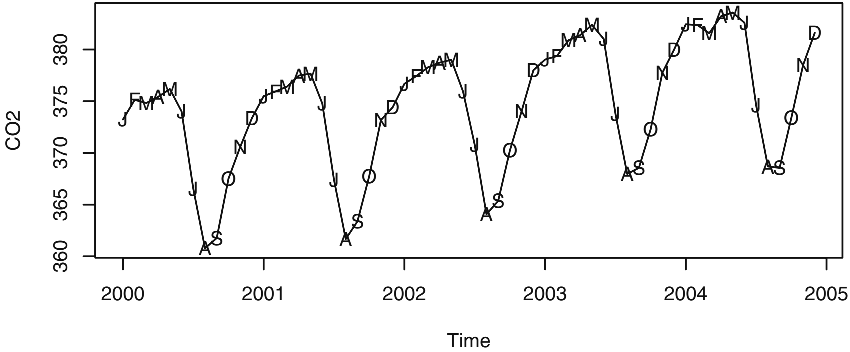
Ch 10: Seasonal models

Motivating example

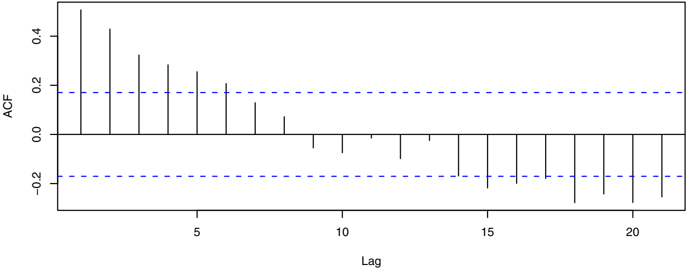
- Carbon dioxide (CO_2) levels, as monitored montly in Canada, near the Artic circle.
- Interesting data! What's going on? How to forecast future outcomes?

Exhibit 10.2 Carbon Dioxide Levels with Monthly Symbols



CO_2 data series: does a simple regression model work?

- If you are familiar with regression analysis, you may consider fitting a model with a time trend and dummies to capture the seasonal variation (e.g., one dummy for each month except for January).
- However, residuals turn out to be autocorrelated, so we need to use time series analysis techniques to account for the autocorrelation in the series.
- We can use seasonal models!
  - Material: Ch.10, material from all sections but 10.3 and 10.5 are not covered in detail.



Seasonal ARIMA models: intro with an example

- Suppose
$$Y_t = e_t - \Theta e_{t-12}, \tag{1}$$
where t here refers to time in months.
- What is ρ_k for $k = 1, 2, \ldots$?
- We find $\rho_k \neq 0$ for $k = 12$ only (when considering $k > 0$), e.g.

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \Theta e_{t-12}, e_{t-1} - \Theta e_{t-13}) = 0, \\ Cov(Y_t, Y_{t-12}) &= Cov(e_t - \Theta e_{t-12}, e_{t-12} - \Theta e_{t-24}) = -\Theta \sigma_e^2. \end{aligned}$$

- You can consider the model in Eq. 1
  - an MA(12) model with $\theta_i = 0$ for $i = 1, 2, \ldots 11$, or
  - a seasonal MA(1) model of order 1 with seasonal period $s = 12$ and only one parameter Θ.

Seasonal MA(Q) models

- A seasonal MA(Q) model of order Q with seasonal period s is defined by

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2\cdot s} - \ldots - \Theta_Q e_{t-Q\cdot s}$$

  with seasonal MA characteristic polynomial

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2\cdot s} - \ldots - \Theta_Q x^{Q\cdot s}.$$

- This corresponds to a non-seasonal MA($Q \cdot s$) model but with a lot less parameters (more parsimonious model representation), e.g. $\theta_i \neq 0$ only for $i = s, 2\cdot s, \ldots, Q\cdot s$.

Seasonal AR(P) models

- A seasonal AR(P) model of order P with seasonal period s is defined by

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2\cdot s} + \ldots + \Phi_P Y_{t-P\cdot s} + e_t,$$

  with seasonal AR characteristic polynomial

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2\cdot s} - \ldots - \Phi_P x^{P\cdot s}.$$

- For these models, $\rho_{k\cdot s} \neq 0$ for $k = 0, 1, 2, \ldots$ only.
- Example: Seasonal stationary AR(1) model with $s = 12$ months:

$$Y_t = \Phi Y_{t-12} + e_t$$

  - Multiply by Y_{t-k}, take expectations, and divide by γ_0 to get

$$\rho_k = \Phi \rho_{k-12} \text{ for } k \geq 1.$$

  - Then $\rho_{12} = \Phi$, $\rho_{24} = \Phi\rho_{12} = \Phi^2$ etc.: $\rho_{k\cdot s} = \Phi^k$ for $k = 1, 2, \ldots$
  - All other ρ's are zero, e.g. $\rho_1 = \Phi\rho_{11}$ and $\rho_{11} = \Phi\rho_1$ which implies $\rho_1 = \rho_{11} = 0$ for $\Phi \neq 0$.
- What if there is autocorrelation at seasonal AND low lags?

Multiplicative Seasonal ARMA models

- Usually, we have not only seasonal autocorrelation but also nonseasonal autocorrelation (for low lags of neighboring values).
- Let's look at parsimonious models that incorporate both: multiplicative seasonal ARMA models.
- These models become a bit complicated to write out in full; easier to use characteristic equations and the backshift operator B.
- Example (and review of B):
  - For a non-seasonal MA(1) model, with MA char. function $\theta(x) = 1 - \theta x$, we can write

$$Y_t = e_t - \theta e_{t-1} = (1 - \theta B) e_t = \theta(B) e_t. \tag{2}$$

  - For a seasonal MA(1) model with $s = 12$, with seasonal MA char. function $\Theta(x) = 1 - \Theta x^{12}$, we write

$$Y_t = e_t - \Theta e_{t-12} = (1 - \Theta B^{12}) e_t = \Theta(B) e_t. \tag{3}$$

  - What happens when we combine both?

Multiplicative Seasonal ARMA(0,1)x$(0,1)_{12}$ model

- For a non-seasonal MA(1) model, with MA char. eq. $\theta(x) = 1 - \theta x$, we can write

$$Y_t = e_t - \theta e_{t-1} = (1 - \theta B)e_t = \theta(B)e_t. \tag{4}$$

- For a seasonal MA(1) model with $s = 12$, with seasonal MA char. eq. $\Theta(x) = 1 - \Theta x^{12}$, we write

$$Y_t = e_t - \Theta e_{t-12} = (1 - \Theta B^{12})e_t = \Theta(B)e_t. \tag{5}$$

- When we combine both as follows, we obtain a multiplicative Seasonal ARMA(0,1)x$(0,1)_{12}$ model:

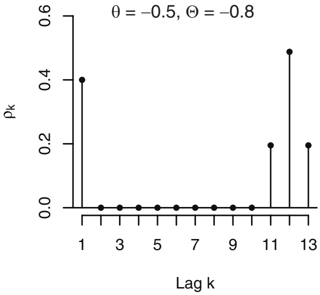
$$\begin{aligned} Y_t &= \theta(B) \cdot \Theta(B)e_t, \\ &= (1 - \theta B)(1 - \Theta B^{12})e_t, \\ &= (1 - \theta B - \Theta B^{12} + \theta\Theta B^{13})e_t, \\ &= e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}. \end{aligned}$$

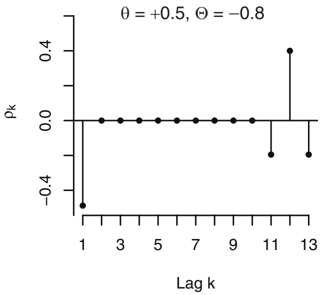
ACF for ARMA(0,1)x$(0,1)_{12}$ model

- The multiplicative Seasonal ARMA(0,1)x$(0,1)_{12}$ model is given by:

$$\begin{aligned} Y_t &= \theta(B) \cdot \Theta(B) e_t, \\ &= e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}. \end{aligned}$$

- Derive autocorrelation function as usual and find that $\rho_k = 0$ for $k \neq 0, 1, 11, 12, 13$.
- Below are example ACFs for different values of the parameters.





Multiplicative Seasonal ARMA models: general definition

- Y_t is a multiplicative ARMA(p, q)x$(P, Q)_s$ process with
  - "mean parameter" (not the mean of Y_t!) θ_0,
  - seasonal period s,
  - AR characteristic polynomial $\phi(x)\Phi(x)$ with

$$\begin{aligned}\phi(x) &= 1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p, \\ \Phi(x) &= 1 - \Phi_s x^s - \Phi_2 x^{2\cdot s} - \ldots - \Phi_P x^{P\cdot s},\end{aligned}$$

  - MA characteristic polynomial $\theta(x)\Theta(x)$ with

$$\begin{aligned}\theta(x) &= 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q, \\ \Theta(x) &= 1 - \Theta_1 x^s - \Theta_2 x^{2\cdot s} - \ldots - \Theta_Q x^{Q\cdot s},\end{aligned}$$

if Y_t is defined as follows:

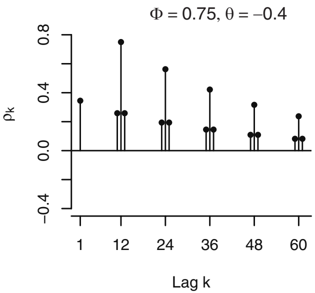
$$\phi(B)\Phi(B)Y_t = \theta_0 + \theta(B)\Theta(B)e_t.$$

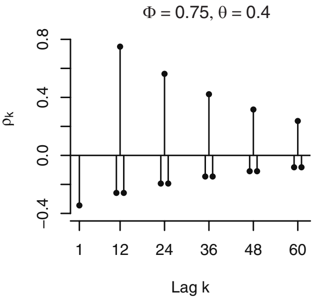
Second example: ARMA$(0,1)\text{x}(1,0)_{12}$

- Write out model expression!
- Using standard techniques, we find that for this model

$$
\begin{aligned}
\gamma_0 &= \left[\frac{1+\theta^2}{1-\Phi^2}\right]\sigma_e^2 \\
\rho_{12k} &= \Phi^k \text{ for } k \geq 1 \\
\rho_{12k-1} = \rho_{12k+1} &= \left(-\frac{\theta}{1+\theta^2}\Phi^k\right) \text{ for } k = 0, 1, 2, \ldots
\end{aligned}
$$

- Below are example ACFs for different values of the parameters.





Back to CO2 data

Exhibit 10.2 Carbon Dioxide Levels with Monthly Symbols

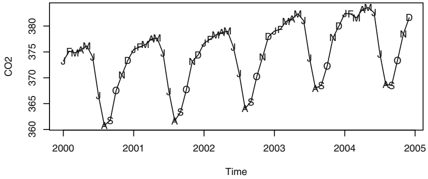
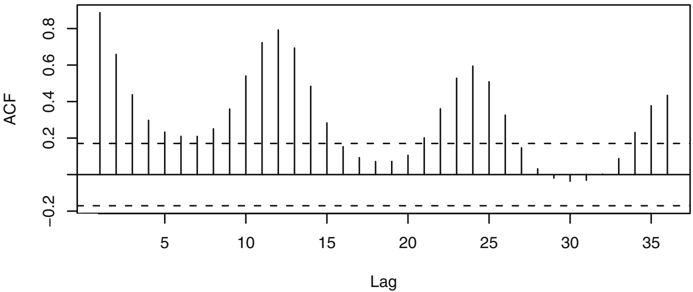


Exhibit 10.5 Sample ACF of CO_2 Levels

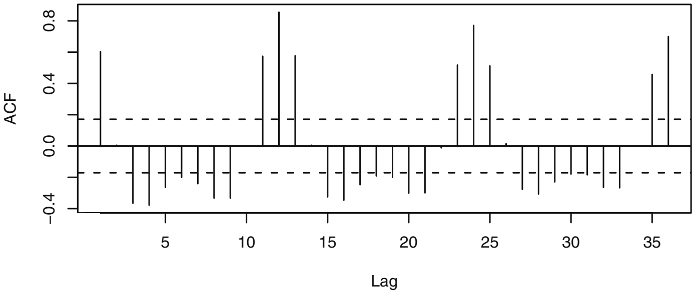


- Does the sample ACF decay exponentially?
- What to do?

Differenced CO2 data

- Difference the series as usual, to remove the time trend: $X_t = \nabla Y_t = Y_t - Y_{t-1}$ (reason for using X_t instead of W_t becomes clear in a bit).
- Sample ACF for X_t is below.
- Can we use a stationary seasonal model now?

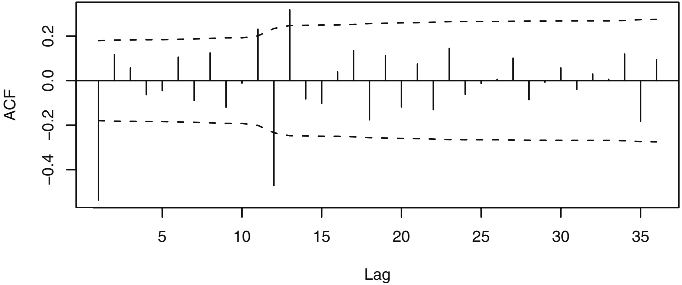
Exhibit 10.7 Sample ACF of First Differences of CO_2 Levels



More differencing...

- After differencing to remove the time trend, $X_t = \nabla Y_t = Y_t - Y_{t-1}$, we find that the sample ACF for lags 12,24,36, ... does not seem to decay exponentially (which we would expect under a seasonal ARMA model).
- What if we apply "seasonal differencing" to X_t: $W_t = \nabla_{12} X_t = X_t - X_{t-12} = \nabla_{12} \nabla Y_t = (Y_t - Y_{t-1}) - (Y_{t-12} - Y_{t-13})$.
- What does the sample ACF of W_t suggest?

Exhibit 10.9 Sample ACF of First and Seasonal Differences of CO_2



Candidate model for CO2 data

- The sample ACF for W_t shows sign. autocorrelation for lags 1 and at/around lag 12, thus an ARMA(0,1)x$(0,1)_{12}$ model (with both a nonseasonal and a seasonal MA(1) part) may be appropriate for W_t:

$$\begin{aligned} W_t &= \theta(B)\Theta(B)e_t, \\ &= e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}. \end{aligned}$$

- This type of model for a differenced series is an example of a non-stationary seasonal ARIMA models:
  - A process Y_t is a multiplicative ARIMA(p, d, q)x$(P, D, Q)_s$ model with seasonal period s, non-seasonal orders p, d, q and seasonal orders P, D, Q, if the differenced series $W_t = \nabla^d \nabla_s^D Y_t$ follows an ARMA(p, q)x$(P, Q)_s$ model.
- What ARIMA(p, d, q)x$(P, D, Q)_s$ model for Y_t does the ARMA(0,1)x$(0,1)_{12}$ model for $W_t = \nabla\nabla_{12} Y_t$ correspond to?

How to fit these multiplicative (non-stationary) seasonal models?

- Remember that seasonal models are special cases of non-seasonal ARIMA models (with many parameters that are equal to zero).
- Use maximum likelihood estimation to obtain parameter estimates.
- Model diagnostics proceed as explained in Ch. 8.

Exhibit 10.10 Parameter Estimates for the CO_2 Model

Coefficient	θ	Θ
Estimate	0.5792	0.8206
Standard error	0.0791	0.1137

$\hat{\sigma}_e^2 = 0.5446$: log-likelihood $= -139.54$, AIC $= 283.08$

```
> m1.co2=arima(co2,order=c(0,1,1),seasonal=list(order=c(0,1,1),
   period=12))
> m1.co2
```

Forecasting multiplicative seasonal ARIMA models

- Same approach is used as discussed for nonseasonal ARIMA models.
- Simple example: ARIMA(0,0,0)x$(0,1,1)_{12}$:

$$\begin{aligned} Y_t - Y_{t-12} &= e_t - \Theta e_{t-12}, \\ Y_{t+g} - Y_{t+g-12} &= e_{t+g} - \Theta e_{t+g-12}, \end{aligned}$$

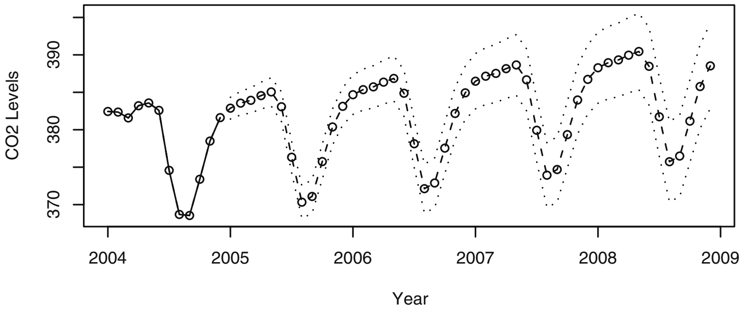
$$\left.\begin{aligned} \hat{Y}_t(1) &= Y_{t-11} - \Theta e_{t-11} \\ \hat{Y}_t(2) &= Y_{t-10} - \Theta e_{t-10} \\ &\vdots \\ \hat{Y}_t(12) &= Y_t - \Theta e_t \end{aligned}\right\}$$

$$\hat{Y}_t(\ell) = \hat{Y}_t(\ell - 12) \text{ for } \ell > 12$$

- After 12 months, the monthly point forecasts do not change anymore!
- Ch. 10.5 gives more examples (optional).

Forecasting the CO2 data series

Exhibit 10.17 Long-Term Forecasts for the CO_2 Model




```
> plot(m1.co2,n1=c(2004,1),n.ahead=48,xlab='Year',type='b',
   ylab='CO2 Levels')
```

Summary

- We discussed multiplicative seasonal ARIMA models.
- There's a bit of notation to get used to, but once we do get used to it, these models give a broad flexible class of time series model to deal with seasonal patterns.
- We focused on applying this type of modeling, using built-in R functions, to the CO2 time series to obtain forecasts that account for seasonal patterns.