1. $u_{xx} + 4u_{xy} + 4u_{yy} + 2u_x = 0$

$$(dy)^2 - 4\,dx\,dy + 4(dx)^2 = 0$$

$D = 0 \Rightarrow$ параб. типа

$$\left(\frac{dy}{dx} - 2\right)^2 = 0 \Rightarrow \begin{matrix} dy = 2\,dx \\ y - 2x = C \end{matrix} \Rightarrow \begin{cases} \xi = y - 2x \\ \eta = x \end{cases}$$

$u_x = -2u_\xi + u_\eta$; $u_{xx} = 4u_{\xi\xi} - 4u_{\xi\eta} + u_{\eta\eta}$

$u_y = u_\xi$; $u_{yy} = u_{\xi\xi}$; $u_{xy} = -2u_{\xi\xi} + u_{\eta\xi}$

$$4u_{\xi\xi} - 4u_{\xi\eta} + u_{\eta\eta} - 8u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\xi\xi} - 4u_\xi + 2u_\eta = 0$$

$$u_{\eta\eta} - 4u_\xi + 2u_\eta = 0$$

$$u = Ve^{\alpha\xi + \beta\eta} \Rightarrow \begin{aligned} u_\eta &= (V_\eta + \beta V)e^{\dots} \\ u_{\eta\eta} &= (V_{\eta\eta} + 2\beta V_\eta + \beta^2 V)e^{\dots} \\ u_\xi &= (V_\xi + \alpha V)e^{\dots} \end{aligned}$$

$$V_{\eta\eta} + 2\beta V_\eta + \beta^2 V - 4V_\xi - 4\alpha V + 2V_\eta + 2\beta V = 0$$

$$V_{\eta\eta} + (2\beta + 2)V_\eta + (\beta^2 - 4\alpha + 2\beta)V - 4V_\xi = 0$$

$\beta = -1$, $\quad -4\alpha - 1 = 0$

$\alpha = -1/4$

$$\boxed{V_{\eta\eta} = 4V_\xi}$$

(2) $$\begin{cases} u_t = u_{xx} + 1 + \cos\frac{x}{2} \\ 0 < x < \pi, t > 0 \\ u_x(0,t) = 1 \\ u(\pi,t) = t + \pi \\ u(x,0) = x \end{cases}$$

Обнуление: $v(x,t) = u(x,t) - t - x$

$u_t = v_t + 1$

$$\begin{cases} v_t = v_{xx} + \cos\frac{x}{2} \\ v_x(0,t) = v(\pi,t) = 0 \\ v(x,0) = 0 \end{cases}$$

$\lambda_n = \left(\frac{\pi(2n-1)}{2\pi}\right)^2$, $X_n(x) = \cos\frac{2n-1}{2}x \Rightarrow$

$\Rightarrow \cos\frac{x}{2}$ - соб. ф-ия $(n=1) \Rightarrow v(x,t) = T(t)\cos\frac{x}{2}$

$$\begin{cases} T' = -\frac{1}{4}T + 1 \\ T(0) = 0 \end{cases} \Rightarrow \quad T = Ce^{-\frac{t}{4}} + 4$$

$T(0) = C + 4 = 0$

$T = 4\cdot(1 - e^{-t/4})$.

$$\boxed{u(x,t) = 4\left(1 - e^{-t/4}\right)\cos\frac{x}{2} + t + x}$$

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$$\begin{cases} u_t = 9u_{xx} + \frac{x^2}{2\pi} + x - \frac{9(t+1)}{\pi} \\ 0 < x < \pi,\ t > 0 \\ u_x(0,t) = t \\ u_x(\pi,t) = 2t+1 \\ u(x,0) = \frac{x^2}{2\pi} + 1 + \cos 3x \end{cases}$$

Обнулеем: $v(x,t) = u(x,t) - \frac{t+1}{2\pi}x^2 - tx$

$$u_t = v_t + x + \frac{x^2}{2\pi}$$

$$u_{xx} = v_{xx} + \frac{t+1}{\pi}$$

$$\begin{cases} v_t = 9v_{xx} \\ v_x(0,t) = v_x(\pi,t) = 0 \\ v(x,0) = u(x,0) - \frac{x^2}{2\pi} = 1 + \cos 3x \end{cases}$$

$\lambda_n = n^2$; $X_n(x) = \cos xn \Rightarrow X_0(x) = 1$ и $X_3(x) = \cos 3x$

– соб. ф-ии

$$\begin{cases} w_t = 9w_{xx} \\ w_x(0,t) = w_x(\pi,t) = 0 \\ w(x,0) = 1 \end{cases} \qquad \begin{cases} z_t = 9z_{xx} \\ z_x(0,t) = z_x(\pi,t) = 0 \\ z(x,0) = \cos 3x \end{cases}$$

$w(x,t) = T(t)$ $\qquad$ $w(x,t) = T(t)\cos 3x$

$$\begin{cases} T' = 0 \\ T(0) = 1 \end{cases} \Rightarrow w(x,t) = 1 \qquad \begin{cases} T' = -81T \\ T(0) = 1 \end{cases} \quad T = Ce^{-81t}, \ T(0) = C = 1$$

$$u(x,t) = 1 + e^{-81t}\cos 3x + \frac{t+1}{2\pi}x^2 + tx$$

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$$\begin{cases} U_t = 4U_{xx} + e^{2t} \\ U(x,0) = 1 + e^{4x} \\ -\infty < x < \infty,\ t > 0 \end{cases}$$

$$\begin{cases} V_t = 4V_{xx} + e^{2t} \\ v(x,0) = 1 \\ -\infty < x < +\infty,\ t > 0 \end{cases}$$

$$\begin{cases} \omega_t = 4\omega_{xx} \\ \omega(x,0) = e^{4x} \\ x \in \mathbb{R},\ t > 0 \end{cases}$$

$$V(x,t) = v(t)$$

$$\begin{cases} v' = e^{2t} \\ v(0) = 1 \end{cases} \Rightarrow \quad v = \frac{e^{2t}}{2} + C \quad v(0) = C + \frac{1}{2} = 1$$

$$v = \frac{e^{2t}+1}{2}$$

$$\omega(x,t) = T(t)e^{4x}$$

$$\begin{cases} T'e^{4x} = 64\,Te^{4x} \\ T(0)e^{4x} = e^{4x} \end{cases}$$

$$T = Ce^{64t}$$

$$T(0) = C = 1$$

$$u(x,t) = \frac{e^{2t}+1}{2} + e^{64t+x}$$