right\}$$

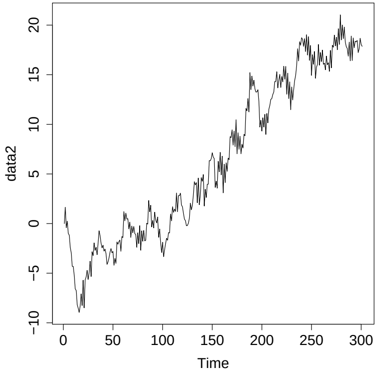
  or through using a recursive approach.
- Based on an observed time series, the sample PACF $\hat{\phi}_{kk}$ estimates ϕ_{kk}, and is obtained by using the equations above, replacing ϕ_{kj} by $\hat{\phi}_{kj}$ and ρ_k by sample ACF r_k for all k, j.

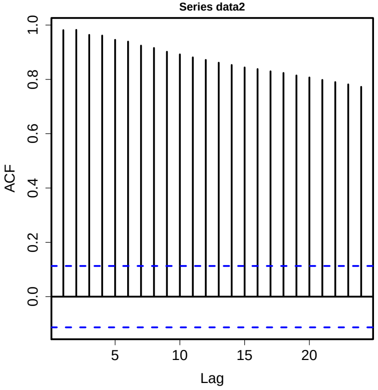
Example of sample PACF for data series 1



More examples: ACF data series 2

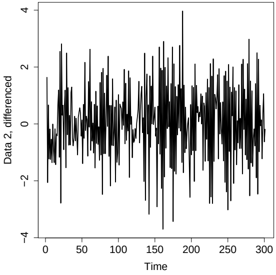
- Something odd going on here?
- The series may not be stationary. Not clear what the autocorrelation structure is.
- What to do?

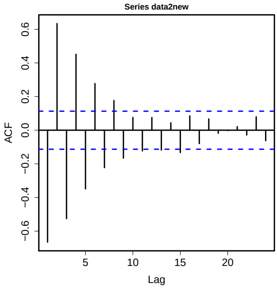


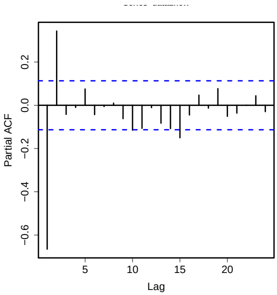


Dealing with non-stationary series

- Differencing or using other transformations may help to obtain a stationary series.
- Below the differenced time series $W_t = \nabla Y_t = Y_t - Y_{t-1}$, and sample ACF and sample PACF for data series 2.







Step 1 summarized, and step 2

- Step 1: Use the sample to investigate/explore/examine/analyze the properties of the underlying time series process.
  - Sample autocorrelation functions provide information for stationary series.
  - If the series is not stationary, differencing and/or transformations may result in a stationary series.
- Step 2: Identify candidate model(s)...
  - Which time series models (specifications of time series processes) can we choose from?
  - If there is more than one, which model(s) to choose?

ARIMA models (processes) are an important class of time series models!

ARMA processes

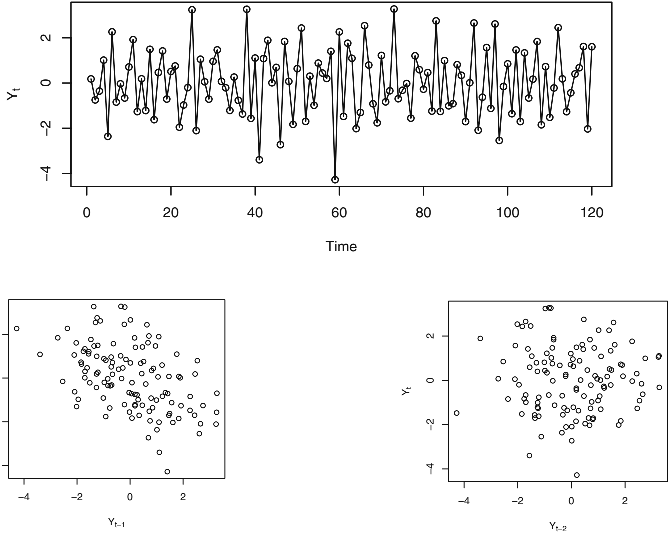
- A (zero-mean) mixed autoregressive moving average process $\{Y_t\}$ of orders p and q, denoted by ARMA(p, q), is defined as:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \end{aligned}$$

  with $e_t \sim WN(0, \sigma_e^2)$ (independent white noise terms).
- Examples:
  - MA(q): $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}$,
  - AR(p): $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t$,
  - ARMA(1,1): $Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$.

Simulations of the MA(1) with $\theta = 0.9$

- We need to set a distribution for the white noise, generally $N(0, \sigma_e^2)$.



Note on simulations

- For MA and AR processes, it's a good learning experience to generate your own simulations
  - Easy for MA processes: just simulate the white noise first.
  - For AR-processes, we can construct simulations through an iterative procedure, starting with a draw from the stationary distribution of $(Y_1, \ldots, Y_p)$.
- Simulations can also be obtained in R through the built-in function "arima.sim", e.g. use

  ```
  thetas = 0.9
  arima.sim(model=list(ma=-thetas),n=400)
  ```

  to obtain a new simulation for the MA(1) process with $\theta = 0.9$ but note that the specification of the MA-coefficients is of opposite sign.
  - IMPORTANT: In R, the θ's are always reported with opposite sign! E.g. in R, think about an MA process as

    $$Y_t = e_t + \theta_1^{(R)} e_{t-1} + \theta_2^{(R)} e_{t-1} + \ldots + \theta_1^{(R)} e_{t-q},$$

    where $\theta_k^{(R)} = -\theta_k$ in the MA notation we use in the class/the book.

Back to ARMA processes

- A (zero-mean) mixed autoregressive moving average process $\{Y_t\}$ of orders p and q, denoted by ARMA(p, q), is defined as:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \end{aligned}$$

  with $e_t \sim WN(0, \sigma_e^2)$ (independent white noise terms).
- What model(s) to use to represent the underlying process in a time series of interest?
- Examine autocorrelation functions of ARMA processes:
  - ACF γ_k, PACF ϕ_{kk}.
- But first: Restrict attention to stationary ARMA processes.

When is an ARMA process stationary?

- The ARMA(p, q) process, with e_t be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$, is stationary if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus):
  If the roots z_i (for $i = 1$ up to p) of the AR characteristic equation

  $$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

  satisfy $|z_i| > 1$.
  - Note:
    If $|z_i| < 1$, we do obtain a stationary process, but with e_t NOT independent of $Y_{t-1}, Y_{t-2} \ldots$, which is called a non-causal process.
  - We focus attention to causal and stationary processes only, with $|z_i| > 1$.
- Let's examine autocorrelation functions
  - ACF γ_k, PACF ϕ_{kk}.

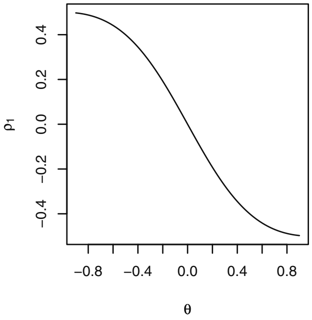
  for a stationary ARMA process.

Autocovariance function for an ARMA process

- How to obtain the (theoretical) autocovariance function $\gamma_k = Cov(Y_t, Y_{t+k})$ and $\rho_k = \gamma_k/\gamma_0$, for an ARMA process?
- For MA(q) processes, $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}$, we can obtain γ_k directly, just plug in the expression for Y_t in $\gamma_k = Cov(Y_t, Y_{t+k})$.

Example for MA(1), $Y_t = e_t - \theta e_{t-1}$:

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-\theta}{1+\theta^2} & \text{for } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$



Autocorrelation function for an ARMA process

- How to obtain the (theoretical) ACF for a zero-mean AR(p) process?
  - Multiply both sides by Y_{t-k}, take expectations and divide by γ_0:

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \phi_3\rho_{k-3} + \ldots + \phi_p\rho_{k-p}. \tag{1}$$

  - Use Eq.(1) to obtain $\rho_1, \ldots, \rho_p$ by solving the following set of (Yule-Walker) equations (based on plugging in $k = 1, \ldots, p$, and noting that $\rho_k = \rho_{-k}$):

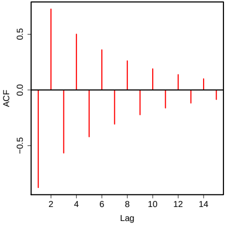
$$\left.\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p
\end{aligned}\right\}$$

- A similar approach is used for an ARMA(p,q) process but the expression becomes more complicated, so we just use software.

Example ACF

```
phis <- c(-0.5,0.3) # AR coefficients
thetas = c(0.9,-0.6,-0.5) # MA coefficients
# to get the help file ?ARMAacf
# note: thetas are with opposite sign in this function
maxlag <- 15
# remove first lag 0 from acf
acf.k <- ARMAacf(ar = phis, ma= -thetas, lag.max = maxlag)[-1
plot(acf.k ~ seq(1, maxlag), type = "h", col = 2,
  xlab = "Lag", ylab = "ACF")
abline(h=0)
```

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2}, \\
&\quad +e_t \\
&\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3}.
\end{aligned}
$$



Partial autocorrelation function for an ARMA process

- How to obtain the (theoretical) PACF ϕ_{kk} for lags $k = 0, 1, \ldots$ for an ARMA(p, q) process?
- As before, ϕ_{kk} is the solution of solving the following set of equations:

$$\left.\begin{array}{rrrrr}
\phi_{k1} + & \rho_1\phi_{k2} + & \rho_2\phi_{k3} & + \cdots + \rho_{k-1}\phi_{kk} & = \rho_1 \\
\rho_1\phi_{k1} + & \phi_{k2} + & \rho_1\phi_{k3} & + \cdots + \rho_{k-2}\phi_{kk} & = \rho_2 \\
\vdots & & & & \\
\rho_{k-1}\phi_{k1} + & \rho_{k-2}\phi_{k2} + & \rho_{k-3}\phi_{k3} & + \cdots + \quad \phi_{kk} & = \rho_k
\end{array}\right\}$$

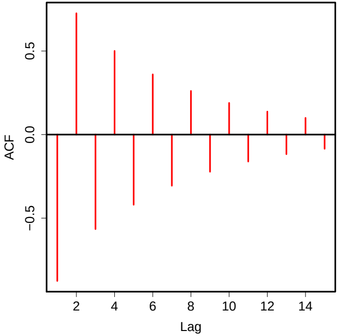
thus after obtaining the ACF (the ρ_k's), the PACF (the ϕ_{kk}'s) can be obtained.

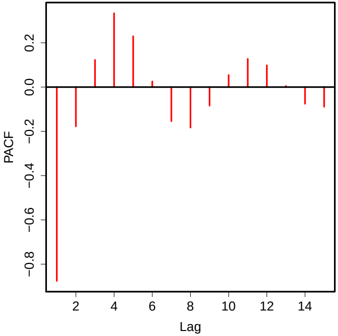
Examples ACF/PACF

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3},$$

with

```
phis <- c(-0.5,0.3) # AR coefficients
thetas = c(0.9,-0.6,-0.5) # MA coefficients
```





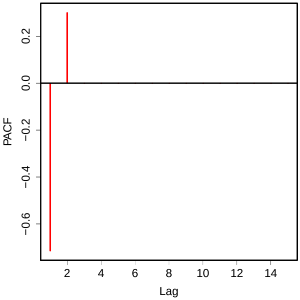
Examples ACF/PACF

- When examining the ACF for MA processes, and the PACF for AR processes, we get some interesting results...

AR(2):

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

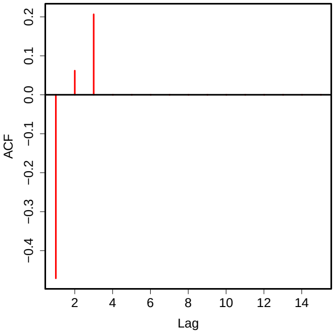
```
phis <- c(-0.5,0.3)
```



MA(3):

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3}.$$

```
thetas = c(0.9,-0.6,-0.5)
```



Summary of ACF and PACFs for ARMA processes

- Overview for theoretical (P)ACF:

Model/process	Features
MA(q)	ACF cuts off after lag q, PACF tails off
AR(p)	PACF cuts off after lag p, ACF tails off
ARMA(p, q)	ACF/PACF tail off

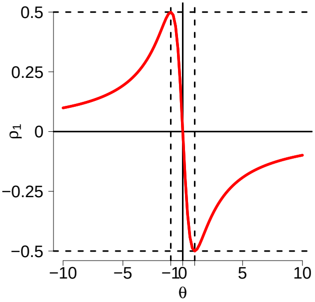
- We can use the sample ACF and sample PACF of an observed (long) time series as exploratory tools for selecting candidate ARMA models:
  - For an MA-process, $E(r_k) = \rho_k$ and the $Var(r_k) \to 0$ as the sample size increases:
    if the underlying process of an observed time series is an MA(q) process, we expect the sample ACF to cut off after lag q.
  - For an AR(p) process, the sample PACF $\hat{\phi_{kk}} \sim N(0, 1/n)$ approximately for large sample sizes:
    if the underlying process of an observed time series is an AR(p) process, we expect the sample PACF to cut off after lag p.
- The extended autocorrelation function (EACF) can be used to identify p and q in a mixed ARMA model but was left out.

One more issues with ARMA models

- We want to make sure that there is one ARMA model corresponding to a specific ACF.
  - If we do not put restrictions on the MA-parameters $\theta_1, \dots, \theta_q$, we run into problems.

Example for MA(1):
ρ_1 is also between 0 and 0.5 for $|\theta_1| > 1$.

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-\theta}{1+\theta^2} & \text{for } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$



- Restrict ARMA processes to invertible (AR(∞)) processes only:
  - An ARMA(p, q) process is invertible if and only if the roots of the MA characteristic equation exceed one in modulus, with MA characteristic equation given by $\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \dots - \theta_q x^q$.

ARMA and ARIMA processes/models

- A mixed autoregressive moving average process $\{Y_t\}$ of orders p and q, denoted by ARMA(p, q), is defined as:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}. \end{aligned}$$

- ARMA processes have diverse (partial) autocorrelation functions, and define an important class of models for time series analysis.
- We discussed how to use sample autocorrelation functions based on an observed time series to identify candidate ARMA models.
- If differencing is applied to Y_t to obtain a stationary ARMA process, the original series Y_t is called an ARIMA process.
  - A process $\{Y_t\}$ is an integrated autoregressive moving average, ARIMA(p, d, q) if the d-th difference $W_t = \nabla^d Y_t$ is a stationary ARMA(p, q) process.

Overview of time series analysis steps

- We discussed how to:
  - Analyze the properties of a time series.
  - Identify candidate (ARIMA) model(s).
- Next:
  - Fit the model through (ML) estimation.
  - Check whether the model "fits well'.
  - Forecast future outcomes.
- Outlook:
  - How to choose between models and how to check systematically for non-stationarity?
  - How to include covariates?
  - ...