Diffusion equation: Stability

$$u = u(\underline{x}, t)$$

$$\frac{\partial u}{\partial t} = \nabla D(x) \nabla u \qquad D(x) = D$$

$$\frac{\partial u}{\partial t} = D \nabla^2 u$$

heat eq: $\frac{\partial T}{\partial t} = \frac{K}{C\varrho} \nabla^2 T$

Forward differencing scheme

$$\eta = \frac{D \Delta t}{\Delta x^2} < \frac{1}{2} \qquad \eta = \frac{K \Delta t}{C\varrho \, \Delta x^2} < \frac{1}{2}$$

cf. random walk in 1D:

$$2 D \Delta t = \Delta x^2$$

$2 D \Delta t < \Delta x^2$ or $\Delta t < \frac{\Delta x^2}{2D}$

i.e. stability requires that grid spacing is larger than the 'diffusion distance' over one step.

Max allowed Δt is the diffusion time across a grid cell

Often interested in features at scale $\lambda \gg \Delta x$, thus

$$\tau \sim \frac{\lambda^2}{D}$$

To see something, we need

$$\sim \frac{\tau}{\Delta t} \approx \frac{\lambda^2}{D} \frac{2D}{\Delta x^2} \approx \frac{\lambda^2}{\Delta x^2}$$

$\rightarrow$ large number of steps!

$\Rightarrow$ Need to find way to make bigger steps Δt!

Relaxation

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho} \nabla^2 T$$

or $$\frac{\partial u}{\partial t} = D \nabla^2 u \qquad u = u(\underline{x}, t)$$

(diffusion equation)

D = diffusion constant

$$\lambda^2 = 2d D \tau$$

Boundary value problem : Laplace

$$0 = \nabla^2 u$$

$$\uparrow$$

$$\frac{\partial u}{\partial t} = 0$$: equilibrium (or steady state)

Eg: $$\frac{\partial u}{\partial t} = D \nabla^2 u \qquad \begin{array}{l} D = 1 \\ \Delta x = \Delta y = \Delta \end{array} \qquad u^{n \leftarrow \text{time}}_{jl \leftarrow \text{space}}$$

$$u^{n+1}_{jl} = u^n_{jl} + \frac{\Delta t}{\Delta^2}\left(u^n_{j+1,l} + u^n_{j-1,l} + u^n_{j,l+1} + u^n_{j,l-1} - 4u^n_{jl}\right)$$

$$\left(\frac{u^{n+1}_{jl} - u^n_{jl}}{\Delta t} = \frac{u^n_{j+1,l} + u^n_{j-1,l} + u^n_{j,l+1} + u^n_{j,l-1} - 4u^n_{jl}}{\Delta^2}\right.$$

Max Δt : in 2D stability: $$\frac{\Delta t}{\Delta^2} \leq \frac{1}{4}$$

choose $\Delta t = 4\Delta^2$

$$u^{n+1}_{jl} = \frac{1}{4}\left(u^n_{j+1,l} + u^n_{j-1,l} + u^n_{j,l+1} + u^n_{j,l-1}\right) \Leftarrow \text{Jacobi scheme!}$$