ST5225: Statistical Analysis of Networks
Lecture 6: Matching Markets

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Outline

- Review
- Graph Partition
- Matching Markets

Review

- Graph Partition
  - Cluster by edge betweenness
  - Hierarchical clustering: Euclidean distance, other dissimilarities; different linkages
  - Modularity
  - Dendrogram
- Bipartite graph
  - Balanced bipartite graph
  - Perfect matching: connectivity and distinction
  - Optimal assignment: maximize the overall valuation

For today:

- Matching Markets: Optimal assignments and Market-clearing prices

The model we start with is called *bipartite matching problem.*

Bipartite graph

For a graph $G = (V, E)$, if the set of vertices V can be divided into two disjoint and independent sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2, then we call G as a *bipartite graph*.

Remark.

- Bipartite graph usually has two types of nodes (dorms and students).
- V_1 and V_2 may have different number of nodes. If they have the same number of nodes, we call it as *balanced*
- Bipartite matching problem is concerning whether there is a *perfect matching* on balanced bipartite graph.

Perfect matching

Perfect Matching

When there are an equal number of nodes on each side of a bipartite graph, a *perfect matching* is an assignment of nodes on the left to nodes on the right, in such a way that

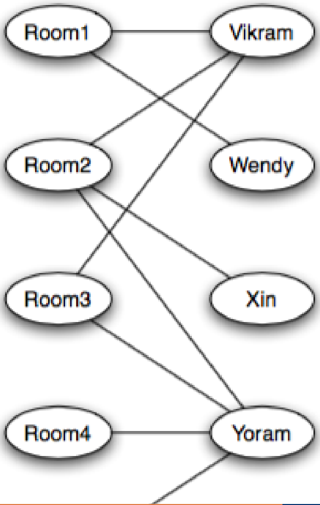
- (Connection) each node is connected by an edge to the node it is assigned to, and
- (distinction) no two nodes on the left are assigned to the same node on the right.

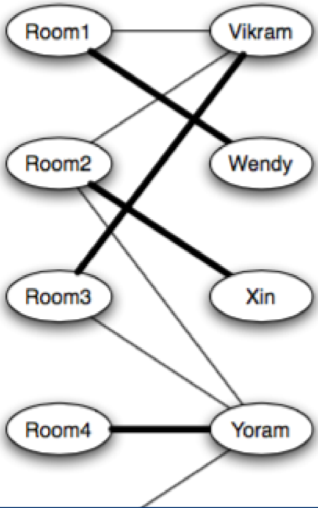
Remark.

- For a perfect matching, each node in V_1 is connected to one and only one "friend" in V_2, and vice versa.

Dorm assignment problem

- 5 dorms available for 5 students; each room for **1** single student
- Each student has a list of acceptable rooms; shown in graph





Question: Are there always be such an assignment?

Constricted sets

A set S is *constricted* if $|S| > |N(S)|$.

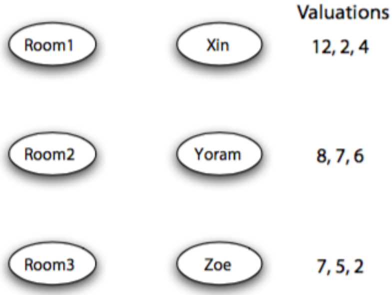
Matching Theorem

If a balance bipartite graph has no perfect matching, then it must contain a constricted set.

- Equivalence between *perfect matching* and *no constricted set*

$$\text{Perfect matching} \iff \text{No constricted sets}$$

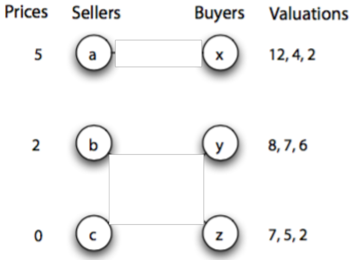
Bipartite Graph with Valuations



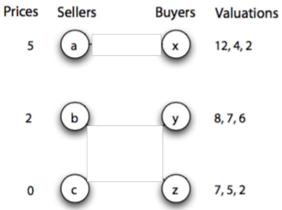
- The three numbers show the evaluation for the three rooms
- We call the assignment that can maximize the overall evaluation score as an *optimal assignment*

Prices

- A university may not have the same charge on the rooms, yet the market won't
- On a market, there are a set of sellers and a set of buyers. Each buyer has valuation on the product of the sellers, and each seller ask for a price.



- 3 houses available for 3 buyers
- Each buyer has his/her valuations for each house. Each seller has a price for his/her house
- Questions
  - Can we still find an "optimal" assignment here?
  - If we can change the price, can we find an even better assignment?



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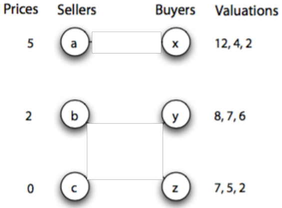

Valuations and prices

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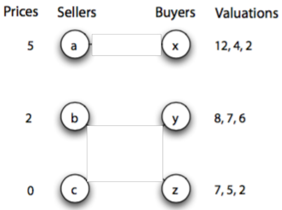
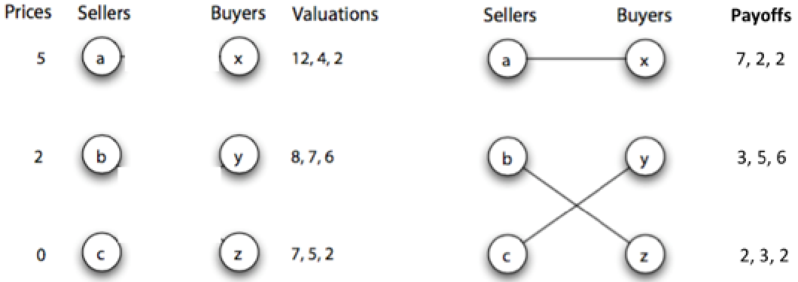
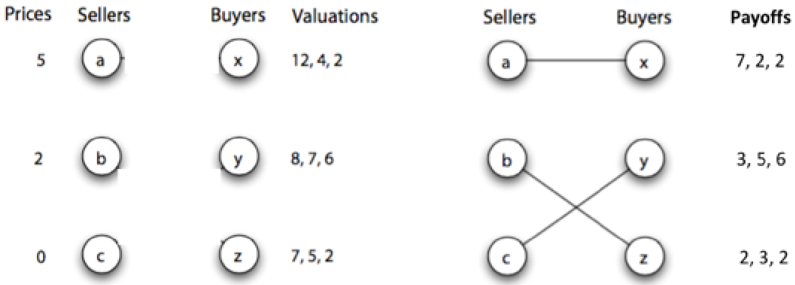


- *Payoffs*: For a buyer and a house, the buyer's payoff is valuation – price of this house
- Therefore, we have a graph with *payoffs* as the previous *valuations*
- Given the prices, we can find an *optimal assignment* with respect to the payoffs
- The optimal assignment could maximize the payoffs



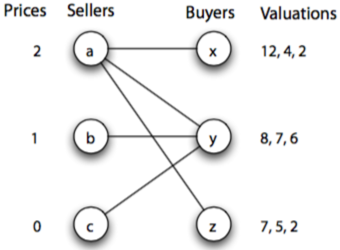
Prices and Payoffs

- For the result, note that each buyer is connected to the seller with largest *payoff*
- *Preferred-seller graph*: Connect each buyer with the sellers that maximizes the payoff of this buyer, and the corresponding graph is called preferred-seller graph, since the seller is the preferred seller of the connected buyers.
- If there is a *perfect matching* in the preferred-seller graph, then this perfect matching is the optimal assignment.

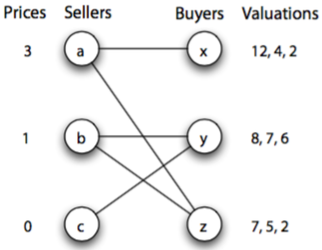


Market-Clearing Prices

- Note: every buyer gets the house that maximizes the payoffs
- We call such a set of prices *market-clearing*, since the prices setting cause each house to get bought by a different buyer–the house market is clear!
- It is possible there is no perfect matching



(c) *Prices that Don't Clear the Market*



(d) *Market-Clearing Prices (Tie-Breaking Required)*

Market Clearing

- *Existence: for any set of buyer valuations, there exists a set of market-clearing prices*
  - For any set of buyers, the sellers can set the prices carefully, so that each buyer gets distinct house

- *Optimality: For any set of market-clearing prices, the optimal assigment has the maximum total valuation of any assignment of sellers to buyers*
  - For an assignment M

    Total payoff of M = Total valuation of M - Sum of all prices
  - Note that sum of all prices don't change. The optimal assignment maximizes the total payoff, so that maximizes the total valuation.

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Market-Clearing Prices: Construction

How to construct a set of market-clearing prices?

- Idea: if one house is in high demand (more than 1 buyers have maximum payoff on this house), then the price increases by 1
- Note: we always set the house with smallest price to be with price 0. It does not affect the result, and it helps to scale the result
- In a real market, the "hot" house usually has a higher price, so that the final buyer is the buyer who likes it most

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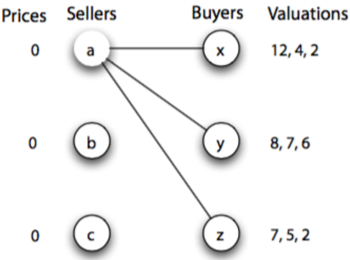
Market-Clearing Prices: Construction

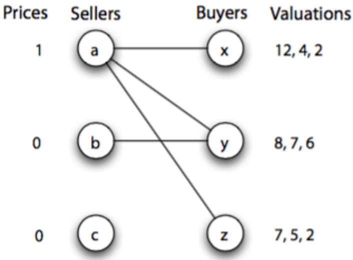
Say there are n sellers (items) and buyers

Start: each item has price 0; each buyer assigns a value to each item

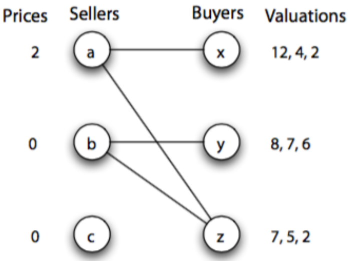
1. Assume smallest value is 0; if not, scale the price so that the smallest is 0
2. Construct the preferred-seller graph and check if there is a perfect matching
3. If yes, done
4. If not, find a constricted set of buyers S
5. Each seller in $N(S)$ increases the price by 1
6. check if the smallest price is 0, if not, subtract the same amount of each price so that the smallest is 0
7. go back to Step 1.

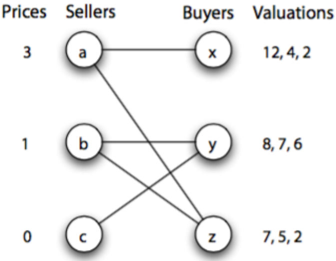


(a) *Start of first round*



(b) *Start of second round*





Market-Clearing Prices: Example II

Prices	Sellers	Buyers	Valuations
0	a	x	5, 7, 1
0	b	y	2, 3, 1
0	c	z	5, 4, 4

Summary

- Bipartite graphs with two sets of nodes V_1 and V_2; each $v_i \in V_1$ has a price; each $v_j \in V_2$ has a valuation for every $v_i \in V_1$
  - Find payoffs of each $v_j \in V_2$ assigned to each $v_i \in V_1$
  - With the payoffs, there is *optimal assignment*
  - For each $v_j \in V_2$, there is a preferred seller in V_1, which maximizes the payoff of v_j. We call this seller as *preferred seller*
  - If the preferred-seller graph has a *perfect matching*, then we call the price as market-clearing price
- Market-clearing prices must exist, and maximizes the total valuations
- Construction of market-clearing prices: begin with 0, and increase the prices of constricted set by 1.

Power and Bargaining on Networks

- Power on networks
- An experiment and the results
- Nash bargaining solution
- Ultimatum game
- Stable outcomes
- Find natural stable outcomes

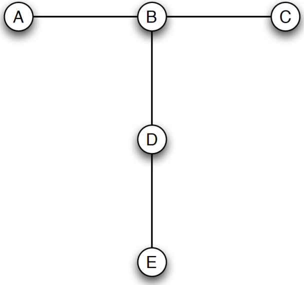
Power on Networks

- In the buyer-seller network, the buyer has payoff as valuation minus price.
- The seller may want to increase the price, so that the payoff splits between the seller and the buyer
- If the seller increases the price only a little bit, then the buyer still transacts with this seller with a fewer payoff, and the seller gets some profit.
- If the seller increases the price too much, the buyer may give up, and no transaction is done.
- *Bargaining on network*
- Bargaining is related to *power*, the popularity of the seller/buyer in the network

We talk about the general cases, not restricted to buyer-seller networks.

- Intuitionally, we may say these two nodes have the same power (related to this graph), since they are symmetric

- Node B in this graph has a larger power, since nodes A and C requires the connection with B to get in touch with D and E



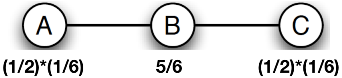
- Power of distributing resources

Set up an experiment to see people's bargaining behavior on networks.

1. Fix a small network. Let each node represent each player. Players that are connected with an edge can exchange information/make offers.
2. Each edge carries a resource (e.g., $100). This resource is split among adjacent nodes if there is a deal between nodes.
3. Each node can only be part of at most one deal (possible that no deal is done).
4. Players can freely negotiate for a fixed amount of time how the resource is split.
5. The experiment is repeated multiple times to get the final result

- For a 2-node network, end up with a fair split. Since A and B has the same power on the transaction, they agree to split the money equally.

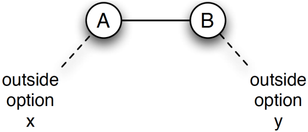
- For a 3-node network, B has higher power, where it can exclude A or C in the transaction. The result shows that A and C are symmetric, while B receives the majority of the money in the transaction.



- In the textbook, more complicated experiments, say, 4-nodes, 5-nodes are also discussed.

Nash bargaining solution

To discuss the problem, we think of each edge. The resources outside this edge is summarized as "outside options", as the following figure



- Assume the total resource to split on this edge is 1.
- Since A can get x from outside if not split with B. So the expectation of A's payoff is at least x
- Similarly, the expectation of B's payoff is at least y
- If $x + y > 1$, impossible to satisfy both A and B
- If $x + y \leq 1$, then there is surplus $s = 1 - x - y$
- *Nash bargaining solution*: both A and B will get half of the surplus, $(1 - x - y)/2$, which means the outcome is

$$x + \frac{1-x-y}{2} = \frac{1+x-y}{2} \text{ to A;} \quad y + \frac{1-x-y}{2} = \frac{1-x+y}{2} \text{ to B.}$$

Interesting remarks:

- The resource is not only money, it can be friendship, collaboration, etc.
- When bargaining, people tries to communicate about the outside option, so that gets a reasonable split
- In real life, for the two endpoints, usually one has a "higher-status" and the other has a "lower-status". For example, a sophomore and a graduate student with high grades.
- It has been found that, people with high-status tend to inflate the size of their outside option; people with lower-status tend to reduce the size of their outside option.
- What's more, the reduced size of outside option will be further underestimated by the higher-status people.
- Do you have the same problem in your job search?

Setup: Two players A and B; no communication allowed; 1-time game

1. Person A is given \$100 and told to propose a division among A and B
2. If B accepts the \$100 are split accordingly; if B rejects, both A and B get nothing

If both A and B are rational,

- B should accept any positive offer, since that's better than nothing
- Since B will accept any positive offer, A, as the person who propose the division, can keep more money
- Solution: A offers \$99 to A, and \$1 to B
- The difference comes from the power of division

The ultimatum game

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Reality:

- In 1/3 of the experiments, A offers 1/2-1/2 split to B
- Moreover, when the offer is very unfair, B rejects to accept the offer – even though he will get nothing
- When the amount of money changes, the result is still the same
- Guess: It hurts B's feelings when offered very small proportion, which behaves as a negative payoff
- Solution: Give B more time to consider, then more people would accept the "unfair" offer

Stable outcomes

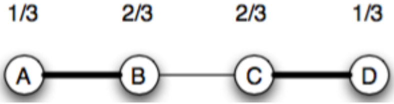
- We need some mathematically defined terms to describe the exchanges on graphs

Outcomes

An *outcome* of a network exchange on a given graph consists of:

- A matching on the set of nodes, specifying who exchanges with whom.
- A number associated with each node, called its *value*, indicating how much this node gets from its exchange. The sum of matched nodes should be 1.

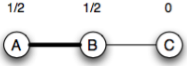
Example.



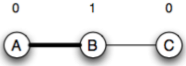
- All discussions are based on the outcomes
- Is the outcome stable?

Stable outcome

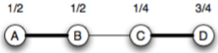
Outcome where no player can make an offer to another player such that both are *better* off, is called a *stable outcome*.



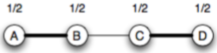
(a) *Not a stable outcome*



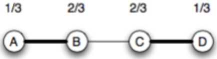
(b) *A stable outcome*



(c) *Not a stable outcome*



(d) *A stable outcome*



(e) *A stable outcome*

Stable outcomes

- *Instability*: if there is an edge not part of the matching, such that the sum of the endpoint values is < 1, then we call this outcome as *an unstable outcome*.
  - Say the endpoints of this edge is A and B, then obviously A and B would prefer to exchange, so that they are better off. The original outcome is not stable.

- If there is no such instabilities, then an outcome is stable.
  - If an outcome is unstable, then there always exists such an edge.
  - It helps us to figure out the stability of an outcome. Just check all the edges and the values of their endpoints, and we know whether it is stable or not

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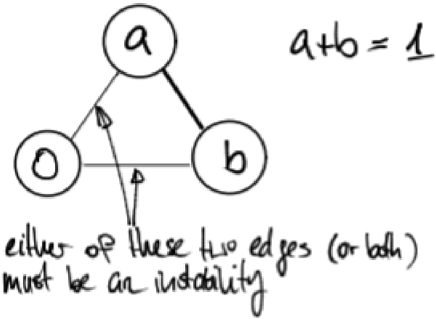
Stable outcomes

Example (restrictions of outcomes):
For a triangle, there are no stable outcomes!



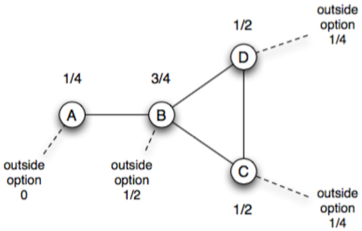
Balanced stable outcomes

- If we have many stable outcomes, then we prefer some "natural" set of outcomes, for which we call *balanced*

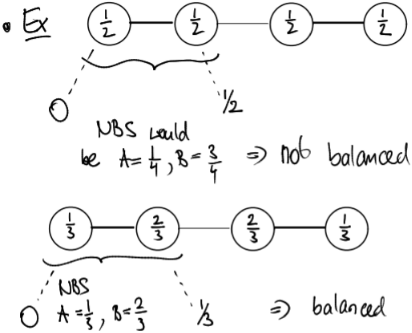
Balanced outcomes

An outcome is called a *balanced outcome* if, for each edge in the matching, the split of money represents the Nash bargaining outcome for the two nodes involved, given the best outside option.

Example: a balanced outcome



- It is closer to what happens in reality



Summary

- Power on networks with experiments
- Nash bargaining solution for two nodes with outside options
- Ultimatum game and the results
- Stable outcomes and natural stable outcomes (Nash bargaining solution required here)