ST5225: Statistical Analysis of Networks
Lecture 5: Matching Markets

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Outline

- Review
- Graph Partition
- Matching Markets

- Eigenvector Centrality
- Cohesion
  - Structure: Cliques, k-cores, coreness
  - Connectivity: k-vertex/edge-connectivity, weakly/strongly connected, bowtie
  - Densities: local density; transitivity (clustering coefficient)

For today:

- Graph partitions: definition and methods

Graph Partition

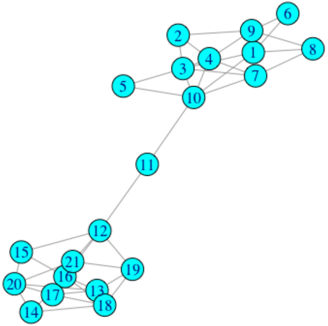
- The graph is not uniformly dense, usually it contains several more cohesive parts

Partition

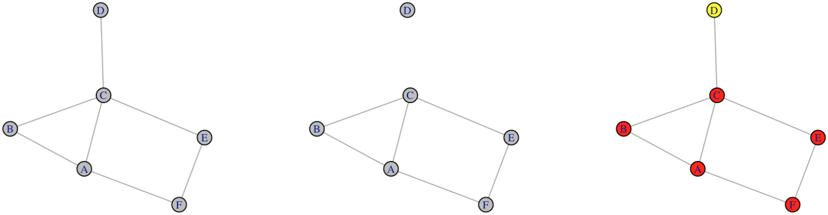
For a set V, a *partition* of V is a set of subsets $\mathfrak{C} = \{C_1, C_2, \cdots, C_K\}$, where $C_1, C_2, \cdots, C_K$ are disjoint, and $\cup_{i=1}^{K} C_i = V$.

- **Goal**: Find a partition of the graph G, so that in each subset C_i, the nodes are *well* connected, and between the subsets, the nodes are *separated*
- Application:
  - Community detection in social networks
  - Identification of possible protein complexes from protein interaction networks
- Motivation for the stochastic block model (introduce in Lecture 9-12)

- Betweenness: How many shortest edges will pass this edge
- If there are two communities, the edges between them should have high betweenness



- If we keep on moving the edges with highest betweenness, we will have more components
- Repeat until we achieve the result we need



Remark.

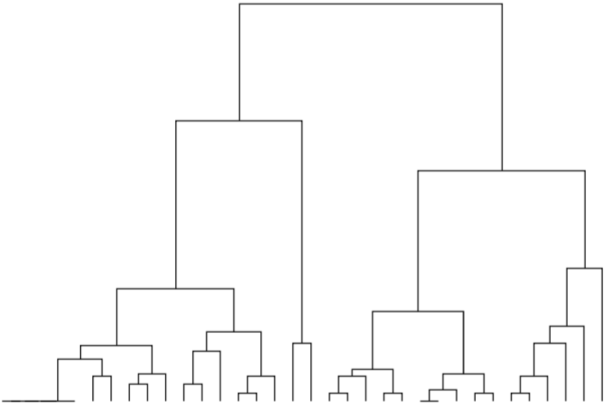
- The process can be repeated until all the nodes are separate (or the result you need according to prior information, say, we already know there are 3 communities)
- However, it is easy to cause the isolation of some nodes.

Questions and Answers:

- How about other measurements, instead of betweenness?
  - Need a *cost function* to *evaluate the quality* of the clustering/partitions
  - Various of cost functions, giving different results
- Dividing nodes, or merging nodes?
  - Both ways work
  - Starting with the original graph, and dividing nodes. We call this as a *divisive* method.
  - Starting with graph $|V|$ clusters (each node is a cluster), and merging nodes. We call this a *agglomerative* method.
- Is this the optimal result?
  - Not necessarily. The optimal result comes from exhaustive search, which is time consuming. This is a greedy search method, which may stuck at local maxima.

Questions and Answers:

- Record the result after each step
  - It will create the entire hierarchy of nested partitions, in the form of a tree. We call it a *dendrogram*.



[1]

[1]SAND, Fig 4.7 Hierarchical clustering of the karate club network.

Cost Functions

Most cost functions is a measure of *(dis)similarity* between sets, based on node-node (dissimilarity).

- Node-node (dis)similarity examples:
  - Euclidean distance:

$$dist(v_i, v_j) = \sqrt{\sum_{k \neq i,j} (A_{ik} - A_{jk})^2}, \quad A_{ij} : (i,j)th \text{ element of adjacency matrix } A$$

    When v_i and v_j have larger distance, the neighbors of them are less shared, and so v_i and v_j are possibly to be separated.
  - Another dissimilarity measurement:

$$dis(v_i, v_j) = \frac{\#\text{unshared neighbors by } v_i \text{ and } v_j}{\text{largest degree} + \text{second largest degree}}$$

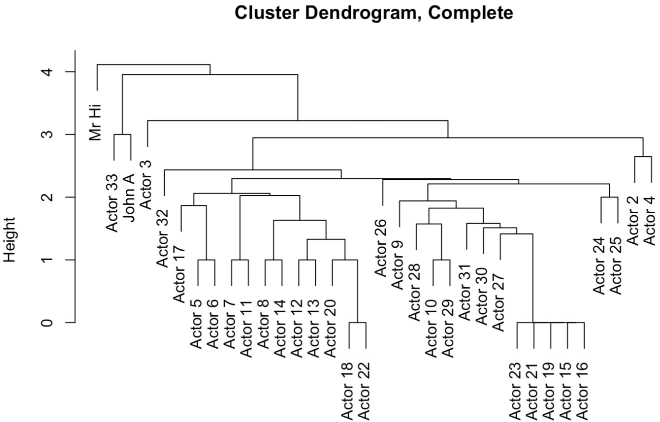
    - Numerator: number of nodes that are either neighbors of v_i only, or neighbors of v_j only
    - Ranges in $(0, 1)$

Cost Functions

Begin with the node-node (dis)similarity, different ways to calculate the (dis)similarity for a partition $\mathfrak{C}$.

- Single Linkage
  For partition $\mathfrak{C} = \{C_1, C_2, \cdots, C_K\}$, the dissimilarity between two elements C_1 and C_2 is defined as
  $$dis(C_1, C_2) = \min_{v_i \in C_1, v_j \in C_2} dis(v_i, v_j),$$
  the *minimal* distance between two nodes in the two subsets.
- Complete Linkage
  The dissimilarity between two subsets C_1 and C_2 is defined as
  $$dis(C_1, C_2) = \max_{v_i \in C_1, v_j \in C_2} dis(v_i, v_j),$$
  the *maximal* distance between two nodes in the two subsets.
- Average Linkage
  The dissimilarity between two subsets C_1 and C_2 is defined as
  $$dis(C_1, C_2) = \frac{1}{|C_1||C_2|} \sum_{v_i \in C_1, v_j \in C_2} dis(v_i, v_j),$$
  the *average* distance between all pairs of nodes in the two subsets.

HC with Complete Linkage: Karate Club



Note: cut off at the number of communities you want

Cost Functions

The cost can be defined on the partition directly, not through the node-node (dis)similarity.
For example,

- *modularity* of a partition $\mathfrak{C} = \{C_1, C_2, \cdots, C_K\}$ of the node set V.
- Define the function $f_{ij}(\mathfrak{C})$, where

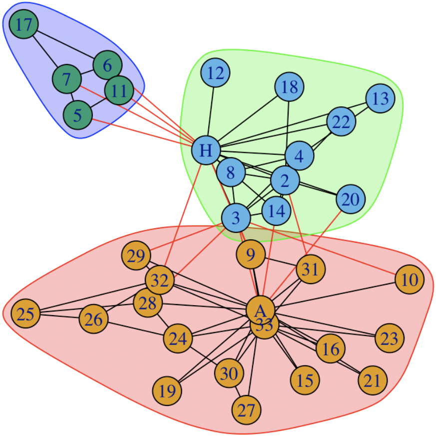
$$f_{ij}(\mathfrak{C}) = \frac{|\{(v_k, v_l) \in E; v_k \in C_i, v_l \in C_j\}|}{|E|},$$

  the fraction of edges that connect a node in C_i and a node in C_j.
- The modularity is

$$mod(\mathfrak{C}) = \sum_{i=1}^{K} [f_{ii}(\mathfrak{C}) - f_{ii}^*], \qquad \text{where } f_{ii}^* = (\sum_{j \in V} f_{ij})(\sum_{j \in V} f_{ji}),$$

- Note:
  - For undirected graphs, $f_{ij} = f_{ji}$

HC with Modularity: Karate Club



Note: 3 communities

Summary: Graph partitions

- A phenomenon in many applications: biology, social network, etc.
- Hierarchical clustering: not sure where to cut off; may stuck at the local minima
- Modularity: computation cost
- Still many other methods:
  - minimal spanning tree
  - k-means
  - Spectral clustering
  - etc.
- The attributes may help

Overview

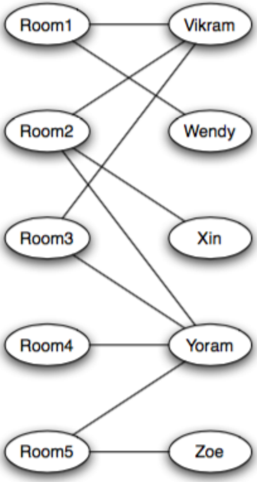
- Bipartite network
- Perfect matching
  - Constricted sets
  - Perfect matching theorem
- Optimal assignments
- Market-clearing prices
  - Properties
  - Construction

Materials

Networks, Crowds, and Markets: Chapter 10.1-10.4

Dorm assignment problem

- 5 dorms available for 5 students; each room for **1** single student
- Each student has a list of acceptable rooms; shown in graph
- Questions:
  - Any assignment so that everyone is happy with the room?
  - Is there always such a "perfect" assignment?
  - How to decide whether this "perfect" assignment exists or not?



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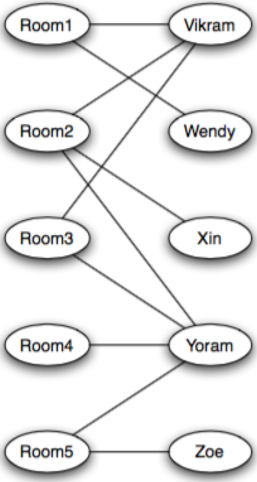

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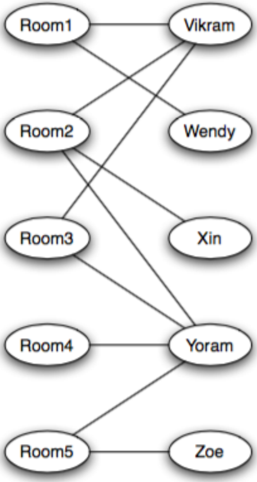

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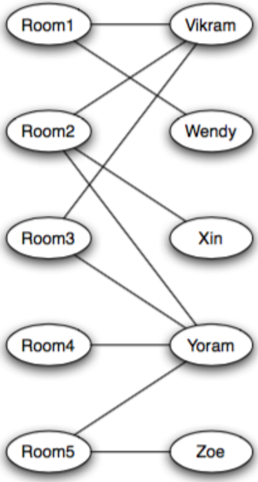

The model we start with is called *bipartite matching problem*.

Bipartite graph

For a graph $G = (V, E)$, if the set of vertices V can be divided into two disjoint and independent sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2, then we call G as a *bipartite graph*.

Remark.

- Bipartite graph usually has two types of nodes (dorms and students).
- It appears frequently in daily life, e.g., network of audience and movies on IMDB, networks of authors and papers, etc.
- Obviously, there won't be self-loops on bipartite graphs. (There might be multi-edges, yet we only consider simple graphs)
- To save space, the adjacency matrix for bipartite graph is usually record as a $|V_1| \times |V_2|$ rectangular matrix, instead of $|V| \times |V|$ matrix.

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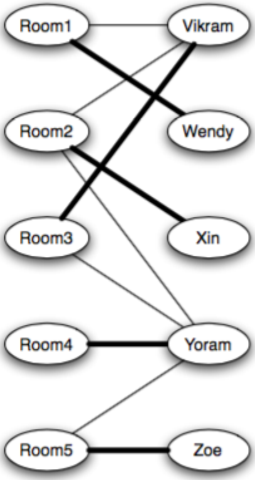
Remark II.

- V_1 and V_2 may have different number of nodes. If they have the same number of nodes, we call it as *balanced*
- Bipartite matching problem is concerning whether there is a *perfect matching* on balanced bipartite graph.

Perfect matching

- Problem: Assignment so that each student is happy with the room
- Graph: Find a sub-graph, so that in this subgraph, each student is assigned a distinct room
- Solution: The darkened edges is such a sub-graph
- We call this solution as a *perfect matching*



Perfect matching

Perfect Matching

When there are an equal number of nodes on each side of a bipartite graph, a *perfect matching* is an assignment of nodes on the left to nodes on the right, in such a way that

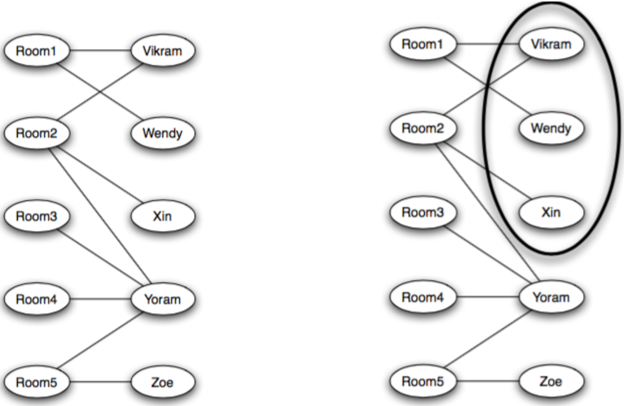
- (Connection) each node is connected by an edge to the node it is assigned to, and
- (distinction) no two nodes on the left are assigned to the same node on the right.

Remark.

- The dorm-student problem is asking for a perfect matching
- For a perfect matching, each node in V_1 is connected to one and only one "friend" in V_2, and vice versa.
- If the perfect matching subgraph is denoted as $H = (V_H, E_H)$, then $V_H = V$, $|E_H| = |V_H|/2$.

Constricted Sets

Are there always be such an assignment?



- New graph: Delete an edge between Vikram and Room 3, so Vikram only has two acceptable rooms
- No proper assignment! (3 students prefer only 2 rooms)

Constricted Sets

- For a set of nodes S, let $N(S)$ denote the collection of neighbors of the nodes in S. For a bipartite graph, note that $N(S)$ is concerning the other group of nodes.

Constricted sets

A set S is *constricted* if $|S| > |N(S)|$.

- *If there's a constricted set S, there is no perfect matching*
- If there is a perfect matching, then with the perfect matching, the neighbors of S equals to $|S|$, which is impossible.

How can we claim whether there is perfect matching or not?

Matching Theorem

If a balance bipartite graph has no perfect matching, then it must contain a constricted set.

Remark.

- Equivalence between *perfect matching* and *no constricted set*

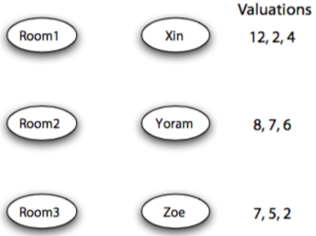
$$\text{Perfect matching} \Longleftrightarrow \text{No constricted sets}$$

- We do not need considering all the possible matchings. The problem reduced to find a set and check whether it is constricted or not

Bipartite Graph with Valuations

Now, each student can also set a score to show *how much* they like the room

- 3 dorms available for 3 students; each room for **1** single student
- The three numbers show the evaluation for the three rooms.
- What is the “perfect” assignment here?



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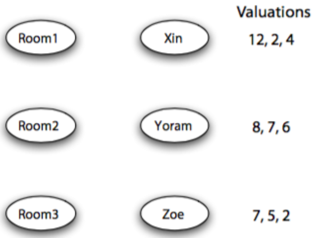

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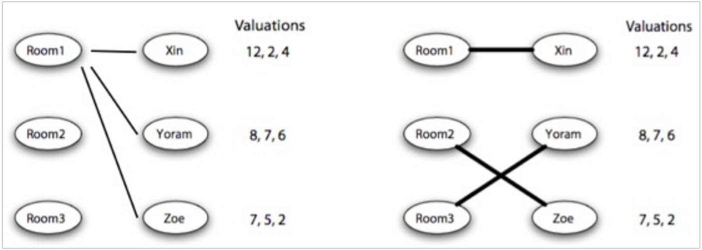
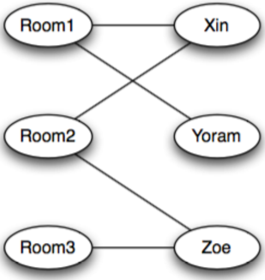
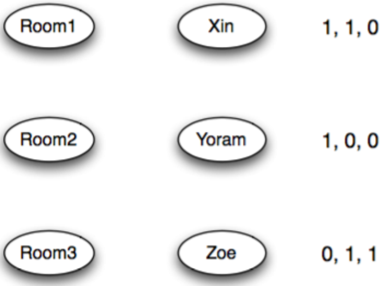


- Left: "Favorite graph": each student is connected with the room with largest evaluation score – No perfect matching!
- Right: Not everyone gets the optimal result, but generally everyone is happy
- Right: The overall valuation is: $12 + 6 + 5 = 23$!
- We call the assignment that can maximize the overall evaluation score as an *optimal assignment*

Bipartite Graph with Valuations, Examples





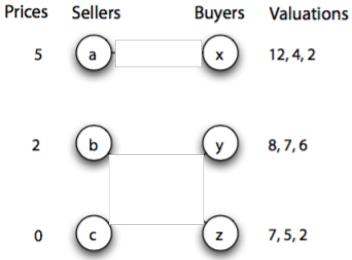
- The "acceptable or not" options can also be seen as a graph with valuations.
- The "perfect matching" is to figure out whether the optimal assignment could reach valuation $|V|$ (=3) or not.

Summary

- Bipartite graph: two sets of nodes V_1 and V_2; edges between V_1 and V_2; no associated values to the nodes/edges
  - Perfect matching
  - Subgraph of the original graph, where each node is associated to one and only one edge
- Bipartite graph with valuations: each $v_j \in V_2$ has valuations for each node in V_1
  - Optimal assignment
  - The assignment could maximize the valuations

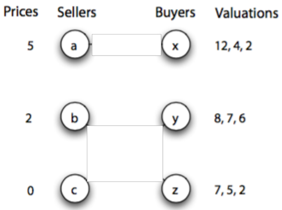
Prices

- A university may not have the same charge on the rooms, yet the market won't
- On a market, there are a set of sellers and a set of buyers. Each buyer has valuation on the product of the sellers, and each seller ask for a price.



Valuations and prices

- 3 houses available for 3 buyers
- Each buyer has his/her valuations for each house. Each seller has a price for his/her house
- Questions
  - Can we still find an “optimal” assignment here?
  - If we can change the price, can we find an even better assignment?



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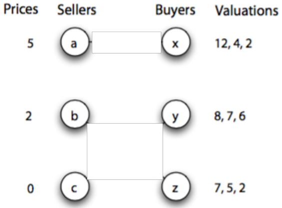

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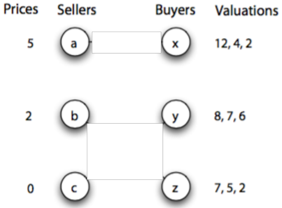
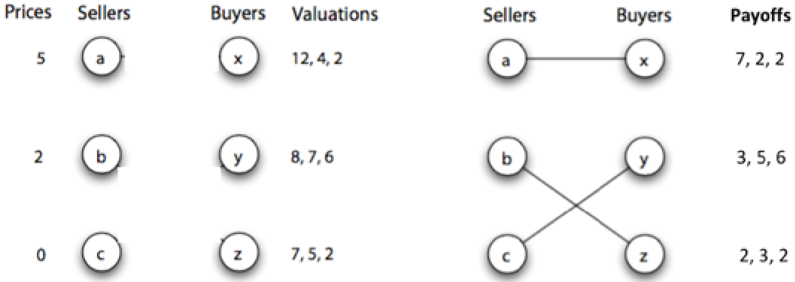
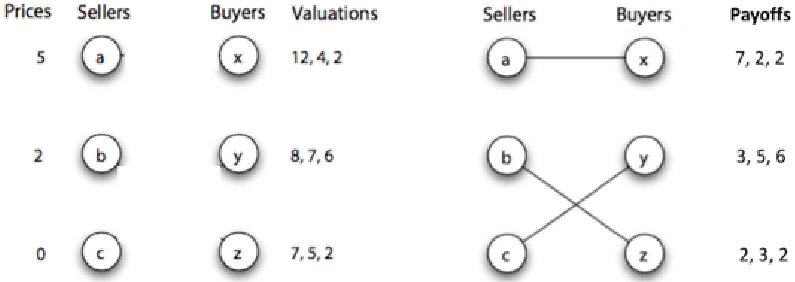


Prices and Payoffs

- *Payoffs*: For a buyer and a house, the buyer's payoff is valuation – price of this house
- Therefore, we have a graph with *payoffs* as the previous *valuations*
- Given the prices, we can find an *optimal assignment* with respect to the payoffs
- The optimal assignment could maximize the payoffs

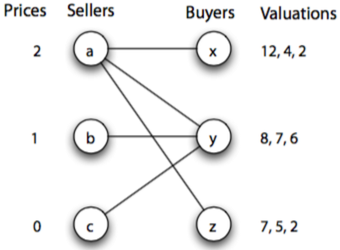


- For the result, note that each buyer is connected to the seller with largest *payoff*
- *Preferred-seller graph*: Connect each buyer with the sellers that maximizes the payoff of this buyer, and the corresponding graph is called preferred-seller graph, since the seller is the preferred seller of the connected buyers.
- If there is a *perfect matching* in the preferred-seller graph, then this perfect matching is the optimal assignment.

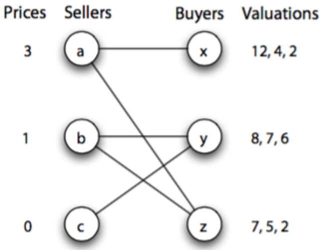


Market-Clearing Prices

- Note: every buyer gets the house that maximizes the payoffs
- We call such a set of prices *market-clearing*, since the prices setting cause each house to get bought by a different buyer–the house market is clear!
- It is possible there is no perfect matching



(c) *Prices that Don't Clear the Market*



(d) *Market-Clearing Prices (Tie-Breaking Required)*

- *Existence: for any set of buyer valuations, there exists a set of market-clearing prices*
  - For any set of buyers, the sellers can set the prices carefully, so that each buyer gets distinct house
- *Optimality: For any set of market-clearing prices, the optimal assigment has the maximum total valuation of any assignment of sellers to buyers*
  - For an assignment M

    Total payoff of M = Total valuation of M - Sum of all prices
  - Note that sum of all prices don't change. The optimal assignment maximizes the total payoff, so that maximizes the total valuation.

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Market-Clearing Prices: Construction

How to construct a set of market-clearing prices?

- Idea: if one house is in high demand (more than 1 buyers have maximum payoff on this house), then the price increases by 1
- Note: we always set the house with smallest price to be with price 0. It does not affect the result, and it helps to scale the result
- In a real market, the "hot" house usually has a higher price, so that the final buyer is the buyer who likes it most

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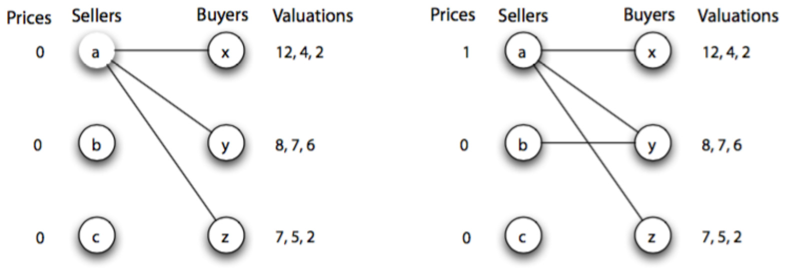
Market-Clearing Prices: Construction

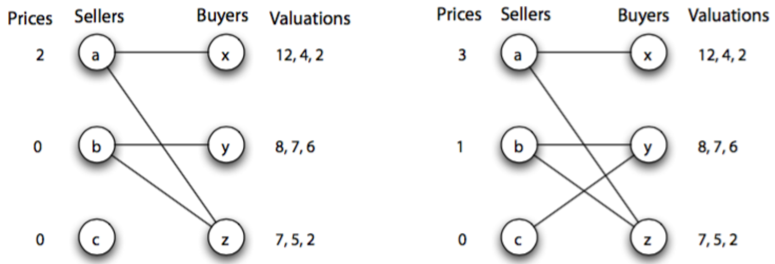
Say there are n sellers (items) and buyers

Start: each item has price 0; each buyer assigns a value to each item

1. Assume smallest value is 0; if not, scale the price so that the smallest is 0
2. Construct the preferred-seller graph and check if there is a perfect matching
3. If yes, done
4. If not, find a constricted set of buyers S
5. Each seller in $N(S)$ increases the price by 1
6. check if the smallest price is 0, if not, subtract the same amount of each price so that the smallest is 0
7. go back to Step 1.



(a) *Start of first round* (b) *Start of second round*



Market-Clearing Prices: Example II

Prices	Sellers	Buyers	Valuations
0	a	x	5, 7, 1
0	b	y	2, 3, 1
0	c	z	5, 4, 4

Summary

- Bipartite graphs with two sets of nodes V_1 and V_2; each $v_i \in V_1$ has a price; each $v_j \in V_2$ has a valuation for every $v_i \in V_1$
  - Find payoffs of each $v_j \in V_2$ assigned to each $v_i \in V_1$
  - With the payoffs, there is *optimal assignment*
  - For each $v_j \in V_2$, there is a preferred seller in V_1, which maximizes the payoff of v_j. We call this seller as *preferred seller*
  - If the preferred-seller graph has a *perfect matching*, then we call the price as market-clearing price
- Market-clearing prices must exist, and maximizes the total valuations
- Construction of market-clearing prices: begin with 0, and increase the prices of constricted set by 1.

- Power on networks
- An experiment and the results
- Nash bargaining solution
- Ultimatum game
- Stable outcomes
- Find natural stable outcomes

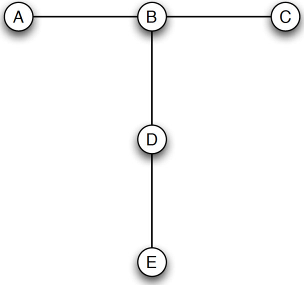
- In the buyer-seller network, the buyer has payoff as valuation minus price.
- The seller may want to increase the price, so that the payoff splits between the seller and the buyer
- If the seller increases the price only a little bit, then the buyer still transacts with this seller with a fewer payoff, and the seller gets some profit.
- If the seller increases the price too much, the buyer may give up, and no transaction is done.
- *Bargaining on network*
- Bargaining is related to *power*, the popularity of the seller/buyer in the network

Power on Networks

We talk about the general cases, not restricted to buyer-seller networks.

- Intuitionally, we may say these two nodes have the same power (related to this graph), since they are symmetric

- Node B in this graph has a larger power, since nodes A and C requires the connection with B to get in touch with D and E



- Power of distributing resources

Bargaining on networks: experiment

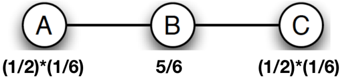
Set up an experiment to see people's bargaining behavior on networks.

1. Fix a small network. Let each node represent each player. Players that are connected with an edge can exchange information/make offers.
2. Each edge carries a resource (e.g., $100). This resource is split among adjacent nodes if there is a deal between nodes.
3. Each node can only be part of at most one deal (possible that no deal is done).
4. Players can freely negotiate for a fixed amount of time how the resource is split.
5. The experiment is repeated multiple times to get the final result

Bargaining on networks: outcomes

- For a 2-node network, end up with a fair split. Since A and B has the same power on the transaction, they agree to split the money equally.

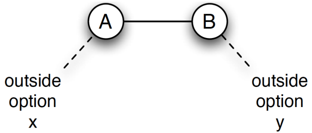
- For a 3-node network, B has higher power, where it can exclude A or C in the transaction. The result shows that A and C are symmetric, while B receives the majority of the money in the transaction.



- In the textbook, more complicated experiments, say, 4-nodes, 5-nodes are also discussed.

Nash bargaining solution

To discuss the problem, we think of each edge. The resources outside this edge is summarized as “outside options”, as the following figure



- Assume the total resource to split on this edge is 1.
- Since A can get x from outside if not split with B. So the expectation of A's payoff is at least x
- Similarly, the expectation of B's payoff is at least y
- If $x + y > 1$, impossible to satisfy both A and B
- If $x + y \leq 1$, then there is surplus $s = 1 - x - y$
- *Nash bargaining solution*: both A and B will get half of the surplus, $(1 - x - y)/2$, which means the outcome is

$$x + \frac{1-x-y}{2} = \frac{1+x-y}{2} \text{ to A;} \quad y + \frac{1-x-y}{2} = \frac{1-x+y}{2} \text{ to B.}$$

Nash bargaining solution

Interesting remarks:

- The resource is not only money, it can be friendship, collaboration, etc.
- When bargaining, people tries to communicate about the outside option, so that gets a reasonable split
- In real life, for the two endpoints, usually one has a "higher-status" and the other has a "lower-status". For example, a sophomore and a graduate student with high grades.
- It has been found that, people with high-status tend to inflate the size of their outside option; people with lower-status tend to reduce the size of their outside option.
- What's more, the reduced size of outside option will be further underestimated by the higher-status people.
- Do you have the same problem in your job search?

The ultimatum game

Setup: Two players A and B; no communication allowed; 1-time game

1. Person A is given \$100 and told to propose a division among A and B
2. If B accepts the \$100 are split accordingly; if B rejects, both A and B get nothing

If both A and B are rational,

- B should accept any positive offer, since that's better than nothing
- Since B will accept any positive offer, A, as the person who propose the division, can keep more money
- Solution: A offers \$99 to A, and \$1 to B
- The difference comes from the power of division

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Reality:

- In 1/3 of the experiments, A offers 1/2-1/2 split to B
- Moreover, when the offer is very unfair, B rejects to accept the offer – even though he will get nothing
- When the amount of money changes, the result is still the same
- Guess: It hurts B's feelings when offered very small proportion, which behaves as a negative payoff
- Solution: Give B more time to consider, then more people would accept the "unfair" offer

Stable outcomes

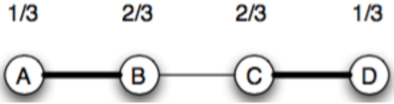
- We need some mathematically defined terms to describe the exchanges on graphs

Outcomes

An *outcome* of a network exchange on a given graph consists of:

- A matching on the set of nodes, specifying who exchanges with whom.
- A number associated with each node, called its *value*, indicating how much this node gets from its exchange. The sum of matched nodes should be 1.

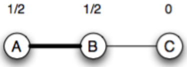
Example.



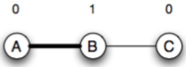
- All discussions are based on the outcomes
- Is the outcome stable?

Stable outcome

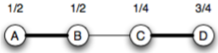
Outcome where no player can make an offer to another player such that both are *better* off, is called a *stable outcome*.



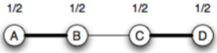
(a) *Not a stable outcome*



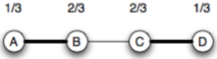
(b) *A stable outcome*



(c) *Not a stable outcome*



(d) *A stable outcome*



(e) *A stable outcome*

Stable outcomes

- *Instability*: if there is an edge not part of the matching, such that the sum of the endpoint values is < 1, then we call this outcome as *an unstable outcome*.
  - Say the endpoints of this edge is A and B, then obviously A and B would prefer to exchange, so that they are better off. The original outcome is not stable.
- If there is no such instabilities, then an outcome is stable.
  - If an outcome is unstable, then there always exists such an edge.
  - It helps us to figure out the stability of an outcome. Just check all the edges and the values of their endpoints, and we know whether it is stable or not

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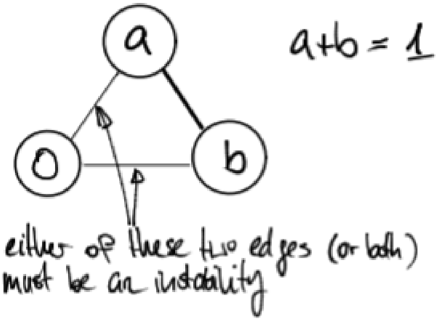
Example (restrictions of outcomes):
For a triangle, there are no stable outcomes!



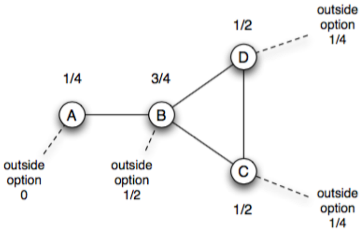
Balanced stable outcomes

- If we have many stable outcomes, then we prefer some "natural" set of outcomes, for which we call *balanced*

Balanced outcomes

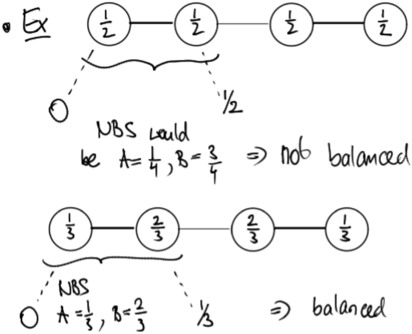
An outcome is called a *balanced outcome* if, for each edge in the matching, the split of money represents the Nash bargaining outcome for the two nodes involved, given the best outside option.

Example: a balanced outcome



Balanced stable outcomes

- It is closer to what happens in reality



- Power on networks with experiments
- Nash bargaining solution for two nodes with outside options
- Ultimatum game and the results
- Stable outcomes and natural stable outcomes (Nash bargaining solution required here)