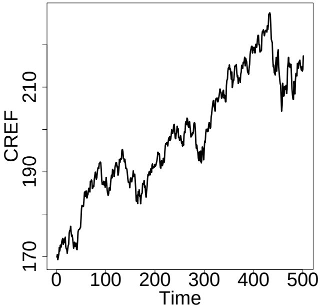
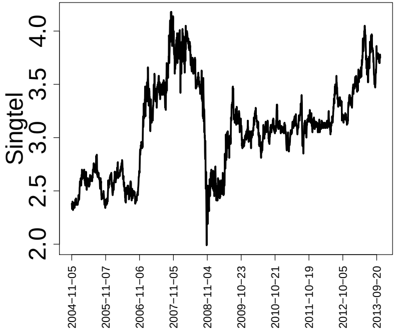
Ch 12: Time series models of heteroscedasticity

- Time series models of heteroscedasticity are a new class of models that form a useful extension to the time series modeling techniques we discussed so far, esp. for financial time series.
- In Part 1, we discuss:
  - for what type of time series modeling heteroscedascity is of interest.
  - one model (ARCH(1)) to do so and derive the conditional and unconditional distribution for the time series based on that model.
- In Part 2:
  - We discuss when/how to use the new models for real time series.
- Part 1 is a bit equation-heavy but should help to get a good understanding what's going on in ARCH(1) and related models.
- Material for CH.12: selected material from Ch 12.1 to 12.5.
  - Note: we use a different R package then explained in the book because it has a better optimization routine and more options to deal with non-normal innovations (discussed later).

Financial time series

- Let p_t be the daily time series of some financial asset.
- Examples: CREF stock fund (fund of several thousands stocks), Singtel stocks, Google stocks (tutorial), ...
- Trading is on working days only but we will analyze the data as if they were equally spaced.

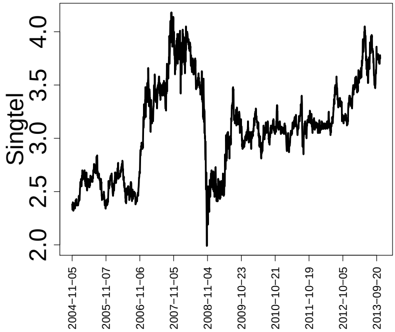


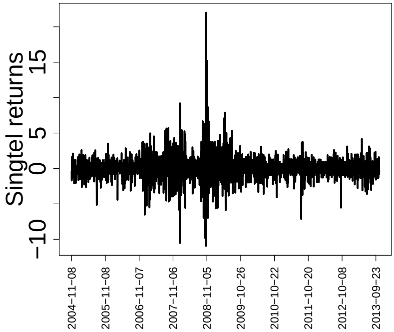


Source Singtel: http://sg.finance.yahoo.com/q?s=ŝti (Oct 30, 2013)

Returns

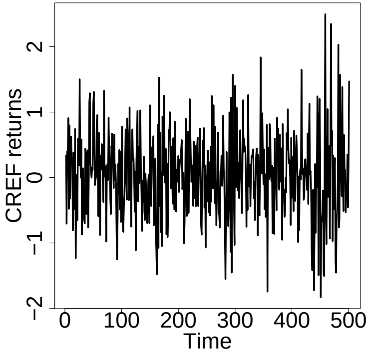
- Returns ($\cdot 100\%$) are defined as $r_t = (\log(p_t) - \log(p_{t-1})) \cdot 100$.
- Wow, what's going on with the returns in 2008!?

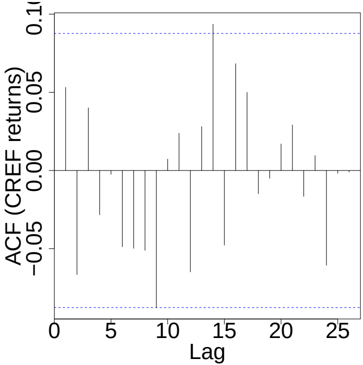




Modeling returns

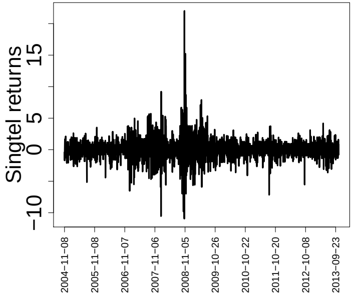
- In an efficient market, mean returns and autocorrelation of returns are often close to zero, so we may not need our class of ARIMA models for forecasting.
- What's of interest is the uncertainty in future returns, in particular the asset volatility.





Examining volatility in returns

- Volatility is a measure of uncertainty in future returns, here defined as the conditional SD or variance, e.g. $Var(r_t | r_{t-1}, r_{t-2}, \ldots)$.
- Asset volatility tends to change over time, e.g., there is high volatility during a financial crisis.



- Volatility clustering refers to the pattern of alternating quiet and volatile periods of substantial duration.
- Volatility forecasting is used for option (derivative) pricing, risk management (e.g. value at risk), asset allocation (e.g. minimum variance portfolio).

Our main question: How to forecast volatility?
Note: a bit tricky because the conditional variance is a latent variable (it is not observed directly!).

Forecasting volatility

- If there is volatility clustering, then r_{t-1}^2 may be informative of $Var(r_t|r_{t-1}, r_{t-2}, \ldots)$:
  - periods of highly variable squared returns may foretell a relatively volatile period.
  - periods of small squared returns may foretell a relatively quiet period of low volatility.
- This motivates the ARCH(1) model for returns:

$$\begin{aligned} r_t &= \sigma_{t|t-1}\varepsilon_t, \\ \sigma_{t|t-1}^2 &= \omega + \alpha r_{t-1}^2. \end{aligned}$$

  where
  - ε_t is a time series of indep. and identically distr. random variables (with ε_t independent of past r_t's) with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = 1$,
  - $\sigma_{t|t-1}^2$ gives the conditional variance for r_t and parameters ω and α are constrained to guarantee positive variance.
- If we assume (for now) that $\varepsilon_t \sim N(0, 1)$, then what distribution for r_t does this model correspond to?
  - Distinguish between the conditional and uncond. distribution of r_t!

ARCH(1):conditional and uncond. distribution of r_t

- ARCH(1) model:

$$\begin{aligned} r_t &= \sigma_{t|t-1}\varepsilon_t, \\ \sigma^2_{t|t-1} &= \omega + \alpha r^2_{t-1}. \end{aligned}$$

- So far, we followed the notation from the book, where r_t and $\sigma^2_{t|t-1}$ refer to both random variables as well as realizations.
- To make it easier to understand derivations related to the conditional distribution of r_t, in the following slides:
  - R_t will refer to the return as random variable and r_t to its realization,
  - $V_{t|t-1}$ will refer to the conditional variance $\sigma^2_{t|t-1}$ as a random variable, $V_{t|t-1} = \omega + \alpha R^2_{t-1}$, and $v^2_{t|t-1}$ to its realization.

  After the derivation, we will go back to the notation from the book.

ARCH(1):conditional and uncond. distribution of r_t

- ARCH(1) model repeated:

$$\begin{aligned} R_t &= \sqrt{V_{t|t-1}}\varepsilon_t, \\ V_{t|t-1} &= \omega + \alpha R_{t-1}^2. \end{aligned}$$

- $E(R_t|R_j = r_j, \text{ for } j = 1, \ldots, t-1) = 0$ (show!),
- For the variance:

$$\begin{aligned} &Var(R_t|R_j = r_j, \text{ for } j = 1, \ldots, t-1), \\ &= E(R_t^2|R_j = r_j, \text{ for } j = 1, \ldots, t-1), \\ &= E(V_{t|t-1}\varepsilon_t^2|R_j = r_j, \text{ for } j = 1, \ldots, t-1), \\ &= v_{t|t-1}E(\varepsilon_t^2|R_j = r_j, \text{ for } j = 1, \ldots, t-1), \\ &= v_{t|t-1}E(\varepsilon_t^2), \\ &= v_{t|t-1} = \omega + \alpha r_{t-1}^2. \end{aligned}$$

- If $\varepsilon_t \sim N(0, 1)$, then

$$R_t|R_{t-1} = r_{t-1} \sim N(0, \omega + \alpha r_{t-1}^2).$$

Unconditional distribution of R_t for an ARCH(1) model

- For the UNconditional distribution of R_t, we find:

$$\begin{aligned} E(R_t) &= E(\sqrt{V_{t|t-1}}\varepsilon_t) = 0, \\ E(R_t^2) &= E(V_{t|t-1}\varepsilon_t^2) \\ &= E(V_{t|t-1})E(\varepsilon_t^2), \\ &= E(\omega + \alpha R_{t-1}^2), \\ &= \omega + \alpha E(R_{t-1}^2). \end{aligned}$$

- If $Var(R_t) = E(R_t^2)$ is constant with time t (which we will discuss holds true when $\alpha < 1$), then $Var(R_t) = \omega/(1-\alpha) = \sigma^2$.
- The R_t's are not autocorrelated; for $k > 0$:

$$\begin{aligned} E(R_t R_{t+k}) &= E(R_t \cdot \sqrt{V_{t|t-1}}\varepsilon_{t+k}), \\ &= E(R_t \cdot \sqrt{V_{t|t-1}})E(\varepsilon_{t+k}) = 0, \end{aligned}$$

thus $Cov(R_t, R_{t+k}) = E(R_t R_{t+k}) - E(R_t)E(R_{t+k}) = 0$.

Unconditional distribution of R_t for an ARCH(1) model

- For the ARCH(1) model with

$$
\begin{aligned}
R_t &= \sqrt{V_{t|t-1}}\varepsilon_t, \\
V_{t|t-1} &= \omega + \alpha R_{t-1}^2.
\end{aligned}
$$

  we derived: $E(R_t) = 0$, $Var(R_t) = \sigma^2$, and the R_t's are not autocorrelated.
- Question: Is R_t normally distributed?
- Answer: R_t is NOT normally distributed:
  - This follows from its definition.
  - The distribution of R_t has fatter tails than a normal distribution with the same variance (its kurtosis, related to its 4th order moment, is greater than 0).

Unconditional distribution of R_t for an ARCH(1) model

- For the ARCH(1) model with $R_t = \sqrt{V_{t|t-1}}\varepsilon_t$, $V_{t|t-1} = \omega + \alpha R_{t-1}^2$, we derived: $E(R_t) = 0$, $Var(R_t) = \sigma^2$, and the R_t's are not autocorrelated.
- Question: Does the zero autocorrelation for the R_t's imply that they are independent?
  - No; uncorrelated does not necessarily imply independence because R_t is not normal.
- The R_t's are dependent because of the specification of the variance terms. We will discuss this in more detail in a bit.
- Additional note related to the definition of white noise:
  - For the ARIMA class of models, we defined white noise e_t to have $E(e_t) = 0$, $Var(e_t) = \sigma_e^2$ (constant) and the e_t's are independent.
  - The R_t's are NOT white noise according to this definition because they are not independent.
  - Often a weaker definition of white noise is used, where the e_t's are uncorrelated. The R_t's would be white noise according to that definition.

ARCH(1) simulation example

- We switch back to the notation of the book, thus r_t and $\sigma^2_{t|t-1}$ refer to random variables or realization, depending on the context.
- ARCH(1) model:

$$
\begin{aligned}
r_t &= \sigma_{t|t-1}\varepsilon_t, \\
\sigma^2_{t|t-1} &= \omega + \alpha r^2_{t-1},
\end{aligned}
$$

  where $\varepsilon_t \sim N(0,1)$, $\omega = 0.01$ and $\alpha = 0.9$.
- R-code:

```
garch01.sim=garch.sim(alpha=c(.01,.9),n=500)
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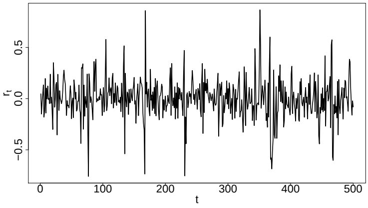
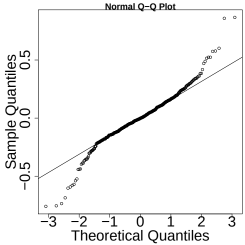
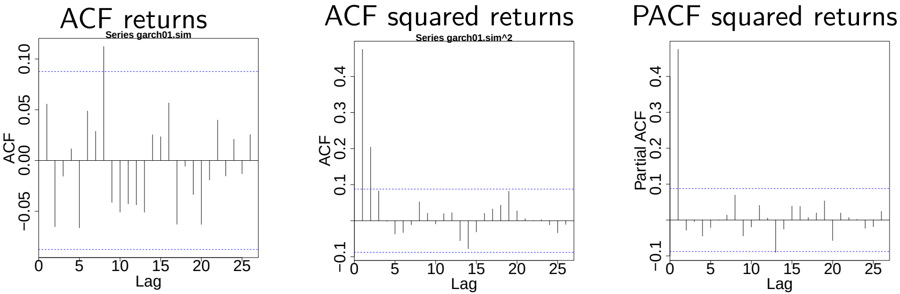




ARCH(1) simulation example (continued)



- The QQ-plot for the returns suggest fatter tails as compared to a normal distribution, as expected.
- The ACF for the returns suggests uncorrelated returns, as expected.
- What's going on with the squared returns.... looks like an AR(1)!

Examining an ARCH(1) process more closely

- Let $\eta_t = r_t^2 - \sigma^2_{t|t-1}$, then

$$\sigma^2_{t|t-1} = r_t^2 - \eta_t.$$

- Plug in this expression for $\sigma^2_{t|t-1}$ into the ARCH(1) model:

$$\begin{aligned} \sigma^2_{t|t-1} &= \omega + \alpha r^2_{t-1}, \\ r_t^2 &= \omega + \alpha r^2_{t-1} + \eta_t, \end{aligned}$$

  such that it looks like r_t^2 has an AR(1) specification ...
  IF the η_t's have mean zero, are not autocorrelated and are not correlated with past squared returns.
- This holds true!
  - To be shown by you in the next tutorial!
- Thus: We can explore the ACF and PACF for r_t^2 for model specification!

Examining an ARCH(1) process more closely

- For the ARCH(1) model:

$$r_t^2 = \omega + \alpha r_{t-1}^2 + \eta_t,$$

  where the η_t's have mean zero, are not autocorrelated and are not correlated with past squared returns.
- This expression can be used to show that variance $Var(r_t) = \sigma^2$ is constant if $0 \leq \alpha < 1$.
- Lastly, more details on the statement earlier that the r_t's are NOT independent:
  - If the r_t's would be independent, the correlation between any transformation of the r_t's would be zero.
  - This does not hold true here, given that the squares returns follow an AR(1) model specification!

Summary

- The goal of Ch.12 is to introduce a new class of models, that are widely used in financial time series analysis.
- So far, we discussed the ARCH(1) model:

$$\begin{aligned} r_t &= \sigma_{t|t-1}\varepsilon_t, \\ \sigma^2_{t|t-1} &= \omega + \alpha r^2_{t-1}, \\ r_t|r_{t-1} &\sim N(0, \omega + \alpha r^2_{t-1}). \end{aligned}$$

- We discussed properties of the (unconditional) distribution of return r_t for $\omega > 0$ and $0 \leq \alpha < 1$:
  - $E(r_t) = 0$, $Var(r_t) = \sigma^2 = \omega/(1-\alpha)$, and the r_t's are not autocorrelated.
  - r_t is NOT normally distributed but has fatter tails than a normal distribution with the same variance.
  - The r_t's are dependent and can be written as $r_t^2 = \omega + \alpha r^2_{t-1} + \eta_t$, where the η_t's have mean zero, are not autocorrelated and are not correlated with past squared returns.

Why bother with ARCH-type models?

- Time series models that account for conditional variance and result in distributions with fat tails are often appropriate for modeling returns.

```
> shapiro.test(r.cref)
data:  r.cref   W = 0.9932, p-value = 0.02412
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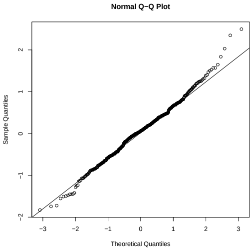

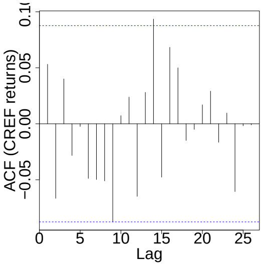

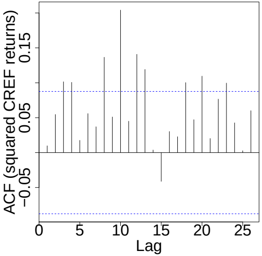

Summary (continued)

- We can consider using an ARCH model for a non-autocorrelated time series
  - with autocorrelation in squared outcomes,
  - where normality does not hold true.
- Next: GARCH models, parameter estimation, forecasting conditional variance.