ST 5203: Experimental Design

(Semester 1, AY 2017/2018)

Text book: *Experiments: Planning, Analysis, and Optimization (2nd. edition)*

by Jeff Wu and Mike Hamada

Topic 2: Experiments with a Single Factor; One way ANOVA

- One-way layout with fixed effects.
- Analysis of Variance (ANOVA).
- Multiple Comparison.
- Contrast Test.
- Quantitative treatment factors and Orthogonal Polynomials.
- Model diagnosis.
- One-way layout with random effects.

An Example to One-way Layout Data

- **Objective:** Study whether some form of of controlled gas atmosphere would provide a more effective packaging environment for meat storage.
- **Response:** Psychrotrophic bacteria [log(count/cm^2)] after 9 days.
- **Factor:** Packaging conditions.
- **Factor levels:** "Commercial plastic wrap"; "Vacuum package"; "1% CO, 40% O_2, 59% N"; "100% CO_2".
- Randomization: use some randomization mechanism to randomly assign 12 beef steaks (units) to four treatment groups, three beef steaks each.

Data Collection

- Table for the experiment outputs:

Data Table for Meat Storage Experiment

Packaging Condition	Psychorotrophic Bacteria Log(count/cm^2)		
Commercial plastic wrap	7.66	6.98	7.80
Vacuum packaged	5.26	5.44	5.80
1% CO, 40% O_2, 59% N	7.41	7.33	7.04
100% CO_2	3.51	2.91	3.66

One-way Layout Model

- Cell means model: $y_{ij} = \mu_i + e_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, where,

  y_{ij} = the jth observation from the ith treatment group.
  μ_i = the mean of the ith treatment population;
  e_{ij} = error, independent $N(0, \sigma^2)$.

- Treatment effects model: $y_{ij} = \eta + \tau_i + e_{ij}$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, where,

  $\eta = \sum_{i=1}^{k} \mu_i / k$, the average of the population mean.
  $\tau_i = \mu_i - \eta$, the treatment effects.
  Note: $\sum_{i=1}^{k} \tau_i = 0$. (Explain)

Model Fitting

- Least Squares (on board)

- Decomposition:

$$\begin{aligned}
\text{(cell means model)} \quad y_{ij} &= \bar{y}_{i\cdot} + (y_{ij} - \bar{y}_{i\cdot}) \\
&= \hat{\mu}_i + \hat{e}_{ij} \\
\text{(treatment effects model)} \quad y_{ij} &= \bar{y}_{\cdot\cdot} + (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}) + (y_{ij} - \bar{y}_{i\cdot}) \\
&= \hat{\eta} + \hat{\tau}_i + \hat{e}_{ij}
\end{aligned}$$

where $\bar{y}_{i\cdot} = \sum_{j=1}^{n_i} y_j / n_i$.

Analysis of Variance (ANOVA)

- Following the treatment effects model decomposition (explain).

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{..})^2 &= \sum_{i=1}^{k} n_i(\bar{y}_{i.}-\bar{y}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i.})^2 \\ \text{SST} &= \text{SSTr} + \text{SSE}\end{aligned}$$

- ANOVA table: $N = \sum_{i=1}^{k} n_i$, the total number of trials in the experiment.

ANOVA Table

Source	D.F.	Sum of Squares	Mean Squares
Treatments	$k-1$	SSTr	$\text{MSTr} = \text{SSTr}/(k-1)$
Error	$N-k$	SSE	$\text{MSE} = \text{SSE}/(N-k)$
Total	$N-1$	SST	

F-test and Its Properties

- The F statistic for testing the significance among the treatments, i.e.,

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_k \quad [(\text{or}) \ \tau_1 = \tau_2 \ldots = \tau_k = 0,]$$

is

$$F = \text{MSTr}/\text{MSE} \sim F_{k-1,N-k}.$$

- Properties of F-test in ANOVA table
  - Its rejection region is one-sided, say, for any given significant level α, p-value $= P(F \geq f_{\text{obs}})$ with $F \sim F_{k-1,N-k}$.
  - If all the assumptions for the model (normal, equal variance and independent) are all satisfied, given any value of α (type I error), the F-test has the smallest probability of a type II error.
  - The F-test is robust against the violations of the normality and constant variance, but not robust against the lack of independence.

ANOVA for Meat Storage Experiment

	D.F.	Sum of Squares	Mean Squares	F
Treatments	3	32.873	10.958	94.584
Error	8	0.927	0.116	
Total	11	33.800		

- p-value $= P(F_{3,8} > 94.584) = 0.000$, thus declaring a significant packaging condition difference in meat storage.
- Further question: among $6 = \binom{4}{2}$ pairs of packaging conditions, what pairs show significant difference? We may need multiple comparisons to answer this question.

Statistical Properties

(Derivations on board)

- $E(\text{MSE}) = \sigma^2$.
- $E(\text{MSTr}) = \dfrac{\sum_{i=1}^{k} n_i(\tau_i - \bar{\tau})^2}{k-1} + \sigma^2$, where $\bar{\tau} = \displaystyle\sum_{i=1}^{k} n_i\tau_i/N$.
- $\text{SSE}/\sigma^2 \sim \chi^2_{N-k}$.
- $\text{SSTr}/\sigma^2 \sim \chi^2_{k-1}$ under $H_0 : \tau_1 = \ldots = \tau_k = 0$.
- SSE and SSTr are independent.
- $F = \text{MSTr}/\text{MSE} \sim F_{k-1,N-k}$ under $H_0 : \tau_1 = \ldots = \tau_k = 0$.

Multiple Comparison

- If we would like to compare only one pair of treatments ($H_{0,ij} : \mu_i = \mu_j$ versus $H_{a,ij} : \mu_i \neq \mu_j$), it is common to use the t-test,

$$T_{ij} = \frac{\bar{y}_{i\cdot} - \bar{y}_{j\cdot}}{\hat{\sigma}\sqrt{1/n_i + 1/n_j}} \sim t_{N-k},$$

  where n_j = number of observations for treatment j, $\hat{\sigma}^2 = MSE$ in ANOVA table (explain); thus, reject "$H_{0,ij} : \mu_i = \mu_j$ at level α" if $|T_{ij}| > t_{N-k,\alpha/2}$.
- Suppose we are simultaneously performing k' tests. *Experiment-wise error rate* (EER) = probability of making at least one type I error.
- In multiple comparison literature, EER is also called *Family-wise error rate* (FWER).

Example

- In our "meat storage experiment", suppose k' tests are performed to test $\{H_{0,ij} : \mu_i = \mu_j, 1 \leq i < j \leq k\}$. There are $k' = 6 = \binom{4}{2}$ pairs to compare.

μ_1 & μ_2	μ_1 & μ_3	μ_1 & μ_4	μ_2 & μ_3	μ_2 & μ_4	μ_3 & μ_4
7.12	0.80	14.82	-6.33	7.70	14.03

  Need to use multiple comparisons to control EER.

- Note: Here $k' = 6$ is just an example, actually, k' can be any integer value. For example, in the above example, if we are only interest in testing $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$ are simultaneously true, then, $k' = 2$. k' **is the number of simultaneous tests you want to perform**.

Bonferroni Method: Theory

- Suppose we want to control EER by α, i.e. we want $P(EER) \leq \alpha$.
- On the other hand

$$\begin{aligned} P(EER) &= P(\text{make AT LEAST 1 type I error out of k' tests}) \\ &= P(\cup_{i=1}^{k'}\{\text{make type I error for ith test}\}) \\ &\leq \sum_{i=1}^{k'} P(\text{make type I error for ith test}) \end{aligned}$$

- How can we make $P(EER) \leq \alpha$ true?
- Bonferroni's idea: get each $P(\text{make type I error for ith test})$ controlled by α/k'.
- In other words: set each test's type I error (significant level) as α/k'.

Bonferroni Method

- Declare "μ_i different from μ_j at level α" if $|t_{i,j}| > t_{N-k,\frac{\alpha}{2k'}}$, where k' = the number of tests.
- For one-way layout with k treatments, if we would like to make multiple comparison on all combinations of μ_i's, then $k' = \binom{k}{2} = k(k-1)/2$. As k increases, k' increases, and the critical value $t_{N-k,\frac{\alpha}{2k'}}$ gets bigger.
- Drawbacks: since k' can be very large, it tends to be conservative in rejecting hypothesis. Actually, Bonferroni method is a very conservative multiple testing approach.
- Advantages: (1). No assumption on the independence; (2). Work for any number k'.
- For our example, choose $\alpha = 0.05$, then, $k' = 6$, $t_{8,0.05/12} = 3.479$. Among the 6 $t_{i,j}$ values, μ_1 & μ_2, μ_1 & μ_4, μ_2 & μ_3, μ_2 & μ_4, μ_3 & μ_4 are bigger than 3.479, thus, these pairs are declared as different at level 0.05.

Tukey Method: Theory

- **Note**: Tukey method is only applicable in comparing difference of treatment effects.
- Tukey method is owing to the following fact:

$$\sqrt{2}\max_{i,j}\frac{|\bar{y}_{i\cdot}-\bar{y}_{j\cdot}|}{\hat{\sigma}\sqrt{1/n_i+1/n_j}}\sim Q_{k,N-k}$$

$$\text{i.e.}\qquad \sqrt{2}\max_{i,j}|T_{ij}|\sim Q_{k,N-k},$$

where $Q_{k,N-k}$ is the Studentized Range statistic, k and $N-k$ are degrees of freedom.

Tukey Method

- Another way to test $\{H_{0,ij} : \mu_i = \mu_j, 1 \le i < j \le k\}$: reject if

$$\max_{i,j} |T_{ij}| \ge q_{k,N-k,\alpha}/\sqrt{2},$$

  where $q_{k,N-k,\alpha}$ is the upper α quantile of the **studentized range distribution** with degrees of freedom k and $N-k$.
- Tukey method for multiple comparison: declare "μ_i different from μ_j at level α" if

$$|T_{ij}| > \frac{1}{\sqrt{2}} q_{k,N-k,\alpha}.$$

- For meat storage experiment $\frac{1}{\sqrt{2}} q_{k,N-k,\alpha} = \frac{1}{\sqrt{2}} q_{4,8,0.05} = \frac{4.53}{\sqrt{2}} = 3.20$. Again, only μ_1 & μ_2 with $t_{1,2} = 0.8$ does not exceed 3.20.
- Tukey method is more powerful than Bonferroni method since 3.20 is smaller than 3.479. (why?)

Tukey Method

	k													
v	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	17.97	26.98	32.82	37.08	40.41	43.12	45.40	47.36	49.07	50.59	51.96	53.20	54.33	55.36
2	6.08	8.33	9.80	10.88	11.74	12.44	13.03	13.54	13.99	14.39	14.75	15.08	15.38	15.65
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.15	10.35	10.52
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74	5.87	5.98	6.09	6.19	6.28
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.98
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39	5.51	5.61	5.71	5.80	5.88
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.71
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.57	5.65
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59

α=upper tail probability, v=degrees of freedom, k=number of treatments

For complete tables corresponding to various values of a refer to Appendix E.

Residual Analysis: Theory

- Define the residual for the ith observation $\mathbf{x}_i$ as

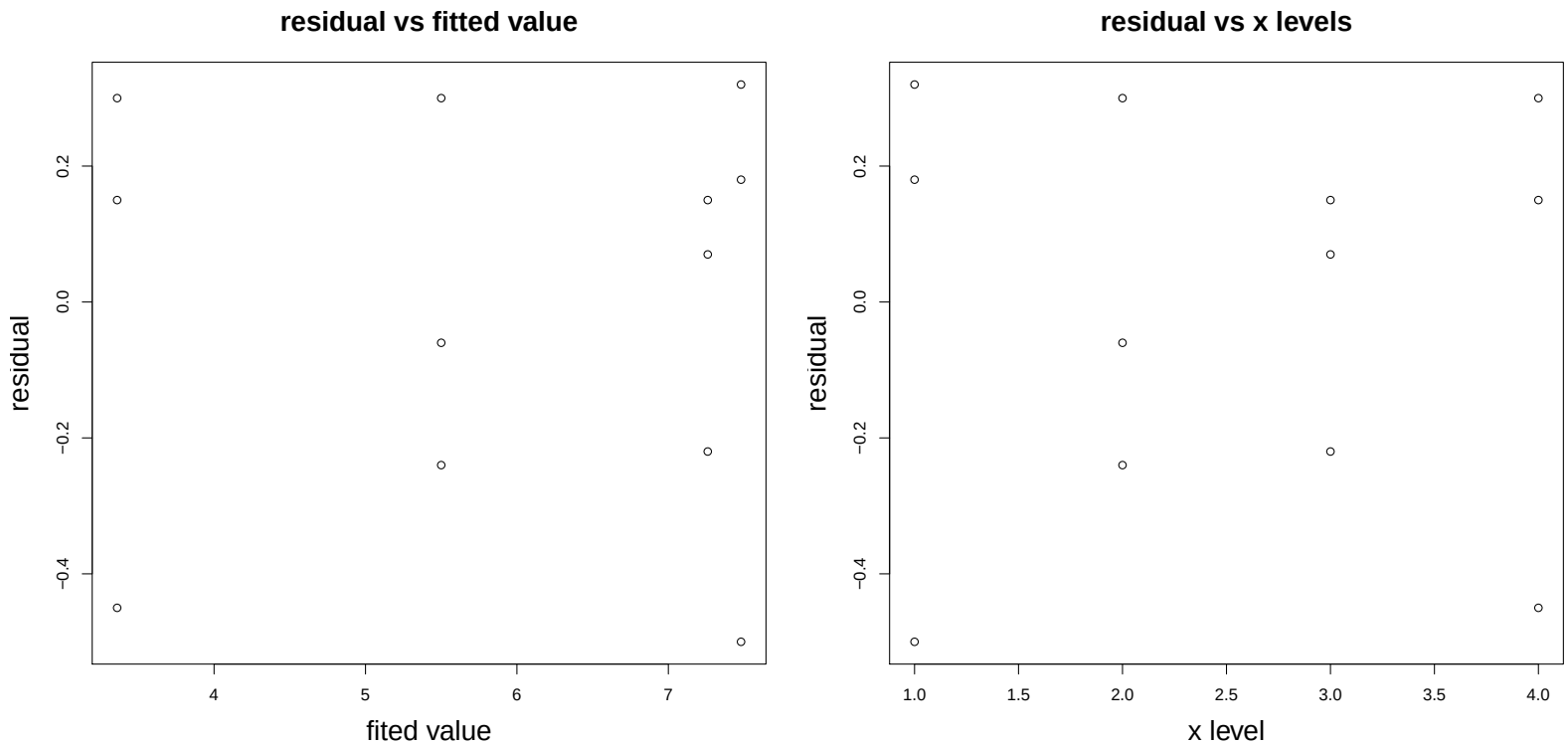
$$r_i = y_i - \hat{y}_i, \quad \hat{y}_i = \mathbf{x}_i' \hat{\boldsymbol{\beta}},$$

  $\hat{y}_i$ contains information given by the model; r_i is the "difference" between y_i(observed) and $\hat{y}_i$(fitted) and contains information on possible model inadequacy.
  Vector of residuals $\mathbf{r} = (r_1, r_2, \ldots, r_N)' = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}$.
- Question: what are the forms of $\mathbf{x}_i$, $\mathbf{X}$, $\mathbf{y}$ and $\hat{\boldsymbol{\beta}}$ for our one-way layout model?
- If all model assumptions are correct, it can be shown that
  - $E(\mathbf{r}) = 0$,
  - $\mathbf{r}$ and $\hat{\mathbf{y}}$ are independent,
  - variances of r_i are nearly constant for "nearly balanced" designs, namely, all treatments have roughly the same number of runs.

Residual Plots

1. Plot r_i vs. $\hat{y}_i$: It should appear as a parallel band around 0. Otherwise, it would suggest some violation of model assumption. For example, if the spread of r_i increases as $\hat{y}_i$ increases, then the error variance of y increases with mean of y, and we need a transformation of y. (Will be explained in Topic 3.)
2. Plot r_i vs. x_i: If not a parallel band around 0, relationships between y_i and x_i are not fully captured. Revise the $\mathbf{X}\beta$ part of the model.
3. Plot r_i vs. time sequence: to see if there is a time trend or autocorrelation over time.
4. Plot r_i from replicates per treatment: to see if error variance depends on treatment.

Plot r_i vs. $\hat{y}_i$ and r_i vs. x_i



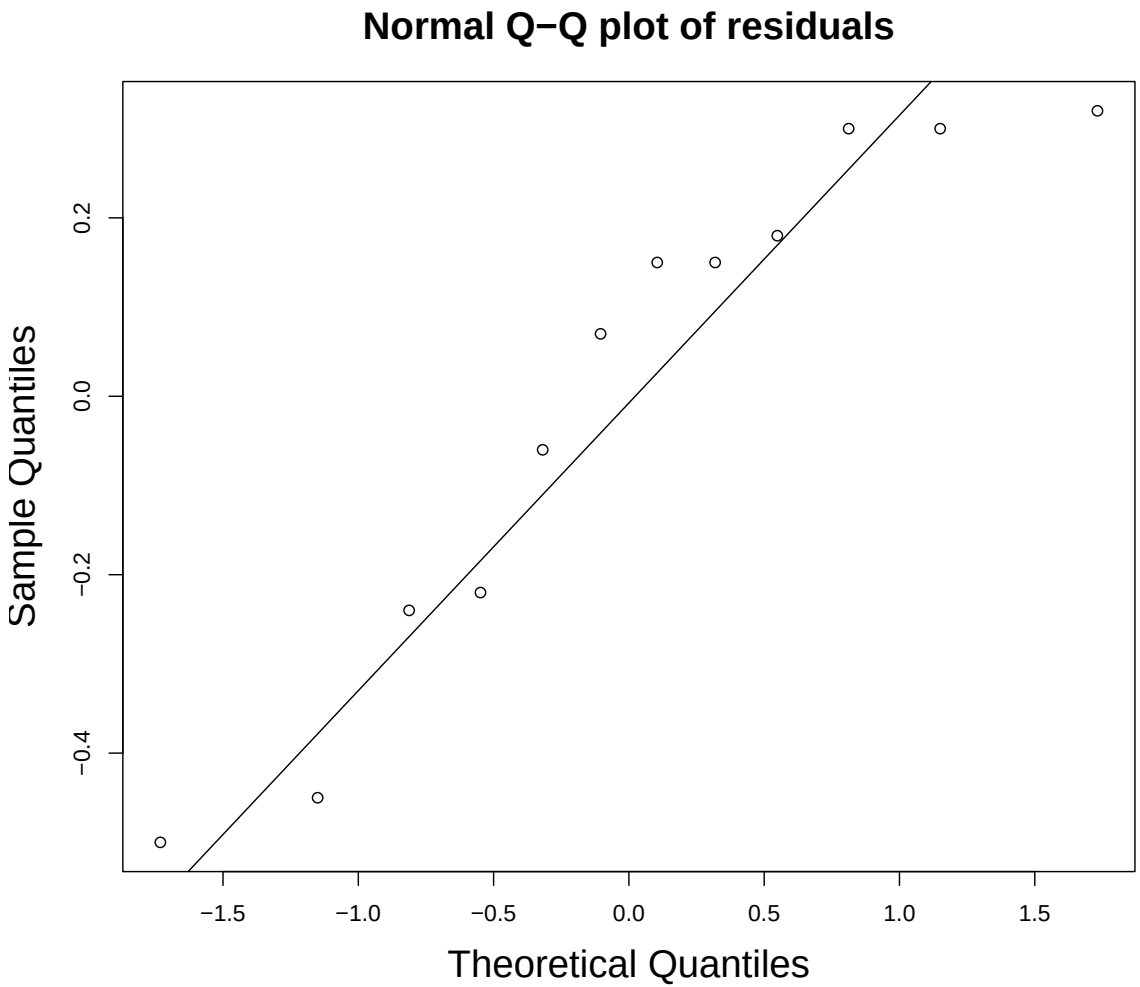
Normal Probability Plot (Q-Q plot)

- Purpose: To visually check if a set of numbers are from a normal distribution. e.g., if the residuals follow a normal distribution, the points should be close to a straight line.
- How to draw: for residuals $r_i, i = 1, 2, \cdots, N$.
  - Order $r_1, r_2, \cdots, r_N$ to have: $r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(N)}$.
  - Find all $(i - 0.5)/N$th standard normal percentiles, say, $z_{\frac{1-0.5}{N}}, z_{\frac{2-0.5}{N}}, \cdots, z_{\frac{N-0.5}{N}}$, where $z_{\frac{i-0.5}{N}}, i = 1, 2, \cdots, N$ are corresponding solutions for

$$P(Z < z_{\frac{i-0.5}{N}}) = (i - 0.5)/N, \quad i = 1, 2, \cdots, N,$$

    where $Z \sim N(0, 1)$.
  - Plot $r_{(i)}$ versus $z_{\frac{i-0.5}{N}}$ $i = 1, 2, \cdots, N$.

QQ plot of r_i


Normal Q−Q plot of residuals
Sample Quantiles
Theoretical Quantiles
0.2
0.0
−0.2
−0.4
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

More on the Constraint Condition

- Without posing constraint condition on $y_{ij} = \eta + \tau_i + e_{ij}$, it is not identifiable. (Explain)
- Mathematically, there exist infinite many possible constraint conditions on the model parameters η and τ_i, such that the model is identifiable. So, which one should we use?
- Constraint conditions should take into account two aspects: mathematical validity (i.e. model identifiability) and meaningful interpretation in real application.
- One good constraint condition we have used so far is the "zero sum constraint condition" $\sum_{i=1}^{k} \tau_i = 0$.
- Another type of widely adopted constraint condition is the "baseline constraint condition".

Baseline Constraint Condition (cont.)

- Recall that we have k treatment groups for one-way lay out design. The basic idea of baseline constraint condition is to treat one group as baseline (controlled group), and the others as treatment groups.
- Example: in a clinical trial, to test which medicine out of 3 is the best,
  - 100 patients are recruited to join the study;
  - One group of 25 randomly selected patients is assigned to take placebo;
  - For the other 3 groups, each group with 25 patients is assigned to one kind of medicine.
- Here, the group taking placebo is usually labeled as the controlled group, whereas the other groups taking medicines are treatment groups.

Baseline Constraint Condition (cont.)

- Instead of modeling τ_i as the difference between levels of the factor and the grand mean, we can model it as the difference between treatment groups and the controlled group.
- Without loss of generality, if we assume group 1 ($i = 1$) as the controlled group,, then τ_i has a clear meaning: the difference between treatment group i and the controlled group. Clearly, now $\tau_1 = 0$, i.e. the difference between the controlled group and itself is 0; η now stands for the population mean of the controlled group.
- In summary, one-way layout model with baseline constraint condition is given by:

$$y_{ij} = \eta + \tau_i + e_{ij},$$

with the baseline constraint condition $\tau_1 = 0$.

- Question: Estimators of parameters? Statistical inference? How is the model above comparable/different to the model with "sum up to zero" constraint?

Contrast Test

- A **contrast** among means is defined as

$$C = \sum_{i=1}^{k} c_i \mu_i = c_1\mu_1 + c_2\mu_2 + \ldots + c_k\mu_k,$$

  where $\sum_{i=1}^{k} c_i = 0$. $\mathbf{c} = (c_1, \ldots, c_k)'$ is the **contrast vector**.

- Examples:

$$\begin{aligned} C_1 &= \mu_1 - \frac{1}{3}(\mu_2 + \mu_3 + \mu_4) \\ C_2 &= \mu_2 - \frac{1}{2}(\mu_3 + \mu_4) \end{aligned}$$

- To test $H_0 : C = 0$ versus $H_a : C \neq 0$, we use the test statistic

$$T_C = \frac{\sum_{i=1}^{k} c_i \hat{\mu}_i}{s.e.(\sum_{i=1}^{k} c_i \hat{\mu}_i)} \sim t_{N-k}.$$

Contrast Test (cont.)

- In one-way layout model, $\hat{\mu}_i = \bar{y}_{i\cdot}$. We have

$$\widehat{C} = \sum_{i=1}^{k} c_i \hat{\mu}_i = \sum_{i=1}^{k} c_i \bar{y}_{i\cdot}$$

$$\text{Var}(\widehat{C}) = \sum_{i=1}^{k} c_i^2 \frac{\sigma^2}{n_i}$$

- Therefore, the t-statistic is given by

$$T_C = \frac{\sum_{i=1}^{k} c_i \bar{y}_{i\cdot}}{\sqrt{\sum_{i=1}^{k} c_i^2 \frac{\hat{\sigma}^2}{n_i}}}$$

  where $\hat{\sigma}^2$ is the estimated error variance (MSE).
- More about contrast can be found in Chapter 3 of *Design and Analysis of Experiments* by Douglas C. Montgomery.

A Quantitative Factor Experiment Example

- Data

  y = gain yields of crop,
  x = plant density at 10, 20, 30, 40 and 50.

Plant Density (x)				
10	20	30	40	50
12.2	16.0	18.6	17.6	18.0
11.4	15.5	20.2	19.3	16.4
12.4	16.5	18.2	17.1	16.6

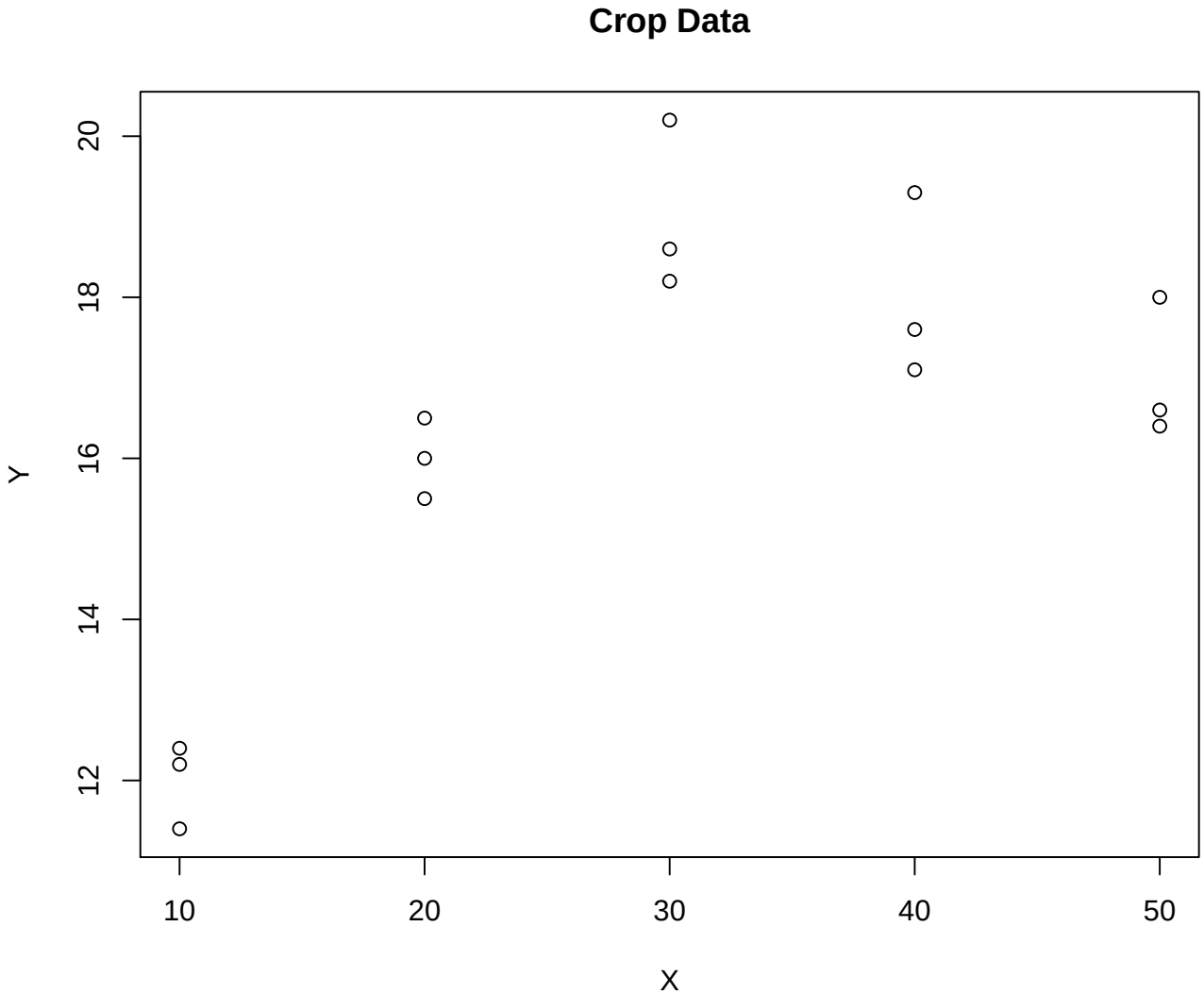
One-way ANOVA

- ANOVA table

Source	D.F.	Sum of Squares	Mean Squares	F
x	4	87.600	21.900	29.278
Residuals	10	7.480	0.748	
Total	14	95.08		

- Conclusion: the plant density has significant effect on the yield.
- However, multiple comparison cannot provide desirable explanation for the effect. (why?)
- The effects of a quantitative factor like the plant density can be decomposed into linear, quadratic, etc.

Scatter Plot



Why Orthogonal Polynomial

- For one-way quantitative factor experiments, the response trend can be better explained by polynomial models.
- If the factor has k levels, the polynomial can be fitted up to the order $k-1$.
- For meaningful explanation of the observed data trend, we can check the orthogonal contrasts among the treatment factor levels that measure the linear, quadratic or higher level polynomial effects.
- If y_1, y_2, y_3 are observed at levels $x = 10, 20, 30$, then
  - $(y_3 - y_2) + (y_2 - y_1) = y_3 - y_1$ is called the *linear contrast*;
  - $(y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$ is called the *quadratic contrast*;
  - Note that their contrast vectors $\mathbf{c}_1 = (-1, 0, 1)'$ and $\mathbf{c}_2 = (1, -2, 1)'$ are orthogonal to each other.

Orthogonal Polynomial

- Polynomial regression model:

$$y = \alpha_0 + \alpha_1 P_1(x) + \ldots + \alpha_{k-1} P_{k-1}(x) + e$$

- The $P_i(x)$ are ith-order orthogonal polynomial.
  - Mean: $P_0(x) = 1$.
  - Linear: $P_1(x) = \lambda_1 \left[\frac{x - m}{\Delta} \right]$
  - Quadratic: $P_2(x) = \lambda_2 \left[\left(\frac{x - m}{\Delta} \right)^2 - \left(\frac{k^2 - 1}{12} \right) \right]$
  - ... ...

  where Δ = distance between factor levels. λ_i makes $P_i(x)$ integer values for all designed x. m is the mean of levels of designed x.
  Values of λ_i and more orthogonal polynomials are given in Appendix G of the textbook.

Advantages of Orthogonal Polynomial

- For factors with numerical nature, it offers predictions not only for designed factor levels but also for other factor values.
- The polynomial functions are well designed, such that the design matrix for regression model:

$$y = \beta_0 P_0(x) + \beta_1 P_1(x) + \beta_2 P_2(x) + \ldots + \beta_{k-1} P_{k-1}(x) + \epsilon$$

is orthogonal. Then..., why good?

1. Low computational complexity.
2. High accuracy in the estimation of β_i, since no multicolinearity.
   - High order polynomials are usually not preferred: why? Parsimonious models are preferred.

Results for the Crop Example

- The estimated equation is

$$\hat{y}_i = 16.4 + 1.2P_1(x_i) - 1.0P_2(x_i) + 0.1P_3(x_i) + 0.1P_4(x_i)$$

- Check the supplementary R codes for how to fit such orthogonal polynomial regression.
- The significance of the effects can be tested by ANOVA table.
- For each x value (not necessarily in the treatment assignment), $\hat{y}$ can be estimated by plugging in x to the above fitted equation.

Example: A Special Case, $k = 3$

- Data:
  - y = bonding strength of composite material.
  - x = laser power at 40, 50 and 60.

Strength Data, Composite Experiment

Laser Power (watts)		
40	50	60
25.66	29.15	35.73
28.00	35.09	39.56
20.65	29.79	35.66

- For three evenly spaced levels, $m - \Delta$, m and $m + \Delta$, the first and second degree polynomials are given by
  - $P_1(x) = \dfrac{x - m}{\Delta}$ has 3 possible values: -1, 0 and 1, respectively corresponding to $x = m - \Delta$, $= m$ and $= m + \Delta$.
  - $P_2(x) = 3\left\{\left(\dfrac{x - m}{\Delta}\right)^2 - \dfrac{2}{3}\right\}$ has 3 possible values: 1, -2 and 1, respectively corresponding to $x = m - \Delta$, $= m$ and $= m + \Delta$.

Example: A Special Case, $k = 3$

- Polynomial regression mode:

$$y = \beta_0 + \beta_1 P_1(x)/\sqrt{2} + \beta_2 P_2(x)/\sqrt{6} + e.$$

  The fitted values are $\hat{\beta}_0 = 31.0322$, $\hat{\beta}_1 = 8.636$ and $\hat{\beta}_2 = -0.381$.

- Fitted model:

$$\hat{y} = 31.0322 + 8.636 P_1(x)/\sqrt{2} - 0.381 P_2(x)/\sqrt{6}.$$

- To compute $\hat{y}$ at any x, plug in the x to the above fitted model. For example $x = 55$, $m = 50$, $\Delta = 10$,

$$\begin{aligned} P_1(55) &= \frac{55-50}{10} = \frac{1}{2} \\ P_2(55) &= 3\left\{\left(\frac{55-50}{10}\right)^2 - \frac{2}{3}\right] = -\frac{5}{4}, \end{aligned}$$

  therefore,

$$\begin{aligned} \hat{y} &= 31.0322 + 8.636(0.3536) - 0.381(-0.5103) \\ &= 34.3803. \end{aligned}$$

Revisiting Meat Storage Example; One-way Random Effects Model

- In the meat storage example, τ_i are so called fixed effects because the study interest is to compare the four specific packaging conditions.
- What if these four packaging conditions were randomly chosen from a population of packaging conditions? The interest would usually be the variation among all packaging conditions in the population.
- If four packaging conditions are randomly selected from the population, the variation among the packaging conditions for the population is referred to as random effects.

One-way Random Effects Model

- One-way random effects model:

$$y_{ij} = \eta + a_i + e_{ij},$$

  - $e_{ij} \overset{i.i.d.}{\sim} N(0, \sigma_e^2)$.
  - a_i are independent $N(0, \sigma_a^2)$.
  - e_{ij} and a_i are assumed to be independent of one another.
  - σ_e^2 and σ_a^2 are the two variance components of the model.
  - The variance among packaging conditions is measured as σ_a^2.
- Question: What is the distribution of y_{ij} now?

ANOVA and Variance Components

- The null hypothesis for fixed effects model: $\tau_1 = \tau_2 = \ldots = \tau_k = 0$ should now be replaced by

$$H_0 : \sigma_a^2 = 0.$$

The F test and the ANOVA table for fixed effects model still hold, i.e.

$$F = \frac{\text{MSTr}}{\text{MSE}} = \frac{\text{SSTrt}/(k-1)}{\text{SSE}/(N-k)} \sim F_{k-1,N-k},$$

and we reject $H_0 : \sigma_a^2 = 0$ if the p-value $P(F > f_{\text{obs}})$ is very small.

- We can still apply the same ANOVA and F test in the fixed effects case for analyzing data. However, we need to compute the expected mean squares under the alternative of $\sigma_a^2 > 0$ to estimated the variance components.

Expected Mean Squares

- For balanced designs ($n_1 = n_2 = \ldots = n_k = n$),

$$E(\text{MSE}) = \sigma_e^2 \quad (1)$$

$$E(\text{MSTr}) = \sigma_e^2 + n\sigma_a^2 \quad (2)$$

- For unbalanced designs, n in (2) is replaced by $\frac{1}{k}\left[\sum_{i=1}^{k} n_i - \frac{\sum_{i=1}^{k} n_i^2}{\sum_{i=1}^{k} n_i}\right]$.

ANOVA Tables (Balanced Designs)

If $n_1 = n_2 = \ldots = n_k = n$,

ANOVA Table

Source	D.F.	S.S	M.S.	E(M.S.)
Treatments	$k-1$	SSTrt	$\text{MSTr} = \text{SSTrt}/(k-1)$	$\sigma_e^2 + n\sigma_a^2$
Error	$N-k$	SSE	$\text{MSE} = \text{SSE}/(N-k)$	σ_e^2
total	$N-1$	SST		

Meat Storage Experiment

	D.F.	S.S.	M.S.	E(M.S.)
Treatments	3	32.873	10.958	$\sigma_e^2 + 3\sigma_a^2$
Error	8	0.927	0.116	σ_e^2
Total	11	33.800		

Estimation of σ_e^2 and σ_a^2

- From (1) and (2), we have the following unbiased estimates of the variance components:

$$\hat{\sigma}_e^2 = \text{MSE} \quad \text{and} \quad \hat{\sigma}_a^2 = \frac{\text{MSTr} - \text{MSE}}{n}.$$

- In the case that $\text{MSTr} \leq \text{MSE}$, $\hat{\sigma}_a^2 \leq 0$. On the other hand, $\text{MSTr} \leq \text{MSE}$ is equivalent to $F \leq 1$. Obviously the null hypothesis $H_0 : \sigma_a^2 = 0$ is not rejected when $F \leq 1$.
- Since the variance cannot be negative, a negative estimate (when $\text{MSTr} < \text{MSE}$) can be replaced by 0. This does not indicate that the σ_a^2 is zero! It only indicates that there is not enough information in the data to get a good estimate of σ_a^2.
- For the meat storage experiment, $\hat{\sigma}_e^2 = 0.116$, $\hat{\sigma}_a^2 = (10.958 - 0.116)/3 = 3.614$.

Inference for Overall Mean η

- In a random effects model, η, the population mean, is often of interest. From $E(y_{ij}) = \eta$, we use the estimate

$$\hat{\eta} = \bar{y}_{\cdot\cdot}$$

- $\text{Var}(\hat{\eta}) = \text{Var}(\bar{a}_{\cdot} + \bar{e}_{\cdot,\cdot}) = \dfrac{\sigma_a^2}{k} + \dfrac{\sigma_e^2}{N}$.
- For balanced designs,

$$\text{Var}(\hat{\eta}) = \frac{\sigma_a^2}{k} + \frac{\sigma_e^2}{nk} = \frac{1}{nk}\left(\sigma_e^2 + n\sigma_a^2\right).$$

  Using Equation (2), an unbiased estimate of $\text{Var}(\hat{\eta})$ is $\dfrac{\text{MSTr}}{nk}$.
- Confidence interval for η is then:

$$\hat{\eta} \pm t_{k-1,\alpha/2}\sqrt{\frac{\text{MSTr}}{nk}}.$$