5.4 Bayesian decision making

Although statistical inference is often just about learning about and understanding the world, it is sometimes used as a tool for making decisions, i.e. for changing the world (however slight the change may be). For instance, for a drug to be approved by the Federal Drug Agency in the US, or in most other countries, requires that the drug be demonstrated over several clinical trials to be efficacious using formal statistical methods—though, sadly, as yet they do not seem to approve of Bayesian methods. It is only natural for statistics to be involved in making difficult decisions, for decisions are difficult only in the presence of uncertainty.

The Bayesian approach to inference naturally lends itself to decision making as it provides a flexible and coherent mechanism for understanding uncertainty: there is no inherent difference between parametric uncertainty, representing lack of knowledge of the rules of a system, and stochasticity, representing lack of knowledge of what might happen even if the system were perfectly understood.

For instance, when patients arrive at the emergency department of a hospital with an apparently mild infection, but with the risk of developing more severe, potentially life threatening complications, the attending clinician has to make a choice on how to triage the patient—send him home, or admit him—without knowing whether he is going to get worse later. In making this decision, she is weighing two uncertainties: (i) if she saw lots of patients exactly like the current one, she doesn't perfectly know the risk of complications (perhaps she thinks it's roughly 10% but because she hasn't seen many patients like that so far, it could be somewhere between 5 and 15%); (ii) even if she was sure there was a 10% chance of complications, she isn't sure if *this* patient will be the 1 out of 10 who will worsen, or one of the 9 out of 10 who will stay in a mild condition. To a Bayesian, there is no real difference between these: the first is uncertainty in the parameter model, as represented by the posterior the clinician has intuitively built up and which we conceptualise as reducing with experience; the second is uncertainty in the data model, which even an experienced clinician with no parametric uncertainty would face. In frequentist statistics, these are fundamentally different.

5.4.1 Utility

Making optimal decisions requires not only that the uncertainties be quantified, but also that the desirability of each potential outcome can be quantified. Since we wish to optimise our decisions, we will need this desirability to be measured in one dimension (optimisation otherwise will be too difficult).

This is called the utility function, which maps from the possibly multidimensional outcome space, with each dimension potentially using different units, to a single dimension on which higher values are better. For the clinician attending to the patient in emergency, the utility function would characterise: (i) the benefit the patient gets from being at home rather than in a ward, including reduced hospitalisation fees (i.e. a mix of financial and non-financial units), (ii) that the patient may improve quicker if hospitalised (quality of life units); (iii) the effect of becoming worse if at home (quality of life units); (iv) the effect of becoming worse if in hospital (ditto, plus financial). Mixing different units is hard to justify since it requires weighting them appropriately, and it is far better to try to use one single, non-controversial dimension if at all possible, such as cost, or quality adjusted life years lost.

Utility we will represent by $u(x, D)$, a function of the random outcome x and the decision D that is made (the distribution of x may depend on the decision D but the notation herein assumes it is not). Since x is random, so is u, and so we shall look to optimise not u, but $\mathrm{E}(u)$, where the expectation is over all forms of uncertainty.

(In financial examples, the utility might not be linear with money, as it is commonly believed that the marginal utility of \$1 decreases with wealth. While this adds extra realism, it should be noted that people usually behave "irrationally" in making decisions in uncertainty: there is a large and entertaining literature on this.)

5.4.2 Optimal decision

To choose the optimal decision, you need to be able to identify the range of possible decisions you could make. For instance, in the triage example, the decision space has two elements: keep the patient in or send him home. A more complicated decision space might be the set containing (i) send him home, (ii) keep him in for now, and if he looks okay tomorrow, send him home, otherwise keep him in until he looks better. Or: (i) send him home, (ii) keep him in until his lab test comes back, if the measurement is more than ϕ, send him home, otherwise keep him in until he looks better.

Once you have that space of decisions identified, the utility function specified, and the distribution of outcomes known (for now, assuming no parametric uncertainty), the optimal decision is

$$D^{\star} = \arg\max_{D} \int_{\mathcal{X}} u(D, x) \mathrm{p}(x)\, \mathrm{d}x$$

that is, the decision which maximises the expected utility over the uncertainty in x. To identify the optimum, if the decision space is discrete (and small enough), a simple search over it could be performed. Otherwise, a subset of decisions could be evaluated, or some other numerical technique to optimise functions over a continuous, possibly complicated, space.

To derive the expected utility for a given decision, the simplest approach is to use simulation. If a large number of Monte Carlo samples are drawn of x from $\mathrm{p}(x)$, the expected utility can be estimated by calculating the utility for each draw and taking the empirical average. As in previous examples in the course, variability between decisions can be minimised by using the same values of x for each decision, and if the decision changes the x (as it might for some applications), the xs might be simulated using something like the inverse CDF method, with uniform building blocks shared across designs.

5.4.3 Multistage decisions

In some situations, you will make a series of decisions: at time t_1 you select D_1, you then observe an outcome that provides additional information x_2 at time t_2 and make a second decision D_2, and then observe an outcome x_3 (which may continue for multiple stages). Although it is still possible to optimise your expected utility, it becomes harder, because to choose D_1 optimally, you need to know how you would react at time t_2 to the data observed then, x_2. The optimal decision of D_2 is, in many cases, a function of x_2, and the optimal choice of D_1 is to maximise utility after x_2 and x_3.

For instance, in the game pontoon (British version), you are dealt a card (from a French deck), which you see, and then must decide on a stake (in money or chips); you are then dealt a second card and must make decide whether to stick or twist (with the aim to get closer to 21 than the bank). D_1 is on the range $\{1c, 2c, 3c, \ldots\}$, x_2 is the combined score on your hand after the second card is dealt, D_2 is on $\{\text{stick}, \text{twist}\}$[2], x_3 is the pair of outcomes: your score after twisting if you choose to twist, and the bank's score. The utility is D_1 if your score beats the bank's and $-D_1$ if not (the bank wins in ties).

The solution to such multistage decisions is to start with the last decision you will get to make: calculate the optimal decision at that time point for each outcome x, and the expected utility for each x. Then go to the decision just before that, work under the premise you will act optimally thereafter, and derive the optimal choice for each scenario you might be in then, then repeat. For the pontoon example, you first decide whether to stick or twist

[2] In fact, it is more complicated as you can decide to twist again and again after seeing each draw. Also, you have the option to buy a card by doubling your stake—which has the complication that the bank does not get to see your hand. There are other complications—see www.pagat.com for full details.

for each hand you might hold (and stake you have already played) based on the chances of beating the bank. You then work out your expected winning for each hand you might hold under the optimal decision. You then decide how high to set your stake having seen only one card and not knowing what the second might be, but knowing what the expected utility would be for each possible second card.

5.4.4 Decision making under uncertainty

The discussion above is more probabilistic than statistical, since we have assumed that the uncertainty lay in the outcomes only, but the rules governing those outcomes were known. However, if the uncertainty in the process can be quantified via a(n informative) prior distribution, this uncertainty can readily be incorporated. Rather than just integrating over the space of possible outcomes, we take the expected utility with respect to the joint distribution of parameters and outcomes. This is more easily achieved by factorising the distribution:

$$\begin{aligned} D^\star &= \arg\max_D \int_\Theta \int_\mathcal{X} u(D,x)\mathrm{p}(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \arg\max_D \int_\Theta \int_\mathcal{X} u(D,x)\mathrm{p}(x|\theta)\mathrm{p}(\theta)\,\mathrm{d}x\,\mathrm{d}\theta \end{aligned}$$

The expected utility for any particular decision can be estimated by drawing parameter values from the prior and then simulating outcomes conditional on those values. For example, we may wish to decide how large a stockpile of antivirals to maintain to mitigate a future influenza pandemic. We have prior distributions for the characteristics of the pandemic, derived from previous outbreaks, and so for a range of stockpile sizes, we simulate characteristics of the pandemic from the prior, simulate pandemic outbreaks given those characteristics, with the outbreak affected by the stockpile size, and then quantify the impact of the outbreak (say in lost disability adjusted life years). We then repeat this many times to obtain a Monte Carlo average for that one stockpile size, and then repeat over stockpiles.

Note: you may encounter so-called "statistical decision theory" in which an artificial utility function is defined for the problem of choosing a point estimator to report from a posterior (if the utility is quadratic in the difference between the reported and the "true" parameter value, the optimal "decision" is to report the mean; if it is the absolute difference, the median; if it is 1 if correct and 0 otherwise, the mode). This is rather old fashioned, ivory tower "statistics".

Outro

These lecture notes have sought to convey the power of the Bayesian paradigm in solving real world problems that non-Bayesian statistics can not always readily manage. Although most of the examples in the course have been biostatistical (and a great many to do with infectious disease), that is a reflexion primarily of the author's research interests, and Bayesian methods should not be taken to fall solely within the domain of medicine and health.

Bayesian statistics spans a broad range of methods and these notes could not aspire to cover it all. Instead, I have tried to cover the most important points in detail (especially MCMC) and cover as many other areas as I could, at least partially. For further reading, I strongly recommend Bayesian Data Analysis by Gelman et al. In writing these notes I have been strongly influenced by that book, by Bayesian Computation in R by Albert, by my former supervisors Gavin Gibson and Glenn Marion, and by current colleagues, especially Leontine Alkema.

There are undoubtedly errors in these notes, and you should not rely on them for anything life threatening. Should you discover any, I would be extremely appreciative if I could be informed.