Review ST3233: Applied time series analysis

- Overview of material:
  - Time series processes, stationarity, (sample) ACF and PACF.
  - (Seasonal) ARIMA processes, parameter estimation, forecasting, model building.
  - Cross-correlation/dynamic regression modeling, GARCH.
- Review is focused on material after midterm (Ch. 10-12): Seasonal models, Model building, Cross-correlation/dynamic regression modeling, GARCH.
  And (on request) a review of "mean" parameters.

Review of ARIMA "mean" parameters

Motivating example for review on mean parameters:

- Suppose Y_t follows an ARIMA$(p, 1, q)$ model, and $W_t = Y_t - Y_{t-1}$.
- If $E(W_t) = \mu \neq 0$, what does that imply for Y_t?
- To discuss:
  - Review: How to formulate/interpret/simulate/estimate/forecast ARMA(p, q) models with non-zero mean μ,
  - How to formulate/interpret/simulate/estimate/forecast ARIMA$(p, 1, q)$ models with non-zero mean μ for $(Y_t - Y_{t-1})$.

ARMA(p, q) models constant term θ_0

- A stationary ARMA(p, q) model can be written compactly as

$$\phi(B)Y_t = \theta_0 + \theta(B)e_t,$$

  with constant term θ_0 and AR and MA characteristic polynomials

$$\begin{aligned} \phi(x) &= 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p, \\ \theta(x) &= 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q. \end{aligned}$$

- Or equivalently

$$\begin{aligned} Y_t &= \theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}. \end{aligned}$$

- With $E(e_t) = 0$, it follows that

$$E(Y_t) = E(\theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p})$$

  and because Y_t is stationary, $E(Y_t) = \mu$, a constant given by:

$$\mu = \theta_0/(1 - \phi_1 - \ldots - \phi_p).$$

Rewriting ARMA(p, q) models with constant term θ_0

- Instead of

$$\begin{aligned} Y_t &= \theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t \\ &\quad - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \end{aligned}$$

  we can also write the ARMA model as

$$\begin{aligned} Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + e_t \\ &\quad - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}. \end{aligned}$$

- Two ways to verify that this expression is correct:
  - Just plug in $\theta_0 = \mu(1 - \phi_1 - \ldots - \phi_p)$ in the first expression
  - Start with $X_t \sim ARMA(p, q)$ with $E(X_t) = 0$:

$$\phi(B)X_t = \theta(B)e_t,$$

    and define $Y_t = X_t + \mu$:

$$\phi(B)(Y_t - \mu) = \theta(B)e_t.$$

- How to estimate μ for a given time series?

Estimating μ and forecasting Y_t for the ARMA(p, q) model

- If $E(Y_t) = \mu \neq 0$, μ is included in the likelihood function, and we can obtain the MLE for μ (see Ch.7).
- The MLE for μ is used for forecasting Y_{t+g} (see Ch.9).
- A note on reading R output:

```
phis <- 0.5
X.t <- arima.sim(model = list(order = c(1,0,1),
ar = phis, ma = 0.8), n = 500)
mu <- 3
Y.t <- X.t + mu
> arima(Y.t, order = c(1,0,1), method="ML")
         ar1     ma1  intercept
      0.4857  0.8027     3.0354
```

- Does “intercept” refer to μ or θ_0?

```
> theta0 <- mu*(1-sum(phis)); theta0
[1] 1.5
> mu
[1] 3
```

ARIMA(p, d, q) models constant term θ_0

- An ARIMA(p, d, q) model can be written compactly as

$$\phi(B)(1-B)^d Y_t = \theta_0 + \theta(B)e_t,$$

  with constant term θ_0 and AR and MA characteristic polynomials

$$\begin{aligned} \phi(x) &= 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p, \\ \theta(x) &= 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q. \end{aligned}$$

- Or equivalently, for $W_t = (1-B)^d Y_t$

$$\begin{aligned} W_t &= \theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_p W_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}, \end{aligned}$$

- It follows that

$$E(W_t) = E(\theta_0 + \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_p W_{t-p})$$

  and because W_t is stationary, $E(W_t) = \mu = \theta_0/(1 - \phi_1 - \ldots - \phi_p)$.
- What is $E(Y_t)$ if $E(W_t) = \mu$?

Example: $E(Y_t)$ for IMA(1,1) model with $\theta_0 \neq 0$

- For the IMA(1,1) model, if $E(W_t) = E(Y_t - Y_{t-1}) = \mu$

$$\begin{aligned}
(1-B)Y_t &= \theta_0 + e_t - \theta e_{t-1}, \\
Y_t - Y_{t-1} &= \mu + e_t - \theta e_{t-1}, \\
Y_t &= \mu + e_t - \theta e_{t-1} + Y_{t-1}.
\end{aligned}$$

- Substituting the expression for Y_{t-1}, Y_{t-2}, etc we find

$$\begin{aligned}
Y_t &= \mu + e_t - \theta e_{t-1} + Y_{t-1}, \\
&= \mu + e_t - \theta e_{t-1} + (\mu + e_{t-1} - \theta e_{t-2} + Y_{t-2}), \\
&= 2\mu + e_t + (1-\theta)e_{t-1} - \theta e_{t-2} + Y_{t-2}, \\
&\dots \\
&= t\mu + e_t + (1-\theta)e_{t-1} + \ldots + (1-\theta)e_1 - \theta e_0 + Y_0.
\end{aligned}$$

- Suppose $Y_0 = 0$, then $E(Y_t) = t \cdot \mu$.

$E(Y_t)$ in an ARIMA$(p, 1, q)$ model with $\theta_0 \neq 0$

- More generally, for an ARIMA$(p, 1, q)$ with $W_t = Y_t - Y_{t-1}$, with $Y_0 = 0$, we find that if $E(W_t) = \mu$ then:

$$\begin{aligned}
E(Y_t) &= E(W_t + Y_{t-1}), \\
&= \mu + E(Y_{t-1}), \\
&= \mu + \mu + E(Y_{t-2}), \\
\dots & \\
&= t \cdot \mu + E(Y_0), \\
&= t \cdot \mu.
\end{aligned}$$

- Even more generally (Ch. 5), $\theta_0 \neq 0$ in an ARIMA(p, d, q) model results in a mean function for Y_t which is a deterministic polynomial of degree d.

Estimating μ and forecasting Y_t for the ARIMA$(p, 1, q)$ model

- Maximum likelihood estimates for all ARIMA model parameters, including μ can be obtained as usual, based on the likelihood function for W_t.
- The MLE for μ is used for forecasting Y_{t+g} (see Ch.9).
  - E.g., for IMA(1,1) use

$$Y_t = Y_{t-1} + \mu + e_t - \theta e_{t-1}.$$

- You can obtain an estimate for μ in an ARIMA(p,1,q) model using the "arima" function in R but it is less straightforward than doing so using the "Arima" function from the "forecast" package... so let's check that one out!

Example simulation/estimation/forecast for ARIMA(1,1,1) process

- How to simulate an ARIMA(1,1,1) process with $\theta_0 \neq 0$?
- Steps:
  - Get $X_t \sim ARMA(1,1)$ with mean zero.
  - Get $W_t = X_t + \mu = Y_t - Y_{t-1}$.
  - Fix $Y_0 = 0$ and get $Y_t = W_t + Y_{t-1}$.

R-code for simu and estimation

```
library(forecast)
mu <- 0.5
phis <- -0.8
Wzeromean <- arima.sim(mode = list(ma = -0.5, ar = phis,
 order = c(1,0,1)), n=200)
W.t <- Wzeromean + mu
Y.t <- diffinv(W.t, xi = 0) # xi is starting value Y_0
mod <- Arima(Y.t, order = c(1,1,1), include.drift = TRUE,
 method="ML")
> summary(mod)
          ar1      ma1   drift
      -0.7809  -0.4860  0.4844
```

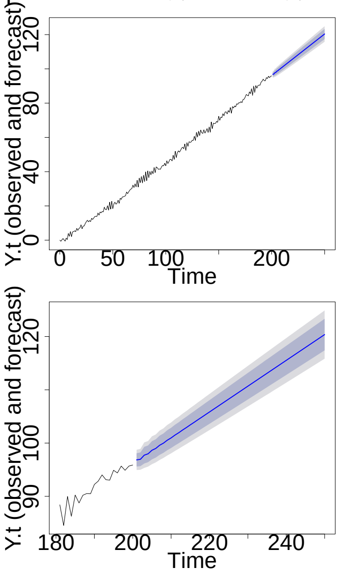
- The drift term refers to μ (NOT θ_0).

Forecasting in R using “forecast”

R-code (main points)

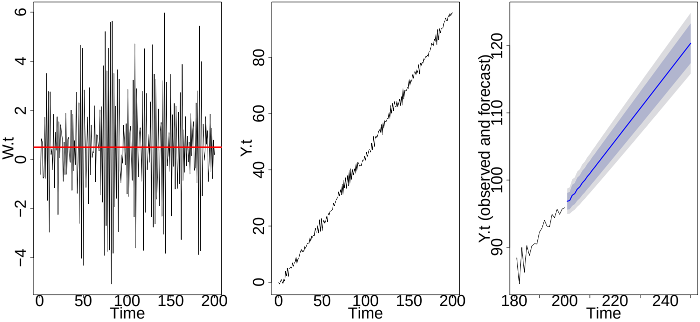
```
mod <- Arima(Y.t,
 order = c(1,1,1),
 include.drift = TRUE,
 method="ML")
fcast <- forecast(mod, h=50)
plot(fcast)
plot(fcast, include = 20)

> fcast$mean[50]-fcast$mean[49]
[1] 0.4843754
> coef(mod)['drift']
    drift
0.484378
```

Nice plots with 80% and 95% PIs!



Summary

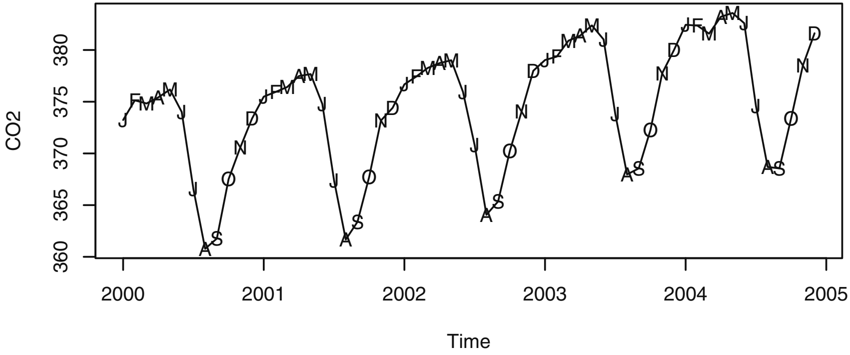
- Suppose Y_t follows an ARIMA$(p, 1, q)$ model, and $W_t = Y_t - Y_{t-1}$.
- If $E(W_t) = \mu \neq 0$, $E(Y_t)$ is a linear function of μ, which we can estimate using ML estimation, and incorporate in the forecast.
- However, do note the implication of including $\mu \neq 0$: decide whether a time trend should be included in the forecast or not!



Seasonal models

- Some time series Y_t have a season associated with them. For example
  - Monthly CO2 data: year (12 months)
  - Monthly airline passenger data: year (12 months)

Exhibit 10.2 Carbon Dioxide Levels with Monthly Symbols



Seasonal models

- Seasonal time series may show seasonal autocorrelation.
  - Example: $Y_t = 0.8Y_{t-12} + e_t$,
  - there is seasonal autocorrelation in Y_t because if Y_t was "relatively low" 12 months ago, it's expected to be low this month too.
- Multiplicative seasonal ARIMA models allow for modeling such seasonal autocorrelation:
  - seasonal autocorrelations are modeled using autoregressive and moving average terms with seasonal lags,
  - non-seasonal autoregressive and moving average terms can be added to capture non-seasonal autocorrelations.
    - This is done in a parsimonious way by multiplying characteristics polynomials, explained on the next slide.
- Seasonal differencing can be applied if the time series is not seasonally stationary (has a seasonal deterministic trend), to obtain a stationary time series.

Differenced multiplicative seasonal models

Y_t is a multiplicative ARIMA(p, d, q)x$(P, D, Q)_s$ process with

- constant term θ_0,
- seasonal period s,
- non-seasonal orders p, q and seasonal orders P, Q,
- AR characteristic polynomial $\phi(x)\Phi(x)$ with

$$\begin{aligned} \phi(x) &= 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p, \\ \Phi(x) &= 1 - \Phi_1 x^s - \Phi_2 x^{2\cdot s} - \ldots - \Phi_P x^{P\cdot s}, \end{aligned}$$

- MA characteristic polynomial $\theta(x)\Theta(x)$ with

$$\begin{aligned} \theta(x) &= 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q, \\ \Theta(x) &= 1 - \Theta_1 x^s - \Theta_2 x^{2\cdot s} - \ldots - \Theta_Q x^{Q\cdot s}, \end{aligned}$$

if Y_t is defined as follows:

$$\phi(B)\Phi(B)(1 - B^s)^D(1 - B)^d Y_t = \theta_0 + \theta(B)\Theta(B)e_t.$$

or equivalently $\phi(B)\Phi(B)W_t = \theta_0 + \theta(B)\Theta(B)e_t$, for $W_t = \nabla_s^D \nabla^d Y_t = (1 - B^s)^D(1 - B)^d Y_t$.

Example: ARMA(0,1)x$(0,1)_{12}$

- The multiplicative Seasonal ARMA(0,1)x$(0,1)_{12}$ model is given by:

$$\begin{aligned} Y_t &= \theta(B) \cdot \Theta(B) e_t, \\ &= (1 - \theta B)(1 - \Theta B^{12}) e_t, \\ &= (1 - \theta B - \Theta B^{12} + \theta\Theta B^{13}) e_t, \\ &= e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}. \end{aligned}$$

- This is also an ARMA(0,13) model with $\theta_1 = \theta$, $\theta_{12} = \Theta$ and $\theta_{13} = -\theta\Theta$, and all other θ_j's are fixed at zero.
- (P)ACFs, parameter estimation and forecasting: follow approach taken for non-seasonal ARIMA processes.
  - E.g., for process above, derive autocorrelation function as usual and find that $\rho_k = 0$ for $k \neq 0, 1, 11, 12, 13$.
- How to do model selection?

Time series model building

What are the tasks involved in selecting candidate ARIMA(p, d, q)x$(P, D, Q)_s$ models for a time series?

- Step 1: Select the order d of non-seasonal and D of seasonal differencing.
  - Informally: does ACF decay on seasonal and non-seasonal lags?
  - Based on tests: Unit root tests (not tested on final exam).
- Step 2: Select the orders p, q, P, Q and decide whether the constant term θ_0 and/or "in-between" predictors (lagged Y's or past white noise terms) should be removed.
  - Informally: Select p, q, P, Q with ACF, PACF (and EACF).
  - Based on information criteria: AICc, BIC.
    - They combine a negative measure of model fit with a penalty for the number of parameters, thus models with lower values are preferred.
    - Models up to $+2$ of the minimum value are usually considered as candidate models.
- Steps 1 and 2 may result in a set of candidate models. If so, we need to compare models (diagnostics and differences in forecasts).

Cross-correlation and dynamic regression models

- Suppose we have a time series of interest $Y_1, Y_2, \ldots, Y_t$ (e.g. changes in the unemployment rate), here denoted by Y, and we want to explore whether/how another time series $X_1, X_2, \ldots, X_t$, here denoted by X (e.g. depression measured in employment-related social media output), relates to Y.
  - E.g. does depression increase before or after unemployment increases? Or is there no relation at all?
  - Other examples: price and sales of an item, weather/climate and dengue outbreaks, ...
- Topics discussed:
  - Summarizing the correlation between X and Y using the (sample) cross-correlation function
  - Modeling Y using X while accounting for autocorrelation in Y: dynamic regression models

Cross-correlation function $\rho_k(X, Y)$

- For jointly stationary X and Y, we define the cross-correlation function $\rho_k(X, Y) = Corr(X_{t+k}, Y_t) = Corr(X_t, Y_{t-k})$.
- The sample ccf, based on pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$, is given by:

$$r_k(X, Y) = \frac{\sum_{t=k+1}^{n}(X_t - \bar{X})(Y_{t-k} - \bar{Y})}{\sqrt{\sum_{t=1}^{n}(X_t - \bar{X})^2}\sqrt{\sum_{t=1}^{n}(Y_t - \bar{Y})^2}}.$$

- If X and Y are stationary processes with ACFs $\rho_k(X)$ and $\rho_k(Y)$, with X independent of Y, then approximately for large n,

$$r_k(X, Y) \sim N(0, V),$$

  for all k, where $V = 1/n(1 + 2\sum_{k=1}^{\infty} \rho_k(X)\rho_k(Y))$.
  - What is V when X and Y are independent white noise processes?

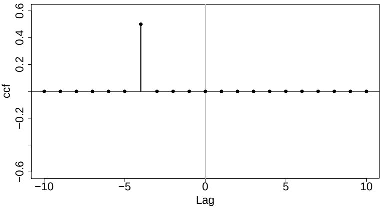
Cross-correlation function: Example

- $Y_t = \beta_0 + \beta_1 X_{t-m} + e_t$, where the X_t's are white noise with $Var(X_t) = \sigma_X^2$, independent of e_t.
  - $m > 0$ is referred to as X leading Y.

$$
\begin{aligned}
\rho_{-m}(X,Y) &= \frac{Cov(X_{t-m}, Y_t)}{\sqrt{Var(X_t)Var(Y_t)}}, \\
&= \frac{Cov(X_{t-m}, \beta_0 + \beta_1 X_{t-m} + e_t)}{\sqrt{\sigma_x^2}\sqrt{\beta_1^2\sigma_X^2 + \sigma_e^2}}, \\
&= \frac{\beta_1 \sigma_X^2}{\sigma_x\sqrt{\beta_1^2\sigma_X^2 + \sigma_e^2}}, \\
&= \frac{\beta_1 \sigma_X}{\sqrt{\beta_1^2\sigma_X^2 + \sigma_e^2}},
\end{aligned}
$$

and $\rho_k(X,Y) = 0$ for $k \neq -m$.

Example: $m = 4$



The CCF $\rho_k(X, Y)$ when X_t's are autocorrelated

- Do we still find that $\rho_k(X, Y) = 0$ for $k \neq -m$ in the model

$$Y_t = \beta_0 + \beta_1 X_{t-m} + Z_t,$$

  where X_t and Z_t are independent from each other, but where X_t is not necessarily white noise?

- Let's check:

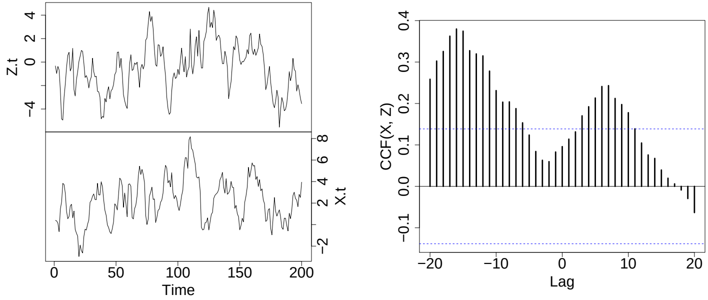
$$\rho_k(X, Y) = \frac{Cov(X_{t+k}, Y_t)}{\sqrt{Var(X_t)Var(Y_t)}} = \frac{Cov(X_{t+k}, \beta_0 + \beta_1 X_{t-m} + Z_t)}{\sigma_X \sigma_Y}$$

  $\rho_k(X, Y)$ can be non-zero for $k \neq -m$ if the X_t's are autocorrelated!

- So how to figure out which lag m is important?

Another issue when trying to figure out what lag(s) to focus on...

Simulation example for the sample CCF when $Y_t = Z_t \sim AR(1)$, $X_t \sim AR(1)$ (independent of Z_t) and $n = 200$.



- What's going on?
- Remember: If X and Y are stationary processes with ACFs $\rho_k(X)$ and $\rho_k(Y)$, with X independent of Y, then approx. for large n, $r_k(X, Y) \sim N(0, V)$ for all k, where $V = 1/n(1 + 2\sum_{k=1}^{\infty} \rho_k(X)\rho_k(Y))$ can get larger than $1/n$!

How to figure out whether some X_{t-m}'s are related to Y_t?

- Suppose we want to decide if any X_{t-m} are related to Y_t in a model in the form of

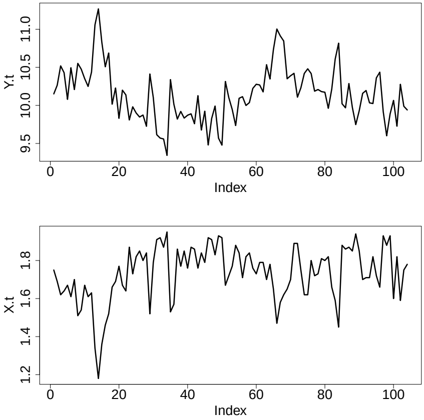
$$Y_t = \sum_{k=-\infty}^{\infty} \beta_k X_{t+k} + Z_t,$$

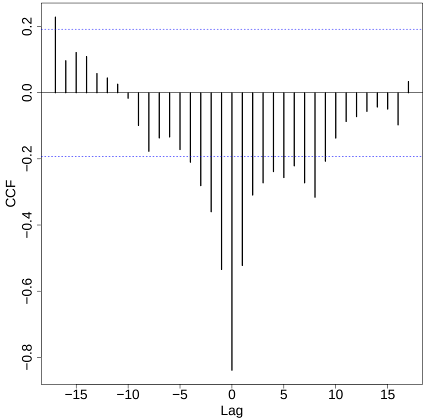
  where X_t and Z_t are time series processes, how can we find out which β_k's are non-zero?
- Approach:
  1. Find filter $\pi(B)$ for X_t such that $\tilde{X}_t = \pi(B)X_t$ is approximately white noise.
     - E.g., if X is an AR(p) process, then $e_t = X_t - \sum_{k=1}^{p} \phi_k X_{t-k}$.
     - The filter for X is given by $\pi(B) = 1 - \sum_{k=1}^{p} \phi_k B^k$.
  2. Examine $r_k(\tilde{X}, \tilde{Y})$ where $\tilde{Y}_t = \pi(B)Y_t$ (assumed to be stationary):
     - $Var(r_k(\tilde{X}, \tilde{Y})) = V = 1/n(1 + 2\sum_{k=1}^{\infty} \rho_k(\tilde{X})\rho_k(\tilde{Y})) = 1/n$, thus if there is no relation between X and Y ($\beta_k = 0 \forall k$), then we expect to find mostly 'insignificant r_k' with $|r_k(\tilde{X}, \tilde{Y})| < 1.96\sqrt{1/n}$.
     - However, if there is a $\beta_m \neq 0$, then $\rho_m(\tilde{X}, \tilde{Y}) \propto \beta_m$, thus we expect to find that r_m is significant.

Example: Q2 in tut 8

Y_t = log(weekly sales) and X_t = sales price, for Bluebird Lite potato chips.

Let's find out if the sales price is related to weekly sales!





First step? Get CCF for prewhitened series!

CCF prewhitened series

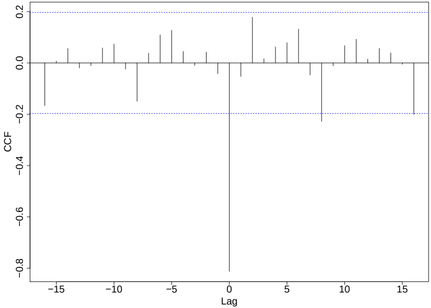
- To obtain the sample CCF for prewhitened series $\tilde{X}_t$ and $\tilde{Y}_t$, the following approach was used in R (to find an ARIMA(p,d,0) model):

  ```
  > auto.arima(X.t, ic = "aicc", approximation = FALSE,
  stepwise = FALSE, max.q = 0)
  Series: X.t
  ARIMA(4,1,0)
  m1=arima(X.t,order=c(4,1,0), include.mean = FALSE)
  prewhiten(x=(X.t),y=(as.vector(Y.t)), x.model=m1)
  ```

- Note that differencing doesn't change the relation between X_t and Y_t because

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 X_t + Z_t, \\ \nabla Y_t &= \beta_1 \nabla X_t + \nabla Z_t. \end{aligned}$$

CCF for whitened series




```
m1=arima(X.t,order=c(4,1,0), include.mean = FALSE)
prewhiten(x=(X.t),y=(as.vector(Y.t)), x.model=m1)
```

- Conclusion? Suggested model: $Y_t = \beta_0 + \beta_1 X_t + Z_t$.
- Can we fit a model $Y_t = \beta_0 + \beta_1 X_t + Z_t$ to try to predict Y_t, or obtain the relation between Y_t and X_t?

Modeling Y_t using X_t

- A model of the form

$$Y_t = \beta_0 + \beta_1 X_{t-m} + Z_t$$

  is called a transfer-function model/distributed-lag model/dynamic regression model.
  - These models may include the covariate at several lags but we discussed only the example with just one lagged covariate.
- After identifying which X_{t-m} to include, how to specify Z_t in the model $Y_t = \beta_0 + \beta_1 X_{t-m} + Z_t$?
- In order to explore the model specification for Z_t, the following approach is used:
  (A) Regress Y_t on X_{t-m} (assume temporarily that $Y_t = \beta_0 + \beta_1 X_{t-m} + e_t$) and obtain residuals $\hat{Z}_t = Y_t - \hat{Y}_t$.
  (B) Explore $\hat{Z}_t$ to specify a candidate model for Z_t
  (C) Fit the complete model $Y_t = \beta_0 + \beta_1 X_{t-m} + Z_t$, where Z_t is specified by the candidate model and check model diagnostics.

Sales example continued

- Model $Y_t = \beta_0 + \beta_1 X_t + Z_t$.
- Step A: Regress Y_t on X_t and obtain residuals.
- Step B: Analyze the residuals to find a candidate model for Z_t.
- Step C: Fit the complete model, here ARIMA(0,1,1) for Z_t:

```
Series: Y.t   ARIMA(0,1,1)
Coefficients:
          ma1      X.t
      -0.7007  -1.9273
s.e.   0.0644   0.1211
```

- Conclusion:
  - $Y_t = \beta_1 X_t + Z_t$ where $\hat{\beta}_1 \approx -1.9$ and $\nabla Z_t = e_t - \theta e_{t-1}$ where $\hat{\theta} \approx 0.7$.
  - Every one unit increase in X at time t is associated with an decrease in Y (log(sales)) at time t of appr. -1.9.
  - Interpretation for relation between X and sales, given by $\exp(Y)$?
    - Every one unit increase in X at time t is associated with a relative change of (multiplying Y by) $\exp(-1.9) \approx 0.15$ in Y.

Summary Ch. 11

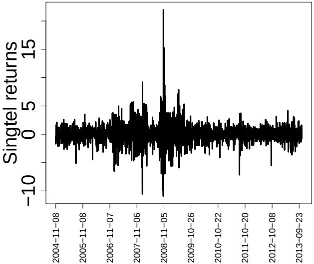
- Suppose we have a time series of interest $Y_1, Y_2, \ldots, Y_t$ (e.g. changes in log-transformed sales), here denoted by Y, and we want to explore whether/how another time series $X_1, X_2, \ldots, X_t$, here denoted by X (e.g. sales price), relates to Y_t.
- We discussed:
  - How to summarize the correlation between X and Y using the (sample) cross-correlation function (CCF),
  - and that the sample CCF can show spurious correlation if

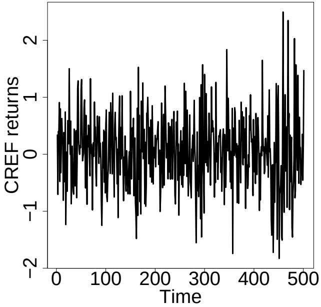
    $$Y_t = \beta_0 + \beta_1 X_{t-m} + Z_t,$$

    if X and Z are both autocorrelated time series.
  - How to prewhiten X, and use the same procedure for Y, to obtain a new sample CCF which is informative of the relation between Y and X.
  - How to fit a dynamic regression model to model Y using X while accounting for autocorrelation in Y.

Ch. 12: GARCH models

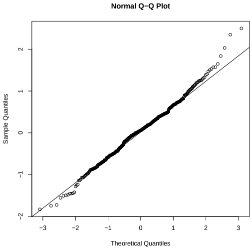
- The class of GARCH(p, q) models is used for estimating and forecasting volatility, which refers to the conditional variance or standard deviation $SD(r_t|r_{t-1}, r_{t-2}, \ldots)$ for some time series r_t (e.g. returns).

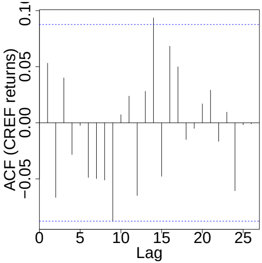


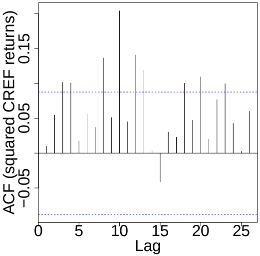


GARCH

- We can consider using a GARCH model for non-autocorrelated time series
  - with autocorrelation in squared or absolute values,
  - where normality does not hold true.







GARCH models

- The generalized autoregressive conditional heteroskedasticity model, GARCH(p, q), for r_t is given by:

$$\begin{aligned} r_t &= \sigma_{t|t-1}\varepsilon_t, \\ \sigma^2_{t|t-1} &= \omega + \beta_1\sigma^2_{t-1|t-2} + \ldots + \beta_p\sigma^2_{t-p|t-p-1} \\ &\quad + \alpha_1 r^2_{t-1} + \alpha_2 r^2_{t-2} + \ldots + \alpha_q r^2_{t-q} \end{aligned}$$

  where the innovations ε_t are iid and independent of the past returns, with $E(\varepsilon_t) = 0$ and constant variance.
- Examples:
  - For the ARCH(1) model:

$$\sigma^2_{t|t-1} = \omega + \alpha r^2_{t-1}.$$

  - For the GARCH(1,1) model:

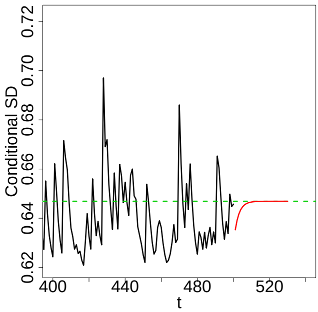
$$\sigma^2_{t|t-1} = \omega + \alpha r^2_{t-1} + \beta\sigma^2_{t-1|t-2}.$$

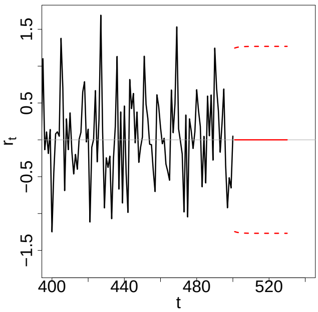
Properties and forecasts

- We discussed how to evaluate (conditional) expectations and the variance of r_t, that r_t^2 satisfies an ARMA model, as well as how to forecast $\sigma^2_{t+h|t}$.
- For these derivations, the following notation was useful:
  - R_t refers to the return as a random variable and r_t to its realization,
  - $V_{t|t-1}$ refers to the conditional variance $\sigma^2_{t|t-1}$ as a random variable and $v_{t|t-1}$ to its realization.
- Example for GARCH(1,1) forecast

$$
\begin{aligned}
\hat{v}_{t+g|t} &= E(R^2_{t+g}|R_j = r_j, j = 1, 2, \ldots t), \\
&= E(V_{t+g|t+g-1}\varepsilon^2_{t+g}|R_j = r_j, \text{ for } j = 1, \ldots, t), \\
&= E(\varepsilon^2_{t+g})E(V_{t+g|t+g-1}|R_j = r_j, \text{ for } j = 1, \ldots, t), \\
&= E(V_{t+g|t+g-1}|R_j = r_j, \text{ for } j = 1, \ldots, t), \\
&= E(\omega + \alpha_1 R^2_{t+g-1} + \beta_1 V_{t+g-1|t+g-2}|R_j = r_j, \text{ for } j = \ldots), \\
&= \omega + \alpha_1 r_t^2 + \beta_1 \hat{v}_{t|t-1}, \text{ for } g = 1, \\
& \ \omega + (\alpha_1 + \beta_1)\hat{v}_{t+g-1|t}, \text{ for } g > 1.
\end{aligned}
$$

Example of GARCH(1,1) forecast: Conditional SD and returns





- If $r_t = \sigma_{t|t-1}\varepsilon_t$ with $\varepsilon_t \sim N(0,1)$, the forecast intervals for r_{t+h} are given by $\hat{r}_{t+h} \pm 1.96\sigma_{t+h|t}$, where $\hat{r}_{t+h} = E(r_{t+h}|r_1, \ldots, r_t) = 0$.
- The conditional SD $\sigma_{t+h|t}$ converges to the unconditional SD for r_t, this follows from the (stationary) ARMA(1,1) representation for r_t... really?

GARCH forecasts

- The conditional SD $\sigma_{t+h|t}$ converges to the unconditional SD for r_t, this follows from the (stationary) ARMA(1,1) representation for R_t:
  - E.g., for GARCH(1,1):

    $$R_t^2 = \omega + (\beta_1 + \alpha_1) R_{t-1}^2 + \eta_t - \beta_1 \eta_{t-1},$$

    where the η_t's have mean zero and not autocorrelated and not correlated with the squared returns.
  - Thus for forecasting: $\hat{R}_t^2(g) \to E(R_t^2)$,
  - where $\hat{R}_t^2(g) = E(R_{t+g}^2 | R_j = r_j, j = 1, 2, \ldots t) = \hat{v}_{t+g|t}$ and $E(R_t^2) = var(R_t)$.

Review ST3233: Applied time series analysis

- Overview of material:
  - Time series processes, stationarity, (sample) ACF and PACF.
  - (Seasonal) ARIMA processes, parameter estimation, forecasting, model building.
  - Cross-correlation/dynamic regression modeling, GARCH.
- The End!