Chapter 5

Testing General Linear Hypothesis

Overview

- Full model: $E\left(\underline{y}\right) = X\underline{\beta}$
- Linear hypothesis : $C\underline{\beta} = \underline{0}$
- Making use of the hypothesis, the model reduced to $E\left(\underline{y}\right) = Z\underline{\alpha}$
- Sum of Squares due to hypothesis $C\underline{\beta} = \underline{0}$ is given by $SSE_H - SSE$
- Test H_0: $C\underline{\beta} = \underline{0}$ against H_1: $C\underline{\beta} \neq \underline{0}$
- Test statistics: $F = \frac{(SSE_H - SSE)/q}{SSE/[n-(p+1)]}$
- Reject H_0 at sig. level α if $F_{\text{obs}} > F_\alpha(q, n-(p+1))$.

5.1 Introduction

- Consider the model $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$
- Suppose that we suspect $\beta_1 = \beta_2$, then the model used should be

$$E(y) = \beta_0 + \beta_1 x_1 + \beta_1 x_2$$
$$= \beta_0 + \beta_1 (x_1 + x_2)$$

- We want to test

$$\mathrm{H}_0: \beta_1 = \beta_2 \text{ against } \mathrm{H}_1: \beta_1 \neq \beta_2$$

Or

$$\mathrm{H}_0: \beta_1 - \beta_2 = 0 \text{ against } \mathrm{H}_1: \beta_1 - \beta_2 \neq 0$$

5.1 Introduction

- The hypothesis $\beta_1 - \beta_2 = 0$ is said to be a linear hypothesis in β's
- A linear function in β's is defined as

$$c_1\beta_1 + c_2\beta_2 + \cdots + c_p\beta_p$$

- A linear hypothesis may consist of more than one statements about the linear functions in β's.
  - For example

$$c_{10}\beta_0 + c_{11}\beta_1 + \cdots + c_{1p}\beta_p = 0$$
$$c_{20}\beta_0 + c_{21}\beta_1 + \cdots + c_{2p}\beta_p = 0$$
$$c_{30}\beta_0 + c_{31}\beta_1 + \cdots + c_{3p}\beta_p = 0$$

5.2 Examples

- In the following examples we consider the **full model**

$$E(Y) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

Example 1

- H_0: $\beta_1 = 0, \beta_2 = 0, \cdots$, and $\beta_p = 0$
  - There are p linear equations specified in H_0
  - These equations are all linearly independent

5.2 Examples

Example 1 (Continued)

- H_0: $\beta_1 = 0, \beta_2 = 0, \cdots$, and $\beta_p = 0$
- The model under H_0 becomes

$$E(y) = \beta_0 + 0x_1 + 0x_2 + \cdots + 0x_p$$

- Hence the model under H_0 can be written as

$$E(y) = \beta_0$$

- The model under H_0 is called the **reduced model**
  - The no. of parameters is **reduced** from $p + 1$ to 1

Example 2

- $H_0: \beta_1 - \beta_2 = 0, \beta_2 - \beta_3 = 0, \cdots$, and $\beta_{p-1} - \beta_p = 0$
  - There are p – 1 linearly independent equations
  - The above p – 1 equations are equivalent to

$$\beta_2 = \beta_1, \beta_3 = \beta_1, \ \cdots, \ \beta_p = \beta_1$$

- Model under H_0 is

$$E(y) = \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \cdots + \beta_1 x_p$$
$$= \beta_0 + \beta_1 (x_1 + \cdots + x_p)$$

- The no. of parameters reduced from p + 1 to 2.
- Note: H_0 is equivalent to $\beta_1 = \beta_2 = \cdots = \beta_p$.

Example 3

$$\mathrm{H}_0: \quad \begin{array}{l} c_{10}\beta_0 + c_{11}\beta_1 + \cdots + c_{1p}\beta_p = 0 \\ c_{20}\beta_0 + c_{21}\beta_1 + \cdots + c_{2p}\beta_p = 0 \\ \quad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ c_{m0}\beta_0 + c_{m1}\beta_1 + \cdots + c_{mp}\beta_p = 0 \end{array} \quad \text{or } C\underline{\beta} = \underline{0}$$

where

$$C = \begin{pmatrix} c_{10} & c_{11} & \cdots & c_{1p} \\ c_{20} & c_{21} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m0} & c_{m1} & \cdots & c_{mp} \end{pmatrix} \text{ and } \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

Example 3 (Continued)

- We assume that the m equations of the $p + 1$ β's are linearly dependent.

Example 3a

- Consider the following 3 equations with 5 β's (β_0, β_1, β_2, β_3 and β_4)

$$\beta_0 + 3\beta_1 - 3\beta_2 = 0, \qquad (1)$$

$$\beta_0 + \beta_1 = 0, \qquad (2)$$

$$\beta_1 - 1.5\beta_2 = 0 \qquad (3)$$

- β_3 and β_4 do not show up in the above equations because the corresponding coefficients, c_3 and c_4 are 0 in all 3 equations.

Example 3 (Continued)

Example 3a

$$\beta_0 + 3\beta_1 - 3\beta_2 = 0, \qquad (1)$$

$$\beta_0 + \beta_1 = 0, \qquad (2)$$

$$\beta_1 - 1.5\beta_2 = 0 \qquad (3)$$

- The third equation can be expressed as half of the difference of the first two equations.
- After eliminating the third equation, the remaining 2 equations are linearly independent
- the no. of parameters = $p + 1 = 5$

  the no. of equations = $m = 3$ and

  the no. of linearly independent equations = $q = 2$.

Example 3 (Continued)

- We assume that m equations of the $p + 1$ β's are linearly dependent.

- Without loss of generality, we assume that the last $m - q$ of them depend upon the first q linearly independent equations.

5.3 Testing a general linear hypothesis $C\underline{\beta} = \underline{0}$

5.3.1 **Full Model**: $E\left(\underline{y}\right) = X\underline{\beta}$,

where $\underline{y}$: $n \times 1$, $X : n \times (p + 1)$, $\underline{\beta}$: $(p + 1) \times 1$.

- LSE of β:

$$\underline{\hat{\beta}} = (X'X)^{-1}X'\underline{y}$$

- $SSE = \underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y}$ with $n - (p + 1)$ d.f.

5.3.2 Reduced Model

- H_0: $C\underline{\beta} = \underline{0}$ provides q linearly independent conditions on the parameters β_0, β_1, ..., and β_p.
- We use these q linearly independent equations to solve for q of the β's in terms of the other $(p + 1) - q$ of them.
- Substituting these solutions back into the original model, we obtain the following reduced model

$$E(\underline{y}) = Z\underline{\alpha},$$

where Z: $n \times [(p+1) - q]$ and $\underline{\alpha}$: $[(p+1) - q] \times 1$

Reduced Model (Continued)

Example 3a (Continued)

$$\beta_0 + 3\beta_1 - 3\beta_2 = 0, \qquad (1)$$

$$\beta_0 + \beta_1 = 0, \qquad (2)$$

$$\beta_1 - 1.5\beta_2 = 0 \qquad (3)$$

- From Eq 3, we have $\beta_1 = 1.5\beta_2$,
- From Eq 2, we have $\beta_0 = -\beta_1$. Hence $\beta_0 = -1.5\beta_2$
- Hence we can express β_0 and β_1 in terms of β_2

Reduced Model (Continued)

Example 3a (Continued)

Full model: $E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$

with 5 parameters $\beta_0, \beta_1, \beta_2, \beta_3$ and β_4

Reduced model:

$$E(y) = -1.5\beta_2 + 1.5\beta_2 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$$

or $E(y) = \beta_2(-1.5 + 1.5x_1 + x_2) + \beta_3 x_3 + \beta_4 x_4$

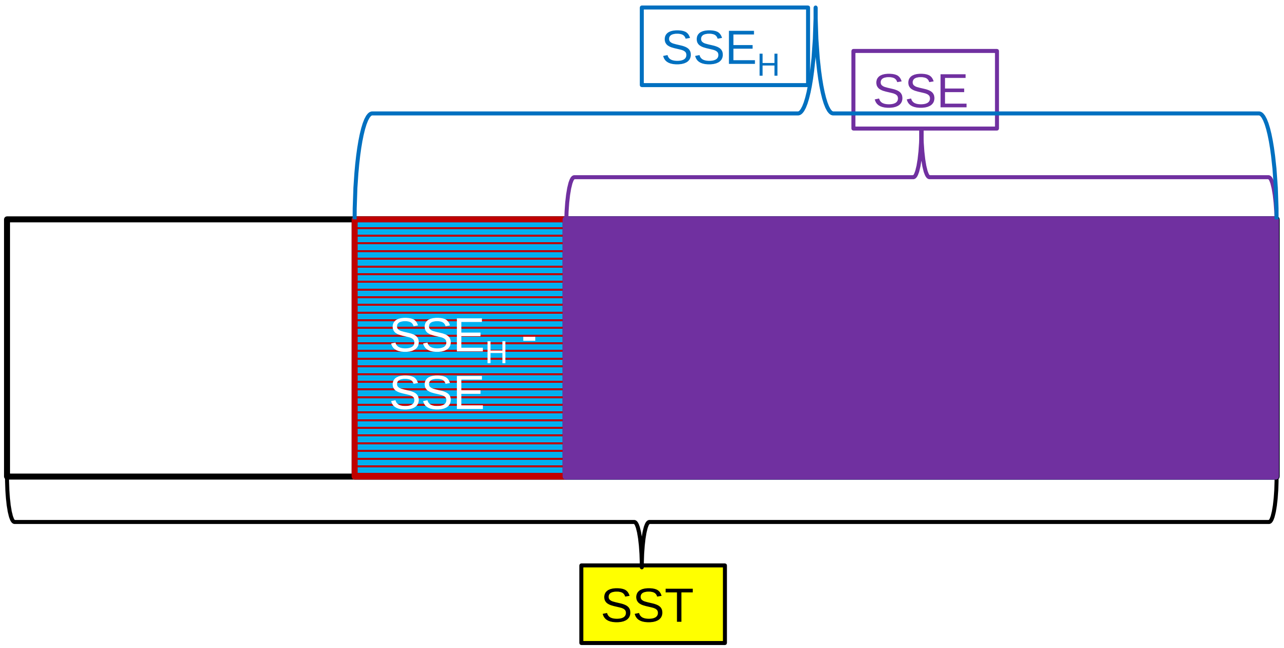
with 3 parameters β_2, β_3 and β_4

5.3.3 Sum of Squares due to $C\underline{\beta} = \underline{0}$

- Reduced model : $\underline{y} = Z\underline{\alpha}$
- LSE of $\underline{\alpha}$:

$$\underline{\hat{\alpha}} = (Z'Z)^{-1}Z'\underline{y}$$

and $SSE_H = \underline{y}'\underline{y} - \underline{\hat{\alpha}}'Z'\underline{y}$ with $n - (p + 1) + q$ d.f.



5.3.3 Sum of Squares due to $C\underline{\beta} = \underline{0}$

- Note: $SSE_H \geq SSE$ since the number of parameters in the reduced model is less than the original (full) model.
  - We expect the model with more parameters explains a larger part of the total sum of squares than the model with less parameters.
- $SSE_H - SSE$ is called the **sum of squares due to the hypothesis $\boldsymbol{C\underline{\beta} = \underline{0}}$** and has $[n - \{(p + 1) - q\}] - [n - (p + 1)] = q$ d.f.
- If SSE_H is not much different from SSE, then it implies that there is not much difference between the reduced model and the full model

5.3.4 Testing $\mathrm{H}_0: C\underline{\beta} = \underline{0}$

- To test $\mathrm{H}_0: C\underline{\beta} = \underline{0}$ vs $\mathrm{H}_1: C\underline{\beta} \neq \underline{0}$,

  we use the test statistic

$$F = \frac{(SSE_H - SSE)/q}{SSE/[n-(p+1)]}$$

- Under H_0, $F \sim F(q, n-(p+1))$.

- H_0 is rejected at α level of significance if $F_{\text{obs}} > F_\alpha(q, n-(p+1))$, where F_{obs} is the observed value of F.

5.4 Examples (Continued)

- Example 1 (cont'd)
- Test H_0: $\beta_1 = 0, \beta_2 = 0, \cdots,$ and $\beta_p = 0$

  Full model:

$$E(y) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

  or $$E(\underline{y}) = X\underline{\beta}$$

- Reduced model:

$$E(y) = \beta_0 + 0x_1 + 0x_2 + \cdots + 0x_p = \beta_0$$

  or $$E\left(\underline{y}\right) = \underline{1}_n \beta_0$$

- i.e. $Z = \underline{1}_n$ and $\underline{\alpha} = \beta_0$.

Example 1 (Continued)

- LSE of $\underline{\alpha}$ (for the reduced model $E(\underline{y}) = Z\underline{\alpha}$)

$$\underline{\hat{\alpha}} = (\underline{1}_n{}'\underline{1}_n)^{-1}\underline{1}_n{}'\underline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

and

$$SSE_H = \underline{y}'\underline{y} - \underline{\hat{\alpha}}'\underline{1}_n{}'\underline{y} = \underline{y}'\underline{y} - n\bar{y}^2 \text{ with } n - 1 \text{ d.f.}$$

For the full model $E(\underline{y}) = X\underline{\beta}$,

- LSE of $\underline{\beta}$:

$$\underline{\hat{\beta}} = (X'X)^{-1}X'\underline{y}$$

and $SSE = \underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y}$ with $n - (p + 1)$ d.f.

Example 1 (Continued)

- $SSE_H - SSE = \underline{\hat{\beta}}'X'\underline{y} - n\bar{y}^2$ with $[n - 1] - [n - (p + 1)] = p$ degrees of freedom
- Let

$$F = \frac{\left(\underline{\hat{\beta}}'X'\underline{y} - n\bar{y}^2\right)/p}{(\underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y})/[n-(p+1)]}$$

- Reject H_0 at the level of significance α if

$$F_{\text{obs}} > F_\alpha(p, n - (p + 1)).$$

Note: The above test is the usual F-test for testing if the model is significant.

Example 2 (Continued)

$$H_0 : C\underline{\beta} = \underline{0}$$

$$\text{where } C = \begin{pmatrix} 0 & 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}$$

$$\text{and } \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

Example 2 (Continued)

- Full model:

$$E(y) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

or $$E(\underline{y}) = X\underline{\beta}$$

- H_0: $\beta_1 - \beta_2 = 0$, $\beta_2 - \beta_3 = 0$, $\cdots$, and $\beta_{p-1} - \beta_p = 0$
- Reduced model:

$$E(y) = \beta_0 + \beta_1 x_1 + \beta_1 x_2 + \cdots + \beta_1 x_p$$
$$= \beta_0 + \beta_1 (x_1 + x_2 + \cdots + x_p)$$

or $$E\left(\underline{y}\right) = Z\underline{\alpha}$$

where $Z = \begin{pmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_n \end{pmatrix}$ wih $z_i = \sum_{j=1}^{p} x_{ji}$ and $\underline{\alpha} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$

Example 2 (Continued)

- LSE of $\underline{\alpha}$ (for the reduced model $E\left(\underline{y}\right) = Z\underline{\alpha}$)

$$\underline{\hat{\alpha}} = (Z'Z)^{-1}Z'\underline{y}$$

and $SSE_H = \underline{y}'\underline{y} - \underline{\hat{\alpha}}'Z'\underline{y}$ with n – 2 d.f.

For the full model $E(\underline{y}) = X\underline{\beta}$,

- LSE of $\underline{\beta}$:

$$\underline{\hat{\beta}} = (X'X)^{-1}X'\underline{y}$$

and $SSE = \underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y}$ with n – (p + 1) d.f.

Example 2 (Continued)

- $SSE_H - SSE = \underline{\hat{\beta}}'X'\underline{y} - \underline{\hat{\alpha}}'Z'\underline{y}$ with $(p - 1)$ d.f.

- Let
$$F = \frac{\left(\underline{\hat{\beta}}'X'\underline{y} - \underline{\hat{\alpha}}'Z'\underline{y}\right)/(p-1)}{(\underline{y}'\underline{y} - \underline{\hat{\beta}}'X'\underline{y})/[n-(p+1)]}$$

- Reject H_0 at the level of significance α if
$$F_{\text{obs}} > F_\alpha(p - 1, n - (p + 1)).$$