№:1.1

Найти ф-цию гармон. в кольце $1<r<2$ такую, что

$$u|_{r=1} = 1+\cos^2\varphi; \quad u|_{r=2} = \sin^2\varphi$$

$$u|_{r=2} = \frac{1-\cos 2\varphi}{2}$$

$$u|_{r=1} = \frac{3}{2} + \frac{\cos 2\varphi}{2}$$

$$u(r,\varphi) = C + C_0 \ln r + \sum_{n=1}^{\infty} \left[r^n (A_n \cos n\varphi + B_n \sin n\varphi) + r^{-n} (C_n \cos n\varphi + D_n \sin n\varphi) \right]$$

$$u(1,\varphi) = \frac{3}{2} + \frac{\cos 2\varphi}{2} = C + \underbrace{C_0 \ln 1}_{=0} + \sum_{n=1}^{\infty} (A_n + C_n)\cos n\varphi + (B_n + D_n)\sin n\varphi \Rightarrow C = \frac{3}{2}$$

$$A_2 + C_2 = \frac{1}{2}$$

$$B_n + D_n = 0 \quad \forall n$$

$$A_n + C_n = 0 \quad \forall n \neq 2$$

$$(2,\varphi) = \frac{1}{2} - \frac{\cos 2\varphi}{2} = \frac{3}{2} + C_0 \ln 2 + \sum_{n=1}^{\infty} 2^n (A_n + C_n)\cos n\varphi + \frac{1}{2^n}(B_n + D_n)\sin n\varphi$$

$$\frac{3}{2} + C_0 \ln 2 = \frac{1}{2} \qquad C_0 \ln 2 = -1 \Rightarrow C_0 = -\frac{1}{\ln 2}$$

$$\begin{cases} -\frac{1}{2} = 4A_2 + \frac{1}{4}C_2 \\ A_2 + C_2 = \frac{1}{2} \end{cases} \qquad \begin{cases} 4A_2 + \frac{1}{4}C_2 = -\frac{1}{2} \\ 4A_2 + 4C_2 = 2 \end{cases} \ominus \Rightarrow \frac{15}{4}C_2 = \frac{5}{2} \Rightarrow C_2 = \frac{2}{3} \qquad A_2 = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$$

$$u(r,\varphi) = \frac{3}{2} - \frac{1}{\ln 2}\ln r + \cos 2\varphi \left(r^2 \left(-\frac{1}{6}\right) + r^{-2}\frac{2}{3} \right)$$

№:2.1

$$\begin{cases} \Delta u = 0 \\ u(0,y) = A \sin \frac{\pi y}{b} \\ u(a,y) = u(x,0) = u(x,b) = 0 \end{cases}$$

$$u(0,0) = 0 \qquad u = v_1 + v_2 + v_3 + v_4 \qquad v_2, v_3, v_4 = 0$$

$$\begin{cases} \Delta v_1 = 0 \\ v_1(0,y) = A \sin \frac{\pi y}{b} \\ v_1(a,y) = v_1(x,0) = v_1(x,b) = 0 \end{cases}$$

$$v_1 = X(x)Y(y) \qquad X''Y + XY'' = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(a) = 0 \end{cases} \qquad \begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = 0 \\ Y(b) = 0 \end{cases} \qquad p^2 - \lambda = 0$$

$$p = \pm\sqrt{\lambda}\, i$$

$$X_n(x) = A_n e^{\sqrt{\lambda}x} + B_n e^{-\sqrt{\lambda}x} = C_n \operatorname{sh}\frac{\pi n}{b}x + D_n \operatorname{sh}\frac{\pi n}{b}(x-a)$$

$$Y = c_1 \sin(\sqrt{\lambda}y) + c_2 \cos(\sqrt{\lambda}y)$$

$$Y(0) = 0 = c_2$$

$$Y = \sin\frac{\pi n}{b}y$$

$$\sum_{n=0}^{\infty} \left(C_n \operatorname{sh}\frac{\pi n}{b}x + D_n \operatorname{sh}\frac{\pi n}{b}(x-a) \right) \sin\frac{\pi n}{b}y$$

$$v_1(0,y) = A \sin\frac{\pi y}{b}$$

$n \neq 1$: $D_n = 0$

$n = 1$: $D_1 \operatorname{sh}\frac{\pi a}{b} = A \qquad D_1 = \frac{A}{\operatorname{sh}\frac{\pi a}{b}}$

$$v_1(a,y) = 0 = \sum_{n=0}^{\infty} C_n \operatorname{sh}\frac{\pi n}{b}a \sin\frac{\pi n y}{b} \Rightarrow C_n = 0$$

$$u(x,y) = \frac{A}{\operatorname{sh}\frac{\pi a}{b}} \operatorname{sh}\frac{\pi(x-a)}{b} \sin\frac{\pi}{b}y$$

No:5.1

$$\begin{cases} u_{tt}-3u_t=u_{xx}+2u_x-3x-2t \\ u(0,t)=0 \\ u(\pi,t)=\pi t \\ u(x,0)=e^{-x}\sin x \\ u_t(x,0)=x \end{cases}$$

$u=v+ax+b$
$0=b$
$\pi t=\pi a+0 \quad a=t$

$u=v+tx$

$$v_{tt}-3v_t\cancel{-3x}=v_{xx}+2v_x+\cancel{2t}-3x-\cancel{2t}$$

$$\begin{cases} v_{tt}=3v_t+v_{xx}+2v_x \\ v(0,t)=v(\pi,t)=0 \\ v(x,0)=e^{-x}\sin x \\ v_t(x,0)=0 \end{cases} \qquad v=e^{\alpha x}w$$

$$e^{\alpha x}w_{tt}-3e^{\alpha x}w_t=\alpha^2e^{\alpha x}w+2\alpha e^{\alpha x}w_x+e^{\alpha x}w_{xx}+2\alpha e^{\alpha x}w+2e^{\alpha x}w_x=(\alpha^2+2\alpha)e^{\alpha x}w+(2\alpha+2)e^{\alpha x}u_x+e^{\alpha x}w_{xx}$$

$\alpha=-1: \quad v=e^{-x}$

$$e^{-x}w_{tt}-3e^{-x}w_t=-e^{-x}w+e^{-x}w_{xx}$$

$$\begin{cases} w_{tt}-3w_t=w_{xx}-w, \\ w(0,t)=w(\pi,t)=0 \\ w(x,0)=\sin x \\ w_t(x,0)=0 \end{cases} \qquad w(x)=X(x)T(t) \qquad X_n=\sin\frac{\pi n x}{\pi}=\sin nx$$

$$n=1: \begin{cases} T''-3T'=-T-T \\ T(0)=1 \\ T'(0)=0 \end{cases} \qquad T''-3T'+2T=0, \quad \lambda^2-3\lambda+2=0, \quad D=9-8=1, \quad \lambda=\frac{3+1}{2}=2, \quad \lambda=1$$

$$w(x,t)=(C_1e^{2t}+C_2e^{t})\sin x$$

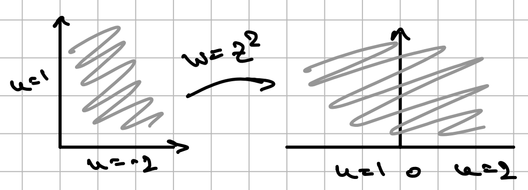
$$\left.\begin{aligned} T(0)&=1=C_1+C_2 \\ T'(0)&=0=2C_1+C_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &C_1-2C_1=1 \\ &C_2=-2C_1 \end{aligned} \qquad \begin{aligned} C_1&=-1 \\ C_2&=2 \end{aligned}$$

$$w(x,t)=(2e^{t}-e^{t})\sin x$$

$$u(x,t)=e^{-x}\sin x\,(2e^{t}-e^{t})+tx$$

No:3.1

$$\begin{cases} \Delta u=0 \\ u(0,y)=1 \\ u(x,0)=-2 \end{cases}$$



$\varphi=\arctan\frac{y}{x}$

$z=x+iy \qquad w=x^2-y^2+2xyi \Rightarrow \xi+i\eta=w, \quad \xi=x^2-y^2 \quad \eta=2xy$

$$u(\xi,\eta)=\frac{\eta}{\pi}\left(\int_{-\infty}^{0}\frac{dt}{(t-\xi)^2+\eta^2}+\int_0^{\infty}\frac{-2\,dt}{(t-\xi)^2+\eta^2}\right)=\frac{\eta}{\pi}\left(\frac{1}{\eta}\arctan\frac{(t-\xi)}{\eta}\Big|_{-\infty}^{0}-\frac{2}{\eta}\arctan\frac{(t-\xi)}{\eta}\Big|_{0}^{+\infty}\right)=$$

$$=\frac{1}{\pi}\left(-\arctan\frac{\xi}{\eta}+\frac{\pi}{2}-2\frac{\pi}{2}-\arctan\frac{\xi}{\eta}\right)=-\frac{1}{\pi}\left(\frac{\pi}{2}+2\arctan\frac{x^2-y^2}{2xy}\right)=-\frac{1}{\pi}\left(\frac{\pi}{2}+2\arctan\frac{\cos2\varphi}{\sin2\varphi}\right)$$

$$=-\frac{1}{2}+\frac{2}{\pi}\arctan\left(-\tan\left(\frac{\pi}{2}+2\varphi\right)\right)=-\frac{3}{2}-\frac{4}{\pi}\arctan\frac{y}{x}$$