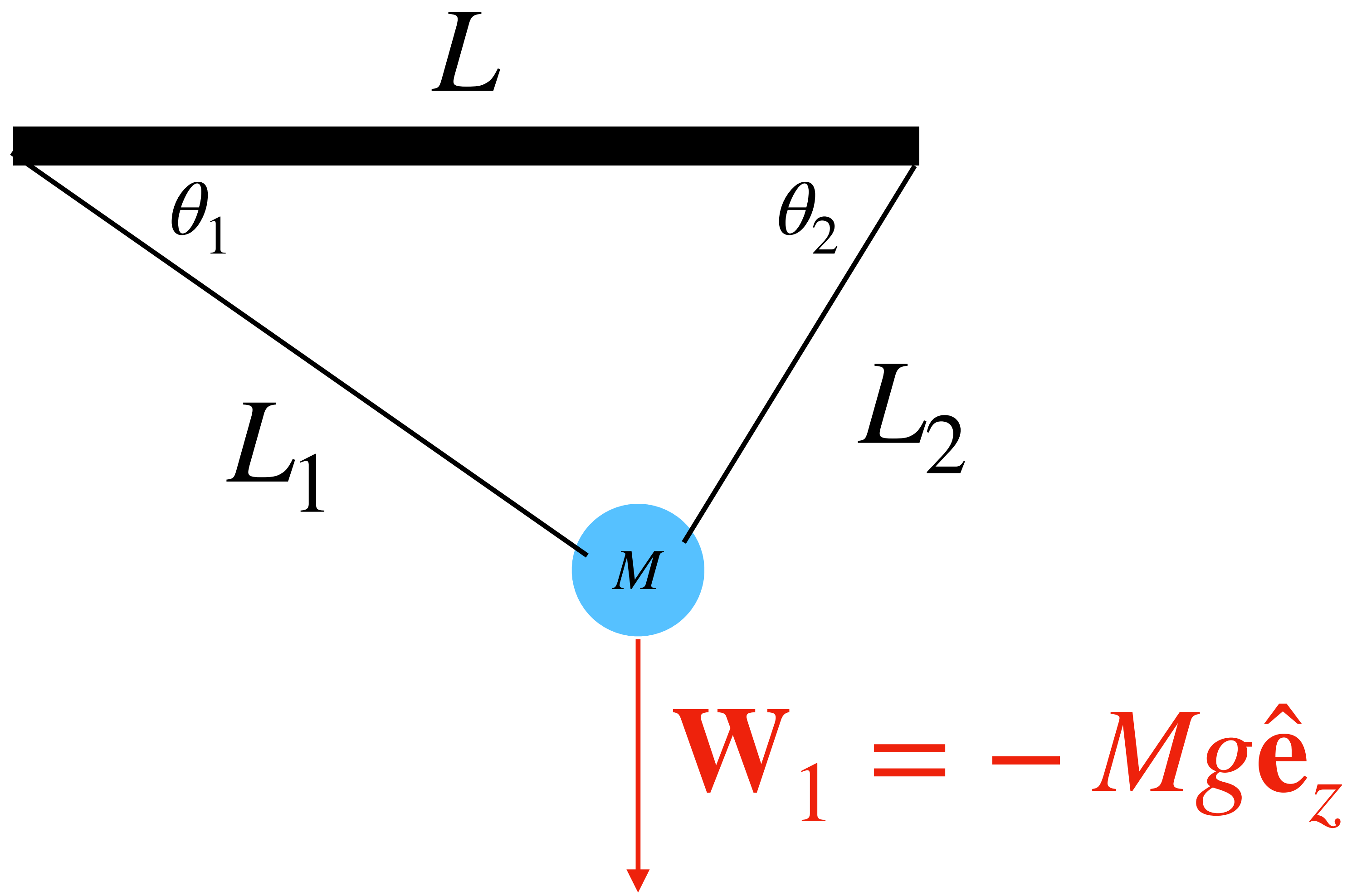
2 masses/3 strings

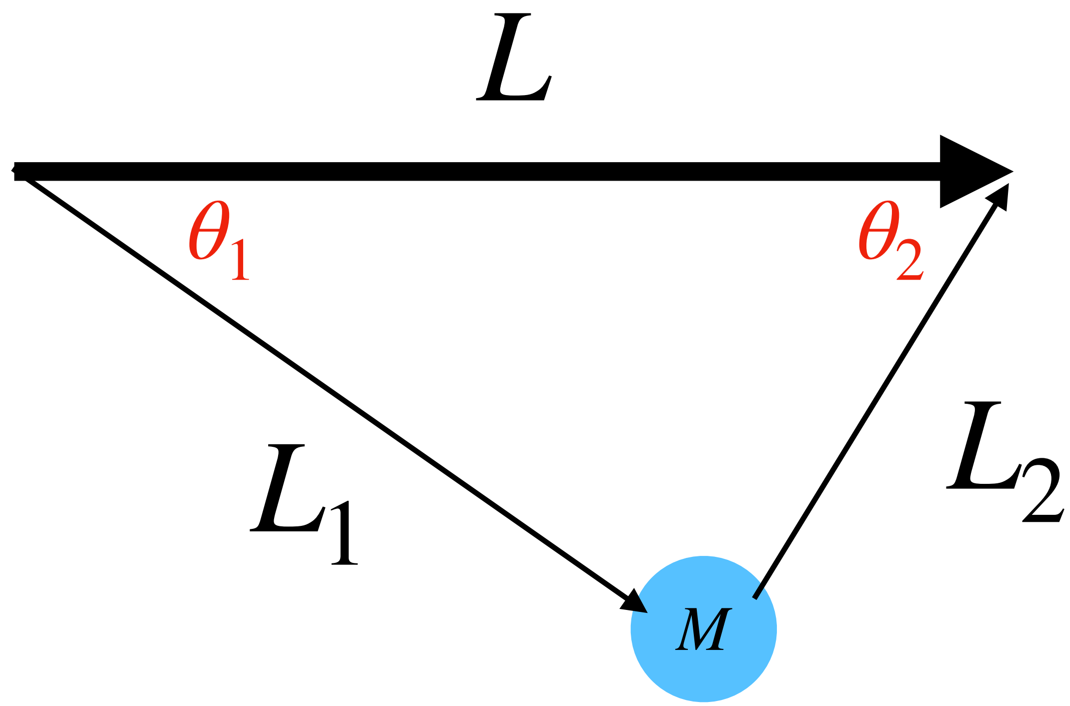
(or: How I learned to love Linear Algebra)

2 strings/1 mass



- given M, L, L_1, L_2 what is the position of the mass?
- easy: triangle with three sides is fully determined

2 strings/1 mass



$$\mathbf{L} = \begin{pmatrix} L \\ 0 \end{pmatrix}$$

$$\mathbf{L}_1 = L_1 \begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}$$

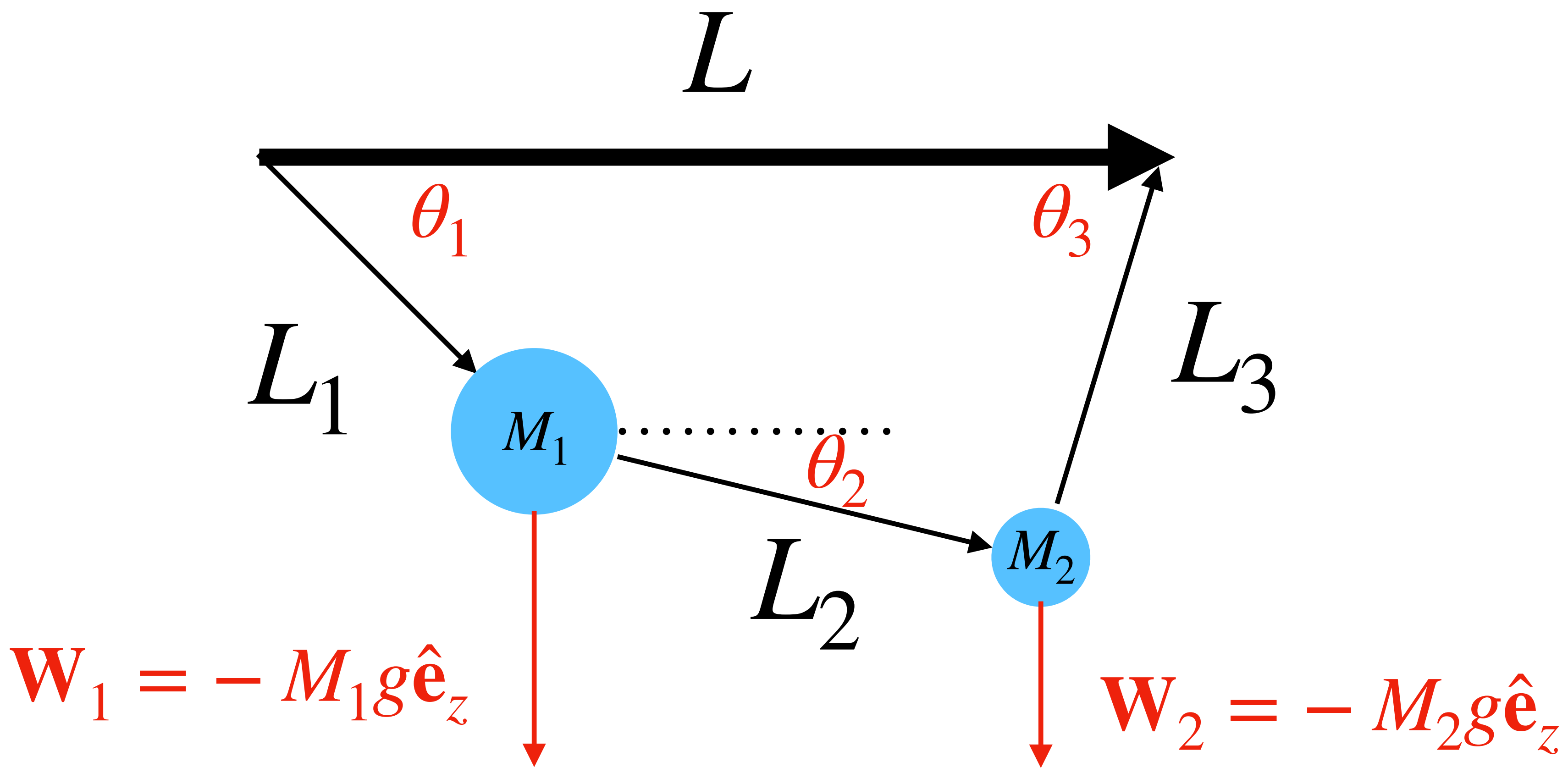
$$\mathbf{L}_2 = L_2 \begin{pmatrix} -\cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$$

2 equations, 2 unknowns

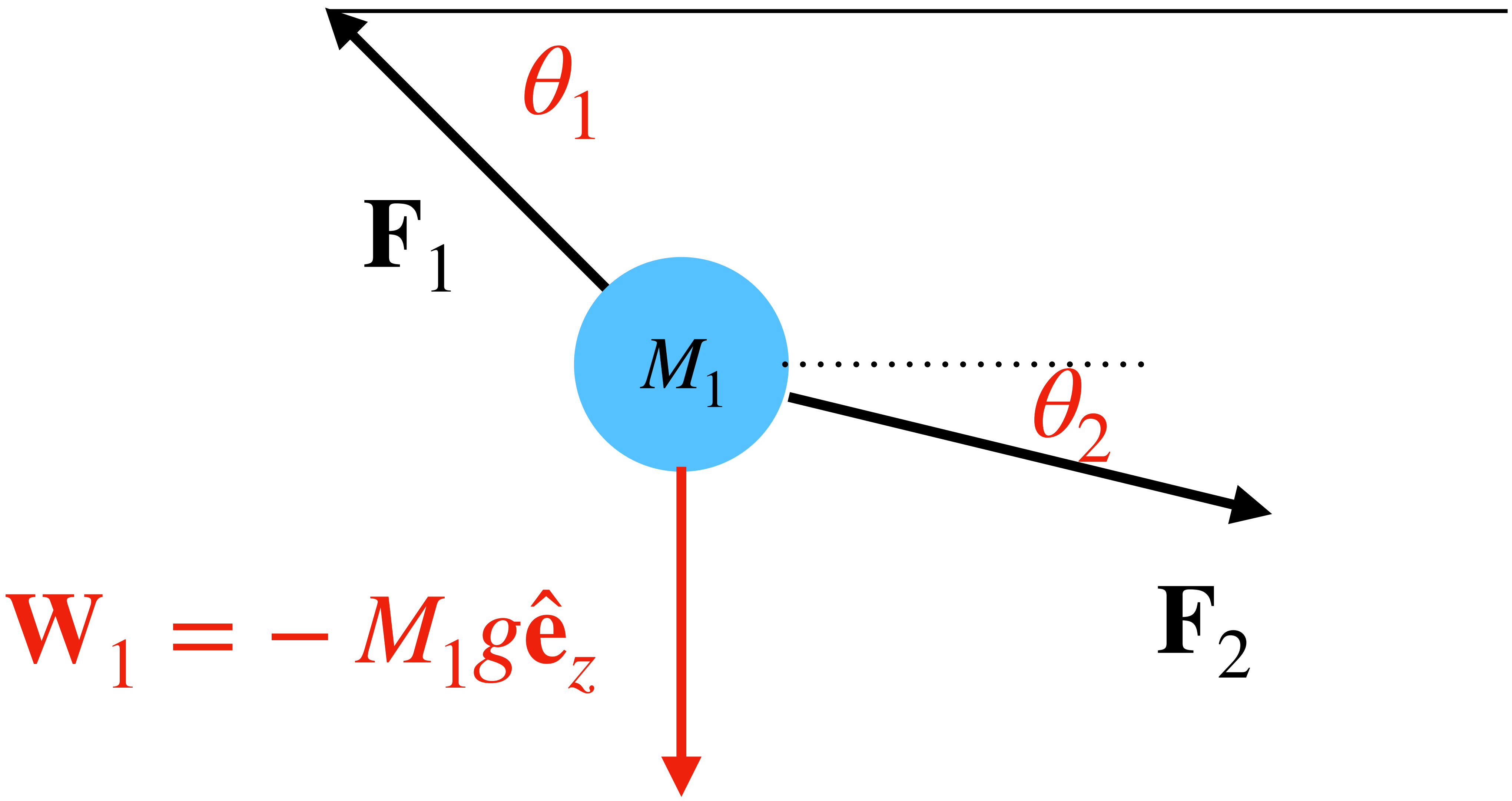
(see Appendix for solution)

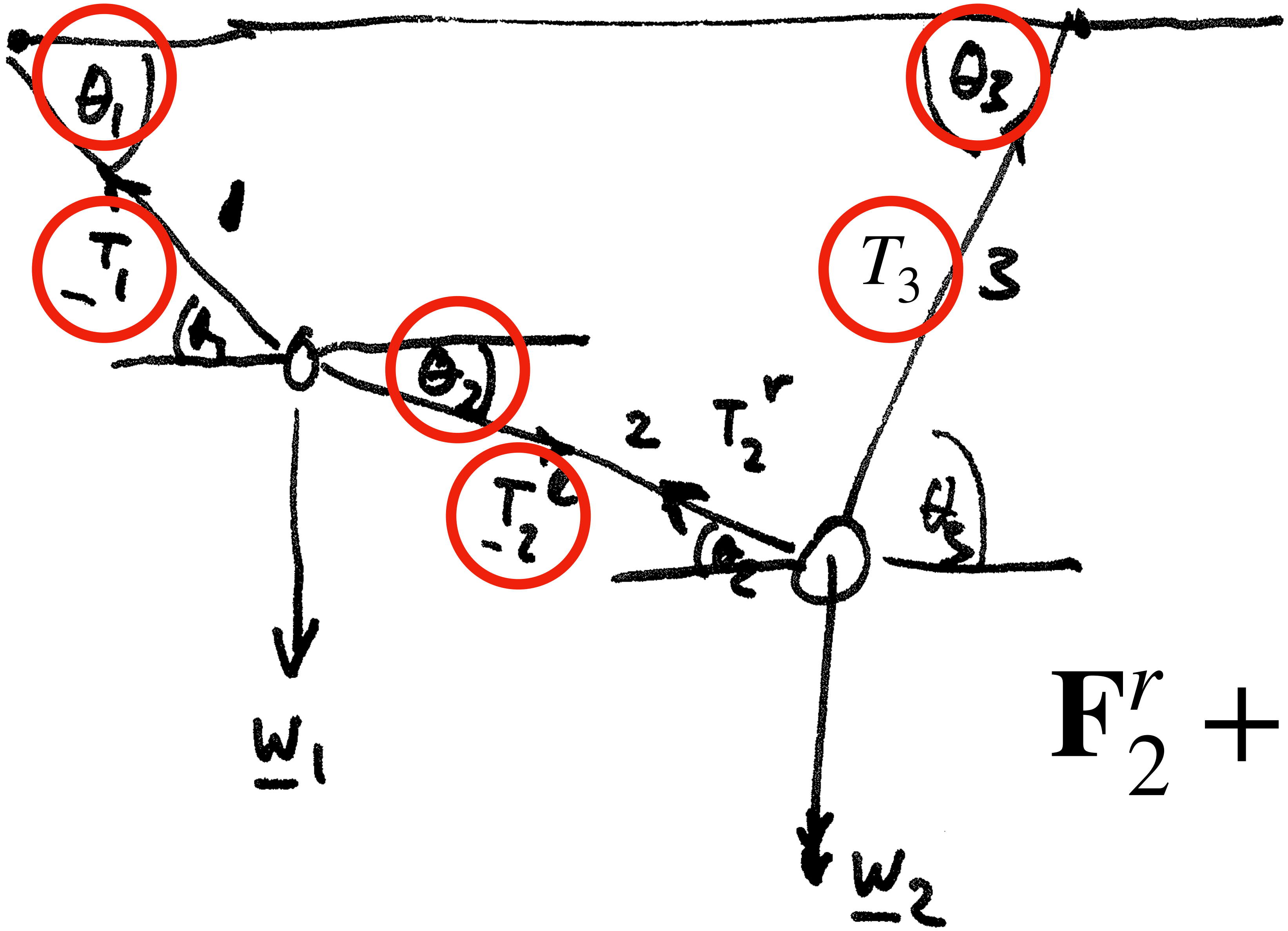
3 strings/2 masses



3 strings/2 masses

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W}_1 = 0$$





unknowns:

- 3 angles θ_i
- 3 tensions T_i

$$\mathbf{F}_2^r + \mathbf{F}_3 + \mathbf{W}_2 = 0$$

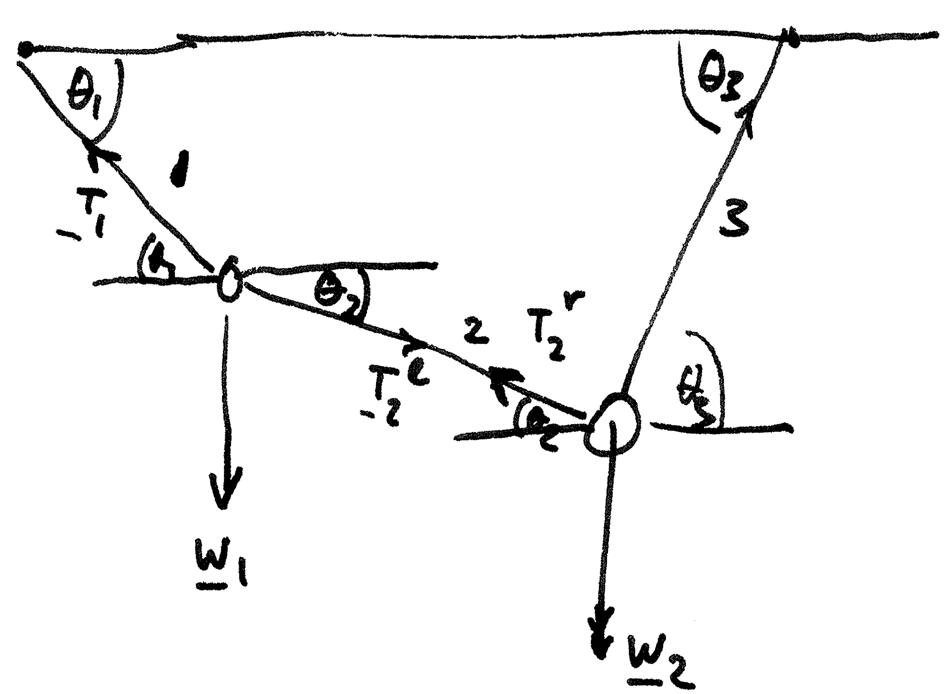
$$\mathbf{F}_1 + \mathbf{F}_2^l + \mathbf{W}_1 = 0$$

Tension forces

$$\underline{F}_1 = T_1 \begin{pmatrix} -\cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$$

$$\underline{F}_2^{\ell} = T_2 \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix} \qquad \underline{F}_2^{r} = -\underline{F}_2^{\ell}$$

$$\underline{F}_3 = T_3 \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}$$



Gravity

$$\underline{W}_1 = \begin{pmatrix} 0 \\ -W_1 \end{pmatrix}$$

$$\underline{W}_2 = \begin{pmatrix} 0 \\ -\underline{W}_2 \end{pmatrix}$$

Geometry

$$\underline{L} = \begin{pmatrix} L \\ 0 \end{pmatrix}$$

$$\underline{L}_1 = L_1 \begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}$$

$$\underline{L}_2 = L_2 \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

$$\underline{L}_3 = L_3 \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}$$

Force equations

M_1

$$\boxed{\underline{F}_1 + \underline{F}_2^{\ell} + \underline{W}_1 = 0}$$

$$\underline{F}_1 = T_1 \begin{pmatrix} -\cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$$

$$\underline{F}_2^{\ell} = T_2 \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix} \qquad \underline{F}_2^{r} = -\underline{F}_2^{\ell}$$

$$\underline{F}_3 = T_3 \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}$$

(1) $-T_1 \cos\theta_1 + T_2 \cos\theta_2 = 0$

(2) $T_1 \sin\theta_1 - T_2 \sin\theta_2 - W_1 = 0$

M_2:

$$\boxed{\underline{F}_2^{r} + \underline{F}_3 + \underline{W}_2 = 0}$$

(3) $-T_2 \cos\theta_2 + T_3 \cos\theta_3 = 0$

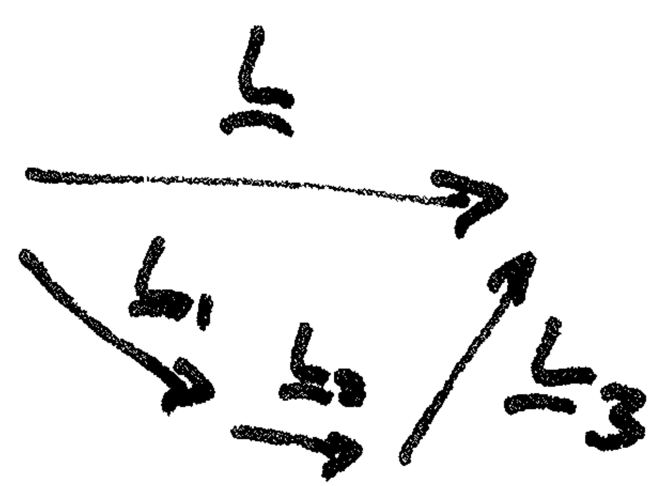
(4) $T_2 \sin\theta_2 + T_3 \sin\theta_3 - W_2 = 0$

Geometry

$$\underline{L} = \underline{L}_1 + \underline{L}_2 + \underline{L}_3$$

$$(5) \quad L_1 \cos\theta_1 + L_2 \cos\theta_2 + L_3 \cos\theta_3 = L$$

$$(6) \quad -L_1 \sin\theta_1 - L_2 \sin\theta_2 + L_3 \sin\theta_3 = 0$$



$$\underline{L} = \begin{pmatrix} L \\ 0 \end{pmatrix}$$

$$\underline{L}_1 = L_1 \begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}$$

$$\underline{L}_2 = L_2 \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

$$\underline{L}_3 = L_3 \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}$$

6 unknowns

- **3 angles**
- **3 tensions *T_i***

6 equations

$$(1)\quad -T_1\cos\theta_1 + T_2\cos\theta_2 = 0$$

$$(2)\quad T_1\sin\theta_1 - T_2\sin\theta_2 - W_1 = 0$$

$$(3)\quad -T_2\cos\theta_2 + T_3\cos\theta_3 = 0$$

$$(4)\quad T_2\sin\theta_2 + T_3\sin\theta_3 - W_2 = 0$$

$$(5)\quad L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3 = L$$

$$(6)\quad -L_1\sin\theta_1 - L_2\sin\theta_2 + L_3\sin\theta_3 = 0$$

Treat cos/sin separately (?!)

$$\cos^2\theta_1 + \sin^2\theta_1 = 1 \qquad (7)$$

$$\cos^2\theta_2 + \sin^2\theta_2 = 1 \qquad (8)$$

$$\cos^2\theta_3 + \sin^2\theta_3 = 1 \qquad (9)$$

- We will treat cos(theta) and sin(theta) as *independent* variables to make our equations simpler.

- But now need 3 more equations: (7) - (9)

$$
\begin{aligned}
-T_1 \cos\theta_1 + T_2 \cos\theta_2 &= 0 \\
T_1 \sin\theta_1 - T_2 \sin\theta_2 - W_1 &= 0 \\
-T_2 \cos\theta_2 + T_3 \cos\theta_3 &= 0 \\
T_2 \sin\theta_2 + T_3 \sin\theta_3 - W_2 &= 0 \\
L_1 \cos\theta_1 + L_2 \cos\theta_2 + L_3 \cos\theta_3 - L &= 0 \\
-L_1 \sin\theta_1 - L_2 \sin\theta_2 + L_3 \sin\theta_3 &= 0 \\
\sin^2\theta_1 + \cos^2\theta_1 - 1 &= 0 \\
\sin^2\theta_2 + \cos^2\theta_2 - 1 &= 0 \\
\sin^2\theta_3 + \cos^2\theta_3 - 1 &= 0
\end{aligned}
$$

$$
\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \\ \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix}
$$

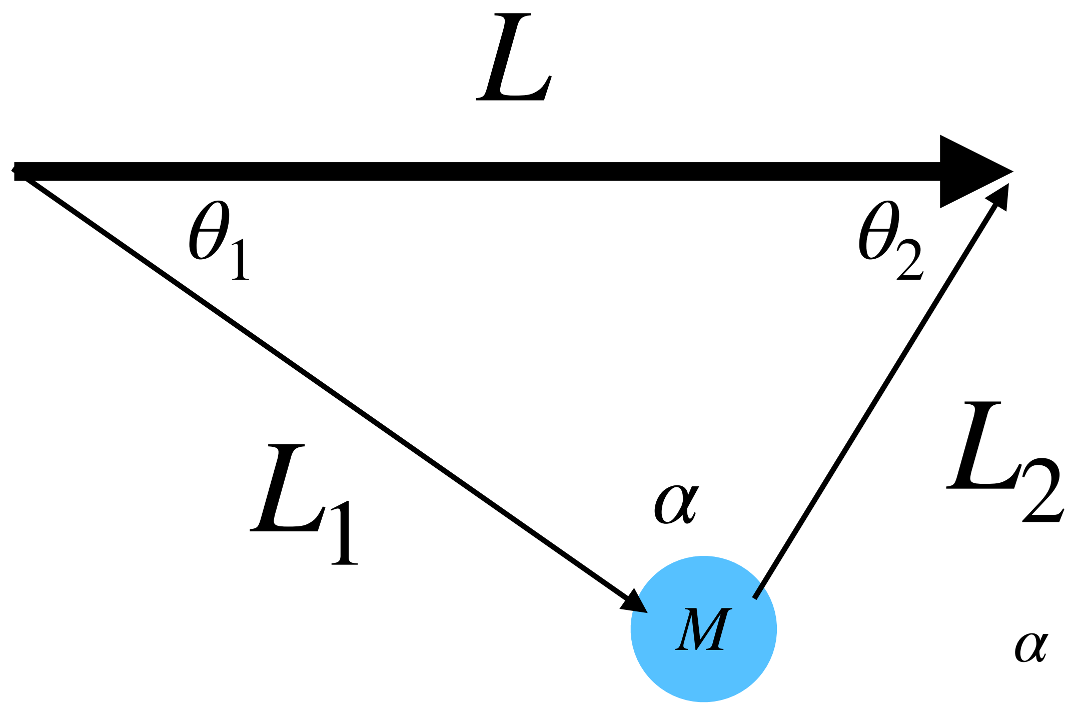
$$
\begin{aligned}
f_0(\mathbf{x}) &= -x_6 x_3 + x_7 x_4 && = 0 \\
f_1(\mathbf{x}) &= x_6 x_0 - x_7 x_1 - W_1 && = 0 \\
f_2(\mathbf{x}) &= -x_7 x_4 + x_8 x_5 && = 0 \\
f_3(\mathbf{x}) &= x_7 x_1 + x_8 x_2 - W_2 && = 0 \\
f_4(\mathbf{x}) &= L_1 x_3 + L_2 x_4 + L_3 x_5 - L && = 0 \\
f_5(\mathbf{x}) &= -L_1 x_0 - L_2 x_1 + L_3 x_2 && = 0 \\
f_6(\mathbf{x}) &= x_0^2 + x_3^2 - 1 && = 0 \\
f_7(\mathbf{x}) &= x_1^2 + x_4^2 - 1 && = 0 \\
f_8(\mathbf{x}) &= x_2^2 + x_5^2 - 1 && = 0
\end{aligned}
$$

$$
\mathbf{L} = \begin{pmatrix} L \\ L_1 \\ L_2 \\ L_3 \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}
$$

$$
\mathbf{f}(\mathbf{x}) = 0
$$

Appendix

2 strings/1 mass



$\alpha = \pi - \theta_1 - \theta_2$

$$\mathbf{L} = \binom{L}{0}$$

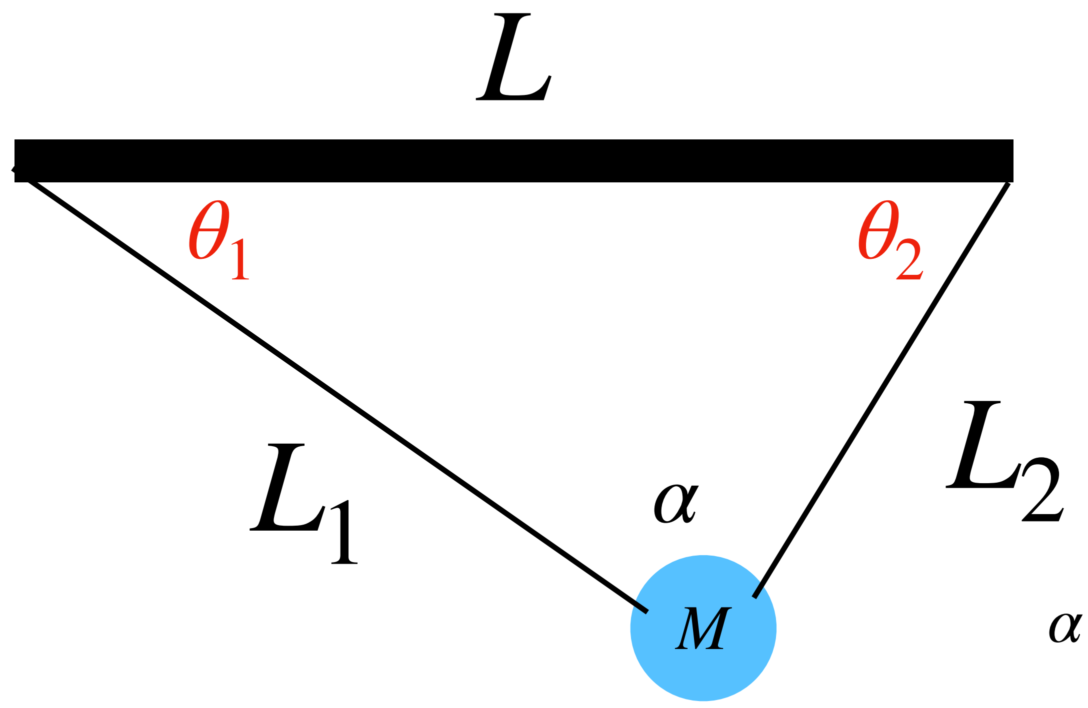
$$\mathbf{L}_1 = L_1 \begin{pmatrix} \cos\theta_1 \\ -\sin\theta_1 \end{pmatrix}$$

$$\mathbf{L}_2 = L_2 \begin{pmatrix} -\cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

$$\boxed{\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2}$$

Geometry is the only necessary stability condition – a triangle is fully defined when the lengths of the three sides are given.

2 strings/1 mass



$\alpha = \pi - \theta_1 - \theta_2$

$$\mathbf{L}_1 = \mathbf{L}_2 - \mathbf{L}$$

$$\mathbf{L}_2 = \mathbf{L}_1 - \mathbf{L}$$

$$L_1^2 = L_2^2 + L^2 - 2\mathbf{L}_2 \cdot \mathbf{L}$$

$$\mathbf{L}_2 \cdot \mathbf{L} = L_2 L \cos\theta_2$$

$$\cos\theta_2 = \frac{L^2 + L_1^2 - L_2^2}{2LL_1}$$

$$L_2^2 = L_1^2 + L^2 - 2\mathbf{L}_1 \cdot \mathbf{L}$$

$$\mathbf{L}_1 \cdot \mathbf{L} = L_1 L \cos\theta_1$$

$$\cos\theta_1 = \frac{L^2 + L_2^2 - L_1^2}{2LL_2}$$