Dynamics in a Stratified Lid-driven Cavity

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1 Problem Description

Consider the flow in a rectangular cavity of depth d and width l, filled with a fluid of kinematic viscosity ν, thermal diffusivity κ, and coefficient of volume expansion β. The top boundary of the cavity is driven horizontally at a constant speed U and is maintained at a fixed temperature T_{top}. The bottom boundary is maintained at a cooler fixed temperature T_{bot}, whilst the two sidewalls are insulated. Gravity g acts in the downward vertical direction. The temperature deviation from the mean temperature, $(T_{\text{top}} + T_{\text{bot}})/2$, is non-dimensionalized with $\Delta T = T_{\text{top}} - T_{\text{bot}}$. Figure 1 shows a schematic of the system. Using d as the length scale and the viscous time d^2/ν as the time scale, the non-dimensional governing equations, employing the Boussinesq approximation, are

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nabla^2 \boldsymbol{u} + Gr\, T \boldsymbol{e}_y\,, \quad \nabla \cdot \boldsymbol{u} = 0\,,$$
$$T_t + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pr}\nabla^2 T\,. \tag{1}$$

The boundary conditions are:

$$\begin{aligned}
\text{top, } (x,y) = (x, 0.5): \quad & (u, v, T) = (Re, 0, 0.5),\\
\text{bottom, } (x,y) = (x, -0.5): \quad & (u, v, T) = (0, 0, -0.5),\\
\text{sides, } (x,y) = (\pm 0.5\gamma, y): \quad & (u, v, T_x) = (0, 0, 0),
\end{aligned} \tag{2}$$

where $\gamma = l/d$ is the aspect ratio, $Gr = RiRe^2$, Pr is the Prandtl number, Gr is the Grashof number and Ri is the Richardson number, which controls the intensity of the buoyancy force. The global energy is used to quantify the flow dynamics inside the cavity, which is defined as

$$E = \frac{1}{Re^2}\int_{y=-0.5}^{y=0.5}\int_{x=-0.5}^{x=0.5}(u^2 + v^2)dxdy\,. \tag{3}$$

2 Approach

The governing equations (1) are discretized using a spectral-collocation method in both spatial directions, which is proposed by [1]. Both velocity components and pressure are approximated by Chebyshev polynomials of the first kind with degree less than M, associated
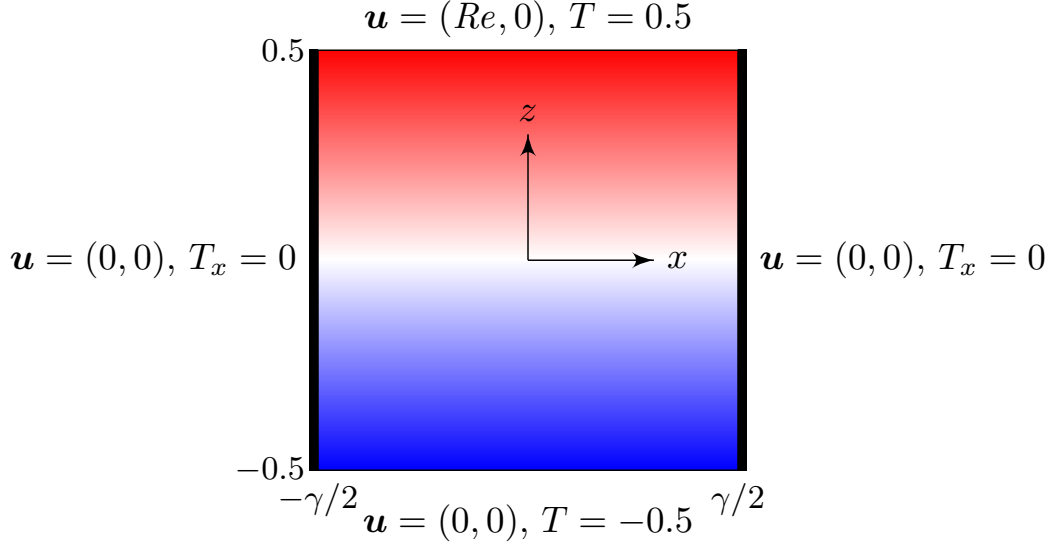
Figure 1: Schematic of the system, indicating the non-dimensional coordinate system and boundary conditions. The top boundary moves to the right (positive x direction) at constant speed Re. The colour map shows the linearly stratified temperature distribution (red is $T = 0.5$, blue is $T = -0.5$, and white is $T = 0$) when $Re = 0$.

with the Chebyshev–Gauss–Lobatto grid. Spatial differentiation is performed via direct matrix-vector multiplication by the pseudo-spectral differentiation matrix. The boundary condition for the x-velocity at the moving top wall, $u(x, 0.5) = Re$, is discontinuous where it meets the two vertical sidewalls, which can lead to Gibbs phenomena when using a global spectral method. In order to resolve this numerical issue, we regularise the top wall boundary condition by using:

$$u(x, 0.5) = Re \left[1 - \exp\left(-(1 - 4x^2)/\delta\right)\right]. \tag{4}$$

3   Objective

In this project, we are going to fix the $Re = 2750$ and vary Ri numbers. For small Ri values, the flow will be steady state with respect to time, but as we increase Ri number, the flow inside the square cavity is expected to lose its stability and become periodic or even quasi-periodic. If we further increase Ri number, the flow is expected to become steady flow again this is due to the buoyancy force is the dominant force which controls the flow dynamics. As we vary the parameter Ri, the flow dynamics will change dramatically, there are different types of bifurcations might occur, therefore, the numerical simulations not only can help us identify the different states of flow, but also can give us enough information that we can classify the bifurcations of the flow as we increase the parameter Ri number. The specific tasks are following:

- The code will be tested without tempture stratification, there are many known solution for $2D$ lid driven cavity problem.

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- with $\boldsymbol{u} = 0$ initial condition, simulate the flow for $Ri \in [0.1, 2]$ with increment of $\Delta Ri = 0.1$. For single case, record the energy time series for each time step.

- Use the solutions obtained in the previous step as initial conditions, do a refinement. Use $\Delta Ri = 0.01$, in total there will be 190 cases.

- Compute the FFT for each time series, and plot the power spectal density diagram.

- Use the power spectal density plot to find the critical Ri where bifurcations occur, and make the bifurcation diagram, which will be quantified by global energy.

- For different flow states, make a $2D$ plot for a steady state solution, a movie for a limit cycle solution.

4 Appendix

A pseudospectal collocation-Chebyshev method is implemented, each variable is expanded in the approximation space P_{MN}, composed of Chebyshev polynomials, T_m and T_n of degrees less or equal than M and N in x, y directions.

$$T(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{T}_{mn} T_m(2x) T_n(2y) \tag{5}$$

$$u(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{M} \hat{u}_{mn} T_m(2x) T_n(2y) \tag{6}$$

$$v(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} \hat{v}_{mn} T_m(2x) T_n(2y) \tag{7}$$

$$p(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{M} \hat{p}_{mn} T_m(2x) T_n(2y) \tag{8}$$

All the partial derivatives are evaluated directly by the Chebyshev differentiation matrix. Since the computational domain is $[-1/2, 1/2]^2$, when we take partial derivatives by use Chebyshev differentiation matrix D, we need to use chain rule, for instance

$$\frac{\partial u}{\partial x} = 2Du \tag{9}$$

$$\frac{\partial^2 u}{\partial x^2} = 4D^2 u \tag{10}$$

Since we use the Chebyshev collocation method, so all the values are directly evaluated at the Chebyshev-Gauss-Lobatto collocation points. Also note that since the aspect ratio we choose is 1, so there is a factor of 2 when we take the first partial derivative, and there is a factor of 4 when we take the second partial derivative.

Temporal scheme:

$$\frac{3\vec{u}^{n+1} - 4\vec{u}^n + \vec{u}^{n-1}}{2\delta t} = -\nabla p^{n+1} - 2NL(\vec{u}^n) + NL(\vec{u}^{n-1}) - \triangle\vec{u}^{n+1} + GrT^{n+1}\vec{e}_y \quad (11)$$

$$\frac{3T^{n+1} - 4T^n + T^{n-1}}{2\delta t} = -2NL(\vec{u}^n, T^n) + NL(\vec{u}^{n-1}, T^{n-1}) + \frac{1}{Pr}\triangle T^{n+1} \quad (12)$$

The time integration used is second order accurate and is based on a combination of Adams-Bashforth and backward differentiation formula (AB2/BDF) schemes.
In order to improve the stability, the viscous term need to be written as

$$L(\vec{u}) = \Delta\vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\nabla \times (\nabla \times \vec{u}) \quad (13)$$

which is due to divergence free.

Code Implementation

Every quantity is evaluated directly on the Chebyshev-Gauss-Lobatto grid points, Let DX,DY denote the Chebyshev differentiation matrix with dimension $(M+1)\times(N+1)$, and use i, j to denote the indices start from 0 to M and N respectively. In our code, define $U(0 : M, 0 : N)$, $V(0 : M, 0 : N)$, $T(0 : M, 0 : N)$, $P(0 : M, 0 : N)$ as the velocity, temperature and pressure variables, $NLu(0 : M, 0 : N)$, $NLv(0 : M, 0 : N)$ as the nonlinear terms in momentum equation, and Define $NLT(0 : M, 0 : N)$ as the nonlinear terms in temperature equation, $Lu(0 : M, 0 : N)$, $Lv(0 : M, 0 : N)$ as the viscous terms in the momentum equation. So these nonlinear terms can be computed as following: let $asp = 2$

$$NLu = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = asp * U * DXU + 2V * UDY^T \quad (14)$$

$$NLv = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = asp * U * DXV + 2V * VDY^T \quad (15)$$

$$NLT = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = asp * U * DXT + 2V * TDY^T \quad (16)$$

$$Lu = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y\partial x} = 4\{U(DY^2)^T\} - 2 * asp * \{(DXV)DY^T\} \quad (17)$$

$$Lv = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x\partial y} = asp^2 * \{(DX^2V)\} - asp * 2 * \{DX(UDY^T)\} \quad (18)$$

The Improved Projection Scheme:

step 1: compute the temperature T^{n+1} from temperature equation

$$(\Delta - \frac{3Pr}{2\delta t})T^{n+1} = Pr\{2NL(\vec{u}^n, T^n) - NL(\vec{u}^{n-1}, T^{n-1}) - \frac{4T^n - T^{n-1}}{2\delta t}\} \quad (19)$$

with boundary conditions:
$T(x, 1/2) = 0.5$, $T(x, -1/2) = -0.5$, $\frac{\partial T}{\partial n}(-1/2, y) = 0$, $\frac{\partial T}{\partial n}(1/2, y) = 0$.

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Formulation of the linear systems:
The indices i, j are in the sets $\{0, 1, 2, ...M\}$ and $\{0, 1, 2, ..., N\}$ respectively.

$$\frac{\partial^2 T^{n+1}}{\partial x^2} + \frac{\partial^2 T^{n+1}}{\partial y^2} - \frac{3Pr}{2\delta t}T^{n+1} = RHS \tag{20}$$

$$RHS = Pr\{2NL(\vec{u}^n, T^n) - NL(\vec{u}^{n-1}, T^{n-1}) - \frac{4T^n - T^{n-1}}{2\delta t}\} \tag{21}$$

$$asp^2 \sum_{k=0}^{M} DX_{i,k}^2 T_{k,j}^{n+1} + 4\sum_{k=0}^{N} T_{i,k}^{n+1}(DY^2)^T - \frac{3Pr}{2\delta t}T_{i,j}^{n+1} = RHS_{i,j}$$

$$asp^2 \sum_{k=1}^{M-1} DX_{i,k}^2 T_{k,j}^{n+1} + asp^2 * DX_{i,0}^2 T_{0,j}^{n+1} + asp^2 * DX_{i,M}^2 T_{M,j}^{n+1} +$$

$$4\sum_{k=1}^{N+1} T_{i,k}^{n+1}(DY^2)_{k,j}^T + 4T_{i,0}^{n+1}(DY^2)_{0,j}^T + 4T_{i,N}^{n+1}(DY^2)_{N,j}^T - \frac{3Pr}{2\delta t}T_{i,j}^{n+1} = RHS_{i,j} \tag{22}$$

$$T_{0,j}^{n+1} = -\sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}}T_{k,j}^{n+1} \tag{23}$$

$$T_{M,j}^{n+1} = -\sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}}T_{k,j}^{n+1} \tag{24}$$

Note that the left and right temperature values at boundary points are expressed as a linear combination of interior points. Plug $T_{0,j}, T_{M,j}$ into the linear equation, we obtain the following linear system:

$$asp^2 \sum_{k=1}^{M-1} DX_{i,k}^2 T_{k,j}^{n+1} - asp^2 * DX_{i,0}^2 \sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}}T_{k,j}^{n+1}$$

$$-asp^2 * DX_{i,M}^2 \sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}}T_{k,j}^{n+1} + 4\sum_{k=1}^{N-1} T_{i,k}^{n+1}(DY^2)_{k,j}^T - \frac{3Pr}{2\delta t}T_{i,j}^{n+1}$$

$$= RHS_{i,j} - 4T_{i,0}^{n+1}(DY^2)_{0,j}^T - 4T_{i,N}^{n+1}(DY^2)_{N,j}^T \tag{25}$$

where the indices i, j start from $\{1, 2, 3, ...M - 1\}$ and $\{1, 2, 3, ..., N - 1\}$.

step 2: compute the preliminary presssure P^*

$$\triangle p^* = \nabla \cdot \{-2NL(\vec{u}^n) + NL(\vec{u}^{n-1}) + GrT^{n+1}\vec{e}_y\} \tag{26}$$

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with boundary conditions:

$$\frac{\partial p^*}{\partial n} = \vec{n} \cdot \{\frac{-3\vec{w} + 4\vec{u}^n - \vec{u}^{n-1}}{2\delta t} - 2NL(\vec{u}^n) + NL(\vec{u}^{n-1}) + GrT^{n+1}\vec{e}_y + 2L(\vec{u}^n) - L(\vec{u}^{n-1})\}$$
(27)

where $\vec{w}$ is the boundary conditions of the velocity field.
Formation of the linear system:

$$asp^2 \sum_{k=0}^{M} DX_{i,k}^2 P_{k,j}^* + 4 \sum_{k=0}^{N} P_{i,k}^* (DY^2)_{k,j}^T = RHS_{i,j}$$
(28)

$$RHS = asp * DX\{-2NL(u^n + NL(u^{n-1})\} + 2\{-2NL(v^n) + NL(v^{n-1}) + GrT^{n+1}\}DY^T$$
(29)

$$asp^2 \sum_{k=1}^{M-1} DX_{i,k}^2 P_{k,j}^* + asp^2 * DX_{i,0}^2 P_{0,j}^* + asp^2 * DX_{i,M}^2 P_{M,j}^*$$
(30)

$$+4 \sum_{k=1}^{N-1} P_{i,k}^* (DY^2)_{k,j}^T + 4P_{i,0}^* (DY^2)_{0,j}^T + 4P_{i,N}^* (DY^2)_{N,j}^T = RHS_{i,j}$$
(31)

(32)

$$P_{0,j}^* = \frac{\frac{1}{asp}rhs1_{0,j}DX_{MM} - \frac{1}{asp}rhs1_{M,j}DX_{0M}}{DX_{00}DX_{MM} - DX_{0M}DX_{M0}} - \sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}} P_{k,j}^*$$
(33)

$$P_{M,j}^* = \frac{\frac{1}{asp}rhs1_{0,j}DX_{M0} - \frac{1}{asp}rhs1_{M,j}DX_{00}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} - \sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} P_{k,j}^*$$
(34)

$$P_{i,0}^* = \frac{\frac{1}{2}rhs2_{i,0}DY_{NN}^T - \frac{1}{2}rhs2_{i,N}DY_{N0}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T} - \sum_{k=1}^{N-1} P_{i,k}^* \frac{DY_{NN}^T DY_{k0}^T - DY_{N0}^T DY_{kN}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T}$$
(35)

$$P_{i,N}^* = \frac{\frac{1}{2}rhs2_{i,0}DY_{0N}^T - \frac{1}{2}rhs2_{i,N}DY_{00}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T} - \sum_{k=1}^{N-1} P_{i,k}^* \frac{DY_{0N}^T DY_{k0}^T - DY_{00}^T DY_{kN}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T}$$
(36)

Plug the boundary P^* values into the poisson equation, the linear system can be written as following:

$$asp^2 \sum_{k=1}^{M-1} DX_{i,k}^2 P_{k,j}^* - asp^2 DX_{i,0}^2 \sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}} P_{k,j}^* \tag{37}$$

$$-asp^2 DX_{i,M}^2 \sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} P_{k,j}^* \tag{38}$$

$$+4 \sum_{k=1}^{N-1} P_{i,k}^* (DY^2)_{k,j}^T - 4 \sum_{k=1}^{N-1} P_{i,k}^* \frac{DY_{NN}^T DY_{k0}^T - DY_{N0}^T DY_{kN}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T} (DY^2)_{0,j}^T \tag{39}$$

$$-4 \sum_{k=1}^{N-1} P_{i,k}^* \frac{DY_{0N}^T DY_{k0}^T - DY_{00}^T DY_{kN}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T} (DY^2)_{N,j}^T \tag{40}$$

$$= RHS_{i,j} - asp * DX_{i,0}^2 \frac{rhs1_{0,j}DX_{MM} - rhs1_{M,j}DX_{0M}}{DX_{00}DX_{MM} - DX_{0M}DX_{M0}} \tag{41}$$

$$-asp * DX_{i,M}^2 \frac{rhs1_{0,j}DX_{M0} - rhs1_{M,j}DX_{00}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} \tag{42}$$

$$-2 * (DY^2)_{0,j}^T \frac{rhs2_{i,0}DY_{NN}^T - rhs2_{i,N}DY_{N0}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T} \tag{43}$$

$$-2 * (DY^2)_{N,j}^T \frac{rhs2_{i,0}DY_{0N}^T - rhs2_{i,N}DY_{00}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T} \tag{44}$$

step 3: compute the predictor velocity field $\vec{u}^*$ from the momentum equation

$$\frac{3\vec{u}^* - 4\vec{u}^n + \vec{u}^{n-1}}{2\delta t} + 2NL(\vec{u}^n) - NL(\vec{u}^{n-1}) = -\nabla p^* + \Delta \vec{u}^* + GrT^{n+1}\hat{e}_y \tag{45}$$

Re-write the above equation and group all the $\vec{u}^*$ terms.

$$(\Delta - \frac{3}{2\delta t})\vec{u}^* = \nabla p^* + 2NL(\vec{u}^n) - NL(\vec{u}^{n-1}) - GrT^{n+1}\vec{e}_y - \frac{4\vec{u}^n - \vec{u}^{n-1}}{2\delta t} \tag{46}$$

with prescribed Dirichlet boundary conditions. Write the above equation in component form $\vec{u} = (u, v)$.

$$asp^2 \sum_{k=0}^{M} DX_{i,k}^2 u_{k,j}^* + 4 \sum_{k=0}^{N} u_{i,k}^* (DY^2)_{k,j}^T - \frac{3}{2\delta t} u_{i,j}^* = RHS1_{i,j} \tag{47}$$

$$asp^2 \sum_{k=0}^{M} DX_{i,k}^2 v_{k,j}^* + 4 \sum_{k=0}^{N} v_{i,k}^* (DY^2)_{k,j}^T - \frac{3}{2\delta t} v_{i,j}^* = RHS2_{i,j} \tag{48}$$

$$RHS1 = \{\frac{\partial p^*}{\partial x} + 2NL(u^n) - NL(u^{n-1}) - \frac{4u^n - u^{n-1}}{2\delta t} - F_x^{n+1}\} \qquad (49)$$

$$= \{asp * DXP^* + 2NLU - NLUold - \frac{4U - Uold}{2\delta t}\}_{i,j} \qquad (50)$$

$$RHS2 = \{\frac{\partial p^*}{\partial y} + 2NL(v^n) - NL(v^{n-1}) - \frac{4v^n - v^{n-1}}{2\delta t} - F_y^{n+1} - GrT^{n+1}\} \qquad (51)$$

$$= \{2P^*(DY)^T + 2NLV - NLVold - \frac{4V - Vold}{2\delta t} - GrT^{n+1}\} \qquad (52)$$

In this step we impose the exact boundary values for the velocity components, and then reduce the system as $M - 1$ by $N - 1$ linear system.

$$\sum_{k=1}^{M-1} DX_{i,k}^2 u_{k,j}^* + \sum_{k=1}^{N-1} u_{i,k}^* (DY^2)_{k,j}^T - \frac{3}{2\delta t} u_{i,j}^*$$
$$= RHS1_{i,j} - asp^2 DX_{i,0}^2 W_{0,j} - asp^2 DX_{i,M}^2 W_{M,j} - 4W_{i,0}(DY^2)_{0,j}^T - 4W_{i,N}(DY^2)_{N,j}^T$$
$$(53)$$

$$\sum_{k=1}^{M-1} DX_{i,k}^2 v_{k,j}^* + \sum_{k=1}^{N-1} v_{i,k}^* (DY^2)_{k,j}^T - \frac{3}{2\delta t} v_{i,j}^*$$
$$= RHS2_{i,j} - asp^2 DX_{i,0}^2 W_{0,j} - asp^2 DX_{i,M}^2 W_{M,j} - 4W_{i,0}(DY^2)_{0,j}^T - 4W_{i,N}(DY^2)_{N,j}^T$$

$$(54)$$

step 4: correct the preliminary pressure and preliminary velocity through evaluating an intermediate variable ϕ:

$$\frac{3\vec{u}^{n+1} - 3\vec{u}^*}{2\delta t} = -\nabla(p^{n+1} - p^*) \qquad (55)$$

Apply the continuity equation and define the intermediate variable ϕ as $\phi = \frac{2\delta t}{3}(p^{n+1} - p^*)$, we obtain following equation:

$$\Delta\phi = \nabla \cdot \vec{u}^* \qquad (56)$$

with pure Neumann boundary condition:

$$\frac{\partial \phi}{\partial n} = 0 \qquad (57)$$

$$RHS = \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \qquad (58)$$

$$RHS = (2DXU^* + 2V^*(DY)^T) \qquad (59)$$

$$4 \sum_{k=0}^{M} DX_{i,k}^2 \phi_{k,j} + 4 \sum_{k=0}^{N} \phi_{i,k}(DY^2)_{k,j}^T = RHS_{i,j} \tag{60}$$

Express the boundary values of ϕ in terms interior points values by the Neumann boundary condition.

$$\phi_{0,j} = -\sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}} \phi_{k,j} \tag{61}$$

$$\phi_{M,j} = -\sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} \phi_{k,j} \tag{62}$$

$$\phi_{i,0} = -\sum_{k=1}^{N-1} \phi_{i,k} \frac{DY_{NN}^T DY_{k0}^T - DY_{N0}^T DY_{kN}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T} \tag{63}$$

$$\phi_{i,N} = -\sum_{k=1}^{N-1} \phi_{i,k} \frac{DY_{0N}^T DY_{k0}^T - DY_{00}^T DY_{kN}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T} \tag{64}$$

Plug in the boundary condition into the above equation, we can form the linear system for the ϕ equation.

$$\sum_{k=1}^{M-1} DX_{i,k}^2 \phi_{k,j} + DX_{i,0}^2 \sum_{k=1}^{M-1} \frac{DX_{MM}DX_{0k} - DX_{0M}DX_{Mk}}{DX_{00}DX_{MM} - DX_{M0}DX_{0M}} \phi_{k,j} \tag{65}$$

$$+ DX_{i,M}^2 \sum_{k=1}^{M-1} \frac{DX_{M0}DX_{0k} - DX_{00}DX_{Mk}}{DX_{M0}DX_{0M} - DX_{00}DX_{MM}} \phi_{k,j} \tag{66}$$

$$+ \sum_{k=1}^{N-1} \phi_{i,k}(DY^2)_{k,j}^T - \sum_{k=1}^{N-1} \frac{DY_{NN}^T DY_{k0}^T - DY_{N0}^T DY_{kN}^T}{DY_{00}^T DY_{NN}^T - DY_{0N}^T DY_{N0}^T}(DY^2)_{0,j}^T \tag{67}$$

$$- \sum_{k=1}^{N-1} \phi_{i,k} \frac{DY_{0N}^T DY_{k0}^T - DY_{00}^T DY_{kN}^T}{DY_{0N}^T DY_{N0}^T - DY_{00}^T DY_{NN}^T}(DY_{N,j}^2) \tag{68}$$

$$= \frac{1}{4}RHS_{i,j} \tag{69}$$

step 5: finally correct the pressure field on the entire domain and update velocity on the entire domain.

$$p^{n+1} = p^* + \frac{3}{2\delta t}\phi \tag{70}$$

$$\vec{u}^{n+1} = \vec{u}^* - \nabla\phi \tag{71}$$

References

[1] S. Hugues and A. Randriamampianina. An improved projection scheme applied to pseudospectral methods for the incompressible Navier–Stokes equations. *Intnl J. Num. Meth. Fluids*, 28:501–521, 1998.