1. Определить тип уравнения и привести к каноническому виду:
$u_{xx} + 4u_{xy} - 5u_{yy} + 2u_x = 0$

$$(dy)^2 - 4\,dx\,dy - 5(dx)^2$$

$$D = 16 + 20 = 36 > 0 \Rightarrow \text{гиперб. тип.}$$

$$\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} - 5 = 0$$

$$\left(\frac{dy}{dx} - 5\right)\left(\frac{dy}{dx} + 1\right) = 0$$

$$\left[\begin{array}{l} dy = 5dx \\ dy = -dx \end{array}\right. \Rightarrow \left[\begin{array}{l} y - 5x = C \\ y + x = C \end{array}\right.$$

$$\begin{cases} \xi = \dfrac{(y+x)-(y-5x)}{2} = 3x \\ \eta = \dfrac{y+x+y-5x}{2} = y - 2x \end{cases} \Rightarrow \begin{array}{l} u_x = 3u_\xi - 2u_\eta \\ u_{xx} = 9u_{\xi\xi} - 12u_{\xi\eta} + 4u_{\eta\eta} \\ u_{xy} = 3u_{\xi\eta} - 2u_{\eta\eta} \\ u_y = u_\eta \\ u_{yy} = u_{\eta\eta} \end{array}$$

$$9u_{\xi\xi} - 12u_{\xi\eta} + 4u_{\eta\eta} + 12u_{\xi\eta} - 8u_{\eta\eta} - 5u_{\eta\eta} + 6u_\xi - 4u_\eta = 0$$

$$9u_{\xi\xi} - 9u_{\eta\eta} + 6u_\xi - 4u_\eta = 0$$

$$u = ve^{\alpha\xi + \beta\eta} \Rightarrow u_\xi = (v_\xi + \alpha v)e^{\cdots}$$

$$u_{\xi\xi} = (v_{\xi\xi} + 2\alpha v_\xi + \alpha^2 v)\, e^{\cdots}$$

$$u_\eta = (v_\eta + \beta v)\, e^{\cdots}$$

$$u_{\eta\eta} = (v_{\eta\eta} + 2\beta v_\eta + \beta^2 v)\, e^{\cdots}$$

$$9v_{\xi\xi} + 18\alpha v_\xi + 9\alpha^2 v - 9v_{\eta\eta} - 18\beta v_\eta - 9\beta^2 v +$$

$$+ 6v_\xi + 6\alpha v - 4v_\eta - 4\beta v = 0$$

$$9v_{\xi\xi} - 9v_{\eta\eta} + (\cancelto{0}{18\alpha + 6})\, v_\xi + (\cancelto{0}{-18\beta - 4})\, v_\eta +$$

$$+ \underbrace{(9\alpha^2 - 9\beta^2 + 6\alpha - 4\beta)}_{=\, 1 - \frac{4}{9} - 2 + \frac{8}{9} \,=\, -\frac{5}{9}}\, v = 0$$

$$\alpha = -\frac{1}{3}, \quad \beta = -\frac{2}{9} \Rightarrow$$

$$\boxed{v_{\xi\xi} - v_{\eta\eta} = \frac{-5}{81}\, v}$$

2.
$$u_t = u_{xx} + 2\sin\frac{2\pi x}{3}, \quad 0<x<3, \quad t>0,$$
$$u(0,t)=1, \quad u(3,t)=0, \quad t\ge 0,$$
$$u(x,0)=1-\frac{x}{3}+\sin(\pi x), \quad 0\le x\le 3.$$

Обнуление: $v(x,t) = u(x,t) + \frac{(x-3)}{3}$

$$\begin{cases} v_t = u_{xx} + 2\sin\frac{2\pi x}{3} \\ v(0,t) = 0 \\ v(3,t) = 0 \\ v(x,0) = 1-\frac{x}{3}+\frac{x}{3}-1+\sin\pi x = \sin\pi x \end{cases}$$

$$\lambda_n = \left(\frac{\pi n}{3}\right)^2 \quad X_n(x) = \sin\frac{\pi n x}{3}$$

1) $$\begin{cases} w_t = w_{xx} + 2\sin\frac{2\pi x}{3} \\ w(0,t) = w(3,t) = 0 \\ w(x,0) = 0 \end{cases}$$

$$w(x,t) = T(t)\sin\frac{2\pi x}{3}$$

$$\begin{cases} T' = -\frac{4}{9}\pi^2 T + 2 \\ T(0) = 0 \end{cases} \qquad T = Ce^{-\frac{4\pi^2}{9}t} + \frac{9}{2\pi^2}$$

$$T(0) = C + \frac{9}{2\pi^2} = 0$$

$$w(x,t) = \left(1-e^{-\frac{4\pi^2}{9}t}\right)\cdot\frac{9}{2\pi^2}$$

2) $\begin{cases} h_t = h_{xx} \\ h(0,t) = h(3,t) = 0 \\ h(x,0) = \sin \pi x \end{cases}$

$$h(x,t) = T(t) \sin \pi x$$

$$\begin{cases} T' = -\pi^2 T \\ T(0) = 1 \end{cases} \Rightarrow \begin{matrix} T = Ce^{-\pi^2 t} \\ T(0) = C = 1 \end{matrix} \Rightarrow h(x,t) = e^{-\pi^2 t} \sin \pi x$$

$$u(x,t) = \left(1 - e^{-\frac{4}{9}\pi^2 t}\right) \frac{9}{2\pi^2} \sin \frac{2\pi x}{3} + e^{-\pi^2 t} \sin \pi x + \frac{3-x}{3}$$

3. $$\begin{aligned} &u_t = u_{xx} + 1 + \cos\frac{x}{2}, \quad 0 < x < \pi, \quad t > 0, \\ &u_x(0,t) = 1, \quad u(\pi,t) = t + \pi, \quad t \geq 0, \\ &u(x,0) = 2x, \quad 0 \leq x \leq \pi. \end{aligned}$$

Замена: $v(x,t) = u(x,t) - t - x$

$$\begin{cases} v_t + 1 = v_{xx} + 1 + \cos\frac{x}{2} \\ v_x(0,t) = v(\pi,t) = 0 \\ v(x,0) = x \end{cases}$$

1) $\begin{cases} w_t = w_{xx} + \cos\frac{x}{2} \\ w_x(0,t) = w(\pi,t) = 0 \\ w(x,0) = 0 \end{cases}$

$$\lambda_n = \left(\frac{\pi(2n-1)}{2\pi}\right)^2$$

$$X_n(x) = \cos\frac{2n-1}{2}x$$

$\cos\frac{x}{2}$ – соб. ф-е

$$w(x,t) = T(t)\cos\frac{x}{2}$$

$$\begin{cases} T' = -\frac{1}{4}T + 1 \\ T(0) = 0 \end{cases} \Longrightarrow T = Ce^{-\frac{t}{4}} + 4$$

$$T(0) = C + 4 = 0$$

$$w(x,t) = 4\left(1 - e^{-t/4}\right)\cos\frac{x}{2}$$

2) $$\begin{cases} h_t = h_{xx} \\ h_x(0,t) = h(\pi,t) = 0 \\ h(x,0) = x \end{cases}$$

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2$$

$$X_n(x) = \cos\frac{2n-1}{2}x$$

$$T_n(t) = C_n e^{-\left(\frac{2n-1}{2}\right)^2 t}$$

$$h(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{2n-1}{2}\right)^2 t} \cdot \cos\frac{2n-1}{2}x$$

$$h(x,0) = \sum_{n=1}^{\infty} C_n \cos\frac{2n-1}{2}x = x$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} x\cos\frac{2n-1}{2}x\,dx = \frac{2}{\pi}\left[\frac{2x}{2n-1}\sin x\cdot\frac{2n-1}{2}\Big|_0^{\pi} - \frac{2}{2n-1}\int_0^{\pi}\sin\frac{2n-1}{2}x\,dx\right] =$$

$$= \frac{2}{\pi}\left[\frac{2(-1)^{n+1}}{2n-1}\cdot\pi + \left(\frac{2}{2n-1}\right)^2\cos\frac{2n-1}{2}x\Big|_0^{\pi}\right] = \frac{4}{2n-1}\left[(-1)^{n+1} - \frac{2}{\pi(2n-1)}\right]$$

$$u(x,t) = t + x + 4\left(1 - e^{-\frac{t}{4}}\right)\cos\frac{x}{2} + \sum_{k=1}^{\infty} C_n e^{-\left(\frac{2n-1}{2}\right)^2 t}\cos\frac{4k-1}{2}x$$

4. $u_t = \frac{1}{4} u_{xx} + 2t + e^t, \quad -\infty < x < +\infty \quad , \quad t > 0 \quad ,$

$u(x,0) = 1 + e^{-x} \quad , \quad -\infty < x < +\infty.$

$$\begin{cases} v_t = \frac{1}{4} v_{xx} + 2t + e^t \\ v(x,0) = 1 \end{cases}$$

$$v(x,t) = v(t)$$

$$\begin{cases} v' = 2t + e^t \\ v(0) = 1 \end{cases}$$

$$v = t^2 + e^t + C$$

$$v(0) = 1 + C = 1$$

$$v(x,t) = t^2 + e^t$$

$$\begin{cases} w_t = \frac{1}{4} w_{xx} \\ w(x,0) = e^{-x} \end{cases}$$

$$w = T(t)\, e^{-x}$$

$$\begin{cases} T' = \frac{T}{4} \\ T(0) = 1 \end{cases}$$

$$T = c e^{\frac{t}{4}}$$

$$T(0) = c = 1$$

$$w(x,t) = e^{t/4 - x}$$

$$u(x,t) = t^2 + e^t + e^{t/4 - x}$$

$$u_t = a^2 u_{xx} - hu \quad , \quad 0 < x < +\infty \quad , \quad t > 0 \quad ,$$

5, $$u(x,0) = \begin{cases} 0 & , 0 < x < c \quad , \\ u_0 , & c < x < +\infty , \end{cases}$$

$$u_x(0,t) = 0.$$

Чётное продолжение: $\Phi(x) = \begin{cases} 0, & |x| < c \\ u_0, & |x| > c \end{cases}$

$$w(x,t) = u(x,t) \cdot e^{ht}$$

$$w_t = (u_t + hu) e^{ht}$$

$$\begin{cases} w_t = a^2 w_{xx} \\ w_x(0,t) = 0 \\ w(x,0) = \Phi(x) \end{cases}$$

$$w(x,t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{+\infty} \Phi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi =$$

$$= \frac{u_0}{2\sqrt{\pi a^2 t}} \left[\int_{-\infty}^{-c} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi + \int_{c}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi \right] = \begin{cases} z = \frac{x-\xi}{2a\sqrt{t}} \\ d\xi = -2a\sqrt{t} \cdot dz \end{cases}$$

$$= \frac{-2u_0 a\sqrt{t}}{2\sqrt{\pi a^2 t}} \left[\int_{+\infty}^{\frac{x+c}{2a\sqrt{t}}} e^{-z^2} dz + \int_{\frac{x-c}{2a\sqrt{t}}}^{-\infty} e^{-z^2} dz \right] =$$

$$= -\frac{u_0}{2} \left[\Phi\left(\frac{x+c}{2a\sqrt{t}}\right) - 1 + (-1) - \Phi\left(\frac{x-c}{2a\sqrt{t}}\right) \right]$$

$$u(x,t) = \frac{e^{-ht} u_0}{2} \left[2 + \Phi\left(\frac{x-c}{2a\sqrt{t}} \right) - \Phi\left(\frac{x+c}{2a\sqrt{t}} \right) \right]$$