Black Hole metrics & phenomenology

topics: time-invariant vacuum solutions to EFEs

symmetries ~~and killing vectors~~

Schwartzschild metric (spherical symmetry)
- gravitational redshift
- photon ring / effective potential
- coordinate singularity / event horizon
- ~~gravitational lensing / deflection~~
- ~~Shapiro delay~~

Kerr metric (azimuthal symmetry)
- ergo regions ~~/ penrose process~~
- frame dragging

~~No-hair theorem~~

~~Area theorem~~

~~Membrane paradigm~~

Black Holes: vacuum solns to EFEs (described by a small number of parameters)

astrophysically relevant metrics

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<u>Schwarzschild</u>: static + spherical symmetry

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$M$: total energy of BH, including binding energy.

Birkhoff's theorem: Schwarzschild is the unique vacuum soln w/ spherical symmetry (& there are no time-dep. solns of this form)

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<u>Kerr</u>: stationary + azimuthal symmetry

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}\left(dt\,d\phi + d\phi\,dt\right)$$
$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

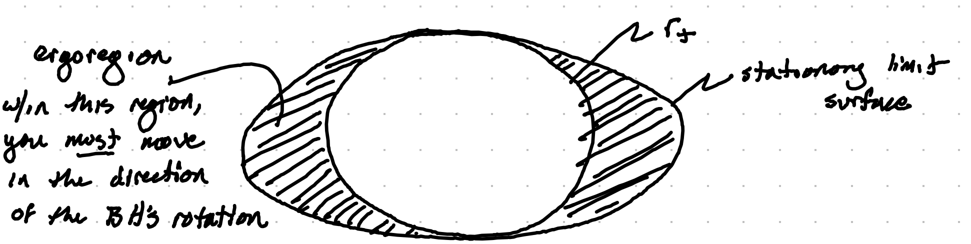
$$\Delta \equiv r^2 - 2GMr + a^2$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$

$$a \equiv J/M \quad \text{(angular momentum per unit mass)}$$

$\Delta$ vanishes at $r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$

note that the Killing vector associated w/ time-invariance @ $r \to \infty$ becomes space-like for some $r > r_+$



Gravitational Redshift (for Schwarzschild)

consider the proper time between 2 events as measured by two stationary observers in a Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

for stationary observers at $r_A$ ; $r_B$

$$\frac{d\tau_A^2 = \left(1 - \frac{2GM}{r_A}\right)dt^2}{d\tau_B^2 = \left(1 - \frac{2GM}{r_B}\right)dt^2} \Rightarrow \frac{d\tau_A}{d\tau_B} = \sqrt{\frac{1 - \frac{2GM}{r_A}}{1 - \frac{2GM}{r_B}}}$$

for $r_A, r_B \gg 2GM$

$$\frac{d\tau_A}{d\tau_B} \approx 1 - \frac{GM}{r_A} + \frac{GM}{r_B} = 1 + \underbrace{\phi_A - \phi_B}_{\text{difference in grav. potentials}}$$

→ the shift in the frequency compensates for the change in the potential energy

⟹ looks like energy conservation

(spacetime has time-like killing vector)

Let's consider some basic orbital dynamics w/ effective potentials

Consider the 2-body problem in Newtonian gravity

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{G m_1 m_2}{|\vec r_1 - \vec r_2|}$$

change coordinates from $\vec r_1, \vec r_2$ to $\vec R = (m_1 \vec r_1 + m_2 \vec r_2)/(m_1+m_2)$

$$\vec r = \vec r_2 - \vec r_1$$

and we obtain

$$\mathcal{H} = \underbrace{\tfrac{1}{2}(m_1+m_2)|\dot{\vec R}|^2}_{\text{conserved}} + \tfrac{1}{2}\frac{m_1 m_2}{(m_1+m_2)}|\dot{\vec r}|^2 - \frac{G m_1 m_2}{|\vec r|}$$

Work in frame with $\dot{\vec R} = \ddot{\vec R} = 0$

decompose $\vec r \Rightarrow (r\cos\phi,\ r\sin\phi,\ 0)$

and we obtain

$$\mathcal{H} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\dot r^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right) r^2 \dot\phi^2 - \frac{G m_1 m_2}{r}$$

which is more commonly expressed as

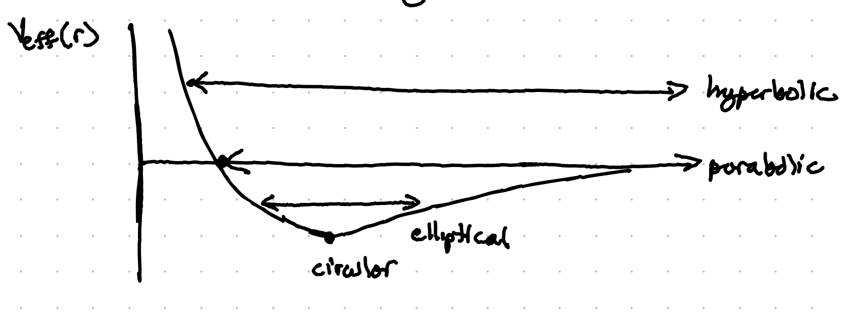
$$\mathcal{H} = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r} = \frac{p_r^2}{2\mu} + \underbrace{V_{eff}(r)}_{\text{effective potential}}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$

$M = m_1 + m_2$

$L = \mu r^2 \dot\phi^2$ = conserved

In general, $V_{eff}$ has the following form:




note that the centrifugal barrier wins at small r

$\Rightarrow$ point particles never collide unless they are perfectly aligned $(L \to 0)$

now, let's consider geodesics in Schwarzschild

trajectories given by $\frac{dx^\mu}{d\lambda}$

conserved quantities: $\varepsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \sim -u^\mu u_\mu = \begin{cases} 0 & \text{massless} \\ 1 & \text{massive} \end{cases}$

$$\left.\begin{aligned} E &= \left(1-\frac{2GM}{r}\right)\frac{dt}{d\lambda} \\ L &= r^2\frac{d\phi}{d\lambda} \end{aligned}\right\} \text{ from Killing Vector}$$

$$\Rightarrow -\varepsilon = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

or

$$\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right)}_{} = E^2$$

this looks like an effective potential!

$$V_{sch}(r) = \varepsilon - \varepsilon\frac{2GM}{r} + \frac{L^2}{r^2} - \frac{2GML^2}{r^3}$$

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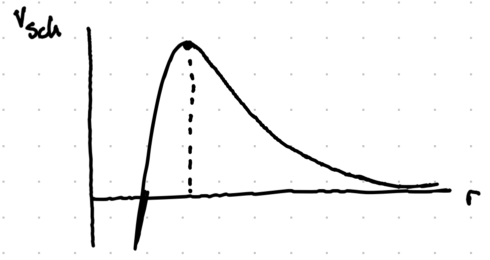
massless: $\varepsilon = 0$




No stable bound orbits
massless particles w/ $L \neq 0$ can still be captured

local maximum at

$$\frac{\partial V}{\partial r} = 0 = -\frac{2L^2}{r^3} + \frac{6GML^2}{r^4}$$

$$\boxed{r = 3GM \longleftarrow \text{light ring}}$$

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massive: $\varepsilon = 1$


particles with $L \neq 0$ can still be captured

there are stable bound orbits, but there is a minimum value of $L$ below which there are no stable orbits

$$\frac{\partial V}{\partial r} = +\frac{2GM}{r^2} - \frac{2L^2}{r^3} + \frac{6GML^2}{r^4} = 0$$

$$\Rightarrow r = \frac{L^2 \pm \sqrt{L^2(L^2 - 12G^2M^2)}}{2GM}$$

if $L^2 < 12G^2M^2$, then the system plunges

when $L^2 = 12G^2M^2 \Rightarrow \boxed{r = 6GM}$

$\boxed{\text{Innermost Stable Circular Orbit}}$

For CBCs, GWs carry away both energy and angular momentum. Circular orbits at $r = (L^2 + \sqrt{L^2(L^2 - 12G^2M^2)})/2GM$ slowly shrink. the only way this can happen is if $L$ decreases. Eventually it will be small enough that there is no longer a stable orbit $\Rightarrow$ plunge. This is one way to define the "inspiral" vs. the "merger" w/in a CBC waveform.

Note that effective potentials like this can also be derived for Kerr, but Kerr does not have spherical symmetry

$\rightarrow$ orbits in the mid-plane stay in the mid plane, but in general they will traverse rosettes in both $\theta$ & $r$

Rindler coordinates and uniformly accelerating observers

consider flat space: $ds^2 = -dt^2 + dx^2$

consider uniform acceleration in $+x$ direction w/ magnitude $a$

$$\Rightarrow \quad t(\tau) = \chi \sinh(a\tau)$$

$$x(\tau) = \chi \cosh(a\tau) \quad \longleftarrow \text{coord. singularity at } \chi = 0$$

with this change of coordinates,

$$ds^2 = \underbrace{-(a\chi)^2}_{} d\tau^2 + d\chi^2$$

position-dependent redshift between nearby observers that follow constant accelerations

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Now, let's look at Schwarzschild near the horizon:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

consider $x^2 \equiv r - 2GM \Rightarrow dr = 2x\,dx$

$$ds^2 = -\left(1 - \frac{2GM}{2GM + x^2}\right) dt^2 + \left(1 - \frac{2GM}{2GM + x^2}\right)^{-1} (2x\,dx)^2$$

$$\approx \underbrace{-\left(\frac{x^2}{2GM}\right) dt^2 + (8G^2M^2)\,dx^2}_{\text{Rindler coordinates}}$$

Membrane Paradigm: what is it? why should we care?

What is an effective field theory?

whenever there is a separation of scales, we can try to "integrate over" one scale (high energy, short times/lengths) to define effective dynamics of the other scales.