Ch 8: Model diagnostics

- How to check if the model "fits the data well"?
- Use residual analysis!
- Material:
  - Ch. 8 (we skip 8.2 for now (analysis of over-parametrized models), but will discuss it later when discussing model selection more formally).
  - You may want to review Ch 3.6 (residual analysis) as well.

Residual analysis

- Remember (e.g., from ST3131) that for a linear regression model:
$$Y_i = \beta_0 + X_{i1}\beta_1 + X_{i2}\beta_2 + \ldots + X_{ip}\beta_p + \varepsilon_i,$$
residuals were defined as $\hat{\varepsilon}_i = Y_i - \hat{Y}_i$, with
$$\hat{Y}_i = \hat{\beta}_0 + X_{1i}\hat{\beta}_1 + X_{2i}\hat{\beta}_2 + \ldots + X_p\hat{\beta}_{pi}.$$
- Same idea in time series analysis:
  - In an AR(p) model
  $$Y_t = \theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t$$
  for $t = 1, \ldots, n$, the residuals $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ are defined as:
  $$\hat{e}_t = Y_t - \hat{\theta}_0 - \hat{\phi}_1 Y_{t-1} - \hat{\phi}_2 Y_{t-2} - \ldots - \hat{\phi}_p Y_{t-p},$$
  - In other words: residual $\hat{e}_t$ = observed Y_t - predicted Y_t.
  More precisely: observed Y_t - one-step-ahead forecast for Y_t (Ch. 9!).
- If the model was correctly specified, and the parameter estimates are reasonably close to the true values, then the residuals $\hat{e}_t$ should have *nearly* the properties of the unobserved white noise $e_t \sim WN(0, \sigma_e^2)$.

One issue with this definition of residuals

- In an AR(p) model, for $t = 1, \ldots, n$, the residuals $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ are defined as:

  $$\hat{e}_t = Y_t - \hat{\theta}_0 - \hat{\phi}_1 Y_{t-1} - \hat{\phi}_2 Y_{t-2} - \ldots - \hat{\phi}_p Y_{t-p}.$$

  But what about $\hat{e}_1, \ldots, \hat{e}_p$?
- Set $Y_t = E(Y_t)$ for $t \leq 0$ and rescale the resulting residuals such that their variance is still approximately σ_e^2.
- Example AR(1) with mean zero, $Y_t = \phi Y_{t-1} + e_t$:
  - For $t > 1$: $\hat{e}_t = Y_t - \hat{\phi}_1 Y_{t-1}$ with $Var(\hat{e}_t) \approx \sigma_e^2$.
  - Set $Y_0 = 0$, then $\hat{e}_1^* = Y_1$, with $Var(\hat{e}_1^*) = Var(Y_1) = \gamma_0 = \sigma_e^2/(1 - \phi^2)$.
  - Define the rescaled residual $\hat{e}_1 = \hat{e}_t^* \cdot \sqrt{(1 - \hat{\phi}^2)}$ with approximate variance σ_e^2.
- Often, standardized residuals are used (for all t) with common variance 1: $\hat{s}_t = \hat{e}_t / \sqrt{\widehat{Var}(\hat{e}_t)}$.

Example: Color data (tut)

Exhibit 1.3 **Time Series Plot of Color Property from a Chemical Process**

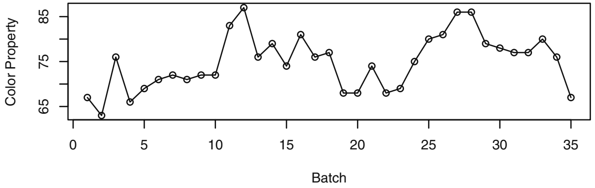
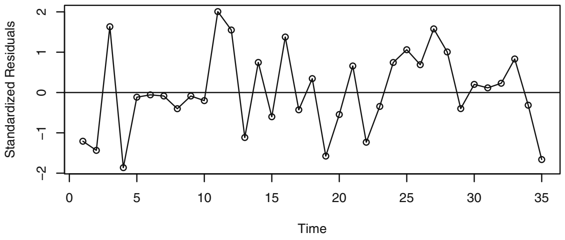


Exhibit 8.1 **Standardized Residuals from AR(1) Model of Color**



Standardized residuals for an ARMA(p, q) model

- What about ARMA(p, q) processes with $q > 0$?
- Same idea: observed Y_t - one-step-ahead forecast for Y_t.
- For a general (invertible) zero-mean ARMA(p, q) model, remember that we can write $Y_t = e_t + \sum_{i=1}^{\infty} \pi_i Y_{t-i}$. Using this definition, we can construct the residuals as

$$\hat{e}_t = Y_t - \hat{\pi}_1 Y_{t-1} - \hat{\pi}_2 Y_{t-2} - \ldots,$$

  where the $\hat{\pi}$'s are functions of the $\hat{\phi}$'s and $\hat{\theta}$'s (that decay to zero as i increases).
- Again, set $Y_t = E(Y_t)$ for $t \leq 0$ and rescale the $\hat{e}_t$'s such that their variance is still approximately σ_e^2.
- What if Y_t is transformed and/or modeled by an ARIMA-model? Examine the residuals after fitting an ARMA model to the transformed and/or differenced series.

Example: Oil data

- We defined the transformed series $W_t = \nabla(\log(Y_t))$

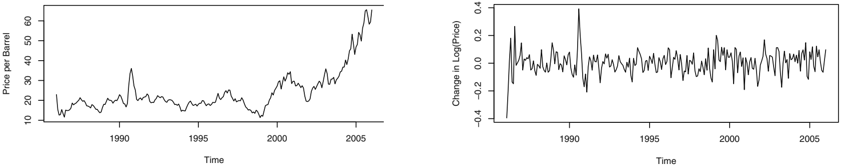
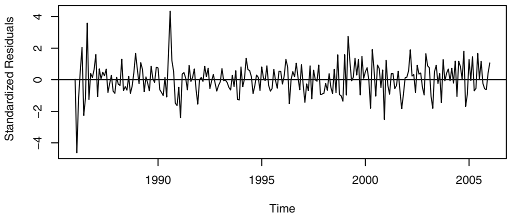


Exhibit 8.3 Standardized Residuals from Log Oil Price IMA(1,1) Model

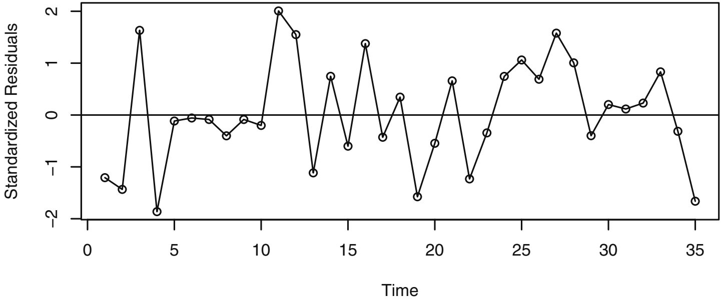


Residual analysis

- If the model was correctly specified, and the parameter estimates are reasonably close to the true values, then the residuals $\hat{e}_t$ should have *nearly* the properties of white noise e_t.
- Things to check:
  - Zero mean
  - Constant variance
  - Outliers
  - Normality (we assume normality of the e_t's by default for MLE)
  - Autocorrelation
- How?
  - Time series plot
  - Normality: QQ-plot and Shapiro-Wilk test
  - Autocorrelation: ACF and Ljung-Box test

Example of residual plot: color time series (tut)

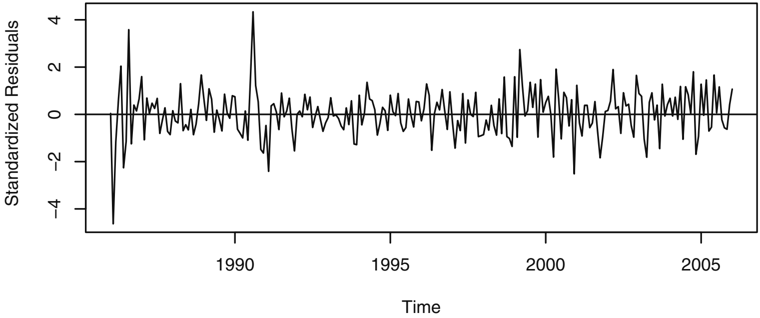
Exhibit 8.1 Standardized Residuals from AR(1) Model of Color



Zero mean, constant variance, any outliers?

Example of residual plot: oil

Exhibit 8.3 Standardized Residuals from Log Oil Price IMA(1,1) Model



Zero mean, constant variance, any outliers?

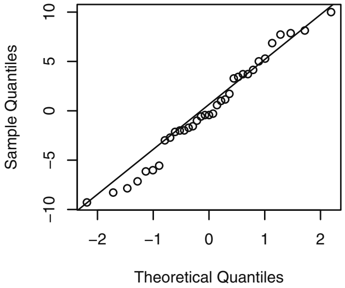
- There are some outlying residuals (with magnitudes exceeding the Bonferroni critical value $z_{1-\alpha/2 \cdot 1/n} = 3.71$ for $n = 241$ and $\alpha = 0.05$) that are worth investigating further.

Normality

- QQ-plot:
  - Plot the ordered residuals $\hat{e}_{(1)}, \hat{e}_{(2)}, \ldots, \hat{e}_{(n)}$ against their expected values if they were the order statistics from a normal distribution.
  - If the residuals are approximately normally distributed, we expect that the points lie on a straight line.
- Shapiro-Wilk normality test:
  - Calculate the squared correlation R^2 between the ordered residuals and their expected values based on normality.
  - Reject normality if the value found for R^2 is smaller than expected (use critical values to do so, e.g. for $n = 200$ $\text{Prob}(R^2 < 0.987) = 0.05$).

Checking normality for color data

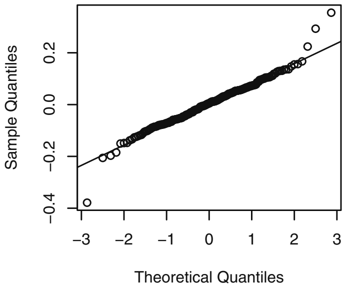
Exhibit 8.4 Quantile-Quantile Plot: Residuals from AR(1) Color Model



- P-value for Shapiro-Wilk normality test around 0.6:
  We don't have evidence to reject the normality assumption.

Checking normality for oil data

Exhibit 8.6 **Quantile-Quantile Plot: Residuals from IMA(1,1) Model for Oil**



- P-value for Shapiro-Wilk normality test around 10^{-5}:
  We reject the normality assumption.

Autocorrelation of the residuals

- How to examine the autocorrelation of the residuals?
- Denote the ACF for the residuals $\hat{e}_t$ with $\hat{r}_k$.
- The asymptotic distributions for the $\hat{r}_k$'s are similar to those for white noise for larger k:

$$\hat{r}_k \sim N(0, 1/n), \text{ approximately, and independent}$$

  but tend to differ for small k:
  - $E(\hat{r}_k) \approx 0$ still holds true,
  - but generally the variance of $\hat{r}_k$ is smaller and the $\hat{r}_k$ are correlated.
  - (Book gives some expression for asymptotic variance of $\hat{r}_k$ for AR(1) and AR(2) processes).
- In practice:
  - We first still plot the ACF with the critical bounds at $\pm 1.96 \cdot \sqrt{1/n}$.
    - Critical value is appropriate for higher lags, and we don't expect to see (many) $\hat{r}_k$'s to be outside at lower lags (because their variance is smaller).
  - We then use a test...

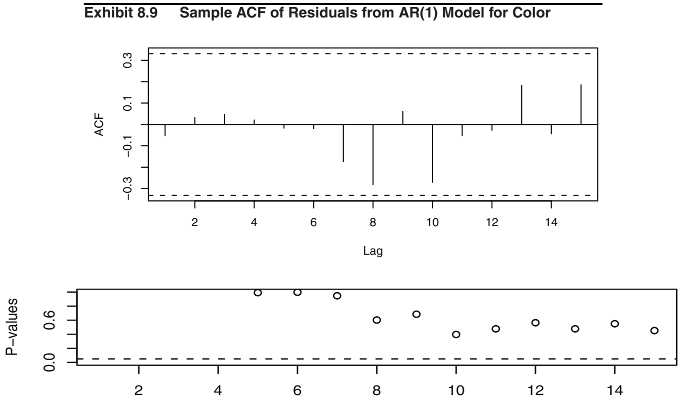
Ljung-Box test

- Test to check if the fitted ARMA model captured the autocorrelation structure accurately.
- The test statistic is based on checking if a set of autocorrelation terms $\hat{r}_1, \ldots, \hat{r}_K$ is "unusual".
- The test statistic is given by

$$Q_* = n(n+2)\left(\frac{\hat{r}_1^2}{n-1} + \frac{\hat{r}_2^2}{n-2} + \ldots + \frac{\hat{r}_K^2}{n-K}\right),$$

  which approximately follows a χ^2_{K-p-q} distribution under H_0.
- We reject the ARMA model if Q_* is larger than expected (if the p-value of the test is less than α).

Example of checking autocorrelation: Color data



- P-values in plot are those for the Ljung-Box test for $K = 5, 6, \ldots, 15$ (no problems detected).

Summary of model diagnostics

- Residuals $\hat{e}_t =$ observed Y_t - one-step-ahead forecast for Y_t.
- We use residuals to check whether the model adequately represents the data:
  If the model was correctly specified, and the parameter estimates are reasonably close to the true values, then the residuals should have *nearly* the properties of normally distributed white noise.
- Things to check: Zero mean; Constant variance; Normality; Outliers; Autocorrelation.
- Tools for checking: Time series plot; QQ-plot and Shapiro-Wilk test; ACF and Ljung-Box test.
- If a model seems appropriate for the data, we can use it for forecasting, which we'll discuss next!