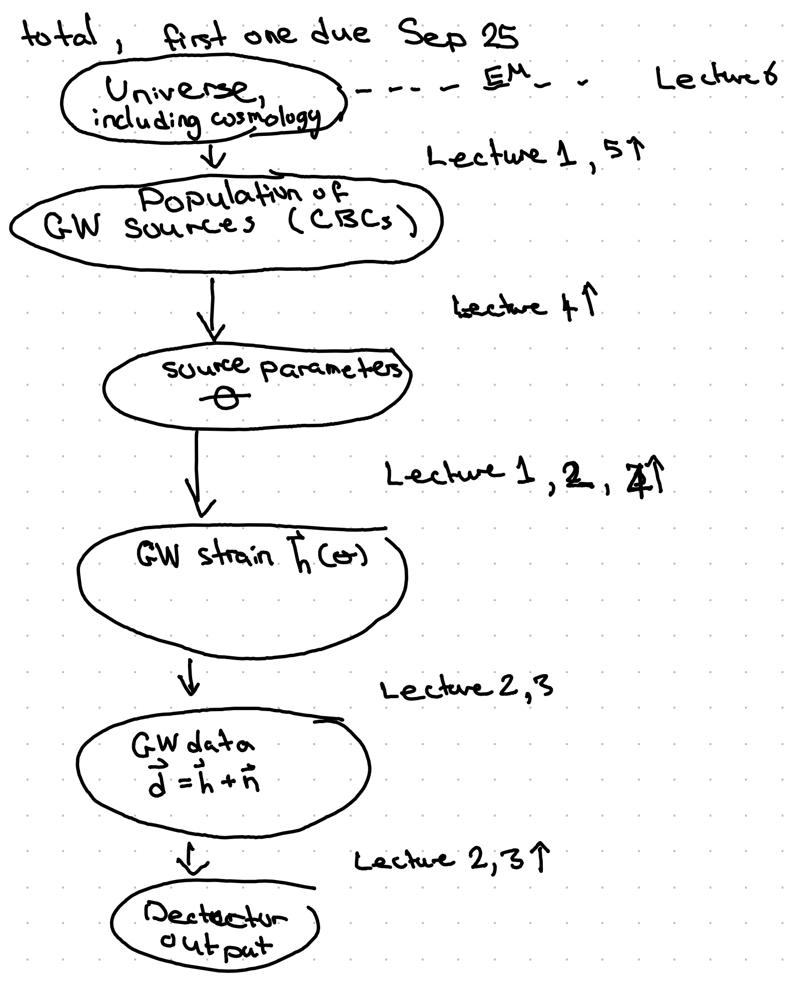
BH + NS: Gravity Lecture 1

1) syllabus – linked on Quercus Page

will redirect you to a GitHub Classroom

4 HW total, first one due Sep 25



2) GW Order of Magnitude

A) strain $h \sim 10^{-21}$ → fractional change in length caused by GW

B) frequency – scales inversely with BH mass

$\sim 100 - 1{,}000$ Hz for stellar mass BBH

C) frequency evolution – chirp – & orbital decay

A) Mass quadrupole $I^{jk} = \int \rho x^j x^k d^3x$, $\rho(\vec{x})$ is a density field.

e.g. 2 point masses

$r_1 = (x_1, y_1)$, m_1 $\qquad$ $r_2 = (x_2, y_2)$, m_2

$$I^{jk} = m_1 r_1^j r_1^k + m_2 r_2^j r_2^k$$

$$I = \begin{pmatrix} m_1 x_1^2 + m_2 x_2^2 & m_1 x_1 y_1 + m_2 x_2 y_2 \\ \ddots & m_1 y_1^2 + m_2 y_2^2 \end{pmatrix}$$

strain $h = \dfrac{2G}{c^4} \dfrac{\ddot{I}}{D}$ ← distance to the source

can get this $\frac{G}{c^4}$ factor by dimensional analysis

$\ddot{I}$ has dimensions $[M][L]^2[T]^{-2}$

h is dimensionless

For mass M, size R, timescale T, $v \sim \frac{R}{T}$

$\ddot{I} \sim \dfrac{MR^2}{T^2} \sim Mv^2$ $\qquad$ v is the part of the motion that deviates from spherical symmetry.

Plugging in some numbers,

$$h \sim 4.8 \times 10^{-21} \left(\frac{M}{10 M_\odot}\right)\left(\frac{100 \text{ Mpc}}{D}\right)\left(\frac{v}{c}\right)^2$$

i.e. fractional change in distance comparable to width of human hair / 4 lyr

B) GW frequency is twice the orbital frequency

From Kepler's third law,

$$f_{orb}^2 = \frac{1}{4\pi^2} \frac{GM}{R^3}$$

For BH, $R_H = \dfrac{2GM}{c^2} \approx 3\text{ km}\left[\dfrac{M}{M_\odot}\right]$, $R_{ISCO} \approx 3R_H$

$$f_{orb}^2 = \left(\frac{1}{4\pi^2}\right)\left(\frac{c^6}{6^3 G^2}\right)\left(\frac{1}{M^2}\right)$$

$$f_{orb} = \frac{1}{2 \cdot 6^{3/2}\pi}\left(\frac{c^3}{G}\right)\left(\frac{1}{M}\right)$$ ← scales inversely with mass

$$\sim 10^3 \left(\frac{M_\odot}{M}\right)$$

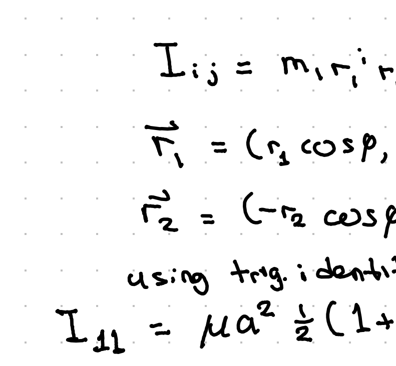
C) Approximate GW luminosity ~ Keplerian energy loss

works in limit that GW luminosity < E_{orb}

GW luminosity $\dfrac{dE_{GW}}{dt} = \dfrac{G}{5c^5}\left\langle \dddot{I}_{ij} \dddot{I}^{ij} \right\rangle$ (Quadrupole approx.)

trace-free

avg. over orbit



$a = r_1 + r_2$

$M = m_1 + m_2$

$\mu = \dfrac{m_1 m_2}{M}$

$\varphi = \omega t$

$r_1 = \dfrac{a m_2}{M}$ $\qquad$ $r_2 = \dfrac{a m_1}{M}$

$$I_{ij} = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j$$

$$\vec{r}_1 = (r_1 \cos\varphi, r_1 \sin\varphi, 0)$$

$$\vec{r}_2 = (-r_2 \cos\varphi, -r_2 \sin\varphi, 0)$$

using trig. identities,

$$I_{11} = \mu a^2 \tfrac{1}{2}(1 + \cos 2\varphi)$$

$$I_{22} = \mu a^2 \tfrac{1}{2}(1 - \cos 2\varphi)$$

$$I_{12} = I_{21} = \mu a^2 \tfrac{1}{2} \sin 2\varphi$$

$$\rightarrow \dddot{I} = 4\mu a^2 \omega^3 \begin{bmatrix} -\sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ trace-free!

$$\left\langle \sum_{ij} (\dddot{I}_{ij})^2 \right\rangle = 16\mu^2 a^4 \omega^6 \left\langle \sin^2 2\varphi + \cos^2 2\varphi + \cos^2 2\varphi + \sin^2 2\varphi \right\rangle$$
$$= 32 \mu^2 a^4 \omega^6$$

$$\dot{E}_{GW} = \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \omega^6$$

$$E_{orb} = \frac{-GM\mu}{2a} \longrightarrow \dot{E}_{orb} = \frac{GM\mu}{2a^2} \dot{a}$$

setting $\dot{E}_{orb} = -\dot{E}_{GW}$,

$$\dot{a} = -\frac{64}{5} \frac{1}{c^5} \frac{\mu}{M} a^6 \omega^6$$

can use Kepler to relate a and ω

$$\omega^2 = \frac{GM}{a^3} \rightarrow \frac{\dot{a}}{a} = -\frac{2}{3} \frac{\dot{\omega}}{\omega}$$

Either solve for $\dot{a}$ (subbing for ω) or $\dot{\omega}$ (subbing a and $\dot{a}$)

$$\dot{a} = -\frac{64}{5} \frac{G^3}{c^5} \mu M^2 a^{-3}$$

$$\Rightarrow T_c \sim a_0^4$$

$$\left(\frac{\dot{a}}{a}\right) a^2 = \frac{-64}{5} \frac{G^3}{c^5} \mu M^2$$

$$\frac{2}{3} \frac{\dot{\omega}}{\omega} \left[\frac{GM}{\omega^2}\right]^{4/3} = \frac{64}{5} \frac{G^3}{c^5} \mu M^2$$

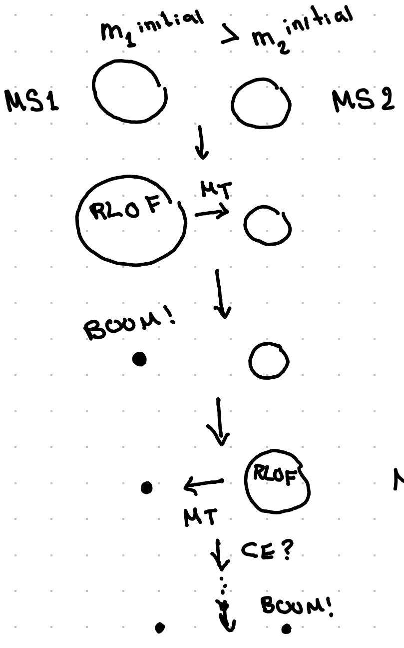
$$\dot{\omega} = \frac{96}{5} \frac{\omega^{11/3}}{c^5} G^{5/3} M^{2/3} \mu$$

chirp mass: $M_c = (\mu^3 M^2)^{1/5}$

$$\boxed{\dot{\omega} = \frac{96}{5} \frac{\omega^{11/3}}{c^5} [GM_c]^{5/3}}$$

"GW chirp"

Formation channels



star 1 may lose H envelope, WR star

MT may be stable or unstable

if $m_2 \gg m_{BH,1}$ leading to CE

MT stability: orbital angular momentum: $J = \frac{M_1 M_2}{M}\sqrt{GMa}$, $M = M_1 + M_2$

$$\frac{\dot{J}}{J} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{1}{2}\frac{\dot{M}}{M} + \frac{1}{2}\frac{\dot{a}}{a}$$

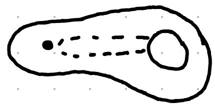
For conservative MT, $\dot{J} = 0$, $\dot{M} = 0$, $\dot{M}_1 = -\dot{M}_2$ [M_2 is donor, M_1 is accretor]

$$\rightarrow \frac{\dot{a}}{a} = \frac{2(M_2 - M_1)}{M_1 M_2}\dot{M}_2 = 2\left(\frac{M_2}{M_1} - 1\right)\frac{\dot{M}_2}{M_2}$$

If $M_2 > M_1$, $\dot{a} < 0$ orbit shrinks

In unstable mass transfer, orbit shrinks faster than the Roche Lobe → more MT → more shrinkage

common envelope (CE)



energy transfer from orbit to CE, if it overcomes CE binding energy, then CE is ejected.

Why do we want low metallicity?

Another example: Dynamical Assembly in GCs

GCs evolve towards energy equipartition (although they never quite reach it)

dynamical friction:

2-body interactions lead to energy exchange

expect velocity dispersion $\sigma \sim m^{-1/2}$

more massive objects sink to the center

→ BH mosh pit

hard binaries get harder

orbital speed > typical velocity dispersion