Sponsored Search Markets

15.1 Advertising Tied to Search Behavior

The problem of Web search, as traditionally formulated, has a very "pure" motivation: it seeks to take the content people produce on the Web and find the pages that are most relevant, useful, or authoritative for any given query. However, it soon became clear that a lucrative market existed within this framework for combining search with advertising, targeted to the queries that users were issuing.

The basic idea behind this is simple. Early Web advertising was sold on the basis of "impressions," by analogy with the print ads one sees in newspapers or magazines: a company like Yahoo! would negotiate a rate with an advertiser, agreeing on a price for showing its ad a fixed number of times. But if the ad you're showing a user isn't tied in some intrinsic way to their behavior, then you're missing one of the main benefits of the Internet as an advertising venue, compared to print or television. Suppose, for example, that you're a very small retailer who's trying to sell a specialized product; say, for example, that you run a business that sells calligraphy pens over the Web. Then paying to display ads to the full Internet-using population seems like a very inefficient way to find customers; instead, you might want to work out an agreement with a search engine that said, "Show my ad to any user who enters the query 'calligraphy pens'." After all, search engine queries are a potent way to get users to express their *intent* – what it is that they're interested in at the moment they issue their query – and an ad that is based on the query catches a user at precisely this receptive moment.

Originally pioneered by the company Overture, this style of *keyword-based advertising* has turned out to be an enormously successful way for search engines to make money. At present it's a business that generates tens of billions of dollars per year in revenue, and it is responsible, for example, for nearly all of Google's revenue. From our perspective, it's also a very nice blend of ideas that have come up earlier in this book: it creates markets out of the information-seeking behavior of hundreds of millions of people traversing the Web, and we will see shortly that it has surprisingly deep connections to the kinds of auctions and matching markets that we discussed in Chapters 9 and 10.

Figure 15.1. Search engines display paid advertisements (shown on the right-hand side of the page in this example) that match the query issued by a user. These appear alongside the results determined by the search engine's own ranking method (shown on the left-hand side). An auction procedure determines the selection and ordering of the ads. (Courtesy of Google Image.)

Keyword-based ads show up on search engine results pages alongside the unpaid ("organic" or "algorithmic") results. Figure 15.1 shows an example of how this currently looks on Google for the query "Keuka Lake," one of the Finger Lakes in upstate New York. The algorithmic results generated by the search engine's internal ranking procedure are on the left, while the paid results (in this case for real estate and vacation rentals) are ordered on the right. There can be multiple paid results for a single query term, which simply means that the search engine has sold an ad on the query to multiple advertisers. Among the multiple slots for displaying ads on a single page, the slots higher up on the page are more expensive, since users click on these at a higher rate.

The search industry has developed certain conventions in the way it sells keyword-based ads, and for thinking about this market it's worth highlighting two of these at the outset.

Paying per Click. Ads such as those shown in Figure 15.1 are based on a *cost-per-click* model. This means that if you create an ad that is shown to users entering the query "Keuka Lake," it will contain a link to your company's Web site – and you only pay when a user actually clicks on the ad. Clicking on an ad represents an even stronger indication of intent than simply issuing a query; it corresponds to a user who issued the query, read your ad, and is now visiting your site. As a result, the amount that advertisers are willing to pay per click is often surprisingly high. For example, to occupy the most prominent spot for "calligraphy pens" costs about $1.70 per click on Google as of this writing; occupying the top spot for "Keuka Lake" costs about $1.50 per click. (For the misspelling "calligaphy pens," the cost is still about $0.60 per click – after all, advertisers are still interested in potential customers even if their query contains a small but frequent typo.)

For some queries, the cost per click can be positively stratospheric. Queries like "loan consolidation," "mortgage refinancing," and "mesothelioma" often reach $50 per click or more. One can take this as an advertiser's estimate that it stands to gain an expected value of $50 for every user who clicks through such an ad to its site.[1]

Setting Prices Through an Auction. There is still the question of how a search engine should set the prices per click for different queries. One possibility is simply to post prices, the way that products in a store are sold. But with so many possible keywords and combinations of keywords, each appealing to a relatively small number of potential advertisers, it would be extremely difficult for the search engine to maintain reasonable prices for each query in the face of changing demand from advertisers.

Instead, search engines determine prices using an auction procedure, in which they solicit bids from the advertisers. If there were just a single slot in which an ad could be displayed, then this would be just a single-item auction such as we saw in Chapter 9, and there we found that the sealed-bid second-price auction has many appealing features. The problem is more complicated in the present case, however, since there are multiple slots for displaying ads, and some are more valuable than others.

We will consider how to design an auction for this setting in several stages.

1. First, if the search engine knew all the advertisers' valuations for clicks, the situation could be represented directly as a matching market in the style that we discussed in Chapter 10: essentially, the slots are the items being sold, and they're being matched with the advertisers as buyers.

2. If we assume that the advertisers' valuations are not known, however, then we need to think about ways of encouraging truthful bidding, or to deal with the consequences of untruthful bidding. This leads us directly to an interesting general question that long predates the specific problem of keyword-based advertising: how do you design a price-setting procedure for matching markets in which truthful bidding is a dominant strategy for the buyers? We will resolve this question using an elegant procedure called the *Vickrey–Clarke–Groves* (VCG) *mechanism* [112, 199, 400], which can be viewed as a far-reaching generalization of the second-price rule for single-item auctions that we discussed in Chapter 9.

3. The VCG mechanism provides a natural way to set prices in matching markets, including those arising from keyword-based advertising. For various reasons, however, it is not the procedure that the search industry adopted. As a result, our third topic will be an exploration of the auction procedure that is used to sell search advertising in practice, the *generalized second-price auction* (GSP). Although GSP has a simple description, the bidding behavior it leads to is very complex, with untruthful bidding and socially nonoptimal outcomes. Trying to understand bidder behavior under this auction turns out to be an interesting case study in the intricacies of a complex auction procedure as it is implemented in a real application.

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[1] Naturally, you may be wondering at this point what mesothelioma is. As a quick check on Google reveals, it's a rare form of lung cancer that is believed to be caused by exposure to asbestos in the workplace. So if you know enough to be querying this term, you may well have been diagnosed with mesothelioma, and are considering suing your employer. Most of the top ads for this query link to law firms.

15.2 Advertising as a Matching Market

Clickthrough Rates and Revenues per Click. To begin formulating a precise description of how search advertising is sold, let's consider the set of available "slots" that the search engine has for selling ads on a given query, like the three advertising slots shown in Figure 15.1. The slots are numbered 1, 2, 3, . . . , starting from the top of the page, and users are more likely to click on the higher slots. We will assume that each slot has a specific *clickthrough rate* associated with it, which is the number of clicks per hour that an ad placed in that slot will receive.

In the models we discuss, we will make a few simplifying assumptions about the clickthrough rates. First, we assume that advertisers know the clickthrough rates. Second, we assume that the clickthrough rate depends only on the slot itself and not on the ad that is placed there. Third, we assume that the clickthrough rate of a slot also doesn't depend on the ads that are in *other* slots. In practice, the first of these assumptions is not particularly problematic since advertisers have a number of means (including tools provided by the search engine itself) for estimating clickthrough rates. The second assumption is an important issue: a relevant, high-quality ad in a high slot will receive more clicks than an off-topic ad, and in fact we will describe how to extend the basic models to deal with ad relevance and ad quality at the end of the chapter. The third assumption – interaction among the different ads being shown – is a more complex issue, and it is still not well understood even within the search industry.

This is the full picture from the search engine's side: the slots are the inventory that it is trying to sell. Now, from the advertisers' side, we assume that each advertiser has a *revenue per click*: the expected amount of revenue it receives per user who clicks on the ad. Here too, we will assume that this value is intrinsic to the advertiser and does not depend on what was being shown on the page when the user clicked on the ad. We will also assume that if an advertiser wants to be shown in a given slot, it wants to be shown there for every single search on the given query. (In practice, advertisers can specify more complex arrangements with the search engine, such as being shown for only a fraction of the queries issued, but we will not consider this here.)

This is all the information we need to understand the market for a particular keyword: the clickthrough rates of the slots, and the revenues per click of the advertisers. Figure 15.2 shows a small example with three slots and three advertisers: the slots have clickthrough rates of 10, 5, and 2, respectively, while the advertisers have revenues per click of 3, 2, and 1, respectively.

Constructing a Matching Market. We now show how to represent the market for a particular keyword as a matching market of the type we studied in Chapter 10. To do this, it is useful to first review the basic ingredients of a matching market from Chapter 10:

- The participants in a matching market consist of a set of buyers and a set of sellers.
- Each buyer j has a *valuation* for the item offered by each seller i. This valuation can depend on the identities of both the buyer and the seller, and we denote it v_{ij}.
- The goal is to match up buyers with sellers, in such a way that no buyer purchases two different items and the same item isn't sold to two different buyers.

clickthrough rates	slots	advertisers	revenues per click
10	a	x	3
5	b	y	2
2	c	z	1

Figure 15.2. In the basic setup of a search engine's market for advertising, there are a certain number of advertising slots to be sold to a population of potential advertisers. Each slot has a *clickthrough rate*: the number of clicks per hour it will receive, with higher slots generally getting higher clickthrough rates. Each advertiser has a *revenue per click*, the amount of money it expects to receive, on average, each time a user clicks on one of its ads and arrives at its site. We draw the advertisers in descending order of their revenue per click; for now, this is purely a pictorial convention, but in Section 15.2 we show that the market in fact generally allocates slots to the advertisers in this order.

To cast the search engine's advertising market for a particular keyword in this framework, we use r_i to denote the clickthrough rate of slot i, and v_j to denote the revenue per click of advertiser j. The benefit that advertiser j receives from being shown in slot i is then just $r_i v_j$, the product of the number of clicks and the revenue per click.

In the language of matching markets, this is advertiser j's valuation v_{ij} for slot i – the value it receives from acquiring slot i. So by declaring the slots to be the sellers, the advertisers to be the buyers, and the buyers' valuations to be $v_{ij} = r_i v_j$, the problem of assigning slots to advertisers is precisely the problem of assigning sellers to buyers in a matching market. In Figure 15.3(a), we show how this conversion is applied to

Figure 15.3. The allocation of advertising slots to advertisers can be represented as a matching market, in which the slots are the items to be sold, and the advertisers are the buyers. (a) An advertiser's valuation for a slot is simply the product of its own revenue per click and the clickthrough rate of the slot; these can be used to determine (b) market-clearing prices for the slots.

the example in Figure 15.2, yielding the buyer valuations shown. As this figure makes clear, the advertising setup produces a matching market with a special structure: since the valuations are obtained by multiplying rates by revenues, we have a situation where all the buyers agree on their preferences for the items being sold and where, in fact, the valuations of one buyer simply form a multiple of the valuations of any other buyer.

When we considered matching markets in Chapter 10, we focused on the special case in which the number of sellers and the number of buyers were the same. This made the discussion simpler in a number of respects; in particular, it meant that the buyers and sellers could be perfectly matched, so that each item is sold and each buyer purchases exactly one item. We will make the analogous assumption here: with slots playing the role of sellers and advertisers playing the role of buyers, we will focus on the case in which the numbers of slots and advertisers are the same. But it is important to note that this assumption is not at all essential, because for purposes of analysis we can always translate a scenario with unequal numbers of slots and advertisers into an equivalent one with equal numbers, as follows. If there are more advertisers than slots, we simply create additional "fictitious" slots of clickthrough rate 0 (i.e., of valuation 0 to all buyers) until the number of slots is equal to the number of advertisers. The advertisers who are matched with the slots of clickthrough rate 0 are then simply the ones who don't get assigned a (real) slot for advertising. Similarly, if there are more slots than advertisers, we just create additional "fictitious" advertisers who have a valuation of 0 for all slots.

Obtaining Market-Clearing Prices. With the connection to matching markets in place, we can use the framework from Chapter 10 to determine *market-clearing prices*. Again, it is worth reviewing this notion from Chapter 10 in a bit of detail as well, since we will use it heavily in what follows. Recall, roughly speaking, that the prices charged by the sellers are market clearing if, with these prices, each buyer prefers a different seller's item. More precisely, the basic ingredients of market-clearing prices are as follows:

- Each seller i announces a price p_i for his item. (In our case, the items are the slots.)
- Each buyer j evaluates her payoff for choosing a particular seller i: it is equal to the valuation minus the price for this seller's item, $v_{ij} - p_i$.
- We then build a *preferred-seller graph* as in Figure 15.3(b) by linking each buyer to the seller or sellers from which she gets the highest payoff.
- The prices are *market-clearing* if this graph has a perfect matching: in this case, we can assign distinct items to all the buyers in such a way that each buyer gets an item that maximizes her payoff.

In Chapter 10, we showed that market-clearing prices exist for every matching market, and we gave a procedure to construct them. We also showed in Chapter 10 that the assignment of buyers to sellers achieved by market-clearing prices always maximizes the buyers' total valuations for the items they get.

Returning to the specific context of advertising markets, market-clearing prices for the search engine's advertising slots have the desirable property that advertisers prefer

different slots, and the resulting assignment of advertisers to slots maximizes the total valuations of each advertiser for what they get. [Again, see Figure 15.3(b).] In fact, it is not hard to work out that when valuations have the special form that we see in advertising markets – each consisting of a clickthrough rate times a revenue per click – then the maximum valuation is always obtained by giving the slot with the highest clickthrough rate to the advertiser with maximum revenue per click, the slot with the second-highest rate to the advertiser with the second-highest revenue per click, and so forth.

The connection with matching markets shows that we can in fact think about advertising prices in the more general case where different advertisers can have arbitrary valuations for slots – they need not be the product of a clickthrough rate and a revenue per click. This allows advertisers, for example, to express how they feel about users who arrive via an ad in the third slot compared with those who arrive via an ad in the first slot. (And indeed, it is reasonable to believe that these two populations of users might have different behavioral characteristics.)

Finally, however, this construction of prices can only be carried out by a search engine if it actually knows the valuations of the advertisers. In the next section we consider how to set prices in a setting where the search engine doesn't know these valuations; it must rely on advertisers to report them without being able to know whether this reporting is truthful.

15.3 Encouraging Truthful Bidding in Matching Markets: The VCG Principle

What would be a good price-setting procedure when the search engine doesn't know the advertisers' valuations? In the early days of the search industry, variants of the first-price auction were used: advertisers were simply asked to report their revenues per click in the form of bids, and then they were assigned slots in decreasing order of these bids, and charged a price per click equal to the full value of their respective bids. Recall from Chapter 9 that when bidders are charged the full value of their bids, they will generally underreport, and this is what happened here. Bids were shaded downward, below their true values. More problematically, because the auctions were running continuously over time, advertisers constantly adjusted their bids by small increments to experiment with the outcome and to try to slightly outbid competitors. This resulted in a highly turbulent market and a huge resource expenditure on the part of both the advertisers and the search engines, as the constant price experimentation led to prices for most queries being updated extremely frequently.

In the case of a single-item auction, we saw in Chapter 9 that these problems are handled by running a second-price auction, in which the single item is awarded to the highest bidder at a price equal to the second-highest bid. As we showed there, truthful bidding is a *dominant strategy* for second-price auctions – it is at least as good as any other strategy, regardless of what the other participants are doing. This dominant-strategy result means that second-price auctions avoid many of the pathologies associated with more complex auctions.

But what is the analogue of the second-price auction for advertising markets with multiple slots? Given the connections to matching markets in the previous section, this

turns out to be a special case of an interesting and fundamental question: how can we define a price-setting procedure for matching markets so that truthful reporting of valuations is a dominant strategy for the buyers? Such a procedure would be a massive generalization of the second-price auction, which – though already fairly subtle – only applies to the case of single items.

The VCG Principle. Since a matching market contains many items, it is hard to directly generalize the literal description of the second-price single-item auction, in which we assign the item to the highest bidder at the second-highest price. However, by viewing the second-price auction in a somewhat less obvious way, we get a principle that does generalize.

This view is the following. First, the second-price auction produces an allocation that maximizes social welfare: the bidder who values the item the most gets it. Second, the winner of the auction is charged an amount equal to the "harm" he causes the other bidders by receiving the item. That is, suppose the bidders' values for the item are $v_1, v_2, v_3, \ldots, v_n$ in decreasing order. Then if bidder 1 were not present, the item would have gone to bidder 2, who values it at v_2. The other bidders still would not get the item, even if bidder 1 weren't there. Thus, bidders 2 through n collectively experience a harm of v_2 because bidder 1 is there – since bidder 2 loses this much value, and bidders 3 through n are unaffected. This harm of v_2 is exactly what bidder 1 is charged. Indeed, the other bidders are also charged an amount equal to the harm they cause to others – in this case zero, since no bidder is affected by the presence of any of bidders 2 through n in the single-item auction.

Again, this approach is a nonobvious way to think about single-item auctions, but it is a principle that turns out to encourage truthful reporting of values in much more general situations: each individual is charged the harm they cause to the rest of the world. To put it another way, each individual is charged a price equal to the total amount everyone would be better off if this individual weren't there. We will refer to this as the Vickrey-Clarke-Groves (VCG) principle, after the work of Clarke and Groves, who generalized the central idea behind Vickrey's second-price auction for single items [112, 199, 400]. For matching markets, we will describe an application of this principle due to Herman Leonard [270] and Gabrielle Demange [128]; it develops a pricing mechanism in this context that causes buyers to reveal their valuations truthfully.

Applying the VCG Principle to Matching Markets. In a matching market, we have a set of buyers and a set of sellers – with equal numbers of each – and buyer j has a valuation of v_{ij} for the item being sold by seller i.[2] We assume here that each buyer knows her own valuations, but that these valuations are not known to the other buyers or to the sellers. Also, we assume that each buyer only cares which item she receives, not about how the remaining goods are allocated to the other buyers. Thus, in the language of auctions, the buyers have *independent, private values*.

Under the VCG principle, we first assign items to buyers so as to maximize total valuation. Then, the price buyer j should pay for seller i's item – in the event she

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[2] As always, we can handle unequal numbers of buyers and sellers by creating "fictitious" individuals and valuations of 0, as in Section 15.2.
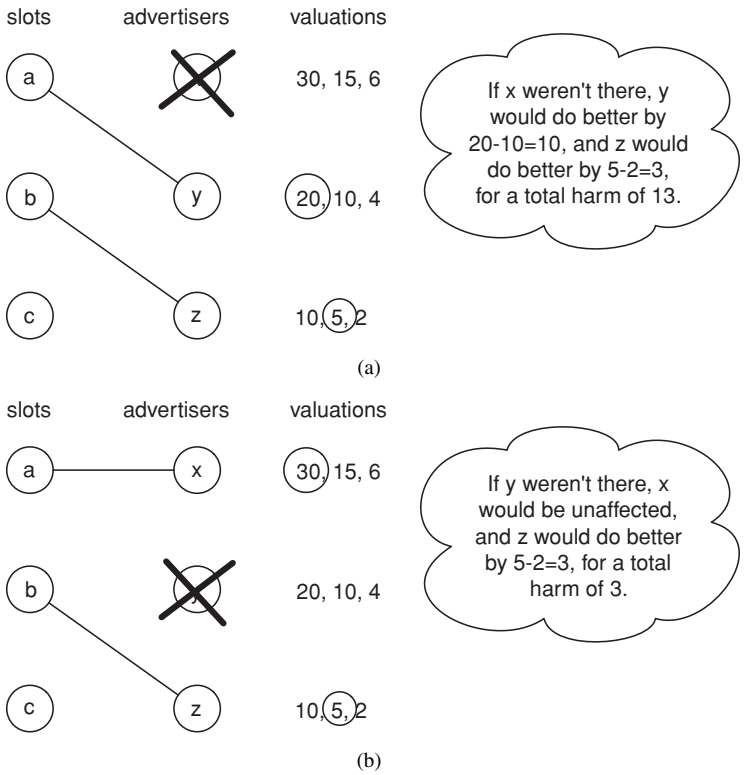
Figure 15.4. The VCG price an individual buyer pays for an item can be determined by working out how much better off all other buyers would be if this individual buyer were not present: (a) determining how much better off y and z would be if x were not present and (b) determining how much better off x and z would be if y were not present.

receives it – is the harm she causes to the remaining buyers through her acquisition of this item. This value is equal to the total boost in valuation everyone else would get if we computed the optimal matching without buyer j present. To give a better sense of how this principle works for matching markets, we first walk through how it would apply to the example in Figure 15.3. We then define the VCG price-setting mechanism in general, and in the next section we show that it yields truth-telling as a dominant strategy: for each buyer, truth-telling is at least as good as any other option, regardless of what the other buyers are doing.

In Figure 15.3, where the buyers are advertisers and the items are advertising slots, suppose we assign items to maximize total valuation: item a to buyer x, item b to buyer y, and item c to buyer z. What prices does the VCG principle dictate for each buyer? We show the reasoning in Figure 15.4.

- First, in the optimal matching without buyer x present, buyer y gets item a and buyer z gets item b. This outcome improves the respective valuations of y and z for their assigned items by $20 - 10 = 10$ and $5 - 2 = 3$, respectively. The total harm caused by x is therefore $10 + 3 = 13$, and so this is the price that x should pay.

- In the optimal matching without buyer y present, buyer x still gets a (so she is unaffected), while buyer z gets item b, for an improved valuation of 3. The total harm caused by y is $0 + 3 = 3$, and so this is the price that y should pay.
- Finally, in the optimal matching without buyer z present, buyers x and y each get the same items they would have gotten had z been there; buyer z causes no harm to the rest of the world, and so her VCG price is 0.

With this example in mind, we now describe the VCG prices for a general matching market. This follows exactly from the principle we've been discussing, but it requires a bit of notation due to the multiple items and valuations. First, let S denote the set of sellers and B denote the set of buyers. Let V_B^S denote the maximum total valuation over all possible perfect matchings of sellers and buyers – simply the value of the socially optimal outcome with all buyers and sellers present.

Now, let $S{-}i$ denote the set of sellers with seller i removed, and let $B{-}j$ denote the set of buyers with buyer j removed. So if we give item i to seller j, then the best total valuation the rest of the buyers could get is V_{B-j}^{S-i}: this is the value of the optimal matching of sellers and buyers when we've taken item i and buyer j out of consideration. On the other hand, if buyer j simply didn't exist but item i were still an option for everyone else, then the best total valuation the rest of the buyers could get is V_{B-j}^S. Thus, the total harm caused by buyer j to the rest of the buyers is the difference between how they'd do without j present and how they do with j present; in other words, it is the difference $V_{B-j}^S - V_{B-j}^{S-i}$. This is the VCG price p_{ij} that we charge to buyer j for item i, so we have the equation

$$p_{ij} = V_{B-j}^S - V_{B-j}^{S-i}. \tag{15.1}$$

The VCG Price-Setting Mechanism. Using the ideas developed so far, we can now define the complete VCG price-setting mechanism for matching markets. We assume that there is a single price-setting authority (an "auctioneer") who can collect information from buyers, assign items to them, and charge prices. Fortunately, this framework works very well for selling advertising slots, where all the items (the slots) are under the control of a single agent (the search engine).

The mechanism works as follows:

1. Ask buyers to announce valuations for the items. (These announcements need not be truthful.)
2. Choose a socially optimal assignment of items to buyers – that is, a perfect matching that maximizes the total valuation of each buyer for what they get. This assignment is based on the announced valuations (since that's all we have access to.)
3. Charge each buyer the appropriate VCG price; that is, if buyer j receives item i under the optimal matching, then charge buyer j a price p_{ij} determined according to Equation (15.1).

Essentially, what the auctioneer has done is to define a game that the buyers play. They must choose a strategy (a set of valuations to announce), and they receive a payoff – their valuation for the item they get, minus the price they pay. What turns out to be true, though it is far from obvious, is that this game has been designed to make truth-telling – in which a buyer announces her true valuations – a dominant strategy. We will prove this in the next section; but before this, we make a few observations.

First, notice that there's a crucial difference between the VCG prices defined here and the market-clearing prices arising from the auction procedure in Chapter 10. The market-clearing prices defined there were *posted prices*, in that the seller simply announced a price and was willing to charge it to any buyer who was interested. The VCG prices here, on the other hand, are *personalized prices*: they depend on both the item being sold and the buyer to whom it is being sold. The VCG price p_{ij} paid by buyer j for item i may well differ, under Equation (15.1), from the VCG price p_{ik} that buyer k would pay if it were assigned item i.[3]

Another way to think about the relationship between the market-clearing prices from Chapter 10 and the VCG prices here is to observe how each is designed to generalize a different single-item auction format. The market-clearing prices in Chapter 10 were defined by a significant generalization of the ascending (English) auction: prices were raised step-by-step until each buyer favored a different item, and we saw in Section 10.5 that one could encode the single-item ascending auction as a special case of the general construction of market-clearing prices.

The VCG prices, on the other hand, are defined by an analogous and equally substantial generalization of the sealed-bid second-price auction. At a qualitative level, we can see that the "harm-done-to-others" principle is behind both the second-price auction and the VCG prices, but in fact we can also see fairly directly that the second-price auction is a special case of the VCG mechanism. Specifically, suppose there are n buyers who each want a single item, and buyer i has valuation v_i for it, where the numbers v_i are sorted in descending order so that v_1 is the largest. Let's turn this into a matching market with n buyers and n sellers by simply adding $n - 1$ fictitious items; all buyers have valuation 0 for each fictitious item. Now, if everyone reports their values truthfully, then the VCG mechanism would assign item 1 (the real item and the only one with any value) to buyer 1 (who has the highest valuation), and all the rest of the buyers would get fictitious items of zero value. What price should buyer 1 pay? According to Equation (15.1), she should pay $V_{B-1}^{S} - V_{B-1}^{S-1}$. The first term is buyer 2's valuation, since with buyer 1 gone the socially optimal matching gives item 1 to buyer 2. The second term is 0 since, with both buyer 1 and item 1 gone, there are no remaining items of any value. Thus, buyer 1 pays buyer 2's valuation, and so we have precisely the pricing rule for second-price sealed-bid auctions.

15.4 Analyzing the VCG Mechanism: Truth-Telling as a Dominant Strategy

We now show that the VCG mechanism encourages truth-telling in a matching market. Concretely, we will prove the following claim.

> *Claim:* If items are assigned and prices computed according to the VCG mechanism, then truthfully announcing valuations is a dominant strategy for each buyer, and the resulting assignment maximizes the total valuation of any perfect matching of items and buyers.

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[3] Despite this, there are deep and subtle connections between the two kinds of prices; we explore this issue further in the final section of this chapter.

The second part of this claim (that the total valuation is maximized) is easy to justify: if buyers report their valuations truthfully, then the assignment of items is designed to maximize the total valuation by definition.

The first part of the claim is more subtle: why is truth-telling a dominant strategy? Suppose that buyer j announces her valuations truthfully, and in the matching we assign her item i. Then her payoff is $v_{ij} - p_{ij}$. We want to show that buyer j has no incentive to deviate from a truthful announcement.

If buyer j decides to lie about her valuations, then one of two things can happen: either this lie affects the item she gets, or it doesn't. If buyer j lies but still gets the same item i, then her payoff remains exactly the same, because the price p_{ij} is computed only using announcements by buyers other than j. So if a deviation from truth-telling is going to be beneficial for buyer j, it has to affect the item she receives.

Suppose, therefore, that buyer j lies about her valuations and gets item h instead of item i. In this case, her payoff would be $v_{hj} - p_{hj}$. Notice again that the price p_{hj} is determined only by the announcements of buyers other than j. To show that there is no incentive to lie and receive item h instead of i, we need to show that

$$v_{ij} - p_{ij} \geq v_{hj} - p_{hj}.$$

If we expand out the definitions of p_{ij} and p_{hj} using Equation (15.1), this is equivalent to showing

$$v_{ij} - \left(V_{B-j}^{S} - V_{B-j}^{S-i}\right) \geq v_{hj} - \left(V_{B-j}^{S} - V_{B-j}^{S-h}\right).$$

Both sides of this inequality contain the term V_{B-j}^{S}, so we can add this to both sides; in this way, the previous inequality is equivalent to showing

$$v_{ij} + V_{B-j}^{S-i} \geq v_{hj} + V_{B-j}^{S-h}. \tag{15.2}$$

We now argue why this last inequality holds. In fact, the left- and right-hand sides each describe the total valuation of different matchings, as shown in Figure 15.5. The matching on the left-hand side is constructed by pairing j with the item i she would get in an optimal matching and then optimally matching the remaining buyers and items. In other words, it is a matching that achieves the maximum total valuation over all possible perfect matchings, so we can write the left-hand side as

$$v_{ij} + V_{B-j}^{S-i} = V_{B}^{S}. \tag{15.3}$$

In contrast, the matching on the right-hand side of Inequality (15.2) is constructed by pairing j with some other item h and then optimally matching the remaining buyers and items. So it is a matching that achieves the maximum total valuation *only over those matchings that pair j with h*. Therefore,

$$v_{hj} + V_{B-j}^{S-h} \leq V_{B}^{S}.$$

The left-hand side of Inequality (15.2) – the maximum valuation with no restrictions on who gets any slot – must be at least as large as the right-hand side – the maximum with a restriction. And this is what we wanted to show.

Nothing in this argument depends on the decisions made by other buyers about what to announce. For example, it doesn't require them to announce their true values; the arguments comparing different matchings can be applied to whatever valuations are
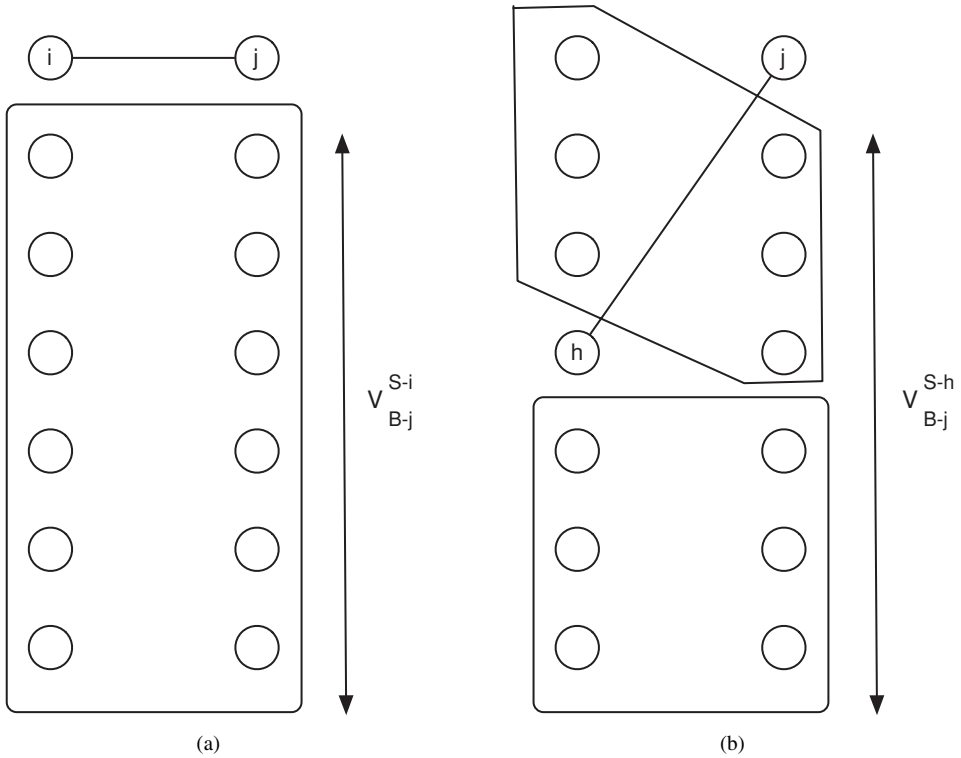
Figure 15.5. The heart of the proof that the VCG mechanism encourages truthful bidding comes down to a comparison of the value of two matchings: (a) $v_{ij} + V_{B-j}^{S-i}$ is the maximum valuation of any matching and (b) $v_{hj} + V_{B-j}^{S-h}$ is the maximum valuation only over matchings constrained to assign h to j.

announced by the other buyers, with the same consequences. Thus, we have shown that truthfully announcing valuations is a dominant strategy in the VCG mechanism.

To close this section, let's go back to the specific case of keyword-based advertising, in which the buyers correspond to advertisers and the items for sale correspond to advertising slots. Our discussion so far has focused on finding and achieving an assignment of advertisers to slots that maximizes the total valuation obtained by advertisers. But of course, this is not what the search engine selling the advertising slots directly cares about. Instead it cares about its revenue: the sum of the prices that it can charge for slots. It is not clear that the VCG mechanism is the best way to generate revenue for the search engine. Determining which procedure maximizes seller revenue is a current topic of research. It could be that the best a seller can do is to use some procedure that generates an optimal matching, and potentially one that is better than VCG at converting more of the total valuation into seller revenue. Or it could be that the seller is better off using a procedure that does not always yield an optimal matching. And it may be that some version of a revenue-equivalence principle – such as we saw for single-item auctions in Chapter 9 – holds here as well, showing that certain classes of auction provide equivalent amounts of revenue to the seller when buyers behave strategically.

In the next sections, we sample the general flavor of some of these revenue issues by considering the alternative to VCG that the search industry has adopted in practice – a

simple-to-describe auction, called the generalized second-price auction, that induces complex bidding behavior.

15.5 The Generalized Second-Price Auction

After some initial experiments with other formats, the main search engines have adopted a procedure for selling advertising slots called the generalized second-price (GSP) auction. At some level, GSP – like VCG – is a generalization of the second-price auction for a single item. However, as we will see, GSP is a generalization only in a superficial sense, since it doesn't retain the nice properties of the second-price auction and VCG.

In the GSP procedure, each advertiser j announces a *bid* consisting of a single number b_j – the price it is willing to pay per click. (This would correspond, for example, to the $1.70 for "calligraphy pens" or $1.50 for "Keuka Lake" that we saw at the beginning of the chapter.) As usual, it is up to the advertiser whether or not its bid is equal to its true valuation per click, v_j. Then, after each advertiser submits a bid, the GSP procedure awards each slot i to the ith highest bidder, at a price per click equal to the $(i + 1)$st highest bid. In other words, each advertiser shown on the results page is paying a price per click equal to the bid of the advertiser just below them.

So GSP and VCG can be viewed in parallel terms, in that each asks for announced valuations from the advertisers, and then each uses these announcements to determine an assignment of slots to advertisers, as well as prices to charge. When there is a single slot, both are equivalent to the second-price auction. But when there are multiple slots, their rules for producing prices are different. VCG's rule is given by Equation (15.1). GSP's rule, when the bids per click are $b_1, b_2, b_3, \ldots$ in descending order, is to charge a cumulative price of $r_i b_{i+1}$ for slot i. This is because the ith highest bidder will get slot i at a price per click of b_{i+1}; multiplying by the clickthrough rate of r_i gives a total price of $r_i b_{i+1}$ for all the clicks associated with slot i.

In the version of GSP we consider here, there is no minimum bid, just as we did not have a minimum bid (or reserve price) when we first introduced second-price auctions in Chapter 9. In practice, search engines do impose a minimum bid, and this minimum bid may increase the revenue generated by the auction, particularly if the smallest clickthrough rate is very low. This minimum bid plays a role similar to that played by the reserve price in a second price auction (as discussed in the final section of Chapter 9).

Analyzing GSP. GSP was originally developed at Google; once it had been in use for a while in the search industry, researchers including Varian [399] and Edelman, Ostrovsky, and Schwarz [144] began working out some of its basic properties. Their analysis formulates the problem as a game, using the definitions from Chapter 6. Each advertiser is a player, its bid is its strategy, and its payoff is its revenue minus the price it pays. In this game, we will consider Nash equilibria: we seek sets of bids so that, given these bids, no advertiser has an incentive to change how it is behaving.[4]

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[4] In order to fit our Nash equilibrium analysis here into the framework defined in Chapter 6, we will assume that each advertiser knows the values of all other advertisers, so that each player (i.e., advertiser) in the bidding

Figure 15.6. An example of a set of advertisers and slots for which truthful bidding is not an equilibrium in the GSP auction. Moreover, this example possesses multiple equilibria, some of which are not socially optimal.

First, we'll see that GSP has a number of pathologies that VCG was designed to avoid: truth-telling may not constitute a Nash equilibrium, there can in fact be multiple possible equilibria, and some of these may produce assignments of advertisers to slots that do not maximize total advertiser valuation. On the positive side, we show in the next section that there is always at least one Nash equilibrium set of bids for GSP, and that among the (possibly multiple) equilibria, there is always one that does maximize total advertiser valuation. The analysis leading to these positive results about equilibria builds directly on the market-clearing prices for the matching market of advertisers and slots, thus establishing a connection between GSP and market-clearing prices.

Hence, while GSP possesses Nash equilibria, it lacks some of the main nice properties of the VCG mechanism from Sections 15.3 and 15.4. However, in keeping with our discussion from the end of the previous section, the search engines ultimately have an interest in choosing a procedure that maximizes their revenue (given the behavior of the advertisers in response to it). Viewed in this light, it is not clear that GSP is the wrong choice, though it is also far from clear that it is the right choice. As mentioned at the end of Section 15.4, understanding the revenue trade-offs among different procedures for selling keyword-based advertising is largely an open question, and the subject of current research.

Truth-Telling May Not Be an Equilibrium. It is not hard to construct an example to show that truth-telling may not be an equilibrium when the GSP procedure is used. One example of this is depicted in Figure 15.6:

- There are two slots for ads, with clickthrough rates of 10 and 4. In the figure, we also show a third fictitious slot of clickthrough rate 0, so as to equalize the number of advertisers and slots.
- There are three advertisers, x, y, and z, with values per click of 7, 6, and 1, respectively.

game knows the full set of payoffs to all players. It is also interesting to consider how this game works when the players do not know each other's payoffs, but only their own; however, we do not consider this version here.

Now, if each advertiser bids its true valuation, then advertiser x gets the top slot at a price per click of 6; since there are 10 clicks associated with this slot, x pays a cumulative price of $6 \times 10 = 60$ for the slot. Advertiser x's valuation for the top slot is $7 \times 10 = 70$, so its payoff is $70 - 60 = 10$. Now, if x were to lower its bid to 5, then it would get the second slot for a price per click of 1, implying a cumulative price of 4 for the slot. Because its valuation for the second slot is $7 \times 4 = 28$, this is a payoff of $28 - 4 = 24$, which is an improvement over the result of bidding truthfully.

Multiple and Nonoptimal Equilibria. The example in Figure 15.6 turns out to illustrate some other complex properties of bidding behavior in GSP auctions. In particular, there is more than one equilibrium set of bids for this example, and among these equilibria are some that produce a socially nonoptimal assignment of advertisers to slots.

First, suppose that advertiser x bids 5, advertiser y bids 4, and advertiser z bids 2. With a little effort, we can check that this forms an equilibrium: checking the condition for z is easy, and the main further things to observe are that x doesn't want to lower its bid below 4 so as to move to the second slot, and y doesn't want to raise its bid above 5 to get the first slot. This is an equilibrium that produces a socially optimal allocation of advertisers to slots, since x gets slot a while y gets b and z gets c.

But one can also check that if advertiser x bids 3, advertiser y bids 5, and advertiser z bids 1, then we also get a set of bids in Nash equilibrium. Again, the main thing to verify is that x doesn't want to raise its bid above y's, and that y doesn't want to lower its bid below x's. This equilibrium is not socially optimal, since it assigns y to the highest slot and x to the second highest.

There is much that is not understood in general about the structure of the suboptimal equilibria arising from the GSP procedure. For example, it is an interesting question to try to quantify how far from social optimality a Nash equilibrium of GSP can be.

The Revenue of GSP and VCG. The existence of multiple equilibria also adds to the difficulty in reasoning about the search engine revenue generated by GSP, since it depends on which equilibrium is selected (potentially from among many) by the bidders. In the example we've been working with, we'll show that, depending on which equilibrium of GSP the advertisers actually use, the revenue to the search engine can be either higher or lower than the revenue it would collect by charging the VCG prices.

Let's start by determining the revenue to the search engine from the two GSP equilibria that we worked out earlier.

- With bids of 5, 4, and 2, the 10 clicks in the top slot are sold for 4 per click, and the 4 clicks in the second slot are sold for 2 per click, for a total revenue to the search engine of 48.
- On the other hand, with bids of 3, 5, and 1, the 10 clicks in the top slot are sold for 3 per click, and the 4 clicks in the second slot are sold for 1 per click, for a total revenue to the search engine of 34.

Now, how do these compare with the revenue generated by the VCG mechanism? To work out the VCG prices, we first need to convert the example from Figure 15.6

Figure 15.7. The example from Figure 15.6 as a matching market, with advertiser valuations for the full set of clicks associated with each slot.

into a matching market, just as we did in Section 15.2: for each advertiser and each slot, we work out the advertiser's valuation for the full set of clicks associated with that slot. We show these valuations in Figure 15.7.

The matching used by the VCG mechanism is the one that maximizes the total valuation of all advertisers for the slot they get; this is achieved by assigning slot a to x, slot b to y, and slot c to z. Now, we work out a price to charge each advertiser for the full set of clicks in the slot it gets, by determining the harm each advertiser causes to all others. The harm x causes to y and z can be computed as follows: without x present, y would move up one slot, obtaining an increased valuation of $60 - 24 = 36$, and z would move up one slot, obtaining an increased valuation of $4 - 0 = 4$. Therefore, x should pay 40 for the full set of clicks in the first slot. Similarly, without y present, z would get 4 instead of 0, so y should pay 4 for the set of clicks in the second slot. Finally, since z causes no harm to anyone, it pays 0. Thus, the total revenue collected by the search engine is 44.

So in this example we find that the answer to the question, "Does GSP or VCG provide more revenue to the search engine?" is indeed that it depends on which equilibrium of GSP the advertisers use. With the first equilibrium of GSP that we identified, the revenue is 48, while with the second, the revenue is 34. The revenue from the VCG mechanism is in between these two values, at 44.

15.6 Equilibria of the Generalized Second-Price Auction

The examples in the previous section give a sense for some of the complex behavior of GSP auctions. Here, we show that there is nonetheless a natural connection between GSP and market-clearing prices: from a set of market-clearing prices for the matching market of advertisers and slots, one can always construct a set of bids in Nash equilibrium – and moreover, one that produces a socially optimal assignment of advertisers to slots. As a consequence, there always exists a set of socially optimal equilibrium bids for the GSP procedure.

To give the basic idea for how to construct an equilibrium, we do it first for the example from Figure 15.6. In fact, we've just seen two equilibria for this example in
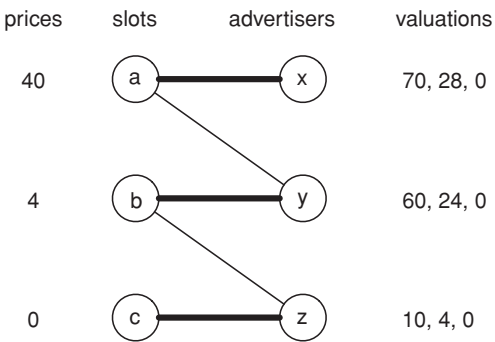
Figure 15.8. Determining market-clearing prices for the example in Figure 15.6, starting with its representation as a matching market.

the previous section, but the point here is to see how a socially optimal one can be easily constructed by following a few simple principles, rather than by trial-and-error or guesswork. We'll then identify the principles from this example that carry over to construct equilibria in general.

An Equilibrium for Figure 15.6. The basic idea is to use market-clearing prices to guide us to a set of bids that produce these prices. To construct market-clearing prices, we first convert the example from Figure 15.6 into a matching market by determining advertisers' valuations for each slot, as we did at the end of the previous section (in Figure 15.7). We then determine market-clearing prices for this matching market, as shown in Figure 15.8.

These market-clearing prices are cumulative prices for each slot – single prices that cover all the clicks associated with that slot. We can easily translate back to prices per click by simply dividing by the clickthrough rate; this produces a price per click of $40/10 = 4$ for the first slot, and $4/4 = 1$ for the second slot. It turns out not to be important how we price the fictitious third slot per click, but it is fine to give it a price of 0.

Next, we find bids that result in these prices per click. This is not hard to do: the prices per click are 4 and 1 for the two slots, so these should be the bids of y and z, respectively. Then the bid of x can be anything as long as it's more than 4. With these bids, x pays 4 per click for the first slot, y pays 1 per click for the second slot, and z pays 0 per click for the (fake) third slot – and the allocation of advertisers to slots is socially optimal.

Having used the market-clearing prices to guide us toward a set of bids, we now use the market-clearing property to verify that these bids form a Nash equilibrium. There are several cases to consider, but the overall reasoning will form the general principles that extend beyond just this example. First, let's argue that x doesn't want to lower its bid. If it drops down to match y's bid, then it can get the second slot at the price that y is currently paying. Similarly, it could match z's bid and get the third slot at the price that z is currently paying. But, since the prices are market clearing, x doesn't want to do either of these things. For similar reasons, y doesn't want to drop its bid to get the third slot at the price z is currently paying.

Next, let's argue that y doesn't want to raise its bid. Indeed, suppose that it raised its bid to get the first slot; to do this, it would need to match x's current bid. But in this case, x becomes the second-highest bidder, and so y would get the first slot at a price per click equal to x's current bid, which is above 4. Because the prices are market clearing, y doesn't prefer the first slot to its current slot at a price per click of 4, so it certainly doesn't prefer the first slot to its current slot at a higher price per click. Thus, y doesn't want to raise its bid. Similar reasoning shows that z doesn't want to raise its bid.

This concludes the analysis: no advertiser wants to raise or lower its current bid, and so the set of bids in this example forms a Nash equilibrium.

It is not hard to carry out the construction and the reasoning used here in general; we show how to do this next.

GSP Always Has a Nash Equilibrium: The General Argument. Now let's consider a general instance in which we have a set of advertisers and a set of slots. By adding fake slots of 0 value if necessary, we will assume that these two sets have the same size.

Let's suppose that the advertisers are labeled $1, 2, \ldots, n$ in decreasing order of their valuations per click, and let's suppose that the slots are labeled $1, 2, \ldots, n$ in decreasing order of their clickthrough rates. We first represent the set of advertisers and slots using a matching market, and we consider any set of market-clearing prices for the slots, denoted $p_1, p_2, \ldots, p_n$, in order. Again, these are prices for the full set of clicks in each slot; we will consider the price per click of each slot below. In Section 15.2, we argued that because a perfect matching in the resulting preferred-seller graph maximizes the total valuation of each advertiser for the slot it gets, it follows that the advertiser with the highest valuation per click gets the top slot, the advertiser with next-highest valuation gets the second slot, and so forth, with advertiser i getting slot i.

We now show how to get this outcome from an equilibrium set of bids in a GSP auction. Our plan is first to construct a set of bids that produces this same set of market-clearing prices, together with the same socially optimal matching of advertisers to slots. Then we will show that these bids form a Nash equilibrium.

Constructing the Bids. For the first step, we start by considering the prices per click that we get from the market-clearing prices: $p_j^* = p_j / r_j$. We start by arguing that these prices per click decrease as we move down the slots: $p_1^* \geq p_2^* \geq \cdots \geq p_n^*$. To see why this is true, let's compare two slots j and k, where j is numbered lower than k, and show that $p_j^* \geq p_k^*$.

Since the prices are market clearing, advertiser k prefers slot k to slot j. In slot k, its total payoff is the product of its payoff per click, $v_k - p_k^*$, times the clickthrough rate r_k. In slot j, its total payoff would be the product of its payoff per click there, $v_k - p_j^*$, times the clickthrough rate r_j. Now, the clickthrough rate is higher in slot j, yet slot k is preferred; so it must be that the payoff per click is smaller in slot j. That is, $v_k - p_j^*$ is smaller than $v_k - p_k^*$, or equivalently, $p_j^* \geq p_k^*$. This inequality is precisely the fact we were looking for.

Now that we have decreasing prices per click, we can construct the bids we're looking for. We simply have advertiser j place a bid of p^*_{j-1} for each $j > 1$, and we have advertiser 1 place any bid larger than p^*_1. Notice that this is exactly what happened when we constructed an equilibrium for the example in Figure 15.6. With these bids, we have all the desired properties: for each j, advertiser j is assigned to slot j and pays a price per click of p^*_j.

Why Do the Bids Form a Nash Equilibrium? To show why these bids form a Nash equilibrium, we adapt the principles that we used in analyzing the equilibrium for Figure 15.6. We first argue that no advertiser will want to lower its bid, and then that no advertiser will want to raise its bid either.

Consider an advertiser j, currently in slot j. If it were to lower its bid, the best it could do is to pick some lower slot k, bid just under the current bid of advertiser k, and thereby get slot k at the price that advertiser k is currently paying. But because the prices are market clearing, j is at least as happy with its current slot at its current price as it would be with k's current slot at k's current price. So in fact, this shows that no advertiser will want to lower its bid.

How about raising a bid? The best advertiser j could do here is to pick some higher slot i, bid just above the current bid of advertiser i, and thereby get slot i. What price would j pay for slot i if it did this? It's forcing advertiser i one slot down, and so it would pay the current *bid* of advertiser i. This is actually larger than what advertiser i is currently paying for slot i: advertiser i is currently paying the bid of advertiser $i + 1$, which is lower. So the upshot is that j would get slot i at a price *higher* than the current price of slot i. Since the market-clearing condition says that j doesn't even prefer slot i at the current price, it certainly wouldn't prefer it at a higher price. This shows that no advertiser wants to raise its bid either, and so the set of bids indeed forms a Nash equilibrium.

15.7 Ad Quality

What we've discussed thus far forms part of the basic framework for thinking about search advertising markets. Of course, there are numerous further issues that come up in the use of this framework by the major search engines, and in this section and the next we briefly discuss a few of these issues. We begin with the issue of ad quality.

The Assumption of a Fixed Clickthrough Rate. One of the assumptions we've made throughout the analysis is that a fixed clickthrough rate r_j is associated with each slot j – in other words, that the number of clicks this slot receives is independent of which ad you place there. But in general this is not likely to be true: users will look at the thumbnail description of an ad placed in a given slot (evaluating, for example, whether they recognize the name of the company placing the ad), and this will affect whether they click on the ad. And *this*, in turn, affects how much money the search engine makes, since it's charging per click, not per impression.

So from the search engine's point of view, the worrisome scenario is that a low-quality advertiser bids very highly, thus obtaining the first slot under the GSP procedure.

Users are then not interested in clicking through on this ad (maybe they don't trust the company, or the ad is only minimally relevant to the query term). As a result, it sits at the top of the list as the high bidder, but the search engine makes almost no money from it because users rarely click on the ad. If the search engine could somehow expel this ad and promote the higher-quality ads, it could potentially make more money.

Again, our model as described can't really address this problem, since it starts from the assumption that an ad in position i will get clicks at rate r_i, regardless of which ad it is. This "pure" version of GSP, using the model from Sections 15.5 and 15.6, is essentially what the company Overture used at the time it was acquired by Yahoo!, and hence what Yahoo! used initially as well. And indeed, it suffers from exactly this problem – advertisers can sometimes occupy high slots without generating much money for the search engine.

The Role of Ad Quality. When Google developed its system for advertising, it addressed this problem as follows. For each ad submitted by an advertiser j, they determine an estimated *quality factor* q_j. This is intended as a "fudge factor" on the clickthrough rate: if advertiser j appears in slot i, then the clickthrough rate is estimated to be not r_i but the product $q_j r_i$. The introduction of ad quality is simply a generalization of the model we've been studying all along. In particular, if we assume that all factors q_i are equal to 1, then we get back the model that we've been using thus far in the chapter.

From the perspective of our matching market formulation, it's easy to incorporate these quality factors: we simply change the valuation of advertiser j for slot i, from $v_{ij} = r_i v_j$ to $v_{ij} = q_j r_i v_j$. The rest of the analysis remains the same, using these new valuations.

Google has adapted the GSP procedure analogously. Rather than assigning advertisers to slots in decreasing order of their bids b_j, it assigns them in decreasing order of the product of their bid and quality factor, $q_j b_j$. This makes sense, since this is the ordering of advertisers by expected revenue to the search engine. The payments change correspondingly. The previous rule – paying the bid of the advertiser just below you – can, in retrospect, be interpreted more generally as paying the minimum bid you would need in order to hold your current position. This rule carries over to the version with quality factors: each advertiser pays the minimum amount it would need to keep its current position, when ranked according to $q_j b_j$.

With these changes, it's possible to go back and perform the analysis of GSP at this more general level. Close analogues of all the previous findings still hold here; although the introduction of quality factors makes the analysis a little bit more complicated, the basic ideas remain largely the same [144, 399].

The Mysterious Nature of Ad Quality. How is ad quality computed? To a significant extent, it's estimated by actually observing the clickthrough rate of the ad when shown on search results pages; this makes sense, since the goal of the quality factor is to act as a modifier on the clickthrough rate. But other factors are taken into account, including the relevance of the ad text and the "landing page" to which the ad links. Just as with the unpaid organic search engine results on the left-hand side of the screen,

search engines are very secretive about how they compute ad quality, and do not reveal the details of this ad-quality computation to the advertisers who are bidding.

One consequence is that the introduction of ad quality factors makes the keyword-based advertising market much more opaque to the advertisers. With pure GSP, the rules were very simple: for a given set of bids, it was clear how the advertisers would be allocated to slots. But because the ad quality factor is under the search engine's control, it gives the search engine nearly unlimited power to affect the actual ordering of the advertisers for a given set of bids.

How does the behavior of a matching market such as this one change when the precise rules of the allocation procedure are being kept secret from the bidders? This is an issue that is actively discussed in the search industry, and a topic for potential research.

15.8 Complex Queries and Interactions among Keywords

At the outset, we observed that markets are being conducted simultaneously for millions of query words and phrases. In our analysis, we've focused the model on what goes on in a single one of these markets, for a single keyword; but in reality, of course, there are complex interactions among the markets for different keywords.

In particular, consider the perspective of a company that's trying to advertise a product using keyword-based advertising. Suppose, for example, that the company is selling ski vacation packages to Switzerland. There are a lot of different keywords and phrases on which the company might want to place bids: "Switzerland," "Swiss vacation," "Swiss hotels," "Alps," "ski vacation," "European ski vacation," and many others (including grammatical permutations of these). With a fixed advertising budget, and some estimates about user behavior and the behavior of other advertisers, how should the company go about dividing its budget across different keywords? This is a challenging problem, and one that is the subject of current research [357].

There's an analogous problem from the search engine's perspective. Suppose advertisers have placed bids on many queries relevant to Swiss ski vacations, and then a user comes and issues the query, "Zurich ski vacation trip December." It's quite likely that very few users have ever issued this exact query before, and also very likely that no advertiser has placed a bid on this exact phrase. If the rules of the keyword-based advertising market are defined too strictly – that the search engine can only show ads for words or phrases that have been explicitly bid on – then it seems as though both the search engine and the advertisers are losing money: there clearly are advertisers who would be happy to be displayed for this query.

The question of which ads to show, however, is quite a difficult problem. A simple rule, such as showing the advertisers that placed the maximum bid for any of the words in the query, seems like a bad idea: probably there are advertisers who have placed very high bids on "vacation" (e.g., companies that sell generic vacation packages) and "ski" (e.g., companies that sell skis), and neither of these seems like the right match to the query. It seems important to take into account the fact that the query, through its choice of terms, is specifying something fairly narrow.

Furthermore, even if relevant advertisers can be identified, how much should they be charged for a click, given that they never expressed a bid on exactly this query? The main search engines tend to get agreements from advertisers that they'll extrapolate from their bids on certain queries to implied bids on more complex queries, such as in this example, but working out the best way to do this is not fully understood. These issues are the subject of active work at search engine companies, and again the subject of some very interesting potential further research.

15.9 Advanced Material: VCG Prices and the Market-Clearing Property

At the end of Section 15.3, we noted some of the differences between the two main ways we've seen to assign prices to items in matching markets: the VCG prices defined in this chapter, and the construction of market-clearing prices from Chapter 10. In particular, we observed that the difference reflected a contrast between *personalized* and *posted* prices. VCG prices are selected only after a matching between buyers and sellers has been determined – the matching that maximizes the total valuation of buyers for what they get. The VCG price of an item thus makes use of information not just about the item itself but also about who is buying it in the matching. Market-clearing prices, in a sense, work the other way around: the prices are chosen first, and they are posted prices that are offered to any buyer who is interested. The prices then cause certain buyers to select certain items, resulting in a matching.[5]

Given these significant differences, one might expect the prices to look different as well. But a comparison of simple examples suggests that something intriguing may be going on. Consider, for instance, the matching market shown in Figures 15.3 and 15.4. In Figure 15.3 we see a set of market-clearing prices constructed using the procedure from Chapter 10. In Figure 15.4, we see that these same prices also arise as the VCG prices.

Nor is it the special structure of prices arising from clickthrough rates and revenues per click that causes this. For instance, let's go back to the example used in Figure 10.6 from Chapter 10, which has valuations with a much more "scrambled" structure. We've redrawn the final preferred-seller graph arising from the auction procedure in Figure 15.9, with the (unique) perfect matching in this graph indicated by bold edges. This is the matching that maximizes the total valuation of buyers for the item they get, so we apply the definitions from earlier in the current chapter to determine the VCG prices. For example, to determine the price that should be charged for seller a's item, we observe the following:

- If neither a nor x were present, the maximum total valuation of a matching between the remaining sellers and buyers would be 11, by matching y to c and z to b.
- If x wasn't present but a was, then the maximum total valuation possible would be 14, by matching y to b and z to a.
- The difference between these two quantities is the definition of the VCG price for item a; it is $14 - 11 = 3$.

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[5] In the discussion that follows, we'll refer to nodes on the left-hand side of the bipartite graph sometimes as "items" and sometimes as "sellers"; for our purposes here, they mean the same thing.
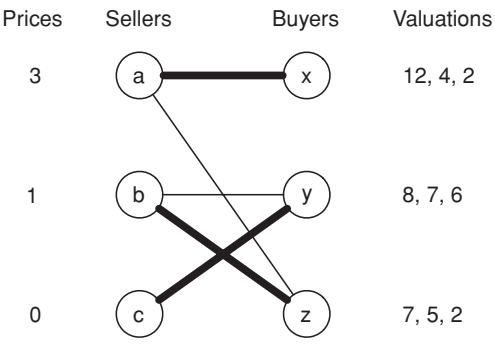
Figure 15.9. A matching market, with valuations and market-clearing prices specified, and a perfect matching in the preferred-seller graph indicated by the bold edges.

We could perform the corresponding analysis to get the VCG prices for items b and c, and we'd see that the values are 1 and 0, respectively. In other words, we again find that the VCG prices are also market-clearing prices.

In this section, we show that the relationship suggested by these examples holds in general. Our main result is that, despite their definition as personalized prices, VCG prices are always market clearing. That is, suppose we were to compute the VCG prices for a given matching market, first determining a matching of maximum total valuation and then assigning each buyer the item they receive in this matching, with a price tailored for this buyer–seller match. Then, however, suppose we go on to post the prices publicly: rather than requiring buyers to follow the matching used in the VCG construction, we allow any buyer to purchase any item at the indicated price. We will see that despite this greater freedom, each buyer will in fact achieve the highest payoff by selecting the item she was assigned when the VCG prices were constructed. This will establish that the prices are market clearing under the definition from Chapter 10.

First Steps Toward a Proof. Let's think for a minute about how you might prove such a fact, once you start to suspect from simple examples that it might be true. It's tempting to start with the very compact formula defining the VCG prices – Equation (15.1) – and then to somehow reason about this formula to show that it has the market-clearing property.

In fact, it's tricky to make this approach work, and it's useful to understand why. Recall that Equation (15.1) says that if item i is assigned to buyer j in the optimal matching then we should charge a price of

$$V_{B-j}^{S} - V_{B-j}^{S-i},$$

where V_{B-j}^{S} is the total valuation of an optimal matching with j removed, and V_{B-j}^{S-i} is the total valuation of an optimal matching with both i and j removed. Now, the term V_{B-j}^{S} is in fact a sum of many smaller terms, each consisting of the valuation of a distinct buyer for the item she is assigned in an optimal matching. Similarly, V_{B-j}^{S-i} is a sum of many terms, but the key conceptual difficulty is the following: V_{B-j}^{S} and V_{B-j}^{S-i} arise from different matchings – potentially very different matchings – and so there is

no direct way to compare the sums that they represent and easily subtract the terms of one from the other.

To make progress, we need to actually understand how the matchings that define these two terms V_{B-j}^{S} and V_{B-j}^{S-i} relate to each other at a structural level. To do this, we will show that matchings achieving these respective quantities can in fact arise from a common set of market-clearing prices: there is a single set of market-clearing prices on the set of items S so that matchings achieving each of V_{B-j}^{S} and V_{B-j}^{S-i} arise as perfect matchings in the preferred-seller graphs of related but slightly different matching markets. This fact will enable us to see how the two matchings relate to each other – and in particular how to build one from the other – in a way that lets us subtract the relevant terms from each other and thus analyze the right-hand side of Equation (15.1).

For all this to work, we need to first understand *which* set of market-clearing prices actually correspond to the VCG prices. There are many possible sets of market-clearing prices, but with some checking, we can see that in our examples the VCG prices have corresponded to prices that are as small as possible, subject to having the market-clearing property. So let's consider the following way to make this precise. Over all possible sets of market-clearing prices, consider the ones that minimize the total sum of the prices. (For example, in Figure 15.9, the total sum of prices is $3 + 1 + 0 = 4$.) We will refer to such prices as a set of *minimum market-clearing prices*. In principle, there could be multiple sets of minimum market-clearing prices, but in fact we will see that there is only one such set of these prices, and they form the VCG prices. This is the crux of the following result, proved by Leonard [270] and Demange [128].

> *Claim:* In any matching market, the VCG prices form the unique set of market-clearing prices of minimum total sum.

This is the statement we will prove in this section.

The proof of this statement is quite elegant, but it is also arguably the most intricate piece of analysis in the book; it has a level of complexity that involves bringing together several non-trivial lines of argument as part of the overall proof. In approaching a proof with this type of structure, it helps to proceed in two stages. First, we will outline a sequence of two key facts that illuminate the structure of the underlying matchings. Each of these two facts needs a proof, but we will first simply state the facts and show how the overall proof of the claim follows directly from them. This provides a high-level overview of how the proof works, in a way that is self-contained and contains the central ideas. After this high-level overview, we will describe how to prove the two facts themselves, which will fill in the remaining details of the proof.

Finally, here is one more point to note before beginning: as in a number of previous places when we discussed matching markets, we will assume that all valuations are whole numbers $(0, 1, 2, \ldots)$ and that all prices are whole numbers as well.

A. High-Level Overview of the Proof

Recall that our basic plan is to understand how matchings defining the quantities V_{B-j}^{S} and V_{B-j}^{S-i} relate to each other, by showing how they arise from a common structure. The
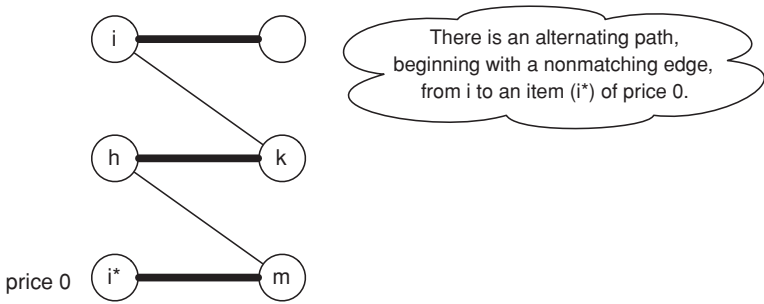
Figure 15.10. The key property of the preferred-seller graph for minimum market-clearing prices: for each item of price greater than 0, there is an alternating path, beginning with a nonmatching edge, to an item of price 0.

first step in doing this to show that the preferred-seller graph for the minimum market-clearing prices contains not only the edges of a perfect matching but also enough extra edges that we can easily assemble other matchings once we begin removing buyers in the ways suggested by the VCG formula.

The First Fact: The Preferred-Seller Graph for Minimum Market-Clearing Prices. The first of our two facts talks about the structure of the preferred-seller graph in the case when a set of market-clearing prices has minimum total sum. As a first step, let's go back to the initial example of market-clearing prices from Chapter 10, and in particular compare preferred-seller graphs for two different sets of market-clearing prices on the same set of valuations, shown in Figures 10.5(b) and 10.5(d). (This latter preferred-seller graph is the one we've redrawn in Figure 15.9.) Notice that the prices in the first of these, Figure 10.5(b), are larger and more "spread out," while the prices in Figure 10.5(d) in fact have minimum total sum. This corresponds to a difference in the structures of the preferred-seller graphs as well. The preferred-seller graph in Figure 10.5(b) is very sparse, with just three separate edges that constitute a perfect matching. The preferred-seller graph in Figure 10.5(d) is much denser: although it too contains only one perfect matching, it has additional edges that seem to serve as supports, "anchoring" the matching in place.

We now show that this anchoring effect is a general one: essentially, whenever a set of market-clearing prices has minimum total sum, the preferred-seller graph must contain not only a perfect matching but also enough other edges to form a path linking each item to an item of price 0. In fact, the paths we construct will be *alternating paths* in the sense defined in Section 10.6: for a given perfect matching in the graph, the edges on the paths will alternate between being part of the matching and not being part of the matching. We refer to these two kinds of edges as *matching edges* and *nonmatching edges*, respectively.

Here is the exact statement of the first fact, shown schematically in Figure 15.10.

> *Fact 1:* Consider the preferred-seller graph for a set of market-clearing prices of minimum total sum, fix a particular perfect matching in this graph, and let *i* be any item whose price is greater than 0. Then there is an alternating path, beginning with a nonmatching edge, that connects *i* to some item of price 0.
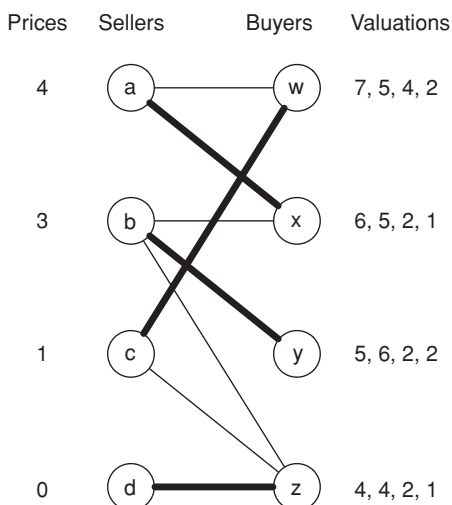
Figure 15.11. A matching market with market-clearing prices of minimum total sum. Note how there is an alternating path from each item, beginning with a nonmatching edge, that leads to the item of zero price.

For example, in Figure 15.9, with the matching indicated in bold, there is an alternating path from b to y to c; this path begins with the nonmatching b-y edge and ends at the zero-priced item c. Similarly, there is a longer alternating path from a through z, b, and y, ending at c. Figure 15.11 shows a larger example, also with market-clearing prices of minimum-total sum, and also with the matching indicated in bold; here too one can find alternating paths from each of items a, b, and c down to the zero-priced item d.

Following our plan, we defer the proof of Fact 1 until later in the section. However, we can give some intuition for the proof as follows. Roughly speaking, if there weren't alternating paths anchoring all the items to items of price 0, then we could find a set of items that were "floating" free of any relation to the zero-priced items. In this case, we could push the prices of these free-floating items down slightly while still preserving the market-clearing property. This would yield a set of market-clearing prices of smaller total sum, contradicting our assumption that we already have the minimum market-clearing prices. This contradiction will show that the minimum market-clearing prices are all anchored via alternating paths to zero-priced items.

The Second Fact: Zeroing Out a Buyer. Our second main fact relates the minimum market-clearing prices to a matching that achieves the value V_{B-j}^S, the first term on the right-hand side of Equation (15.1).

To explain how this fact works, we start with a useful way to think about the quantity V_{B-j}^S. Formally, V_{B-j}^S is the maximum total valuation of any matching in the market where j has been removed, but where all items have been kept. But here's a different, equivalent way to define V_{B-j}^S. Suppose that we were to change j's valuations for every item to 0; we'll call this the version of the matching market in which j has been *zeroed out*. To find an optimal matching in this market with j zeroed out, we note that it doesn't matter which item j gets (since j values them all at zero); therefore, we can
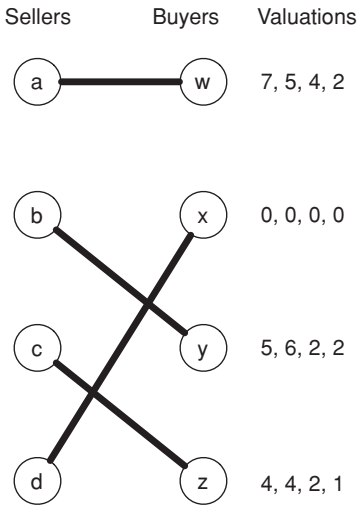
Figure 15.12. If we start with the example in Figure 15.11 and zero out buyer x, the structure of the optimal matching changes significantly.

first optimally match all the other buyers with items and then give j whatever is left over. The value of the resulting matching is V^S_{B-j}. In other words, V^S_{B-j} is the value of the optimal matching in the market where j is zeroed out: she is still present, but all her valuations are now equal to 0.

Now, an optimal matching in the market with j zeroed out may have a structure very different from that of an optimal matching in the original market; different buyers may get completely different items. For example, Figure 15.12 shows the unique optimal matching in the market from Figure 15.11 after we zero out x: other than buyer y, who still gets item b, the assignment of items to all other buyers has changed completely. This is another reflection of the difficulty in reasoning about Equation (15.1): when we remove buyers or items, the matchings can rearrange themselves in complex ways.

Despite this difficulty, there is an important connection between the original market and the zeroed-out market: *the minimum market-clearing prices for the original market are also market-clearing for the zeroed-out market.* We illustrate this for our example in Figure 15.13. Keeping the same prices that were used in Figure 15.11, we see that the preferred-seller graph still has a perfect matching even after x has been zeroed out, which means that the prices are still market clearing. Moreover, we can observe some additional features of this example. First, x now receives an item of price 0. Second, consider the *payoff* of each other buyer, defined as the valuation minus the price of the item she gets. For each other buyer, the payoff is the same in Figures 15.11 and 15.13.

Our second fact shows that all these observations hold in general.

> *Fact 2:* Consider any matching market with minimum market-clearing prices p, and let j be any buyer.
>
> (i) The prices p are also market clearing for the market in which j is zeroed out.

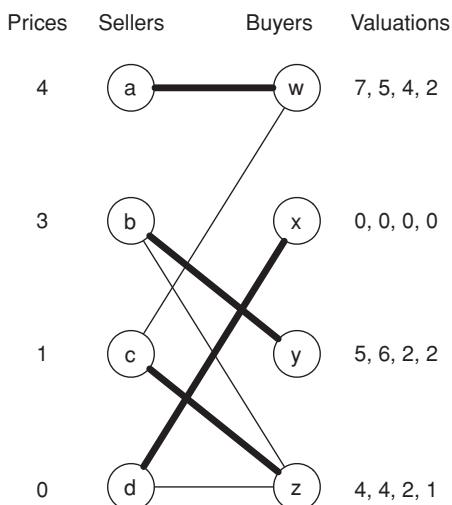Prices   Sellers   Buyers   Valuations

Figure 15.13. However, even after we zero out buyer x, the same set of prices remains market clearing. This principle is true not just for this example, but in general.

Moreover, for any perfect matching in the preferred-seller graph of the zeroed-out market,

   (ii)   buyer j receives a zero-priced item, and
   (iii) each buyer other than j obtains the same payoff that she did in the original market.

   Again, we defer the proof of Fact 2 to later in the section, but it is not difficult to establish the proof using Fact 1. Essentially, when we zero out j, we look at the item i that j formerly got in the original market, before she was zeroed out. We follow the alternating path provided by Fact 1 from i down to an item i^* of price 0. We then show that assigning item i^* to j, and shifting the assignment to all other buyers using the edges on this alternating path gives us a perfect matching in the preferred-seller graph of the zeroed-out market at the same prices. This shows that the same prices are in fact market clearing for the zeroed-out market and will establish parts (ii) and (iii) of the claim as well.

Proving the Claim Using Facts 1 and 2. With Facts 1 and 2 in place, we can finish the proof of our main claim – that the minimum market-clearing prices are defined by the VCG formula.

   To start, let's review some notation. As before, let v_{ij} denote the valuation that a buyer j has for an item i. Let p_i be the price charged for item i in our market-clearing prices (for the original market, before any buyers are zeroed out), and let P be the sum of the prices of all items. Suppose that buyer j is matched to item i in the perfect matching in the preferred-seller graph. Buyer j receives a payoff of $v_{ij} - p_i$ from this item i; we will use z_j to denote this payoff,

$$z_j = v_{ij} - p_i, \tag{15.4}$$

and Z to denote the sum of the payoffs of all buyers from the items they are matched with.

Next, let's recall two basic observations that were made in earlier sections. First, each buyer j achieves a payoff of $v_{ij} - p_i$ from the item to which she is matched. As we noted in Chapter 10, if we add these expressions up over all buyers, we get the following relationship for the matching M of buyers to items:

$$\text{Total Payoff of } M = \text{Total Valuation of } M - \text{Sum of All Prices}.$$

In our current notation, this is

$$Z = V_B^S - P. \tag{15.5}$$

Second, we argued in Section 15.4 that, if i is matched to j in an optimal matching, then

$$v_{ij} + V_{B-j}^{S-i} = V_B^S. \tag{15.6}$$

This is Equation (15.3) from Section 15.4, and it follows simply because one way to achieve an optimal matching is to first pair i with j (obtaining a valuation of v_{ij}) and then optimally match all the remaining buyers and items.

Finally, let's consider this same formula,

$$\text{Total Payoff of } M = \text{Total Valuation of } M - \text{Sum of All Prices},$$

for the market in which j has been zeroed out, using the same set of market-clearing prices and a perfect matching in the preferred-seller graph that Fact 2 provides. The total valuation of this matching is V_{B-j}^S, as we argued earlier. The prices haven't changed, so their total sum is still P. Finally, what's the total payoff? By part (ii) of Fact 2, the payoff for buyer j has dropped from z_j, which it was in the original market, to 0. By part (iii) of Fact 2, the payoff for every other buyer has remained the same. Therefore, the total payoff in the zeroed-out market is $Z - z_i$. Putting all these together, we have the equation

$$Z - z_i = V_{B-j}^S - P. \tag{15.7}$$

Since we now have equations that relate the two terms on the right-hand side of Equation (15.1) to a common set of quantities, we can finish the proof using a small amount of algebraic manipulation. Let's first subtract Equation (15.7) from Equation (15.5), which gives us

$$z_i = V_B^S - V_{B-j}^S.$$

Next, let's expand z_i using Equation (15.4) and expand V_B^S using Equation (15.6), which gives us

$$v_{ij} - p_i = v_{ij} + V_{B-j}^{S-i} - V_{B-j}^S.$$

Canceling the common term of v_{ij} and negating everything, we get

$$p_i = V_{B-j}^S - V_{B-j}^{S-i},$$

which is the VCG formula we were seeking. This shows that the market-clearing prices of minimum total sum are defined by the VCG formula, and hence proves the claim.
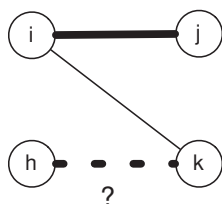
Figure 15.14. In order for a matching edge from a buyer k to an item h to leave the preferred-seller graph when the price of i is reduced by 1, it must be that k now strictly prefers i. In this case, k must have previously viewed i as comparable in payoff to h, resulting in a nonmatching edge to i.

B. Details of the Proof

The discussion so far provides a complete proof, assuming that we take Facts 1 and 2 as given. To finish the proof, therefore, we need to provide proofs of Facts 1 and 2. The crux of this is proving Fact 1, which will consist of an analysis of alternating paths in the style of Section 10.6. After this analysis, establishing Fact 2 is relatively quick using Fact 1.

A First Step Toward Fact 1. To prove Fact 1, we consider a set of minimum market-clearing prices and an item i whose price is greater than 0, and we try to construct an alternating path (beginning with a nonmatching edge) from i to some zero-priced item.

As a first step, to convey the idea at the heart of the argument, let's show something simpler: that this item i, of price $p_i > 0$, is connected to at least one nonmatching edge (in addition to its matching edge to the buyer j that obtains it). Clearly it is necessary to establish the presence of such a nonmatching edge in any case, if we want ultimately to show that i has an alternating path all the way down to a zero-priced item.

So suppose, by way of contradiction, that i is not connected to a nonmatching edge: its only edge is the matching edge to buyer j. In this case, we claim that we can subtract 1 from the price p_i, and the resulting modified prices would still be market clearing. This would be a contradiction, since we assumed our market-clearing prices have minimum total sum.

Clearly if we subtract 1 from p_i, it is still nonnegative, so we just need to show that the preferred-seller graph still contains a perfect matching. In fact, we'll show the stronger fact that the preferred-seller graph still contains all the matching edges that it used to have. Indeed, how could a matching edge leave the preferred-seller graph after the price reduction? The only item that became more attractive was item i, so for a matching edge to leave the preferred-seller graph, it must be that some buyer k other than j, who used to be matched to an item h, drops its edge to h because it now strictly prefers i. This situation is pictured in Figure 15.14. Now, since i's price was only reduced by 1, and since all prices and valuations are whole numbers, if k now strictly prefers to i to h after the price reduction, it must have formerly viewed them as tied. But this means that, before the reduction in i's price, k had a preferred-seller edge to i. Since k was matched to h, this k-i edge would be a nonmatching edge in the preferred-seller graph, which is not possible since we assumed at the outset that i's only edge in the preferred-seller graph was its matching edge to j. This
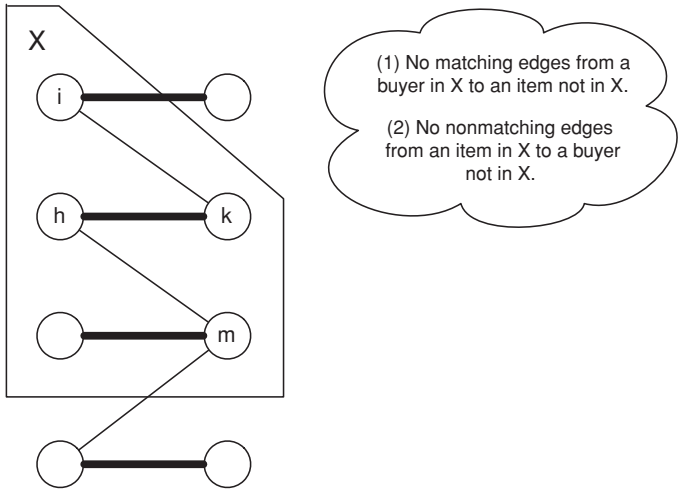
Figure 15.15. Consider the set X of all nodes that can be reached from i using an alternating path that begins with a nonmatching edge. As we argue in the text, if k is a buyer in X, then the item to which she is matched must also be in X. Also, if h is an item in X, then any buyer to which h is connected by a nonmatching edge must also be in X. An equivalent way to phrase this is as follows: there cannot be a matching edge connecting a buyer in X to an item not in X, or a nonmatching edge connecting an item in X to a buyer not in X.

completes the chain of conclusions we need: no matching edge can leave the preferred-seller graph when i's price is reduced by 1, so the reduced prices are still market clearing, and this contradicts the assumption that we had minimum market-clearing prices.

A Proof of Fact 1. The preceding argument is the key to proving Fact 1; for the complete proof, we need to move from simply showing the existence of a nonmatching edge out of i to a full alternating path, beginning with such an edge, all the way to a zero-priced item.

To do this, we start at the item i, and we consider the set X of all nodes in the bipartite graph (both items and buyers) that can be reached from i, using an alternating path that begins with a nonmatching edge. Here are two simple observations about the set X.

(a) For any buyer k who is in X, the item h to which she is matched is also in X. Figure 15.15 helps make clear why this must be true. The alternating path that reached k from i must have ended on a nonmatching edge, so by adding the matching edge to h to the end of this path, we see that h must also be in X.

(b) For any item h that is in X, and any buyer m connected to h by a nonmatching edge in the preferred-seller graph, the buyer m must also be in X. This is a direct companion to the previous fact and is also illustrated by Figure 15.15: the alternating path that reached h from i must have ended on a matching edge, so by adding the nonmatching edge to m to the end of this path, we see that m must also be in X.
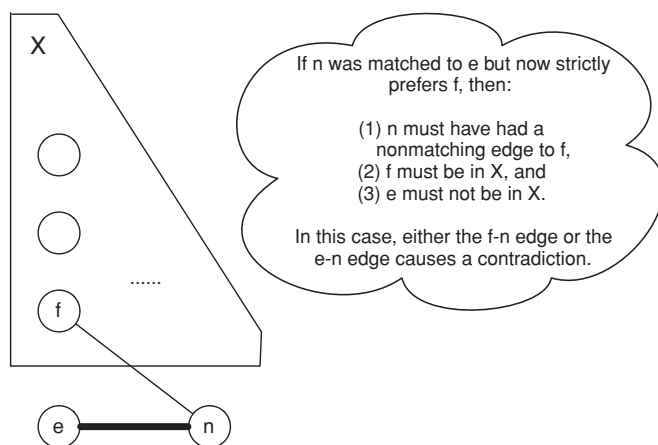
Figure 15.16. We can reduce the prices of all items in X by 1 and still retain the market-clearing property. As argued in the text, the only way this can fail is if some matching edge connects a buyer in X to an item not in X, or some nonmatching edge connects an item in X to a buyer not in X. Either of these possibilities would contradict the facts in Figure 15.15.

If this set X contains an item of price 0, we are done; we have the path we want. If this set X doesn't contain an item of price 0, then we complete the proof using the same price-reduction idea we saw earlier, in our warmup to the proof of Fact 1: we will reduce the price of each item in X by 1, show that the resulting prices are still market clearing, and thereby contradict our assumption that we had the minimum market-clearing prices. It will follow that X must contain a zero-priced item.

Here is the main thing we need to show.

> Suppose we reduce the price of each item in X by 1. Then all matching edges that were in the preferred-seller graph before the price reduction remain in the preferred-seller graph after the price reduction.

The argument is essentially the same as the one we used earlier when reducing the price of just item i. We ask: how could a matching edge leave the preferred-seller graph after the reduction? Figure 15.16 shows what must happen for this to be possible: a buyer n was formerly matched to an item e, and now some other item f has strictly higher payoff after the price reduction. Because all valuations, prices, and payoffs are whole numbers, and no price changed by more than 1, it must be that e and f used to be tied for the highest payoff to n (so n had edges to both of them in the preferred-seller graph before the reduction) and f is in the set X while e is not (so f had its price reduced while e's price remained the same).

Now we get a contradiction to one of our basic observations (a) and (b) about the set X. Since n was matched to e, and e is not in X, observation (a) says that n must not be in X, but since n was not matched to f, and f is in X, observation (b) says that n must be in X. This contradiction – n must be both in X and not in X – shows that no matching edge can leave the preferred-seller graph after the price reduction. And this in turn establishes that the reduced prices are still market clearing after the price
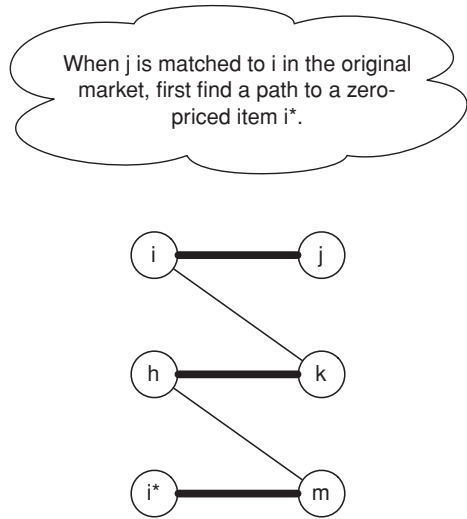
Figure 15.17. The first step in analyzing the market with j zeroed out: find an alternating path from item i – to which buyer j was matched in the original market – to a zero-priced item i^*.

reduction, contradicting our assumption that they were the minimum market-clearing prices.

This concludes the proof; if we look back at how it worked, we can see that it bears out our intuition for how the nonmatching edges serve to anchor all the items via alternating paths to the items of price 0. Specifically, if this anchoring did not happen, then there would be a set X that was floating free of any connections to zero-priced items, and in this case the prices of all items in X could be pushed further downward. This can't happen if the market-clearing prices are already as low as possible.

A Proof of Fact 2. To prove Fact 2, we start with a matching market with minimum market-clearing prices p, and we consider the preferred-seller graph for these prices. Now, suppose that we zero out a buyer j but keep the prices the same. The resulting preferred-seller graph is now different, but we'd like to show that it still contains a perfect matching.

How does the preferred-seller graph change when we zero out j, keeping the prices fixed? For buyers other than j, their edges remain the same, since they have the same valuations and observe the same prices. For j, on the other hand, the zero-priced items are now the only items that give her a nonnegative payoff, so her edges in the preferred-seller graph now go to precisely this set of zero-priced items. Notice, for example, that this is what happens to the preferred-seller graph as we move from Figure 15.11 to Figure 15.13: the zeroed-out buyer x has its preferred-seller edge shift from item b to the zero-priced item d.

Because we know that the preferred-seller graph in the original market has the structure guaranteed by Fact 1, we can view this change to the preferred-seller graph in the way suggested by Figures 15.17 and 15.18. Before zeroing out j, when it is matched to some item i, there is an alternating path in the preferred-seller graph, beginning with
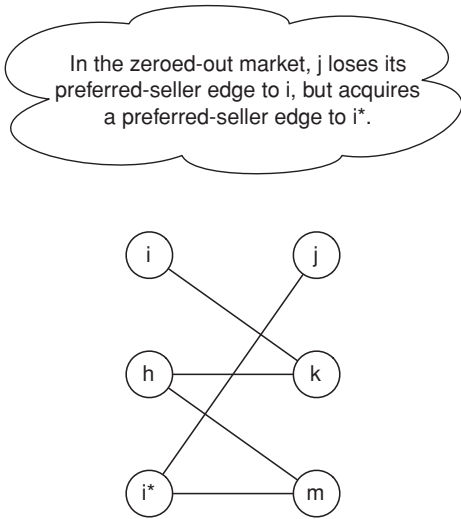
Figure 15.18. The second step in analyzing the market with j zeroed out: build the new preferred-seller graph by rewiring j's preferred-seller edges to point to the zero-priced items.

a nonmatching edge, from i to a zero-priced item i^*. After zeroing out j, there is a preferred-seller edge from j directly to i^* (and to any other zero-priced items as well).

It is easy to see from this pair of pictures how to find a perfect matching in the preferred-seller graph after this change to its structure. This is shown in Figure 15.19: for each buyer other than j who is involved in the alternating path from i to i^*, we simply shift her edge "upward" along the alternating path. This shift makes room for j to match with i^*, restoring the perfect matching.
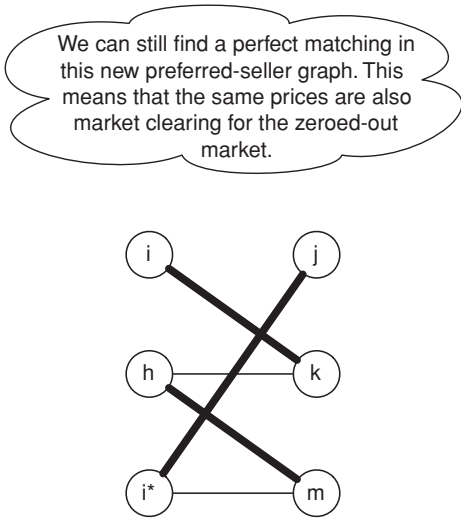

Figure 15.19. The third and final step in analyzing the market with j zeroed out: observe that the rewired preferred-seller graph still contains a perfect matching, in which j is now paired with i^*.

Since the preferred-seller graph has a perfect matching, this establishes that the prices are still market clearing for the zeroed-out market. We can also establish parts (ii) and (iii) of Fact 2 directly from our construction. Part (ii) follows simply from the fact that j only has edges to zero-priced items in the preferred-seller graph. For part (iii), note first of all that it is a statement about the payoffs that buyers receive. Even when there are potentially multiple perfect matchings in a preferred-seller graph, any given buyer obtains the same payoff in every one of these perfect matchings, since all of her edges in the preferred-seller graph yield the same payoff. As a result, it is enough to establish part (iii) for the perfect matching we just constructed, and it then applies to the payoff properties of every perfect matching in the preferred-seller graph. So consider the matching we just constructed, and let k be any buyer other than j. Either k gets the same item she had in the perfect matching for the original market – in which case she gets the same payoff – or else k shifts from one item to another along the alternating path. In this latter case, since k had edges to both of these items in the preferred-seller graph of the original market, she receives the same payoff from each of them, and so again k's payoff remains the same. This completes the proof of Fact 2, and hence fills in the final details needed to complete the proof of the overall claim.

15.10 Exercises

1. Suppose a search engine has two ad slots that it can sell. Slot a has a clickthrough rate of 10 and slot b has a clickthrough rate of 5. There are three advertisers who are interested in these slots. Advertiser x values clicks at 3 per click, advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click.
   Compute the socially optimal allocation and the VCG prices for it. Give a brief explanation for your answer.

2. Suppose a search engine has three ad slots that it can sell. Slot a has a clickthrough rate of 6, slot b has a clickthrough rate of 5, and slot c has a clickthrough rate of 1. There are three advertisers who are interested in these slots. Advertiser x values clicks at 4 per click, advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click. Compute the socially optimal allocation and the VCG prices for it. Give a brief explanation for your answer.

3. Suppose a search engine has three ad slots that it can sell. Slot a has a clickthrough rate of 5, slot b has a clickthrough rate of 2, and slot c has a clickthrough rate of 1. There are three advertisers who are interested in these slots. Advertiser x values clicks at 3 per click, advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click.
   Compute the socially optimal allocation and the VCG prices for it. Give a brief explanation for your answer.

4. Suppose a search engine has two ad slots that it can sell. Slot a has a clickthrough rate of 4 and slot b has a clickthrough rate of 3. There are three advertisers who are interested in these slots. Advertiser x values clicks at 4 per click, advertiser y values clicks at 3 per click, and advertiser z values clicks at 1 per click.

   (a) Suppose that the search engine runs the VCG mechanism to allocate slots. What assignment of slots will occur and what prices will the advertisers pay? Give an explanation for your answer.

(b) Now the search engine is considering the creation of a third ad slot which will have a clickthrough rate of 2. Let's call this new ad slot c. Suppose that search engine does create this slot and again uses the VCG mechanism to allocate slots. What assignment of slots will occur and what prices will the advertisers pay? Give an explanation for your answer.

(c) What revenue will the search engine receive from the VCG mechanism in parts (a) and (b)? If you were running the search engine, given this set of advertisers and slots, and could choose whether to create slot c, what would you do? Why? (In answering this question, assume that you have to use the VCG mechanism to allocate any slots you create.)

5. Suppose a search engine has two ad slots that it can sell. Slot a has a clickthrough rate of 12 and slot b has a clickthrough rate of 5. There are two advertisers who are interested in these slots. Advertiser x values clicks at 5 per click and advertiser y values clicks at 4 per click.

   (a) Compute the socially optimal allocation and the VCG prices for it.
   (b) Suppose the search engine decides not to sell slot b. Instead, it sells only slot a using a sealed-bid, second-price auction. What bids will the advertisers submit for slot a, who will win, and what price will they pay?
   (c) Which of these two possible procedures, (a) or (b), generates the greater revenue for the search engine? By how much?
   (d) Now let's see if the result in part (c) is general or not. That is, does it depend on the clickthrough rates and values? Suppose there are two slots and two advertisers; the clickthrough rates are r_a for slot a and r_b for slot b, with $r_a > r_b > 0$; and the advertisers' values are v_x and v_y, with $v_x > v_y > 0$. Can you determine which of the two procedures generates the greater revenue for the search engine? Explain.

6. Chapter 15 discusses the relationship between the VCG principle and second-price auctions. In particular, we saw that the VCG principle is a generalization of the idea behind second-price auctions to a setting in which there is more than one object being sold. In this problem we explore this relationship in an example. Suppose that a seller has one item, which we call item x. There are three buyers, whom we call a, b, and c. The values that these buyers (a, b, and c) have for the item are 6, 3, and 1, respectively.

   (a) Suppose that the seller runs a second-price auction for the item. Which buyer will win the auction and how much will this buyer pay?
   (b) Now let's suppose that the seller uses the VCG mechanism to allocate the item. Remember that the first step in the running the VCG mechanism when there are more buyers than items is to create fictional items, which each buyer values at 0, so that the number of items to be allocated is the same as the number of bidders. Let's call these additional (fictional) items y and z. Find the allocation that results from running the VCG mechanism. What are the prices charged to each buyer for the item that they receive? Explain why the price that buyer a pays is the harm that he causes to the remaining bidders by taking the item he is assigned.