Вариант 2

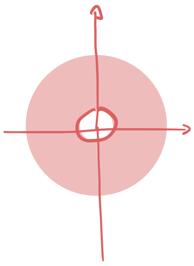
1. Решить задачу:

$\Delta u(r,\varphi)=0,\quad 1\le r<\infty,\quad 0\le\varphi<2\pi.$

$u(r=1,\varphi)=2\cos^2 2\varphi = 1+\cos 4\varphi$

$\cos 2\varphi = 2\cos^2\varphi - 1$

$2\cos^2 2\varphi = \cos 4\varphi + 1$



$$u(r,\varphi) = A_0 + \underbrace{B_0 \ln\frac{1}{r} + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi)}_{\text{неогр. при } r\to\infty \;\Rightarrow\; A_n,\, B_n = 0} + \sum_{n=1}^{\infty} r^{-n}(C_n \cos n\varphi + D_n \sin n\varphi)$$

$$u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} r^{-n}(C_n \cos n\varphi + D_n \sin n\varphi)$$

$$u(1,\varphi) = A_0 + \sum_{n=1}^{\infty}(C_n \cos n\varphi + D_n \sin n\varphi) = 1 + \cos 4\varphi$$

$$A_0 = 1,\quad C_4 = 1,\quad A_{n\neq 0} = C_{n\neq 4} = D_n = 0$$

$$u(r,\varphi) = 1 + \frac{\cos 4\varphi}{r^4}$$

2. Решить задачу:

$\Delta u(x,y)=\sin y,\quad 0<x<\infty,\quad 0<y<\pi.$

$u(x,y=0)=0,\quad u(x,y=\pi)=0,\quad u(x=0,y)=\sin 2y$

$$U_{\text{частное}} = A\cdot\sin y$$

$$\Delta U_{ч} = -A\sin y = \sin y \;\Rightarrow\; A=-1 \;\Rightarrow\; U_{ч} = -\sin y$$

$$u(x,y) = v(x,y) - \sin y$$

$$\begin{cases} \Delta v(x,y) = 0 \\ v(x,y=0) = 0 \\ v(x,y=\pi) = 0 \\ v(x=0,y) = \sin 2y + \sin y \end{cases}$$

$$v(x,y) = X(x)\cdot Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \Rightarrow \begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases} \qquad \lambda_n = n^2, \quad Y_n(y) = \sin ny$$

$$X''_n(x) - \lambda X_n(x) = 0$$

$$X_n(x) = A_n e^{-nx} + \underbrace{B_n e^{nx}}_{=0 \text{ т.к. } e^{nx} \xrightarrow[x\to\infty]{} \infty}$$

$$v(x,y) = \sum_{n=1}^{\infty} A_n e^{-nx} \sin ny$$

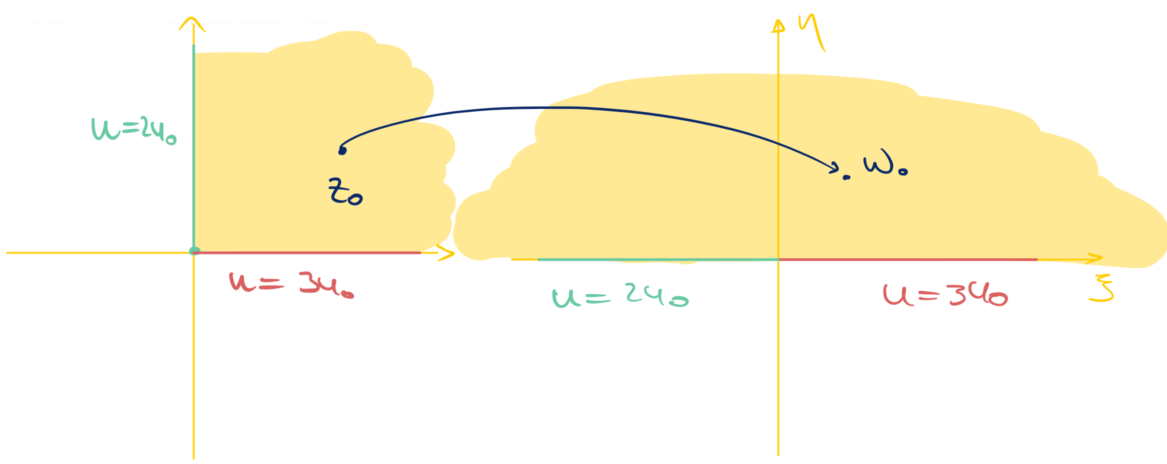
$$v(0,y) = \sum_{n=1}^{\infty} A_n \sin ny = \sin y + \sin 2y \qquad A_1 = A_2 = 1, \quad A_3 = A_4 = \ldots = 0$$

$$u(x,y) = e^{-x} \sin y + e^{-2x} \sin 2y - \sin y$$

3. **Решить задачу методом конформного отображения:**

$\Delta u(x,y) = 0$, $0 < x < \infty$, $0 < y < \infty$,

$u(x=0,y) = 2\cdot u_0$, $u(x,y=0) = 3\cdot u_0$.



$$w = z^2 = (x+iy)^2 = x^2 - y^2 + i\cdot 2xy = \xi + i\eta$$

$$u(\xi_0, \eta_0) = \frac{\eta_0}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)\, d\xi}{(\xi-\xi_0)^2 + \eta_0^2} =$$

$$= \frac{2u_0}{\pi} \int_{-\infty}^{-\frac{\xi_0}{\eta_0}} \frac{d\left(\frac{\xi-\xi_0}{\eta_0}\right)}{\left(\frac{\xi-\xi_0}{\eta_0}\right)^2 + 1} + \frac{3u_0}{\pi} \int_{-\frac{\xi_0}{\eta_0}}^{+\infty} \frac{d\frac{\xi-\xi_0}{\eta_0}}{t^2+1} =$$

$$= \frac{u_0}{\pi}\left(-2\,\mathrm{arctg}\frac{\xi_0}{\eta_0} + 2\cdot\frac{\pi}{2} + 3\cdot\frac{\pi}{2} + 3\cdot\mathrm{arctg}\frac{\xi_0}{\eta_0}\right) =$$

$$= \frac{u_0}{\pi}\,\mathrm{arctg}\frac{x_0^2 - y_0^2}{2x_0 y_0} + \frac{5u_0}{2}$$

4. Решить задачу:

$$u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0,$$

$$u_x(0,t) = 0, \quad u(\pi,t) = t, \quad t > 0,$$

$$u(x,0) = 0, \quad u_t(x,0) = 4, \quad 0 < x < \pi.$$

$$u(x,t) = v(x,t) + t$$

$$\begin{cases} v_{tt} = v_{xx} \\ v_x(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = 0 \\ v_t(x,0) = 3 \end{cases}$$

$$v(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(\pi) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2 ; \quad X_n = \cos\frac{2n-1}{2}x$$

$$T'' + \lambda T = 0 \qquad T_n = A_n \cos\frac{2n-1}{2}t + B_n \sin\frac{2n-1}{n}t$$

$$v(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{2n-1}{2}t + B_n \sin\frac{2n-1}{2}t\right)\cos\frac{2n-1}{2}x$$

$$v(x,0) = \sum_{n=1}^{\infty} A_n \cos\frac{2n-1}{2}x = 0 \implies A_n = 0$$

$$v_t(x,0) = \sum_{n=1}^{\infty}\frac{2n-1}{2}B_n \cos\frac{2n-1}{2}x = 3$$

$$B_n = \frac{6}{\pi}\int_0^{\pi}\frac{2n-1}{2}\cos\frac{2n-1}{2}x\,dx = \frac{6}{\pi}\sin\frac{2n-1}{2}x\Big|_0^{\pi} =$$

$$= \frac{2}{\pi}(-1)^{n+1}$$

$$u(x,t) = t + \sum_{n=1}^{\infty}\frac{2}{\pi}(-1)^{n+1}\cos\frac{2n-1}{2}x \cdot \sin\frac{2n-1}{2}t$$

5

Решить задачу:

$u_{tt} = u_{xx}, \quad 0 < x < +\infty, \quad t > 0,$

$u(x,t=0) = 0, \quad u_t(x,t=0) = -u_0 k\cos kx,$

$u(x=0,t) = 0, \quad t > 0$. Построить график функции $u\left(x = \frac{2\pi}{k}, t\right)$ в зависимости от t.

Нечётное прод. $\Rightarrow$ $\Psi(x,t) = \begin{cases} -u_0 k\cos kx & x>0 \\ u_0 k\cos kx & x<0 \end{cases}$

$$u(x,t) = \begin{cases} \frac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(\alpha)\,d\alpha, & x-at>0 \\ \frac{1}{2}\left(\varphi(x+t) - \varphi(x-t)\right) + \frac{1}{2}\int_{t-x}^{x+t} \Psi(\alpha)\,d\alpha, & x-at<0 \end{cases}$$

$$= \begin{cases} -\frac{u_0}{2} \sin k\alpha \Big|_{\alpha=x-t}^{\alpha=x+t} \\ \frac{u_0}{2} \sin k\alpha \Big|_{\alpha=t-x}^{\alpha=x+t} \end{cases}$$

$u(x,t_0) = \dots$