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Statics and Dynamics of a Helical Spring

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The theory of the extension and twisting of a helical spring has been given quite completely in a recent paper by Krebs and Weidlich. This theory is reviewed and its applicability as an experiment in advanced mechanics is described. From measurements of extension and twisting under static load, and also from simple observation of the coupled oscillations of these degrees of freedom it is possible to calculate Poisson's ratio and the shear modulus. The two independent procedures give consistent results. The experiment provides an excellent demonstration of the geometry and elastic deformation of a helix, its static equilibrium and its coupled oscillations.

1. INTRODUCTION

STUDY of the oscillations of a coupled system forms an important part of any mechanics course. An excellent example of such a system would appear to be a loaded helical spring. Here the two degrees of freedom are the extensional and twisting displacements. Coupling between them gives rise to a continuing exchange of energy. The frequencies of the two modes can be altered to produce resonance. As Sommerfeld[1] has shown, the resonance condition reduces to a simple relation involving Poisson's ratio for the spring material.

The basis is thus provided for an interesting experiment, suitable for an upper division laboratory. Indeed, a well-known manufacturer at one time supplied equipment for this experiment based on Sommerfeld's treatment. In our experience, the results were always outside the theoretical limits for Poisson's ratio.

The purpose of this article is to call to the attention of others who may have had similar experiences, an article by K. Krebs and W. Weidlich[2] that treats the problem completely and satisfactorily. Moreover, the complete treatment provides a basis for an even more interesting experiment. The remainder of this discussion is a brief review of their treatment and a summary of experimental results obtained in our laboratory.

2. ENERGY OF THE SPRING

Krebs and Weidlich proceed by finding the potential energy of the helical spring. The spring is assumed to have been given initially a curvature κ_0 and a torsion τ_0, i.e., these describe its configuration with no load. The energy to produce further bending and twisting is then shown to be

$$U_B=\tfrac{1}{2}EI_1L(\kappa-\kappa_0)^2, \tag{1}$$

$$U_T=\tfrac{1}{2}GI_2L(\tau-\tau_0)^2, \tag{2}$$

where E is Young's modulus, G is the shear modulus, I_1, I_2 are second moments of area of the perpendicular section of the wire (assumed circular) about a diameter and a normal axis through the center, and L is the stretched-out length of the wire. It is convenient to be able to express κ and τ in terms of measurable quantities, and for this purpose z and θ, the coordinates of the free end of the spring are introduced. Here, θ is measured in the plane perpendicular to the axis of the helix, and $\theta=2\pi$ (number of turns). We find, from standard geometric definitions and equations of the helix

$$\kappa=\theta(L^2-z^2)^{\frac{1}{2}}/L^2, \tag{3}$$

$$\tau=z\theta/L^2. \tag{4}$$

If the additional bending and twisting are caused by suspension of a mass M from the free end of the spring, the total energy is then

$$U=U_B+U_T-Mgz, \tag{5}$$

neglecting the mass of the spring.

3. STATIC EQUILIBRIUM

If the spring end is in equilibrium, the two equations

$$\partial U/\partial z=U_z=0, \tag{6}$$

$$\partial U/\partial\theta=U_\theta=0 \tag{7}$$

[1] A. Sommerfeld, J. Opt. Soc. Am. 7, 529 (1923).
[2] K. Krebs and W. Weidlich, Z. angew. Phys. 5, 260 (1953).

are satisfied. The second of these relations is simplified on the assumptions that

$$(z/L)^2 \ll 1$$

and

$$(\theta-\theta_0)/\theta_0 \ll 1.$$

It then becomes

$$(\theta-\theta_0)/\theta_0 = [zz_0 - z^2(1-\nu)/2 - z_0^2(1+\nu)/2]/L^2(1+\nu), \quad (8)$$

where ν is Poission's ratio. This is a parabolic relation between θ and z, implying that as mass is added to the spring, its end twists in one direction at first, stops, and then twists in the opposite direction. The value of z for which the angular deflection is a maximum is given by the equation

$$z_{\max} = z_0/(1-\nu). \quad (9)$$

This relation forms the basis for a static measurement of ν.

In the neighborhood of z_0, the angular deflection is linear in z and has the form

$$(\theta-\theta_0)/\theta_0 = \nu z_0 z/L^2(1+\nu). \quad (10)$$

This is equivalent to the equations given by Sommerfeld. It is thus apparent that his treatment is correct only when the spring extensions lie in a limited region near the unloaded length. Furthermore, the helix must be an open one in its pre-stressed condition, and not depend on its own mass for its openness.

4. SPRING CONSTANT

By the term "spring constant" is meant gdM/dz when the spring is stretched by the addition of weights to its lower end. Equation (6) must hold for mass M and also for mass $M+dM$, so we can conclude that

$$dU_z = U_{zz}dz + U_{z\theta}d\theta + U_{zm}dM = 0, \quad (11)$$

and

$$dU_\theta = U_{\theta z}dz + U_{\theta\theta}d\theta = 0. \quad (12)$$

Combining these, Krebs and Weidlich find

$$gdM/dz = U_{zz} - U^2{}_{z\theta}/U_{\theta\theta}. \quad (13)$$

The second derivatives can be evaluated using the assumptions of Part 3, leading to the result that

$$gdM/dz = GI_2\theta_0^2\{1+(z_0/L)^2[(1+2\nu)/(1+\nu)]K(\nu,z/z_0)\}/L^3. \quad (14)$$

Here $K(\nu,z/z_0)$ is a slowly varying function which is plotted by Krebs and Weidlich in Fig. 4 of their paper. It is seen that although gdM/dz is not constant with z it does not vary a great deal. Another interesting result is found for $z=z_{\max}$, for in this case $U_{z\theta}=0$ and $gdM/dz=U_{zz}$. Therefore, if the spring constant is measured at $z_{\max}$ (either statically or from the period of small oscillations) and U_{zz} is evaluated from Eq. (5), one may determine G and E since ν is known.

5. COUPLED OSCILLATIONS

We expand Eq. (5) in terms of small displacements ζ and φ about an equilibrium configuration z, θ obtaining

$$U = \tfrac{1}{2}U_{zz}\zeta^2 + U_{z\theta}\zeta\varphi + \tfrac{1}{2}U_{\theta\theta}\varphi^2, \quad (15)$$

where it is understood that the derivatives are evaluated at the equilibrium point. The equations of motion are:

$$M\ddot{\zeta} = -\partial U/\partial\zeta = -U_{zz}\zeta - U_{z\theta}\varphi, \quad (16)$$

$$I\ddot{\varphi} = -\partial U/\partial\varphi = -U_{z\theta}\zeta - U_{\theta\theta}\varphi, \quad (17)$$

where M is the total mass of the system and I is its moment of inertia. The coupling is due to the term $U_{z\theta}$ and vanishes for $z=z_{\max}$. The normal frequencies then involve the two uncoupled frequencies

$$\omega_z^2 = U_{zz}/M, \quad (18)$$

$$\omega_\theta^2 = U_{\theta\theta}/I, \quad (19)$$

and the coupling constant

$$\omega_c^2 = U_{z\theta}/\sqrt{(IM)} \quad (20)$$

and are found by standard means. We are interested in the case of small coupling and at resonance where

$$\omega_z^2 = \omega_\theta^2 = \omega^2 \gg \omega_c^2. \quad (21)$$

The coupled frequencies are then

$$\gamma_\pm = \omega[1 \pm \tfrac{1}{2}\omega_c^2/\omega^2] \quad (22)$$

and since they are not very different, each degree of freedom exhibits beats with frequency

$$\gamma_B = \omega_c^2/\omega \tag{23}$$

and the number of oscillations per beat is

$$n = \omega/\gamma_B = (U_{zz}U_{\theta\theta})^{\frac{1}{2}}/U_{z\theta}. \tag{24}$$

As Sommerfeld has shown, n is a maximum at resonance. The approximate relation

$$n = \frac{\sqrt{(1+\nu)}}{(z/L)[1-\nu-z_0/z]} \tag{25}$$

holds under these conditions. For $z=z_0$ this reduces to the equation derived by Sommerfeld to determine ν. In seeking the resonant condition it is valuable to note that

$$U_{\theta\theta}/U_{zz} = I/M. \tag{26}$$

Hence for a given z there is a definite value of I/M which produces resonance (i.e., gives a maximum number of revolutions per beat). Since changing M affects z and hence $U_{\theta\theta}$, U_{zz}, it proves simpler to vary I until the number of oscillations is a maximum. If, at the same time, ω is measured, Eq. (18) can be used to find G.

6. APPARATUS AND EXPERIMENTAL RESULTS

Although Krebs and Weidlich give a complete description of their springs and the results they obtained, we will present data obtained in our laboratory. Our spring was constructed of piano wire of radius 0.04 cm and consisted of 246 turns with an unloaded length $z_0 = 152$ cm. The projected radius of the helix was 0.465 cm.

Static measurements were made by attaching a cardboard pointer of over-all length about 10 in. to the free end of the spring. A sheet of graph paper mounted inside a large glass cylinder surrounded the pointer. Masses from 250 grams to 905 grams caused the pointer to swing through and well past a maximum twist of about 45° (for a load of 655 grams). The 905-gram load produced an extension of 60 cm. Figure 1 shows a graph of the deflection of the pointer end against spring extension. The extension was closely linear with load up to the greatest value used.

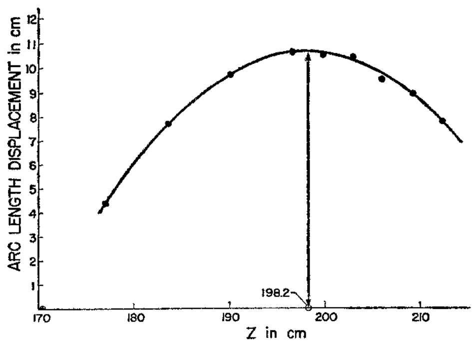

FIG. 1. Arc length displacement as a function of spring length for various loads. The parabolic character of the angular displacement agrees with the prediction of Eq. (8). The value of z for maximum displacement is used in Eq. (9) to obtain a static measurement of Poisson's ratio.

TABLE I. Conditions at resonance for three values of M.

M	n maximum oscillations per beat	T_B beat period	T_ω oscillation period
232 g	35	28.9 sec	0.82 sec
311	43	39.4	0.92
390	54	55.2	1.02

Dynamic measurements were made by replacing the pointer with a thin rod to serve as weight-holder. Weights were made from thin brass cylinders bored out to slip over the rod. In this way the moment of inertia I was kept small so that appreciable increments to it could be made by adding wide pieces of tinfoil which contributed only negligibly to M. Resonance was obtained for three different values of M.

Table I gives the means of three trials for each value of M.

At $z=z_{\max}$ the translational period was found to be 1.342 sec.

From the theory it has been seen that each of ν, gdM/dz, and G can be computed from static and dynamic measurements. Table II gives results from the two methods.

A comparison of these values with tabulated ones is not particularly meaningful since the composition and treatment of the samples being compared is almost certainly different. Moreover, recent measurements described in the literature are likely to be at ultrasonic frequencies. Various possibilities for systematic error suggest themselves; among these are nonuniformity of the spring and excitation of extraneous modes of motion. The good agreement between the two quite distinct methods here presented seems to argue against any large systematic error. Krebs and Weidlich point out that the static method is simpler and probably

TABLE II. Comparison of static and dynamic measurements.

	Static	Dynamic
ν	0.23 ± 0.00	0.23 ± 0.03
gdM/dz	$15\,200 \pm 140$ dyne/cm	$15\,100 \pm 60$ dyne/cm
G	5.8 ± 1.2 dyne/cm^2 $\times 10^{-11}$	5.4 ± 1.2 dyne/cm^2 $\times 10^{-11}$

more accurate, if one desires measurements of elastic properties.

However, the emphasis in the present paper is on the pedagogical features which make this a rich and attractive lesson. These include the geometry of the helix, the calculation of its elastic energy of deformation, its static equilibrium and the coupled oscillations of its two degrees of freedom.

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Concept of Energy as the Theme of a General Education Course in Physics

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The paper describes an experimental course in general education physics based on the concept of energy as the central theme. The course has been given to undergraduate liberal arts students with little or no previous education in science. Handled entirely by the class discussion method, the course has sought to stress energy as not only the premier concept of physical science but also the key to other vast areas of human experience.

INTRODUCTION: GENERAL EDUCATION IN PHYSICS

A CONSIDERABLE difference of opinion still exists about the kind of college course in physics which is appropriate for the general education of liberal arts students. Many apparently feel that some form of the standard general physics course as ordinarily presented to physics majors provides after all the effective answer. This is based on the so-called "standard" general physics text, the number of which is now legion; each distills the wisdom of an experienced teacher and contains a few novel wrinkles, but somehow manages to devote some 600 pages (often a couple of hundred more) to essentially the same range of topics as its competitors and in more or less the same order. The protagonists of this point of view evidently believe that if the student is to know anything about physics he had better study it as if he were going actually to use it as a basis for further work in science; only in this way, it is held, can the fundamental concepts be so pounded into him that some of them will stick. This view, to say the least, has the merit of its conviction that the way to learn a thing is to do it!

However, there is by no means universal agreement with the method just described. General education has been the subject of considerable experimentation in the past ten years, and whatever scientists may think of the results in the humanities and social studies, it is clear that the humanists and social scientists consider physicists in general to be entirely too conservative in their approach. This betrays indeed ignorance of the fact that new physics courses *have* been devised for the nonmajor, ranging all the way from the block-gap type in which certain aspects of the subject are rather intensively treated while others are left out entirely, to the historical approach in which the evolution of certain important ideas is traced. These have met with varying degrees of success, according to their sponsors. It is not the purpose of the present author to embark on a critique of them, but rather to describe a course of this general type with which he himself has been experimenting during the past few years.