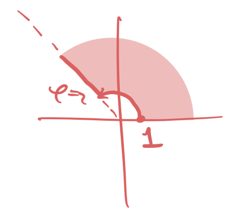
Вариант 3.

1. Решить задачу:

$\Delta u(r,\varphi)=0,\ 1\le r<\infty,\ \ 0\le\varphi<2,$

$u(r,\varphi=0)=0,\ \ u(r,\varphi=2)=0,$

$u(r=1,\varphi)=2\sin\pi\varphi+3\sin 4\pi\varphi.$



$$u(r,\varphi)=\sum_{n=1}^{\infty}\left(\underbrace{A_n\left(\frac{r}{a}\right)^{\frac{\pi n}{\alpha}}}_{\text{неогр при } r\to\infty}+B_n\left(\frac{r}{a}\right)^{-\frac{\pi n}{\alpha}}\right)\sin\frac{\pi n}{\alpha}\varphi=$$

$\Rightarrow A_n=0$

$$=\sum_{n=1}^{\infty}B_n\, r^{-\frac{\pi n}{2}}\sin\frac{\pi n\varphi}{2}$$

$$u(r=1,\varphi)=\sum_{n=1}^{\infty}B_n\sin\frac{\pi n\varphi}{2}=2\sin\pi\varphi+3\sin 4\pi\varphi$$

$$B_2=2,\quad B_8=3$$

$$u(r,\varphi)=2r^{-\pi}\sin\pi\varphi+3r^{-4\pi}\sin 4\pi\varphi$$

2. Решить задачу:

$\Delta u(x,y)=\sin 3x,\ \ 0<x<\pi,\ \ 0<y<\infty,$

$u(x,y=0)=\sin 4x,\ u(x=0,y)=0,\ \ u(x=\pi,y)=0.$

$$u_r=A\cdot\sin 3x$$

$$\Delta u_r=-9A\sin 3x=\sin 3x\quad\Rightarrow\quad u_r=-\frac{1}{9}\sin 3x$$

$$u(x,y) = v(x,y) - \frac{1}{9}\sin 3x$$

$$\begin{cases} \Delta v(x,y) = 0 \\ v(x, y=0) = \sin 4x + \frac{1}{9}\sin 3x \\ v(x=0, y) = 0 \\ v(x=\pi, y) = 0 \end{cases}$$

$$v(x,y) = X(x)Y(y) \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} \qquad \lambda_n = n^2, \quad X_n = \sin nx$$

$$Y'' - \lambda Y = 0 \Rightarrow Y(y) = A_n e^{-ny} + \underbrace{B_n e^{ny}}_{\text{неогр. при } y \to \infty \Rightarrow B_n = 0}$$

$$v(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx$$

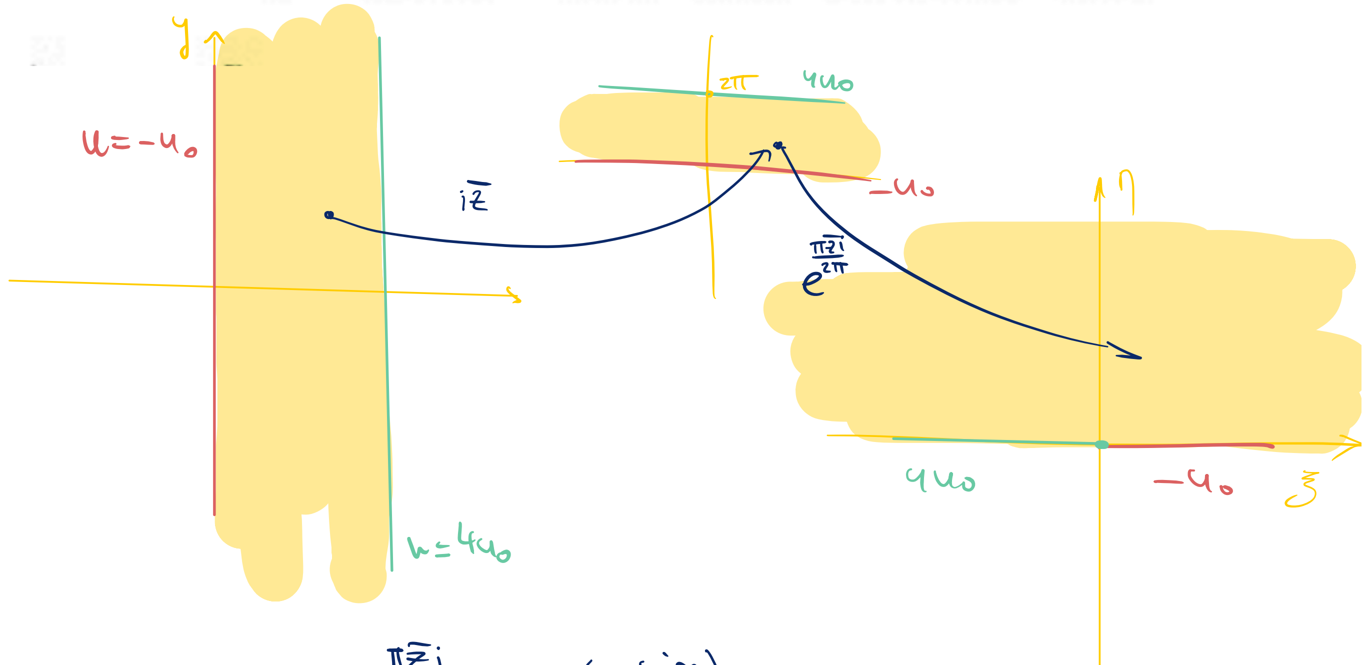
$$v(x, y=0) = \sum_{n=1}^{\infty} A_n \sin nx = \frac{1}{9}\sin 3x + \sin 4x$$

$$v(x,y) = \frac{1}{9} e^{-3y} \sin 3x + e^{-4y} \sin 4x$$

3. Решить методом конформного отображения:

$$\Delta u(x,y)=0,\quad 0<x<2\pi,\quad -\infty<y<\infty,$$

$$u(x=0,y)=-u_0,\quad u(x=2\pi,y)=4u_0.$$



$$w=\xi+i\eta=e^{\frac{\pi\bar z i}{2\pi}}=e^{\frac{(y+ix)}{2}}=e^{\frac{y}{2}+\frac{ix}{2}}=$$

$$=e^{\frac{y}{2}}\left(\cos\frac{x}{2}+i\sin\frac{x}{2}\right)\Rightarrow \begin{aligned}\xi&=e^{y/2}\cos\frac{x}{2}\\ \eta&=e^{y/2}\sin\frac{x}{2}\end{aligned}$$

$$u(\xi_0,\eta_0)=\frac{\eta_0}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)\,d\xi}{(\xi-\xi_0)^2+\eta_0^2}=$$

$$=\frac{4u_0}{\pi}\int_{-\infty}^{\frac{\xi_0}{\eta_0}}\frac{dt}{t^2+1}-\frac{u_0}{\pi}\int_{\frac{\xi_0}{\eta_0}}^{+\infty}=\frac{u_0}{\pi}\left[4\operatorname{arctg}\frac{\xi_0}{\eta_0}+2\pi-\frac{\pi}{2}+\operatorname{arctg}\ldots\right]$$

$$= \frac{u_0}{\pi}\left[\frac{5\pi}{2} - \frac{5x_0}{2} + \frac{3\pi}{2} \right] = 4u_0 - \frac{5u_0 x_0}{2}$$

4. Решить задачу:

$$u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad t > 0,$$
$$u_x(0,t) = \pi t, \quad u_x(\pi,t) = 3\pi t, \quad t > 0,$$
$$u(x,0) = x, \quad u_t(x,0) = \pi x, \quad 0 < x < \pi.$$

$$v(x,t) = u(x,t) - \pi t x - x^2 t$$
$$v_x(0,t) = u_x(0,t) - \pi t = 0$$
$$v_x(\pi,t) = 3\pi t - \pi t - 2\pi t = 0$$
$$v(x,0) = x$$
$$v_t(x,0) = \pi x - \pi x - x^2 = -x^2$$

$$\begin{cases} v_{xx} = v_{tt} - 2t \\ v_x(0,t) = v_x(\pi,t) = 0 \\ v(x,0) = x \\ v_t(x,0) = -x^2 \end{cases}$$

2 задачи

1 $$\begin{cases} w_{tt} = w_{xx} + 2t \\ w_x(0,t) = w_x(\pi,t) = 0 \\ w(x,0) = w_t(x,0) = 0 \end{cases}$$

2 $$\begin{cases} h_{xx} = h_{tt} \\ h_x(0,t) = h_x(\pi,t) = 0 \\ h(x,0) = x \\ h_t(x,0) = -x^2 \end{cases}$$

2 $$h(x,t) = X(x) \cdot T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \Rightarrow \quad \lambda_n = n^2, \quad X_n(x) = \cos nx$$

$$T'' + \lambda T = 0$$

$$T_n = A_n \cos nt + B_n \sin nt$$

$$h(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt\right) \cos nx$$

$$h(x,0) = \sum_{n=1}^{\infty} A_n \cos nx = x$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} \underset{u}{x} \underset{dv}{\cos nx\,dx} = \frac{2}{\pi} \left[\frac{x \sin nx}{n} \Big|_{x=0}^{x=\pi} - \right.$$

$$\left. - \frac{1}{n} \int_0^{\pi} \sin nx\,dx \right] = \frac{2}{\pi n^2} \cos nx \Big|_0^{\pi} = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] =$$

$$= \begin{cases} \frac{2}{\pi(2k+1)^2}, & n = 2k+1 \\ 0, & n = 2k \end{cases} \qquad A_0 = \frac{1}{\pi} \int_0^{\pi} x\,dx = \frac{\pi}{2}$$

$$h_t(x,0) = \sum_{n=1}^{\infty} n B_n \cos nx = -x^2$$

$$B_n = -\frac{2n}{\pi} \int_0^{\pi} x^2 \cos nx\,dx = \begin{cases} -\frac{4}{n^3}, & n = 2k \\ \frac{12}{n^3}, & n = 2k+1 \end{cases}$$

$$B_0 = \frac{1}{\pi} \int_0^{\pi} -x^2\,dx = -\frac{\pi^2}{3}$$

1 $$\begin{cases} w_{tt} = w_{xx} + 2t \\ w_x(0,t) = w_x(\pi,t) = 0 \\ w(x,0) = w_t(x,0) = 0 \end{cases}$$

$$\lambda_0 = 0 \Rightarrow X_0 = \cos 0 = 1 \Rightarrow w = T(t) \cdot 1$$

$$\begin{cases} T'' = 2t \\ T(0) = T'(0) = 0 \end{cases} \Longrightarrow \quad T = at + bt + c$$

$$T'' = 6a = 2 \Rightarrow a = \frac{1}{3}$$

$$T'(0) = b = 0$$

$$T(0) = c = 0$$

$$w(x,t) = \frac{t^3}{3}$$

$$u(x,t) = \frac{t^3}{3} + x^2 t + \pi t x + \frac{\pi}{2} - \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt \right) \cos nx$$