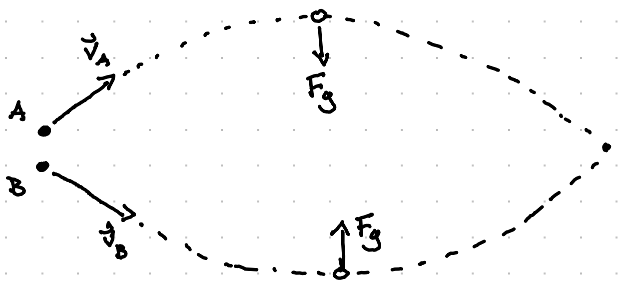
Lecture 3: Linearized Gravity and Gravitational Waves

topics: metric theories of gravity & geometric interpretation
- parallel transport
  - strong & weak equivalence principles
- Einstein Field Equations
- linearization & gage freedom
  - cannonical polarizations (TT gage)
  - non-tensor polarizations
- quadrupole approximation
  - 0PN waveform (Peters & Mathiews)
  - key properties of the source

Metric theories of Gravity & Geometric Interpretation

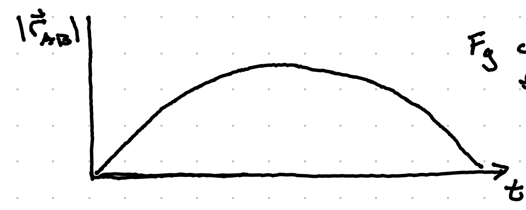
consider 2 massive particles in flat space time



Newton tells us that they will not follow straight lines b/c there is a <u>force</u> acting on them that causes an acceleration

$$F_g = \frac{G\, m_A m_B}{|\vec{r}_{AB}|^2}$$

consider the separation btwn objects as a function of time



F_g cause the objects to fall towards one another

In this picture, there is a miracle. the grav. force couples to all objects proportionally to their <u>inertial mass</u>. Therefore, the trajectory of an object in an external grav. field does not depend on it's mass.

$\Rightarrow$ equivalence principle

- Weak Equiv. Principle

  the motion of a point mass in a grav. field depends on on it's initial position, velocity, and not its mass

- Einstein Equiv. Principle

  the outcome of any local, non-grav. experiment in a freely-falling laboratory is indep. of the velocity & position of the lab.

  -or-

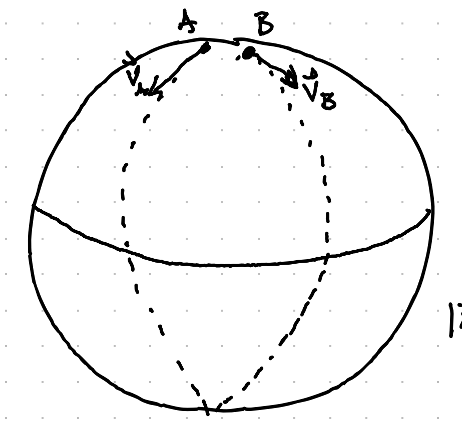
  uniform acceleration is equivalent to a uniform grav. field

- Strong Equiv. Principle

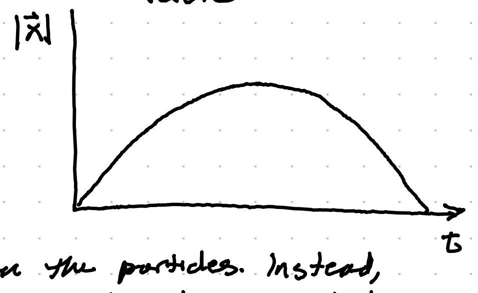
  Einstein $\oplus$ the motion does not depend on the object's composition

General Relativity produces this in a different way (force-free)

consider the same 2 objects as before, but now they live on a sphere.



Each mass follows a "straight line" (geodesic) and there is no force acting between them. However, if we plot the separation btwn the objects, we find a similar curve to before



Here, there is <u>no force</u> between the particles. Instead, it is the curvature of the space they traverse that causes them to fall toward each other.

$\Rightarrow$ Because we believe all objects will follow straight lines in the absence of external forces, regardless of their internal composition or mass, we naturally get the Strong Eq. Principle "for free" w/in this picture.

But how do we measure the separation between objects?

Euclidean (flat) space: $ds^2 = dx^2 + dy^2 + dz^2$
$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

Minkowski (flat) spacetime: $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

in general, we can write $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

Riemannian Metric encodes the <u>geometry</u> of the manifold

When we do integrals over spacetime, we want the measure to be indep. of the particular coordinate choice

$$\therefore \int d(\text{spacetime volume}) [\cdots] = \int \sqrt{-g}\, dx^0 dx^1 \cdots dx^n [\cdots]$$

so the action becomes

$$S = \int \sqrt{-g}\, dx^\mu \left[\mathcal{L}_{\text{matter+fields}} \right]$$

gravity couples universally to all particles & fields

Einstein Field Equations, Linearized Gravity, & Gage Freedom

Einstein postulated that gravity is connected to the stress-energy tensor of "normal matter & fields"

"Geometry" = "energy"

$$\boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

where

$$R = R^{\mu}{}_{\mu} \quad \& \quad R_{\mu\nu} = R_{\mu\nu}(g_{\mu\nu})$$

for a fluid in equilibrium w/ 4-velocity u_μ

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$$

this has a special form b/c it guarantees conservation of energy

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) = \emptyset$$

$$\Rightarrow \quad \nabla^{\mu} T_{\mu\nu} = \emptyset$$

Special case: Minkowski, zero-pressure, slow speeds

$$\nabla^{\mu} = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)$$

$$T_{\mu 0} = \rho u_\mu u_0 \qquad \text{(zero pressure)}$$

$$u_\mu \approx \left(1, \frac{v_x}{c}, \frac{v_y}{c}, \frac{v_z}{c}\right) \qquad \text{(slow speeds)}$$

$$T_{\mu 0} \sim \left(\rho, \rho\frac{\vec{v}}{c}\right)$$

$$\therefore \quad \nabla^{\mu} T_{\mu 0} \sim \underbrace{\frac{1}{c}\partial_t(\rho) + \vec{\nabla}\cdot\left(\rho\frac{\vec{v}}{c}\right) = \emptyset}_{\text{conservation of mass}}$$

Now, what are Gravitational Waves? <u>Perturbations of the Metric</u>

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{background metric}} + \underbrace{h_{\mu\nu}}_{\text{perturbation}} \qquad \text{where "} h \ll \eta \text{"}$$

it is convenient to define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

Expanding the EFE's in vacuum ($T_{\mu\nu} = \emptyset$), we obtain

$$\left(\Box = \nabla_\mu \nabla^\mu\right) \bar{h}_{\alpha\beta} = \emptyset \qquad \underline{\text{WAVE EQ}}$$

$$\sim \left(-c^2\partial_t^2 + \nabla^2\right) \bar{h}_{\alpha\beta} = 0$$

In the transverse gage (coordinate system) can express

$$h_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_b + h_+ & h_\times & h_{vx} \\ 0 & h_\times & h_b - h_+ & h_{vy} \\ 0 & h_{vx} & h_{vy} & h_\ell \end{bmatrix}$$

w/in GR, only h_+ & $h_\times$ are allowed

→ 2 polarizations of "spin-2" field propagating at the speed of light

$$h^{TT}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sources of Gravitational Radiation

In the limit of a distance, slowly moving source, we obtain the **Quadrupole Approximation** (See any GR textbook for a derivation)

$$\bar{h}_{ij} = \frac{2G}{r} \left.\frac{d^2 I_{ij}}{dt^2}\right|_{t-r}$$

where $$I_{ij} = \int d^3x \; x_i x_j \, T^{00}(t,\vec{x})$$

$$\approx \int d^3x \; x_i x_j \, \rho(t,\vec{x})$$

Why is the leading-order source the quadrupole moment?

Consider lower orders:

$$\int d^3x \, \rho(t,\vec{x}) \quad : \text{ total mass}$$

$$\int d^3x \, x_i \, \rho(t,\vec{x}) \quad : \text{ center of mass}$$

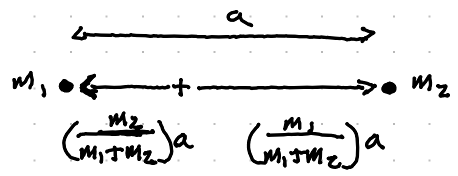
both these are protected by conservation laws (they're conserved charges) and therefore their time-derivatives vanish

$\therefore$ the quadrupole terms are the lowest order w/ non-vanishing time derivatives. However, there can also be radiation from higher-order modes (octopole, etc)

However, compact binary coalescences are typically driven primarily by quadrupole emission.

Orders of magnitude:

consider a circular Keplerian binary with $(m_2 \leq m_1)$



$$\Omega^2 = \frac{G(m_1+m_2)}{a^3}$$

$$\rho = m_1 \, \delta\!\left(x + \left(\frac{m_2}{m_1+m_2}\right) a \cos \Omega t\right) \delta\!\left(y + \left(\frac{m_2}{m_1+m_2}\right) a \sin \Omega t\right)$$
$$+ \; m_2 \, \delta\!\left(x - \left(\frac{m_1}{m_1+m_2}\right) a \cos \Omega t\right) \delta\!\left(y - \left(\frac{m_1}{m_1+m_2}\right) a \sin \Omega t\right)$$

$$I_{ij} = \left(\frac{m_1 m_2}{m_1+m_2}\right) a^2 \begin{bmatrix} 2\cos^2 \Omega t & 2\cos\Omega t \sin \Omega t \\ 2\cos \Omega t \sin \Omega t & 2\sin^2 \Omega t \end{bmatrix}$$

$$= \left(\frac{m_1 m_2}{m_1+m_2}\right) a^2 \begin{bmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) \\ \sin(2\Omega t) & 1-\cos(2\Omega t) \end{bmatrix}$$

$$\boxed{\therefore \; \bar{h}_{ij} = \frac{4G}{r}\left(\frac{m_1 m_2}{m_1+m_2}\right) a^2 \Omega^2 \begin{bmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) \\ -\sin(2\Omega t) & +\cos(2\Omega t) \end{bmatrix}}$$

$$a^2 = \left(\frac{G(m_1+m_2)}{\Omega^2}\right)^{2/3}$$

$$|\bar{h}_{ij}| \sim \frac{4G^{5/3}}{r} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \Omega^{2/3}$$

$$= \frac{4}{r}\left(G \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \Omega\right)^{5/3} \Omega^{-1}$$

$$\mathcal{M} \equiv \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \quad : \text{ chirp mass}$$