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The Stern–Gerlach Experiment 10

The discussion in Chap. 9 shows that the theory of measurement in quantum mechanics is a complex subject. Although widely studied, there is, however, no agreement as to what is the crux of a quantum measurement. Given the central importance of measurements in quantum mechanics, this chapter studies the *Stern–Gerlach experiment*, which is one of the few experiments that can be examined in great detail and can help to further our understanding of the subtleties of quantum measurements.

Spin is a quantum degree of freedom with only two possible outcomes for its z-component and is one of the simplest quantum system; the Stern–Gerlach experiment measures σ_z, the z-component of the spin of an electron.

The mathematics of this experiment is comparatively simple, leaving us to focus on the physics of measurement. The Stern–Gerlach experiment is modeled by a simplified Hamiltonian proposed by Gottfried and Yan [15]; the advantage of this Hamiltonian is that all the results can be obtained exactly. The measurement process is studied by using the Schrödinger equation to evolve the state vector of the electron through all the stages of the experiment.

This chapter is more technical than others because a qualitative discussion cannot address the controversies that surround the problem of measurement; instead, there is a need to quantitatively study the problem so that one makes generalizations that have a precise mathematical basis.

10.1 The Experiment

An electron consists of its position and spin degrees of freedom, with the term "electron" referring to its degrees of freedom. The Stern–Gerlach experiment measures the z-component of the electron's spin, which are eigenvalues of the eigenstates of σ_z, given in Sect. 8.1. For ease of reference, the eigenvalues and eigenstates of σ_z are given below:

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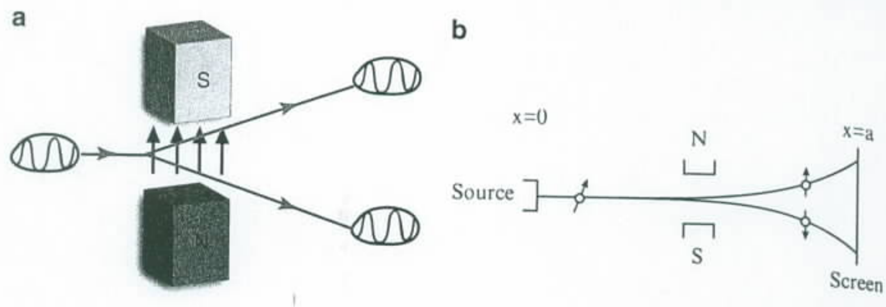

Fig. 10.1 The Stern–Gerlach experiment. The *arrows* pointing up represent an inhomogeneous magnet field. (**a**) The incoming electron is represented by a wave packet. (**b**) The incoming electron's spin is a superposed state; after crossing the magnetic field, the up and down spin eigenstates are well separated (published with permission of © Belal E. Baaquie 2012. All Rights Reserved)

$$|\text{up}\rangle = |+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad |\text{down}\rangle = |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\Rightarrow\; \sigma_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

where

$$\sigma_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The trans-empirical superposed state vector for spin $1/2$, from (8.3), is given by

$$|\Psi\rangle = \alpha|+\rangle + \beta|-\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \quad \langle\Psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \tag{10.1}$$

Figure 10.1 shows a schematic representation of the process of measurement: The incoming state vector is an electron wave packet traveling in an inhomogeneous magnetic field, as shown in Fig. 10.1a; the spin is in a superposed state $|\Psi\rangle$ of the up and down spin states. The magnet field separates the up and down spin states, and the electron's state vector finally hits the detector, which is a screen that records the collapse of the electron's state vector, and in effect yielding the position of the electron at screen, as shown in Fig. 10.1b.

The intuitive concept of the Stern–Gerlach experiment is the following. The state vector subjected to an inhomogeneous magnet field propagates (on the average) along two paths: For "up" spin, case the state vector goes upwards and for the "down" spin case, goes downwards. Measuring the trajectory of the electron's wave packet is equivalent to determining the electron's spin. This is the essence of the Stern–Gerlach experiment, and the remaining sections express this intuitive idea in the mathematical framework of quantum mechanics.

The Stern–Gerlach experiment has the following arrangement:

- The electrons travel along the x-axis and with $z = 0$. The electron's state vector is a wave packet with a well-localized position.
- An inhomogeneous magnetic field points in the z-direction, with the field getting stronger for increasing z. The electron is in the magnetic field for a distance a along the horizontal direction.
- After leaving the magnetic field, the electron's state vector is found by the detectors at the screen to be either above or below $z = 0$.
- The final vertical position z of the electron is measured at horizontal position $x = a$, as shown in Fig. 10.1, where a screen is placed.
- The distance a that the electron travels in the magnetic field is adjusted so that there is a clear separation of the up and down paths of the electron. In particular, the separation of the two wave packets is much greater than the spread of the wave packets.
- From the position measurement of the electron's state vector at the screen, it is concluded whether the electron's spin is pointing up or down.
- The probabilities P_u and P_d are obtained by repeating the experiment many times and counting how many times the electron's state vector is found at the screen in the up or down positions.

In more general terms, the electron has two distinct degrees of freedom, namely, its position degrees of freedom x, y, z and its spin degree of freedom. The Stern–Gerlach experiment uses the electron's position degree of freedom z as the macroscopic variable that is measured in the laboratory and plays the role of the macroscopic counter reader.

The microscopic values of the z-component of spin, namely, $\pm\frac{\hbar}{2}$, are inferred from the position readings. In essence, due to the effect of the magnetic field, once the two trajectories for the "up" and "down" states have separated out more than the spread of the incident wave packet, the position of the final state yields the value of the observed eigenvalue of the spin operator σ_z.

10.2 Classical and Quantum Predictions

As shown in Fig. 10.1, electrons produced at the source are collimated into an electron beam that travels across an inhomogeneous magnetic field and is finally observed on a screen. The electrons coming out of the source have their spins in an arbitrary superposed state. In the classical picture, the spins are pointing in arbitrary directions. Classical physics predicts that the position of the electron on the screen will lie *everywhere* between the maximum up position, for a spin pointing up, and the minimum down position for a spin pointing down. Classical physics predicts that the observed electrons will *lie on a continuous line*, as shown in Fig. 10.2a.

The quantum mechanical solution for a spin moving in an inhomogeneous magnetic field is dramatically unlike the classical result. The electron coming out of the source has a state vector given by the quantum superposition of the up and down spin states and given by (8.3), namely,

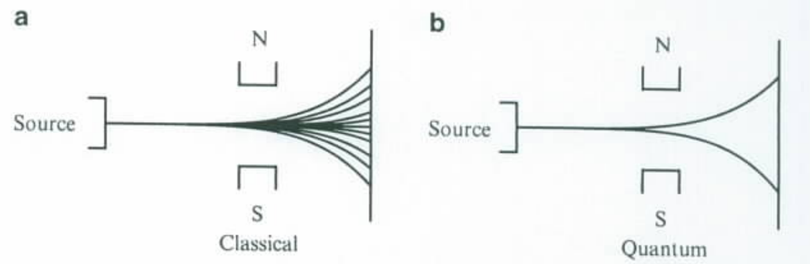

Fig. 10.2 (a) Classical prediction for the Stern–Gerlach experiment. (b) Quantum prediction for the Stern–Gerlach experiment (published with permission of © Belal E. Baaquie 2012. All Rights Reserved)

$$|\Psi\rangle = \alpha|+\rangle + \beta|-\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \qquad |\alpha|^2 + |\beta|^2 = 1$$

Since the electrons coming from the source have an arbitrary orientation, the coefficients α, β can take all the allowed values on the Bloch sphere discussed in Sect. 4.4. The electron is in a trans-empirical state, simultaneously being in both the up and down state, and *each* electron has an indeterminate and trans-empirical path, simultaneously propagating along two different possible paths, as shown in Fig. 10.2b and later in Fig. 10.5b.

Quantum superposition predicts that every experiment will only obtain the eigenvalues of the components of the superposed state. The analysis for the quantum superposition of the spin 1/2 degree of freedom is discussed in Sect. 8.1. For the spin 1/2 case, every measurement of σ_z, the z-component of spin, will result in *either* the up value of the spin *or* the down value, namely, $\hbar/2$ or $-\hbar/2$, and with *no other value* for the spin (that is in between the up and down values), as shown in Fig. 10.2b; this follows from the principle of superposition discussed in Sect. 8.1.

In other words, on the screen, all the electrons will be observed at *only two points*, either in the up position for the case of σ_z equal to $\hbar/2$ or in the down position for the case of σ_z equal to $-\hbar/2$, and nowhere else.

The predictions of classical and quantum mechanics are in stark contrast and shown in Fig. 10.2a, b. Experiments confirm the prediction of quantum superposition.

The average value of the z-component of spin, as in (8.4), is given by

$$\langle\Psi|\sigma_z|\Psi\rangle = \frac{\hbar}{2}\left[|\alpha|^2 - |\beta|^2\right]$$

10.3 The Stern–Gerlach Hamiltonian

The electron's spin is measured by greatly amplifying the (different) effect of the experimental device on the two spin states. The process of amplification is realized

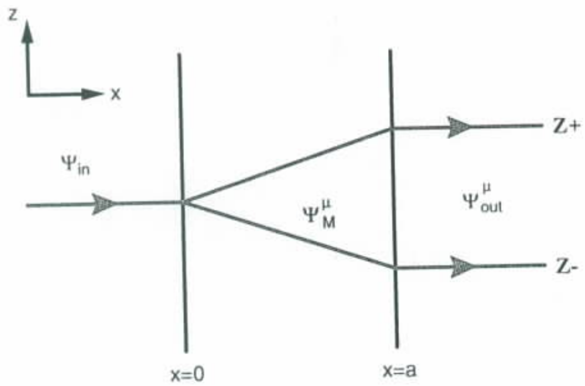

Fig. 10.3 Two possible trajectories of the wave packet under the influence of the magnetic field: cross-sectional view (published with permission of © Belal E. Baaquie 2012. All Rights Reserved)

by the electron's spin interacting with the apparatus that is represented by a quantum Hamiltonian.

The process of measuring the electron's spin is modeled by a linearized version of the Stern–Gerlach Hamiltonian—so that all the steps can be carried out exactly. For ease of notation, we set $\hbar = 1$ and we will restore its value if necessary. Following Gottfried and Yan [15], the Stern–Gerlach Hamiltonian is given by

$$H = -\frac{1}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] - i\alpha\sigma_z f(x)\frac{\partial}{\partial z}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad f(x) = \begin{cases} 1 & x \in [0,a] \\ 0 & \text{otherwise} \end{cases} \tag{10.2}$$

where the coupling of the magnetic field to the spin is given by α.[1] The term $\partial/\partial z$ reflects the increasing strength of the magnetic field with increasing horizontal distance z. The function $f(x)$ reflects the fact that the inhomogeneous magnetic field is nonzero only for the horizontal distance $x \in [0,a]$.

The classical solution to this Hamiltonian has two possible trajectories, with the classical spin initially pointing up or down and with the z-coordinate of the particle linearly rising or falling in the interval $x \in [0,a]$, as is shown in Fig. 10.3, for the two spin cases, respectively.

Consider a steady flux of electrons going through the apparatus, one by one. The expected (average) position of the electron follows a time-dependent trajectory, being incident as a free particle on the magnetic field from the left, at $x = 0$, propagating in the magnetic field until the point of $x = a$, and then propagating as a free particle along the x-axis for $x > a$. The time-dependent Schrödinger equation, for $\mathbf{r} = (x,y,z)$, is given in (2.4) ($\hbar = 1$)

[1] $\alpha = \mu_e B$, where μ_e is the magnetic moment of the electron and B has dimension of the magnetic field.

$$i\frac{\partial\psi(t,\mathbf{r})}{\partial t} = H\psi(t,\mathbf{r}) \tag{10.3}$$

and needs to be solved for the Stern–Gerlach system.

The time-dependent state function $\psi(t,x,y,z)$ is expanded in the basis provided by the stationary eigenstates of H. Let

$$\xi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \xi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \sigma_z\xi_\mu = \frac{\mu}{2}\xi_\mu; \quad \mu = \pm 1$$

The energy eigenstates of the Hamiltonian H [given in (10.2)] are labeled by $\mathbf{p} = (p_x, p_y, p_z)$ and μ, shown in Fig. 10.3, and are given by

$$H\psi_E^\mu(\mathbf{r};\mathbf{p})\xi_\mu = E(\mu,\mathbf{p})\psi_E^\mu(\mathbf{r};\mathbf{p})\xi_\mu; \quad \mu = \pm 1 \tag{10.4}$$

$$\psi_E^\mu(\mathbf{r};\mathbf{p}) = \begin{cases} \Psi_{\text{in}}(\mathbf{r};\mathbf{p}); & x < 0 \\ \Psi_{\text{M}}^\mu(\mathbf{r};\mathbf{p}); & 0 < x < a \\ \Psi_{\text{out}}^\mu(\mathbf{r};\mathbf{p}); & x > a \end{cases} \tag{10.5}$$

For $x < a$ and $x > a$, there is no potential, and hence the eigenstates are plane waves. It can be verified that the energy eigenstates are given by

$$\begin{aligned} \Psi_{\text{in}}(\mathbf{r}) &= \exp \mathrm{i}\{xp_x + yp_y + zp_z\}; \quad x < 0 \\ \Psi_{\text{M}}^\mu(\mathbf{r}) &= \exp \mathrm{i}\{x\tilde{p}_x + yp_y + zp_z\}; \quad 0 < x < a \\ \Psi_{\text{out}}^\mu(\mathbf{r}) &= \exp \mathrm{i}\{xp_x + yp_y + (z - \bar{z}_\mu)p_z\}; \quad x > a \end{aligned} \tag{10.6}$$

where

$$\begin{aligned} E(\mu,\mathbf{p}) &= \frac{1}{2m}[p_x^2 + p_y^2 + p_z^2] = \frac{1}{2m}[\tilde{p}_x^2 + p_y^2 + p_z^2] + \alpha\mu p_z \\ \tilde{p}_x &= \sqrt{p_x^2 - 2m\alpha\mu p_z} \end{aligned}$$

Note that the incoming eigenfunction $\Psi_{\text{in}}(\mathbf{r})$ does not depend on μ. The effect of the magnetic field is encoded in the eigenstate $\Psi_{\text{M}}^\mu(\mathbf{r})$, with its momentum $\tilde{p}_x$ being modified from the free particle case. The constant phase $\bar{z}_\mu$ in $\Psi_{\text{out}}^\mu(\mathbf{r})$ is fixed by requiring continuity of the state function as a function of t and $\mathbf{r}$.[2]

An arbitrary time-dependent state vector that satisfies the Schrödinger equation given in (10.3) has the following expansion:

[2]The value of $\bar{z}_\mu$ is given in (10.17).

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$$\psi(t,\mathbf{r}) = \sum_{\mu=\pm 1} c_\mu \xi_\mu \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \psi_E^\mu(\mathbf{r};\mathbf{p}) \mathrm{e}^{-\mathrm{i}tE(\mu,\mathbf{p})/2m} \tag{10.7}$$

The coefficients c_μ and $g(\mathbf{p})$ are fixed by the initial condition at initial time $t = 0$.

10.4 Electron's Time Evolution

To analyze the electron's propagation in time, let t_* be the time required by the electron to travel across the region of the magnetic field, from $x = 0$ to $x = a$. The following notation is used for the state vector and for the different intervals of time and regions of space.

$$\psi(t,\mathbf{r}) = \begin{cases} \psi_{\mathrm{in}}(t,\mathbf{r}); & t < 0;\ x < 0 \\ \psi_{\mathrm{M}}(t,\mathbf{r}); & 0 < t < t_*;\ 0 < x < a \\ \psi_{\mathrm{out}}(t,\mathbf{r}); & t > t_*;\ x > a \end{cases} \tag{10.8}$$

An initial incoming electron wave packet with energy $E = \mathbf{p}^2/2m$ is given by

$$\psi_{\mathrm{in}}(t,\mathbf{r}) = \chi(t,\mathbf{r}) \sum_{\mu=\pm 1} c_\mu \xi_\mu;\ x < 0$$

$$\chi(t,\mathbf{r}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \mathrm{e}^{\mathrm{i}xp_x + \mathrm{i}yp_y + zp_z} \mathrm{e}^{-\mathrm{i}t\mathbf{p}^2/2m} \tag{10.9}$$

Note the position and spin degrees of freedom for $\psi_{\mathrm{in}}(t,\mathbf{r})$ are *not entangled* since they are in the form of a product: The state vectors for the spin and position degrees of freedom of freedom are completely *factorized* in $\psi_{\mathrm{in}}(t,\mathbf{r})$.

As indicated in Fig. 10.1, the incident state function needs to be a wave packet that is well localized in space so that the detector can observe the motion of the center of the wave packet as it traverses the magnetic field—with the motion of the center of the wave packet yielding information about the spin of the electron.

To create the incident wave packet that is well localized in space, the following function $g(\mathbf{p})$ is chosen:

$$g(\mathbf{p}) = \left(\frac{2\pi}{\beta^2}\right)^{3/2} \mathrm{e}^{-\frac{1}{2\beta^2}(\mathbf{p}-\mathbf{K})^2};\quad \mathbf{K} = (K,0,0) \tag{10.10}$$

The wave packet is chosen to be traveling in the x-direction. Choosing $\beta \ll K$ ensures that the momentum of the wave packet is peaked around $\mathbf{K}$ and does not spread as it traverses the magnetic field. The average momentum K is chosen to be large enough so that there is *no reflected* component of the state vector at $x = 0$. For this choice of $g(\mathbf{p})$, at $t = 0$, the wave packet is well localized around $x = y = z = 0$.

Wave Packet Inside the Magnetic Field: Formation of Entanglement

For $0 < t < t_*$, the wave packet is inside the magnetic field and yields the following time-dependent state function:

$$\psi_{\mathrm{M}}(t,\mathbf{r}) = \sum_{\mu} c_\mu \xi_\mu \chi_\mu(t,\mathbf{r}); \quad x > 0;\ 0 < t < t_*$$

$$\begin{aligned}\chi_\mu(t,\mathbf{r}) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \Psi^{\mu}_{\mathrm{M}}(\mathbf{r}) \mathrm{e}^{-it\mathbf{p}^2/2m};\ 0 < x < a \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \mathrm{e}^{\mathrm{i}x\tilde{p}_x + \mathrm{i}(yp_y + zp_z)} \mathrm{e}^{-it\mathbf{p}^2/2m} \end{aligned} \tag{10.11}$$

As the electron propagates along the x-axis, the position and spin degrees of freedom for $\psi_{\mathrm{M}}(t,\mathbf{r})$ become entangled. We study how this entanglement is brought about by the interaction Hamiltonian H given in (10.2).

The momentum of the incident wave packet, due to $g(\mathbf{p})$, is peaked at K along the x-direction; hence, in the momentum integral given in (10.11), p_x can be expanded about the momentum K and yields for $\tilde{p}_x$ the following:

$$\tilde{p}_x = \sqrt{p_x^2 - 2m\alpha\mu p_z} \simeq p_x - m\mu\alpha p_z/p_x \simeq p_x - \mu\alpha p_z/v \tag{10.12}$$

$$v = p_x/m \simeq K/m$$

Equation (10.12) yields, in (10.11), the following:

$$\mathrm{e}^{\mathrm{i}x\tilde{p}_x}\mathrm{e}^{\mathrm{i}zp_z} \simeq \mathrm{e}^{\mathrm{i}xp_x}\mathrm{e}^{\mathrm{i}p_z(z-\mu\alpha x/v)} \tag{10.13}$$

Equation (10.13) is the key to understanding the mechanism of entanglement. Due to the magnetic field, the propagation in the z-direction now has a contribution $\mu\alpha x/v$; furthermore, the propagation is now *entangled* with the spin of the electron, since the trajectory for $\mu = 1$ separates out from the one for $\mu = -1$; in other words, the position and spin degrees of freedom become entangled due to the interaction of the electron's degrees of freedom with the magnetic field.

Performing the Gaussian integrations in (10.11) using (10.10) and (10.13) yields, for $0 < x < a$ and $0 < t < t_*$, the following[3]:

$$\begin{aligned}\chi_\mu(t,\mathbf{r}) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(\frac{2\pi}{\beta^2}\right)^{3/2} \mathrm{e}^{-\frac{1}{2\beta^2}(\mathbf{p}-\mathbf{K})^2} \mathrm{e}^{\mathrm{i}xp_x + \mathrm{i}yp_y} \mathrm{e}^{\mathrm{i}p_z(z-\mu\alpha x/v)} \mathrm{e}^{-it\mathbf{p}^2/2m} \\ &= \mathcal{N}' \exp\left\{-\frac{m\beta^2}{2(m+it\beta^2)}\left[y^2 + \left(z - \frac{\mu\alpha x}{v}\right)^2\right]\right\} \mathrm{e}^{\mathrm{i}Kx} \int \mathrm{d}p\mathrm{e}^{-\frac{1}{2\beta^2}p^2 - \frac{it}{2m}(p+K)^2} \mathrm{e}^{\mathrm{i}px}\end{aligned}$$

[3] $\mathcal{N}', \mathcal{N}$ are normalization constants.

$$= \mathcal{N} \exp\left\{-\frac{m\beta^2}{2(m + \mathrm{i}t\beta^2)}\left[y^2 + \left(z - \frac{\mu\alpha x}{v}\right)^2 + (x - vt)^2\right]\right\} \mathrm{e}^{\mathrm{i}Kx}\mathrm{e}^{-\frac{\mathrm{i}t}{2m}K^2} \tag{10.14}$$

The average positions are given by (10.14) and yield, to leading order,

$$E_\chi[x] = \int_0^a \mathrm{d}x \int dy\mathrm{d}z\, x\, |\chi_\mu(t, \mathbf{r})|^2 \simeq vt$$

$$E_\chi[z] = \int_0^a \mathrm{d}x \int \mathrm{d}y\mathrm{d}z\, z\, |\chi_\mu(t, \mathbf{r})|^2 \simeq \frac{\mu\alpha}{v} E_\chi[x] = \mu\alpha t$$

The trajectory of the wave packet is obtained by the expectation value of the position of the wave packet; for $\mathbf{r} = (x, y, z)$, the equations above yield, to leading order, the expected position, given by $\bar{\mathbf{r}}_\mu(t)$, as follows:

$$\bar{\mathbf{r}}_\mu(t) = \langle\chi(t)|\mathbf{r}|\chi(t)\rangle = E_\chi[\mathbf{r}] = (vt, 0, \mu\alpha t) \tag{10.15}$$

The trajectory of the wave packet is the motion of its expected position $\bar{\mathbf{r}}_\mu(t)$. For $0 < x < a$, the trajectories $\bar{\mathbf{r}}_+(t)$ and $\bar{\mathbf{r}}_-(t)$, given in Fig. 10.3, are the upward and downward sloping straight lines, as expected from (10.15).

Wave Packet After Crossing the Magnetic Field: Entangled State

The domain of the magnetic field, defined by $0 < x < a$, is chosen so that—on crossing the distance a—the separation of the electron wave packet for $\mu = \pm 1$ is much greater than its spread, schematically shown in Fig. 10.1.

On reaching $x = a$ in time $t_* = a/v$, the wave packet is out of the magnetic field, and there is no longer any average motion in the z-direction. The trajectory for $t > t_* = a/v$ is determined by the state function $\Psi^\mu_{\mathrm{out}}(\mathbf{r})$ given by (10.6), which together with (10.5) yields

$$\psi_{\mathrm{out}}(t, \mathbf{r}) = \sum_\mu c_\mu \xi_\mu \zeta_\mu(t, \mathbf{r});\ \ x > a;\ \ t > t_* = a/v$$

$$\zeta_\mu(t, \mathbf{r}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \Psi^\mu_{\mathrm{out}}(\mathbf{r}) \mathrm{e}^{-\mathrm{i}t\mathbf{p}^2/2m}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} g(\mathbf{p}) \exp \mathrm{i}\{xp_x + yp_y + (z - \bar{z}_\mu)p_z\} \mathrm{e}^{-\mathrm{i}t\mathbf{p}^2/2m} \tag{10.16}$$

Recall the constant phase $\bar{z}_\mu$ is fixed by the requirement that the state function $\psi(t,\mathbf{r})$ is continuous at $x = a$ and for time $t_* = a/v$; hence, from (10.14),

$$\bar{z}_\mu = \frac{\mu\alpha a}{v} \tag{10.17}$$

Similar to the derivation given in (10.14), in the approximation that $K \gg \beta$, from (10.16), the state function for $x > a$ and $t > t_*$ is given by

$$\zeta_\mu(t,\mathbf{r}) = \mathcal{N} \mathrm{e}^{-\frac{m\beta^2}{2(m+it\beta^2)}\left[y^2+(z-\frac{\mu\alpha a}{v})^2+(x-vt)^2\right]} \mathrm{e}^{\mathrm{i}Kx} \mathrm{e}^{-\frac{it}{2m}K^2} \tag{10.18}$$

$$\Rightarrow \bar{\mathbf{r}}_\mu(t) = \int_a^\infty \mathrm{d}x \int \mathrm{d}y\mathrm{d}z\, \mathbf{r}\, |\zeta_\mu(t,\mathbf{r})|^2 \simeq \left(vt, 0, \frac{\mu\alpha a}{v}\right) \tag{10.19}$$

Note that the two state functions ζ_μ are orthogonal, that is, $\langle\zeta_v|\zeta_\mu\rangle = \delta_{\mu-v}$.

The trajectory of the wave packet is given by the motion of its expected position, given by $\bar{\mathbf{r}}_\mu(t)$; after time t_*, the trajectories $\bar{\mathbf{r}}_+(t)$ and $\bar{\mathbf{r}}_-(t)$ *travel in a straight line* parallel to the x-axis and at a height of $\bar{z}_\pm$—and are shown, in Fig. 10.3 for $x > a$, as two lines parallel to the x-axis.

10.5 Entanglement of Spin and Device

To understand the measurement process, recall from (10.9) that the incident state function is given by

$$\psi_{\text{in}}(t,\mathbf{r}) = \chi(t,\mathbf{r}) \sum_{\mu=\pm 1} c_\mu \xi_\mu; \quad x < 0;\ t < 0$$

The incident state function is a product state—of the space state function $\chi(t,\mathbf{r})$ multiplied by the superposed state of the spins $\sum_\mu c_\mu \xi_\mu$. There is no correlation between the position of the electrons and its spin; measuring the position of the electron yields no information about the spin of the electron.

The Hamiltonian given in (10.2) introduces an interaction of the spin with the external magnetic field and creates a macroscopic amplification between the two microscopic spin eigenstates. The up spin eigenstate has an average trajectory, given by $\bar{\mathbf{r}}_+(t)$, that is clearly different from the average trajectory of the down spin eigenstate $\bar{\mathbf{r}}_+(t)$.

After passing through the magnet, the electron's position degree of freedom becomes *entangled* with its spin degree of freedom, with the deflection of the electron in the z-direction depending on its spin state defined by μ. More precisely, from (10.16),

$$\psi_{\text{out}}(t,\mathbf{r}) = \sum_\mu c_\mu \xi_\mu \zeta_\mu(t,\mathbf{r}); \quad x > a;\ t > t_* = a/v$$

$$\Rightarrow |\psi_{\text{out}}\rangle = \sum_{\mu} c_{\mu}|\xi_{\mu}\rangle|\zeta_{\mu}\rangle; \qquad \langle\zeta_{\nu}|\zeta_{\mu}\rangle = \delta_{\mu-\nu} \tag{10.20}$$

On leaving the magnetic field, the electron is in an entangled state, for which the electron takes the up and down paths that are *exactly correlated* with the state of the spin: Measuring the position of the electron is tantamount to measuring its spin. Or put differently, due to entanglement, immediately after the position measurement of $\bar{z}_{\mu}$, the spin is certain to be in a spin eigenstate ξ_{μ}; if an independent measurement of spin is carried out immediately *after* the measurement of the position yields $\bar{z}_{\mu}$, then the spin is certain to be found in the state ξ_{μ}.

Hence, a measurement of position in effect is also a measurement of spin and can be represented by the projection to state function $|\xi_{\mu}\rangle|\zeta_{\mu}\rangle$; (10.20) then yields

$$|\langle\zeta_{\mu}|\langle\xi_{\mu}|\psi_{\text{out}}\rangle|^2 = |c_{\mu}|^2$$

Equivalently, note that the Stern–Gerlach experiment does not directly observe the spin states of the electron. Hence, a measurement of only the position degree of freedom results in a partial trace over the spin degrees of freedom and yields the following reduced density matrix:

$$\rho_{\text{R}} = tr_{spin}\big(|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|\big) = \sum_{\mu}|c_{\mu}|^2|\zeta_{\mu}\rangle\langle\zeta_{\mu}| \tag{10.21}$$

The reduced density matrix ρ_{R} represents a classical random system. After it crosses the magnetic field, the probability of finding the electron in device state $|\zeta_{+}\rangle$, with the pointer at position $\bar{z}_{+}$ is P_{u} and, similarly, the probability of finding the device in state $|\zeta_{-}\rangle$, with pointer at position $\bar{z}_{-}$, is P_{d}; (10.21) then states that

$$P_{\text{u}} = |c_{+}|^2; \qquad P_{\text{d}} = |c_{-}|^2$$

The experiment determines the average value of the z-component of spin, as in (8.4), and is given by

$$\langle\Psi|\sigma_z|\Psi\rangle = \frac{\hbar}{2}\Big[|c_{+}|^2 - |c_{-}|^2\Big] = \frac{\hbar}{2}\Big[P_{\text{u}} - P_{\text{d}}\Big]$$

10.6 Summary of Spin Measurement

We recapitulate the process of measurement to highlight its conceptual underpinnings and connect the specific example of the spin measurement to the general features of a quantum measurement discussed in Sects. 9.6–9.8.

The Stern–Gerlach experiment measures the spin magnetic moment, in short the spin, of the quantum spin degree of freedom. The measurement is carried out by entangling the spin degree of freedom with the degrees of freedom of the *experimental device*. The z-coordinate of the electron's position degree of freedom is, in fact, a representation of the experimental device, since the z-position of the electron is what the device measures. Hence, the z-coordinate degree of freedom of the electron plays the role of the degree of freedom of the device.

The counterposition of the device is the point on the screen at which the electron's state vector is detected. Hence, the position of the electron on the screen is the degree of freedom of the device. The state of the device has the following three values:

- The counterposition $z = 0$ indicates the neutral position of the counter and contains no information about the spin of the system and is represented by state $|D_0\rangle$.
- The counterposition $z = \bar{z}_+ = \alpha a/v$ shows that the spin is in the up state and is represented by state $|D_+\rangle$.
- The counterposition $z = \bar{z}_- = -\alpha a/v$ shows that the spin is in the down state and is represented by state $|D_-\rangle$.

The three states of the device, represented by the state vectors $|D_0\rangle, |D_+\rangle, |D_-\rangle$, are given by

$$|D_0\rangle = \chi(t,\mathbf{r}); \qquad |D_+\rangle = \zeta_+(t,\mathbf{r}); \qquad |D_+\rangle = \zeta_-(t,\mathbf{r})$$

The measurement process is constituted by the following stages:

- The initial quantum state is prepared at the source to be in a superposed state of the spin degree of freedom. The initial quantum state and device state are in a joint product state.

$$|\psi_{\text{in}}\rangle = |D_0\rangle \Big(c_1|\xi_1\rangle + c_2|\xi_2\rangle\Big) : \quad \text{device and spin not entangled}$$

- After the interaction of the device and the electron's spin, the final state is an entangled state given by

$$|\psi_{\text{in}}\rangle \to |\psi_{\text{out}}\rangle = c_1|\xi_1\rangle|D_+\rangle + c_2|\xi_2\rangle|D_-\rangle : \quad \text{device and spin } \textit{entangled}$$

- The third stage in measuring the spin of the electron is to perform the measurement by *recording* the quantum state of the electron—hence bringing about an irreversible change in the spin-device system by collapsing the state function. This process yields the mixed density matrix ρ_{M} given by

$$\rho = |\psi_{\text{out}}\rangle\langle\psi_{\text{out}}| \to \text{Measurement} \to \rho_{\text{M}}$$
$$\rho_{\text{M}} = |c_1|^2|\xi_1\rangle\langle\xi_1| \otimes |D_+\rangle\langle D_+| + |c_2|^2|\xi_2\rangle\langle\xi_2| \otimes |D_-\rangle\langle D_-|$$

- Since the value of the spin degree of freedom is not measured, the Stern–Gerlach in effect performs a partial trace over the spin degree of freedom and yields the reduced density matrix ρ_{R} that gives the final result:

$$\rho_R = \mathrm{tr}_{\mathrm{spin}}(\rho_M) = |c_1|^2 |D_+\rangle\langle D_+| + |c_2|^2 |D_-\rangle\langle D_-|$$

10.7 Irreversibility and Collapse of State Vector

As discussed in Chap. 9, measurement involves four steps, namely, preparation, amplification, entanglement, and irreversibility. The process of measurement is brought to a *closure* by recording the outcome of the measurement, and this process of recording the result brings up about an irreversible change in the detector as well as in the quantum system being observed.

Recall for the Stern–Gerlach experiment, the applied magnetic field brought about an *amplification* of the up and down quantum states of the electron's spin by separating these states into macroscopically separated electron (average) trajectories. Furthermore, the electron's position became *entangled* with its spin degree of freedom, thus allowing for the unambiguous determination of the state of the spin degree of freedom by measuring only the position of the electron.

However, both amplification and entanglement do not bring an irreversible change in the system. As shown schematically in Fig. 10.4, one can apply a reversed magnetic field to the entangled state of the electron and disentangle it and deamplify the separation of the quantum states so that the original state of the electron is restored. According to Wigner [12], the passage of the electron through the magnetic field is not a complete measurement, and it is only when the position of the electron is *recorded*, for example, by a photographic plate as in Fig. 9.1, that an *irreversible* change is made and the process of measurement is completed.

Registering the electron's position is a process that *causes* a collapse of the state vector and brings about an irreversible change in the system—called decoherence. The concept of quantum entropy as discussed in (6.34) provides an appropriate mathematical description of irreversibility in quantum mechanics.

This view of Wigner has been contested by some physicists pointing to the necessity replacing the classical magnetic field with the quantized electromagnetic field [31]; suffice to say, the relevant point in this discussion is that an irreversible change needs to be made for completing the measurement process. If the preparation of the state vector or its subsequent propagation through the magnetic field brings about such an irreversible change, then it proves Wigner's point.

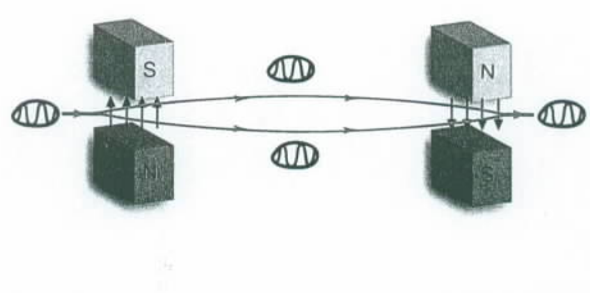

Fig. 10.4 Reversing amplification and entanglement (published with permission of © Belal E. Baaquie 2012. All Rights Reserved)

10.8 Interpretation of Spin Measurement

The empirical and trans-empirical interpretation of the Stern–Gerlach experiment is given in Fig. 10.5.

Figure 10.5a–c is a representation of the different aspects of the measurement process of the electron's spin. Figure 10.5a is the usual drawing of the experiment indicating both the possible paths for the electron; Fig. 10.5b and c resolves the experiment into two components: the empirical part shown in Fig. 10.5b and the trans-empirical part shown in Fig. 10.5c.

The stages of the measurement process are the following:

- At the "source," the quantum state to be measured is prepared together with the experimental device that performs the measurement and is shown in Fig. 10.5a. $\psi(\mu, \mathbf{r})$ is the state function of the electron that is emitted by the source, where $\mu = \pm 1$ (up, down) is the spin of the electron; $D(z_0)$ is the state function of the detector with z_0 being the initial value of the detector pointer. The preparation results in a product state $\psi(\mu, \mathbf{r}) \otimes D(z)$.
- Figure 10.5b and c is an empirical and trans-empirical interpretation of the experiment. The experimental device is the "screen," which is in the empirical domain, where both the source and screen (detectors) are placed and shown in Fig. 10.5b. The superposed state of the electron's spin is simultaneously in both the up and down states and hence is trans-empirical. Figure 10.5c shows the trans-empirical state of the electron that does not exist in (physical) space but rather exists in Hilbert space.
- During the transit of the electron from the source to the screen, the state vectors of the electron and the device become entangled, as shown in Fig. 9.6; moreover, the difference between the up and down state of the electron's

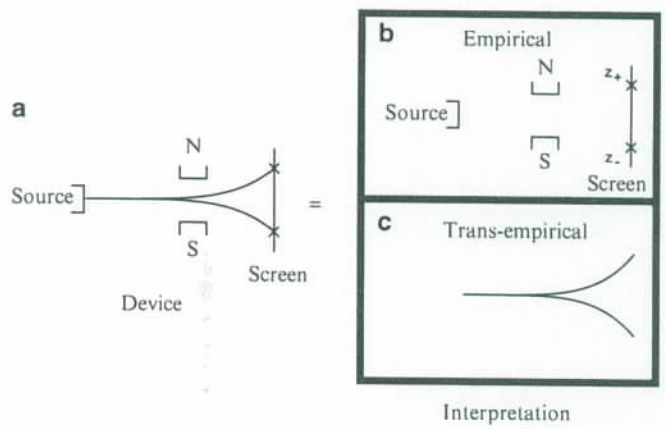

Fig. 10.5 (a) Stern–Gerlach experiment. (b) Empirical and (c) trans-empirical interpretation of the Stern–Gerlach experiment (published with permission of © Belal E. Baaquie 2012. All Rights Reserved)

spin is vastly amplified. The trans-empirical entangled state vector is given by $\sum_{s=\pm} c_s \psi(s,\mathbf{r}) \otimes D(z_s)$.

- *Both* the eigenstates arrive at the screen as a trans-empirical superposed state vector.
- When the entangled state vector ψ hits the screen, it makes a discontinuous and irreversible transition from its trans-empirical form ψ to its empirical form $|\psi|^2$. *Only one* of the trans-empirical states is actualized at either z_+ or z_-, with a likelihood given by $|c_+|^2$ or $|c_-|^2$, respectively.
- Suppose the electron's state vector is detected at position z_+ that is shown in Fig. 10.5b in the empirical setup of the experiment. The detector's state vector is put into a definite state $D(z_+)$ when the electron's state vector is detected; due to entanglement, one can conclude that the electron's spin is also put into a definite state $\psi(+,z_+)$.
- Repeating the experiment many times gives the probability for the electron's spin to be in the different possible eigenstates, namely, yields $|c_\pm|^2$.

10.9 Summary

Measurement of the properties of a quantum degree of freedom lies at the heart of quantum mechanics. The measurement of the state vector of a spin 1/2 degree of freedom was studied in great detail, using the Stern–Gerlach experiment, to examine each step in the process of measurement.

A model Hamiltonian was used to obtain the state vector of the electron and evolve it through the experimental apparatus, in particular, from the source of electrons, through the inhomogeneous magnetic field and finally to the screen where the electron's state vector is detected.

The time-dependent state vector yields explicit expressions on how the entanglement of the spin degree of freedom with the position degree of freedom develops due to the interaction of the electron's degrees of freedom with the magnetic field. The state vector also demonstrates how the amplification of the microscopic difference between the up and down spin states is a function of time, with the time spent in the magnetic field determining the degree of macroscopic separation of the two possible paths of the electron.

Recording the collapse of the state vector of the electron on the screen causes the state vector to collapse. This collapse of the state vector has to be *postulated* and brings to a conclusion the Stern–Gerlach experiment.

Measuring only the position of the electron is mathematically realized by a partial trace of the electron's density matrix over the spin degree of freedom and yields the reduced density matrix, which in turn yields the likelihood of finding the spin to be in the up or down state.

The Stern–Gerlach experiment was lastly analyzed to determine the empirical and trans-empirical aspects of the experiment. It was seen that only the preparation of the initial state and the measurement of the final state are empirical events, with the evolution of the electron's state vector, the formation of entanglement, and the amplification of the physical effect of the spin degree of freedom all being trans-empirical processes.

In summary, the Stern–Gerlach experiment illustrates all four ingredients of a quantum measurement, namely, the preparation of the quantum state, entanglement of the degree of freedom being measured with the device, amplification of the quantum quantity to a macroscopic magnitude, and the irreversible collapse of the state vector.