Lecture 3: Neutron Star structure and the Equation of State

topics: Newtonian Stars (hydrostatic equilibrium)

~~Lane-Emden Eqn for polytropes~~

Degeneracy pressure: (non)relativistic Fermi gas @ $T=0$

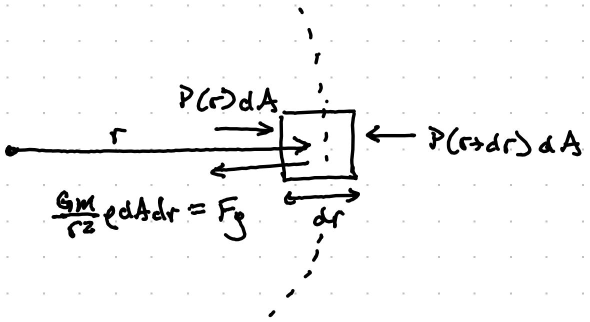
TOV Equations
- stability criteria

Mass-Radius relations ; exotic phenomenology
- connection btwn phase transitions & twin stars

Nuclear Physics (npe matter)
- saturation density
- symmetry energy
- "neutron drip"

Electromagnetic signals from NS
- radio pulsars
- x-ray pulsars
- accreting systems
- mergers:
  - GRB + afterglow
  - kilonova

Hydrostatic Equilibrium (Newtonian Gravity)



Free body diagram:

$$P(r)\,dA - P(r+dr)\,dA - \frac{Gm\rho}{r^2}\,dA\,dr = 0$$

$$\Rightarrow \quad -\frac{Gm\rho}{r^2} = \frac{P(r+dr) - P(r)}{dr} = \frac{dP}{dr}$$

enclosed mass: $\quad \frac{dm}{dr} = 4\pi r^2 \rho$

We can close this system of equations w/ an Equation of State (EOS)

$P = P(\rho) \iff$ barotropic

$P = K\rho^n$ polytropic

Quantum degeneracy pressure (non-interacting Fermi gas at $T=0$)

predicts $\quad P \sim \rho^{5/3}$ if particles are nonrelativistic

$\quad\quad\quad\quad \sim \rho^{4/3}$ if particles are relativistic (marginally stable)

Basic physical rules: $\quad \underbrace{0 \le \frac{dP}{d\rho}}_{\text{thermodynamic stability}} \underbrace{= c_s^2 \le c^2}_{\text{causality}}$

These equations give us a map from central pressure (P_c) to the macroscopic properties of stars.

Initial conditions:

P_c, ρ_c

$m(r=0) = 0$

$\left.\frac{dP}{dr}\right|_{r=0} = 0$

Termination conditions

$\rho(R) = 0$

Stability condition $\quad \frac{dM}{dP_c} > 0$

if this is violated, then the star can collapse to a more compact state.

Tolman-Oppenheimer-Volkov (TOV) equations.

Assume perfect, stationary fluid.

$$T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$

p: pressure
ε: energy density

we assume stationary flow so that u_μ only has a non-zero component in the time-like direction

assume static, spherical metric

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

EFE's become

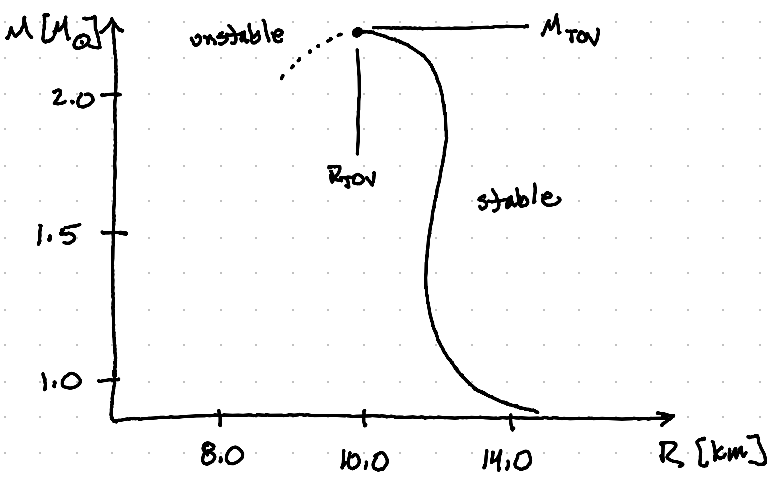
$$\frac{dm}{dr} = 4\pi r^2 \varepsilon$$

$$\frac{dp}{dr} = - \frac{(\varepsilon + p)(Gm + 4\pi G r^3 p)}{r(r - 2Gm)}$$

w/similar intial/termination/stability conditions as Newtonian gravity
The exterior of the star is Schwarzschild.

Here, "m" is the total grav. mass enclosed
→ includes both rest-mass and grav. potential and corresponds to the Schwarzschild parameter "M"

Although there can be a strong dependence on the EoS, we often get mass-radius curves that look like



Basic Nuclear Physics (npe matter)

charge neutrality: $n_e = n_p$

β-equilibrium $\mu_n = \mu_p + \mu_e$ ← for cold, degenerate fermions, these depend only on the number densities

Pauli-exclusion ⇒ fill energy levels

electrons @ high densities have high fermi momenta ⇒ eventually favorable to add more neutrons rather than additional protons & electrons

We can describe the thermodynamic properties of bulk npe-matter w/ the energy per particle (E/N) as a function of the baryon number density $(n = n_n + n_p)$ and the proton fraction $(x = n_p/n)$

Common References

Symmetric Nuclear Matter: $\frac{E_{SNM}}{N}(n) = \frac{E}{N}(n, x=\frac{1}{2})$

Pure Neutron Matter $\frac{E_{PNM}}{N}(n) = \frac{E}{N}(n, x=0)$

Empirical Observations

E_{SNM}/N has a local minimum (sets density scale for nuclei)

$$\frac{E_{SNM}}{N} = E_0 + \frac{1}{2} K_0 \left(\frac{n-n_0}{3n_0}\right)^2 + \dots$$

$n_0 \sim 0.164 \pm 0.007 \ \text{fm}^{-3}$

$E_0 \sim -15.86 \pm 0.57$ MeV

$K_0 \sim 215 \pm 40$ MeV

Common Approximations

$$\frac{E}{N}(n,x) = \frac{E_{SNM}}{N}(n) + (1-2x)^2 S(n)$$

these are important parameters for nuclear physics!

$S(n) = \frac{E_{PNM}}{N}(n) - \frac{E_{SNM}}{N}(n)$ Symmetry Energy

$L \equiv 3n \left.\frac{dS}{dn}\right|_{n_0}$ slope parameter

$K_{sym} \equiv 9n^2 \left.\frac{d^2S}{dn^2}\right|_{n_0}$ curvature / compressibility

Why do we care?

→ S sets x in β-equilibrium

"energy cost of changing protons to neutrons"

→ if S is small, x is small (few protons)
S is large, x is large (many protons)

→ L gives the pressure of PNM at n_0

$$\frac{E_{PNM}}{N} = \frac{E_{SNM}}{N} + S$$

$$V = \frac{1}{n} \qquad dV = -\frac{1}{n^2}\,dn$$

$$P_{PNM} = -\left.\frac{\partial E_{PNM}}{\partial V}\right|_N = n^2 \left.\frac{\partial \bar{E}_{PNM}}{\partial n}\right|_N$$

$$P_{PNM}(n_0) = n_0^2 \left(\cancel{\frac{d\,\bar{E}_{SNM}}{dn}} + \frac{dS}{dn} \right)\Big|_{n_0}$$

$\varnothing$

$$= \frac{1}{3} n_0 L \approx P_{NSM}(n_0)$$

pressure of Neutron Star Matter @ saturation density

How do we extract (S, L, K_{sym}) from astro observations of cold, slowly rotating NSs?

Stellar structure is determined by $\mathcal{E}_\beta(p)$: β-equilib. energy density

there is a 1-to-1 mapping btwn $\mathcal{E}_\beta(p)$ & M-R, etc

→ if we measure M-R well, we measure $\mathcal{E}_\beta(p)$ well

How do we go from $\mathcal{E}_\beta \to (S, L, K_{sym})$ at n_0?

$$\left(\frac{E}{N}\right)_\beta = \frac{\mathcal{E}_\beta - \mathcal{E}_e}{n} - m_N$$

rest mass of nucleon ~ 939 MeV

$$\begin{pmatrix} m_p = 938.3 \text{ MeV} \\ m_n = 939.6 \text{ MeV} \end{pmatrix}$$

$$\mu_p = \frac{\partial}{\partial n_p}\left[n\left(\frac{E}{N} + m_p\right)\right]\Big|_V$$

$$= \frac{\partial}{\partial n}(\cdot)\frac{\partial n}{\partial n_p} + \frac{\partial}{\partial x}(\cdot)\frac{\partial x}{\partial n_p}$$

$$= n\frac{\partial (E/N)}{\partial n} + \frac{\partial (E/N)}{\partial x}(1-x) + \frac{E}{N} + m_p$$

$$\mu_n = \frac{\partial}{\partial n_n}\left[n\left(\frac{E}{N} + m_n\right)\right]\Big|_V$$

$$= n\frac{\partial (E/N)}{\partial n} - \frac{\partial (E/N)}{\partial x}x + \frac{E}{N} + m_n$$

β-equilib $\Rightarrow$ $\mu_n = \mu_p + \mu_e$

$$\frac{\partial (E/N)}{\partial x} = m_n - m_p - \mu_e$$

$\exists$ standard expressions for $(\mu_e, \mathcal{E}_e)$ as functions of $n_e = x n$

In the neighborhood of n_0, where we expect our expansions to hold reasonably well, we obtain

$$\left.\begin{aligned} \left(\frac{E}{N}\right)_\beta &= \frac{\mathcal{E}_\beta - \mathcal{E}_e}{n} - m_N = \frac{E_{SNM}}{N} + S(1-2x_\beta)^2 \\ \frac{\partial (E/N)_\beta}{\partial x} &= -4S(1-2x_\beta) = m_n - m_p - \mu_e \end{aligned}\right\} \text{ solve for } (S, x_\beta)$$

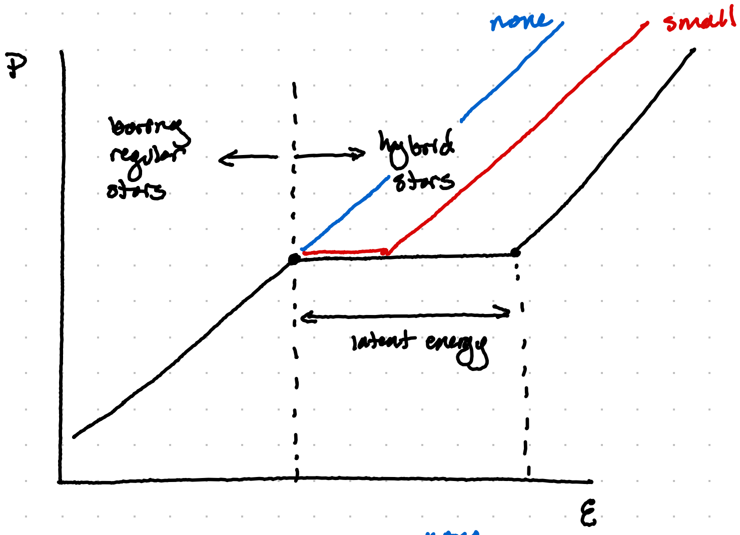
We obtain $S(n)$ in the neighborhood of n for arbitrary $\mathcal{E}_\beta(n)$

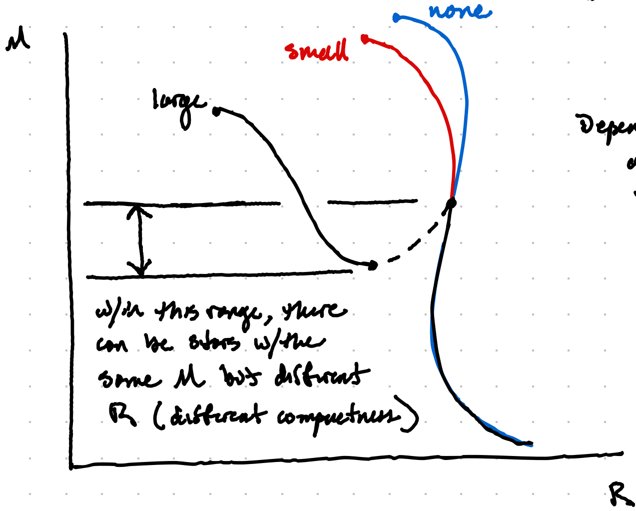
→ extract (L, K_{sym}) directly via definitions (derivatives)

Exotic NS phenomenology

Phase Transition Phenomenology with Nonparametric Representations of the Neutron Star Equation of State
PRD 108, 043013 (2023)
arXiv: 2305.07411

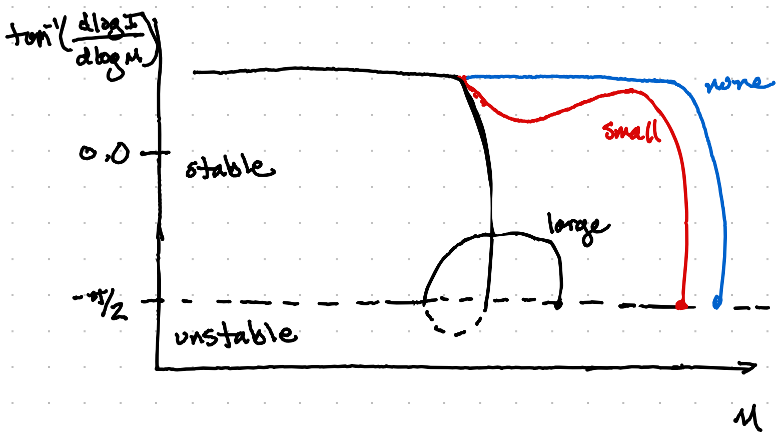
Classic picture of 1st order transition for appearance of new particle





Depending on latent energy and the onset pressure, the star may lose stability thru the phase transition

However, there are plenty of models of phase transitions that look a lot messier than this. Can we identify physical properties systematically w/out an underlying parametrization?



We can look for spikes + dips in this space to identify all kinds of phase transitions

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