$$\alpha_1 = \sqrt{\frac{ma^2 E_1}{2\hbar^2}} \simeq 1.9 \quad \Longrightarrow \quad E_1 \simeq \frac{7.22\hbar^2}{ma^2}. \tag{4.113}$$

4.8 The Harmonic Oscillator

The harmonic oscillator is one of those few problems that are important to all branches of physics. It provides a useful model for a variety of vibrational phenomena that are encountered, for instance, in classical mechanics, electrodynamics, statistical mechanics, solid state, atomic, nuclear, and particle physics. In quantum mechanics, it serves as an invaluable tool to illustrate the basic concepts and the formalism.

The Hamiltonian of a particle of mass m which oscillates with an angular frequency ω under the influence of a one-dimensional harmonic potential is

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2. \tag{4.114}$$

The problem is how to find the energy eigenvalues and eigenstates of this Hamiltonian. This problem can be studied by means of two separate methods. The first method, called the *analytic method*, consists in solving the time-independent Schrödinger equation (TISE) for the Hamiltonian (4.114). The second method, called the *ladder* or *algebraic method*, does not deal with solving the Schrödinger equation, but deals instead with operator algebra involving operators known as the *creation* and *annihilation* or *ladder* operators; this method is in essence a matrix formulation, because it expresses the various quantities in terms of matrices. In our presentation, we are going to adopt the second method, for it is more straightforward, more elegant and much simpler than solving the Schrödinger equation. Unlike the examples seen up to now, solving the Schrödinger equation for the potential $V(x) = \frac{1}{2}m\omega x^2$ is no easy job. Before embarking on the second method, let us highlight the main steps involved in the first method.

Brief outline of the analytic method

This approach consists in using the power series method to solve the following differential (Schrödinger) equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x), \tag{4.115}$$

which can be reduced to

$$\frac{d^2\psi(x)}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{x^2}{x_0^4}\right)\psi(x) = 0, \tag{4.116}$$

where $x_0 = \sqrt{\hbar/(m\omega)}$ is a constant that has the dimensions of length; it sets the length scale of the oscillator, as will be seen later. The solutions of differential equations like (4.116) have been worked out by our mathematician colleagues well before the arrival of quantum mechanics (the solutions are expressed in terms of some special functions, the Hermite polynomials). The occurrence of the term $x^2\psi(x)$ in (4.116) suggests trying a Gaussian type solution[3]: $\psi(x) =$

[3] Solutions of the type $\psi(x) = f(x)\exp(x^2/2x_0^2)$ are physically unacceptable, for they diverge when $x \longrightarrow \pm\infty$.

$f(x)\exp(-x^2/2x_0^2)$, where $f(x)$ is a function of x. Inserting this trial function into (4.116), we obtain a differential equation for $f(x)$. This new differential equation can be solved by expanding $f(x)$ out in a power series (i.e., $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n is just a coefficient), which when inserted into the differential equation leads to a recursion relation. By demanding the power series of $f(x)$ to terminate at some finite value of n (because the wave function $\psi(x)$ has to be finite everywhere, notably when $x \longrightarrow \pm\infty$), the recursion relation yields an expression for the energy eigenvalues which are discrete or quantized:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad (n = 0, 1, 2, 3, \ \ldots). \tag{4.117}$$

After some calculations, we can show that the wave functions that are physically acceptable and that satisfy (4.116) are given by

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\,2^n n!\, x_0}} e^{-x^2/2x_0^2} H_n\left(\frac{x}{x_0}\right), \tag{4.118}$$

where $H_n(y)$ are nth order polynomials called *Hermite polynomials*:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}. \tag{4.119}$$

From this relation it is easy to calculate the first few polynomials:

$$\begin{aligned} &H_0(y) = 1, && H_1(y) = 2y, \\ &H_2(y) = 4y^2 - 2, && H_3(y) = 8y^3 - 12y, \\ &H_4(y) = 16y^4 - 48y^2 + 12, && H_5(y) = 32y^5 - 160y^3 + 120y. \end{aligned} \tag{4.120}$$

We will deal with the physical interpretations of the harmonic oscillator results when we study the second method.

Algebraic method

Let us now show how to solve the harmonic oscillator eigenvalue problem using the algebraic method. For this, we need to rewrite the Hamiltonian (4.114) in terms of the two Hermitian, dimensionless operators $\hat{p} = \hat{P}/\sqrt{m\hbar\omega}$ and $\hat{q} = \hat{X}\sqrt{m\omega/\hbar}$:

$$\underline{\hat{H} = \frac{\hbar\omega}{2}(\hat{p}^2 + \hat{q}^2)}, \tag{4.121}$$

and then introduce two non-Hermitian, dimensionless operators:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}). \tag{4.122}$$

The physical meaning of the operators $\hat{a}$ and $\hat{a}^\dagger$ will be examined later. Note that

$$\hat{a}^\dagger\hat{a} = \frac{1}{2}(\hat{q} - i\hat{p})(\hat{q} + i\hat{p}) = \frac{1}{2}(\hat{q}^2 + \hat{p}^2 + i\hat{q}\hat{p} - i\hat{p}\hat{q}) = \frac{1}{2}(\hat{q}^2 + \hat{p}^2) + \frac{i}{2}[\hat{q},\ \hat{p}], \tag{4.123}$$

where, using $[\hat{X}, \hat{P}] = i\hbar$, we can verify that the commutator between $\hat{q}$ and $\hat{p}$ is

$$[\hat{q},\ \hat{p}] = \left[\sqrt{\frac{m\omega}{\hbar}}\hat{X}, \frac{1}{\sqrt{\hbar m\omega}}\hat{P}\right] = \frac{1}{\hbar}\left[\hat{X}, \hat{P}\right] = i; \tag{4.124}$$

$$P = i\hbar \frac{d}{dx}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X}\sqrt{\frac{m\omega}{\hbar}} + \frac{i\hat{P}}{\sqrt{m\hbar\omega}}\right)$$

hence

$$\hat{a}^\dagger \hat{a} = \frac{1}{2}(\hat{q}^2 + \hat{p}^2) - \frac{1}{2} \tag{4.125}$$

or

$$\frac{1}{2}(\hat{q}^2 + \hat{p}^2) = \hat{a}^\dagger \hat{a} + \frac{1}{2}. \tag{4.126}$$

Inserting (4.126) into (4.121) we obtain

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \qquad \text{with} \quad \hat{N} = \hat{a}^\dagger \hat{a},} \tag{4.127}$$

where $\hat{N}$ is known as the *number* operator or *occupation number* operator, which is clearly Hermitian.

Let us now derive the commutator $[\hat{a}, \hat{a}^\dagger]$. Since $[\hat{X}, \hat{P}] = i\hbar$ we have $[\hat{q}, \hat{p}] = \frac{1}{\hbar}[\hat{X}, \hat{P}] = i$; hence

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2}\left[\hat{q} + i\hat{p}, \hat{q} - i\hat{p}\right] = -i\left[\hat{q}, \hat{p}\right] = 1 \tag{4.128}$$

or

$$\boxed{[\hat{a}, \hat{a}^\dagger] = 1.} \tag{4.129}$$

4.8.1 Energy Eigenvalues

Note that $\hat{H}$ as given by (4.127) commutes with $\hat{N}$, since $\hat{H}$ is linear in $\hat{N}$. Thus, $\hat{H}$ and $\hat{N}$ can have a set of joint eigenstates, to be denoted by $\mid n\rangle$:

$$\hat{N} \mid n\rangle = n \mid n\rangle \tag{4.130}$$

and

$$\hat{H} \mid n\rangle = E_n \mid n\rangle; \tag{4.131}$$

the states $\mid n\rangle$ are called energy eigenstates. Combining (4.127) and (4.131), we obtain the energy eigenvalues at once:

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega.} \tag{4.132}$$

We will show later that n is a *positive integer*; it cannot have negative values.

The physical meaning of the operators $\hat{a}$, $\hat{a}^\dagger$, and $\hat{N}$ can now be clarified. First, we need the following two commutators that can be extracted from (4.129) and (4.127):

$$\boxed{[\hat{a}, \hat{H}] = \hbar\omega\hat{a}, \qquad [\hat{a}^\dagger, \hat{H}] = -\hbar\omega\hat{a}^\dagger.} \tag{4.133}$$

These commutation relations along with (4.131) lead to

$$\hat{H}\left(\hat{a} \mid n\rangle\right) = (\hat{a}\hat{H} - \hbar\omega\hat{a}) \mid n\rangle = (E_n - \hbar\omega)\left(\hat{a} \mid n\rangle\right), \tag{4.134}$$

$$\hat{H}\left(\hat{a}^\dagger \mid n\rangle\right) = (\hat{a}^\dagger\hat{H} + \hbar\omega\hat{a}^\dagger) \mid n\rangle = (E_n + \hbar\omega)\left(\hat{a}^\dagger \mid n\rangle\right). \tag{4.135}$$

Thus, $\hat{a} \mid n\rangle$ and $\hat{a}^\dagger \mid n\rangle$ are eigenstates of $\hat{H}$ with eigenvalues $(E_n - \hbar\omega)$ and $(E_n + \hbar\omega)$, respectively. So the actions of $\hat{a}$ and $\hat{a}^\dagger$ on $\mid n\rangle$ generate new energy states that are lower and higher by one unit of $\hbar\omega$, respectively. As a result, $\hat{a}$ and $\hat{a}^\dagger$ are respectively known as the *lowering* and *raising* operators, or the *annihilation* and *creation* operators; they are also known as the *ladder* operators.

Let us now find out how the operators $\hat{a}$ and $\hat{a}^\dagger$ act on the energy eigenstates $\mid n\rangle$. Since $\hat{a}$ and $\hat{a}^\dagger$ do not commute with $\hat{N}$, the states $\mid n\rangle$ are eigenstates neither to $\hat{a}$ nor to $\hat{a}^\dagger$. Using (4.129) along with $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$, we can show that

$$\boxed{[\hat{N}, \hat{a}] = -\hat{a}, \qquad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger;} \tag{4.136}$$

hence $\hat{N}\hat{a} = \hat{a}(\hat{N} - 1)$ and $\hat{N}\hat{a}^\dagger = \hat{a}^\dagger(\hat{N} + 1)$. Combining these relations with (4.130), we obtain

$$\hat{N}\left(\hat{a} \mid n\rangle\right) = \hat{a}(\hat{N} - 1) \mid n\rangle = (n-1)\left(\hat{a} \mid n\rangle\right), \tag{4.137}$$

$$\hat{N}\left(\hat{a}^\dagger \mid n\rangle\right) = \hat{a}^\dagger(\hat{N} + 1) \mid n\rangle = (n+1)(\hat{a}^\dagger \mid n\rangle). \tag{4.138}$$

These relations reveal that $\hat{a} \mid n\rangle$ and $\hat{a}^\dagger \mid n\rangle$ are eigenstates of $\hat{N}$ with eigenvalues $(n - 1)$ and $(n + 1)$, respectively. This implies that when $\hat{a}$ and $\hat{a}^\dagger$ operate on $\mid n\rangle$, respectively, they decrease and increase n by one unit. That is, while the action of $\hat{a}$ on $\mid n\rangle$ generates a new state $\mid n-1\rangle$ (i.e., $\hat{a} \mid n\rangle \sim\mid n-1\rangle$), the action of $\hat{a}^\dagger$ on $\mid n\rangle$ generates $\mid n+1\rangle$. Hence from (4.137) we can write

$$\hat{a} \mid n\rangle = c_n \mid n-1\rangle, \tag{4.139}$$

where c_n is a constant to be determined from the requirement that the states $\mid n\rangle$ be normalized for all values of n. On the one hand, (4.139) yields

$$\left(\langle n \mid \hat{a}^\dagger\right) \cdot \left(\hat{a} \mid n\rangle\right) = \langle n \mid \hat{a}^\dagger\hat{a} \mid n\rangle = |c_n|^2 \langle n-1 \mid n-1\rangle = |c_n|^2 \tag{4.140}$$

and, on the other hand, (4.130) gives

$$\left(\langle n \mid \hat{a}^\dagger\right) \cdot \left(\hat{a} \mid n\rangle\right) = \langle n \mid \hat{a}^\dagger\hat{a} \mid n\rangle = n\langle n \mid n\rangle = n. \tag{4.141}$$

When combined, the last two equations yield

$$|c_n|^2 = n. \tag{4.142}$$

This implies that n, which is equal to the norm of $\hat{a} \mid n\rangle$ (see (4.141)), *cannot be negative*, $n \geq 0$, since the norm is a positive quantity. Substituting (4.142) into (4.139) we end up with

$$\boxed{\hat{a} \mid n\rangle = \sqrt{n} \mid n-1\rangle.} \tag{4.143}$$

This equation shows that repeated applications of the operator $\hat{a}$ on $\mid n\rangle$ generate a sequence of eigenvectors $\mid n-1\rangle, \mid n-2\rangle, \mid n-3\rangle, \ldots$. Since $n \geq 0$ and since $\hat{a} \mid 0\rangle = 0$, this sequence has to terminate at $n = 0$; this is true if we start with an integer value of n. But if we start with a noninteger n, the sequence will not terminate; hence it leads to eigenvectors with negative values of n. But as shown above, since n cannot be negative, we conclude that n has to be a *nonnegative integer*.

Now, we can easily show, as we did for (4.143), that

$$\boxed{\hat{a}^\dagger \mid n\rangle = \sqrt{n+1} \mid n+1\rangle.} \tag{4.144}$$

This implies that repeated applications of $\hat{a}^\dagger$ on $\mid n\rangle$ generate an infinite sequence of eigenvectors $\mid n+1\rangle, \ \mid n+2\rangle, \ \mid n+3\rangle, \ \ldots$. Since n is a positive integer, the energy spectrum of a harmonic oscillator as specified by (4.132) is therefore *discrete*:

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad (n = 0, 1, 2, 3, \ \ldots).} \tag{4.145}$$

This expression is similar to the one obtained from the first method (see Eq. (4.117)). The energy spectrum of the harmonic oscillator consists of energy levels that are equally spaced: $E_{n+1} - E_n = \hbar\omega$. This is Planck's famous equidistant energy idea—the energy of the radiation emitted by the oscillating charges (from the inside walls of the cavity) must come only in bundles (quanta) that are integral multiples of $\hbar\omega$—which, as mentioned in Chapter 1, led to the birth of quantum mechanics.

As expected for bound states of one-dimensional potentials, the energy spectrum is both discrete and nondegenerate. Once again, as in the case of the infinite square well potential, we encounter the zero-point energy phenomenon: the lowest energy eigenvalue of the oscillator is not zero but is instead equal to $E_0 = \hbar\omega/2$. It is called the *zero-point energy* of the oscillator, for it corresponds to $n = 0$. The zero-point energy of bound state systems cannot be zero, otherwise it would violate the uncertainty principle. For the harmonic oscillator, for instance, the classical minimum energy corresponds to $x = 0$ and $p = 0$; there would be no oscillations in this case. This would imply that we know simultaneously and with absolute precision both the position and the momentum of the system. This would contradict the uncertainty principle.

4.8.2 Energy Eigenstates

The algebraic or operator method can also be used to determine the energy eigenvectors. First, using (4.144), we see that the various eigenvectors can be written in terms of the ground state $\mid 0\rangle$ as follows:

$$\mid 1\rangle = \hat{a}^\dagger \mid 0\rangle, \tag{4.146}$$

$$\mid 2\rangle = \frac{1}{\sqrt{2}}\hat{a}^\dagger \mid 1\rangle = \frac{1}{\sqrt{2!}}\left(\hat{a}^\dagger\right)^2 \mid 0\rangle, \tag{4.147}$$

$$\mid 3\rangle = \frac{1}{\sqrt{3}}\hat{a}^\dagger \mid 2\rangle = \frac{1}{\sqrt{3!}}\left(\hat{a}^\dagger\right)^3 \mid 0\rangle, \tag{4.148}$$

$$\vdots$$

$$\mid n\rangle = \frac{1}{\sqrt{n}}\hat{a}^\dagger \mid n-1\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n \mid 0\rangle. \tag{4.149}$$

So, to find any excited eigenstate $\mid n\rangle$, we need simply to operate $\hat{a}^\dagger$ on $\mid 0\rangle$ n successive times.

Note that any set of kets $\mid n\rangle$ and $|n'\rangle$, corresponding to different eigenvalues, must be orthogonal, $\langle n' \mid n\rangle \sim \delta_{n',n}$, since $\hat{H}$ is Hermitian and none of its eigenstates is degenerate.

Moreover, the states $|\,0\rangle, |\,1\rangle, |\,2\rangle, |\,3\rangle, \ldots, |\,n\rangle, \ldots$ are *simultaneous eigenstates* of $\hat{H}$ and $\hat{N}$; the set $\{|\,n\rangle\}$ constitutes an orthonormal and complete basis:

$$\langle n' \mid n\rangle = \delta_{n',n}, \qquad \sum_{n=0}^{+\infty} |\,n\rangle\langle n\,| = 1. \tag{4.150}$$

$$\frac{d}{dx} = \frac{i\hat{P}}{x_0\sqrt{m\hbar\omega}}$$

4.8.3 Energy Eigenstates in Position Space

Let us now determine the harmonic oscillator wave function in the position representation.

Equations (4.146) to (4.149) show that, knowing the ground state wave function, we can determine any other eigenstate by successive applications of the operator $a^\dagger$ on the ground state. So let us first determine the ground state wave function in the position representation.

The operator $\hat{p}$, defined by $\hat{p} = \hat{P}/\sqrt{m\hbar\omega}$, is given in the position space by

$$\hat{p} = -\frac{i\hbar}{\sqrt{m\hbar\omega}}\frac{d}{dx} = -ix_0\frac{d}{dx}, \tag{4.151}$$

$$-ix_0\frac{d}{dx} = p = \frac{\hat{P}}{\sqrt{m\hbar\omega}}$$

where, as mentioned above, $x_0 = \sqrt{\hbar/(m\omega)}$ is a constant that has the dimensions of length; it sets the length scale of the oscillator. We can easily show that the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, defined in (4.122), can be written in the position representation as

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{x_0} + x_0\frac{d}{dx}\right) = \frac{1}{\sqrt{2}x_0}\left(\hat{X} + x_0^2\frac{d}{dx}\right), \tag{4.152}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{x_0} - x_0\frac{d}{dx}\right) = \frac{1}{\sqrt{2}x_0}\left(\hat{X} - x_0^2\frac{d}{dx}\right). \tag{4.153}$$

Using (4.152) we can write the equation $\hat{a}\mid 0\rangle = 0$ in the position space as

$$\langle x|\hat{a}\mid 0\rangle = \frac{1}{\sqrt{2}x_0}\langle x|\hat{X} + x_0^2\frac{d}{dx}\mid 0\rangle = \frac{1}{\sqrt{2}x_0}\left(x\,\psi_0(x) + x_0^2\frac{d\psi_0(x)}{dx}\right) = 0; \tag{4.154}$$

hence

$$\frac{d\psi_0(x)}{dx} = -\frac{x}{x_0^2}\psi_0(x), \tag{4.155}$$

where $\psi_0(x) = \langle x \mid 0\rangle$ represents the ground state wave function. The solution of this differential equation is

$$\psi_0(x) = A\exp\left(-\frac{x^2}{2x_0^2}\right), \tag{4.156}$$

where A is a constant that can be determined from the normalization condition

$$1 = \int_{-\infty}^{+\infty} dx\,|\psi_0(x)|^2 = A^2\int_{-\infty}^{+\infty} dx\,\exp\left(-\frac{x^2}{x_0^2}\right) = A^2\sqrt{\pi}x_0; \tag{4.157}$$

hence $A = (m\omega/(\pi\hbar))^{1/4} = 1/\sqrt{\sqrt{\pi}x_0}$. The normalized ground state wave function is then given by

$$\boxed{\psi_0(x) = \frac{1}{\sqrt{\sqrt{\pi}x_0}} \exp\left(-\frac{x^2}{2x_0^2}\right).} \tag{4.158}$$

This is a Gaussian function.

We can then obtain the wave function of any excited state by a series of applications of $\hat{a}^\dagger$ on the ground state. For instance, the first excited state is obtained by one single application of the operator $\hat{a}^\dagger$ of (4.153) on the ground state:

$$\begin{aligned} \langle x \mid 1\rangle &= \langle x|\hat{a}^\dagger \mid 0\rangle = \frac{1}{\sqrt{2}x_0}\left(x - x_0^2\frac{d}{dx}\right)\langle x \mid 0\rangle \\ &= \frac{1}{\sqrt{2}x_0}\left(x - x_0^2\left(-\frac{x}{x_0^2}\right)\right)\psi_0(x) = \frac{\sqrt{2}}{x_0}x\,\psi_0(x) \end{aligned} \tag{4.159}$$

or

$$\psi_1(x) = \frac{\sqrt{2}}{x_0}x\,\psi_0(x) = \sqrt{\frac{2}{\sqrt{\pi}x_0^3}}\,x\exp\left(-\frac{x^2}{2x_0^2}\right). \tag{4.160}$$

As for the eigenstates of the second and third excited states, we can obtain them by applying $\hat{a}^\dagger$ on the ground state twice and three times, respectively:

$$\langle x \mid 2\rangle = \frac{1}{\sqrt{2!}}\langle x|\left(a^\dagger\right)^2 \mid 0\rangle = \frac{1}{\sqrt{2!}}\left(\frac{1}{\sqrt{2}x_0}\right)^2\left(x - x_0^2\frac{d}{dx}\right)^2\psi_0(x), \tag{4.161}$$

$$\langle x \mid 3\rangle = \frac{1}{\sqrt{3!}}\langle x|\left(a^\dagger\right)^3 \mid 0\rangle = \frac{1}{\sqrt{3!}}\left(\frac{1}{\sqrt{2}x_0}\right)^3\left(x - x_0^2\frac{d}{dx}\right)^3\psi_0(x) \tag{4.162}$$

or

$$\psi_2(x) = \frac{1}{\sqrt{2\sqrt{\pi}x_0}}\left(\frac{2x^2}{x_0^2} - 1\right)\exp\left(-\frac{x^2}{2x_0^2}\right), \quad \psi_3(x) = \frac{1}{\sqrt{3\sqrt{\pi}x_0}}\left(\frac{2x^3}{x_0^3} - \frac{3x}{x_0}\right)\exp\left(-\frac{x^2}{2x_0^2}\right). \tag{4.163}$$

Similarly, using (4.149), (4.153), and (4.158), we can easily infer the energy eigenstate for the nth excited state:

$$\langle x \mid n\rangle = \frac{1}{\sqrt{n!}}\langle x|\left(a^\dagger\right)^n \mid 0\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}x_0}\right)^n\left(x - x_0^2\frac{d}{dx}\right)^n\psi_0(x), \tag{4.164}$$

which in turn can be rewritten as

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}\frac{1}{x_0^{n+1/2}}\left(x - x_0^2\frac{d}{dx}\right)^n\exp\left(-\frac{x^2}{2x_0^2}\right).} \tag{4.165}$$

In summary, by successive applications of $\hat{a}^\dagger = (\hat{X} - x_0^2 d/dx)/(\sqrt{2}x_0)$ on $\psi_0(x)$, we can find the wave function of any excited state $\psi_n(x)$.

Oscillator wave functions and the Hermite polynomials

At this level, we can show that the wave function (4.165) derived from the algebraic method is similar to the one obtained from the first method (4.118). To see this, we simply need to use the following operator identity:

$$e^{-x^2/2}\left(x-\frac{d}{dx}\right)e^{x^2/2}=-\frac{d}{dx} \quad \text{or} \quad e^{-x^2/2x_0^2}\left(x-x_0^2\frac{d}{dx}\right)e^{x^2/2x_0^2}=-x_0^2\frac{d}{dx}. \tag{4.166}$$

An application of this operator n times leads at once to

$$e^{-x^2/2x_0^2}\left(x-x_0^2\frac{d}{dx}\right)^n e^{x^2/2x_0^2}=(-1)^n(x_0^2)^n\frac{d^n}{dx^n}, \tag{4.167}$$

which can be shown to yield

$$\left(x-x_0^2\frac{d}{dx}\right)^n e^{-x^2/2x_0^2}=(-1)^n(x_0^2)^n e^{x^2/2x_0^2}\frac{d^n}{dx^n}e^{-x^2/x_0^2}. \tag{4.168}$$

We can now rewrite the right-hand side of this equation as follows:

$$\begin{aligned}(-1)^n(x_0^2)^n e^{x^2/2x_0^2}\frac{d^n}{dx^n}e^{-x^2/x_0^2} &= x_0^n e^{-x^2/2x_0^2}\left[(-1)^n e^{x^2/x_0^2}\frac{d^n}{d(x/x_0)^n}e^{-x^2/x_0^2}\right] \\ &= x_0^n e^{-x^2/2x_0^2}\left[(-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}\right] \\ &= x_0^n e^{-x^2/2x_0^2}H_n(y),\end{aligned} \tag{4.169}$$

where $y=x/x_0$ and where $H_n(y)$ are the Hermite polynomials listed in (4.119):

$$H_n(y)=(-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}. \tag{4.170}$$

Note that the polynomials $H_{2n}(y)$ are even and $H_{2n+1}(y)$ are odd, since $H_n(-y)=(-1)^n H_n(y)$.

Inserting (4.169) into (4.168), we obtain

$$\left(x-x_0^2\frac{d}{dx}\right)^n e^{-x^2/2x_0^2}=x_0^n e^{-x^2/2x_0^2}H_n\left(\frac{x}{x_0}\right); \tag{4.171}$$

substituting this equation into (4.165), we can write the oscillator wave function in terms of the Hermite polynomials as follows:

$$\boxed{\psi_n(x)=\frac{1}{\sqrt{\sqrt{\pi}2^n n! x_0}}e^{-x^2/2x_0^2}H_n\left(\frac{x}{x_0}\right).} \tag{4.172}$$

This wave function is identical with the one obtained from the first method (see Eq. (4.118)).

Remark

This wave function is either even or odd depending on n; in fact, the functions $\psi_{2n}(x)$ are even (i.e., $\psi_{2n}(-x)=\psi_{2n}(x)$) and $\psi_{2n+1}(x)$ are odd (i.e., $\psi_{2n}(-x)=-\psi_{2n}(x)$) since, as can be inferred from Eq (4.120), the Hermite polynomials $H_{2n}(x)$ are even and $H_{2n+1}(x)$ are odd. This is expected because, as mentioned in Section 4.2.4, the wave functions of even one-dimensional potentials have definite parity. Figure 4.9 displays the shapes of the first few wave functions.