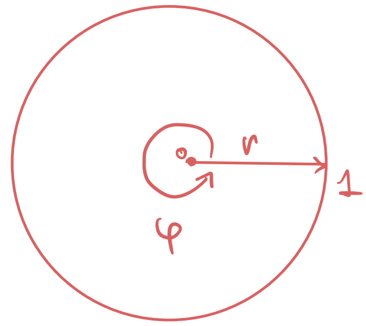
Вариант 1

1 Решить задачу
$\Delta u(r,\varphi) = \sin 3\varphi,\ 0 \le r < 1,\ 0 \le \varphi < 2\pi.$
$u(r=1,\varphi) = \cos 2\varphi$



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \sin 3\varphi$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \varphi^2} = r^2 \sin 3\varphi$$

$$u_{частное}(r,\varphi) = A r^2 \sin 3\varphi$$

$$\frac{\partial u_{част}}{\partial r} = 2Ar\sin 3\varphi$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u_{част}}{\partial r}\right) = 4Ar^2\sin 3\varphi \Rightarrow 4A - 9A = 1$$

$$A = -\frac{1}{5}$$

$$u_{част} = -\frac{r^2}{5}\sin 3\varphi$$

$$u(r,\varphi) = V(r,\varphi) - \frac{r^2}{5}\sin 3\varphi$$

$$\begin{cases} v(r,\varphi) = 0 \\ V(r=1,\varphi) = \cos 2\varphi + \dfrac{\sin 3\varphi}{5} \end{cases} \quad 0 \le r < 1, \quad 0 \le \varphi < 2\pi$$

$$v(r,\varphi) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\varphi + B_n \sin n\varphi)$$

$$v(1,\varphi) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi) = \cos 2\varphi$$

$$A_2 = 1, \quad A_{n\neq 2} = 0, \quad B_3 = \frac{1}{5}, \quad B_{n\neq 3} = 0$$

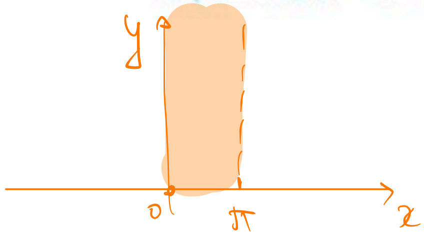
$$u(r,\varphi) = r^2\cos 2\varphi + \frac{r^3}{5}\sin 3\varphi - \frac{r^2}{5}\sin 3\varphi$$

2

Решить задачу

$\Delta u(x,y)=0,\ 0<x<\pi,\ 0<y<\infty,$

$u(x=0,y)=0,\ u(x=\pi,y)=0,\ u(x,y=0)=\sin 2x$



$$u = X(x)\cdot Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X(0) = X(\pi) = 0 \\ X'' + X\lambda = 0 \end{cases}$$

$$\lambda_n = \left(\frac{\pi n}{e}\right)^2 = n^2 \Rightarrow X_n = \sin nx$$

$$Y''(y) - n^2 Y(y) = 0$$

$$Y_n(y) = \underbrace{A_n e^{ny}}_{\substack{\text{неогр при } y\to\infty \\ \Rightarrow A_n = 0}} + B_n e^{-ny} = B_n e^{-ny}$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin nx\, e^{-ny}$$

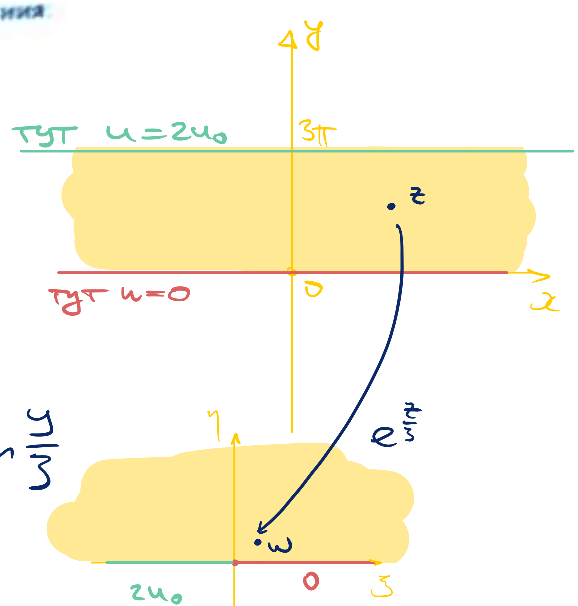
$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx = \sin 2x$$

$$B_2 = 1;\quad B_{n\neq 2} = 0$$

$$u(x,y) = \sin 2x \cdot e^{-2y}$$

3

Решить задачу методом конформного отображения
$\Delta u(x,y)=0,\quad -\infty<x<\infty,\quad 0<y<3\cdot\pi.$
$u(x,y=0)=0,\ u(x,y=3\cdot\pi)=2\cdot u_0$



$$w=\xi+i\eta=e^{\frac{\pi z}{3\pi}}=e^{\frac{z}{3}}=$$

$$=e^{\frac{x}{3}}\left(\cos\frac{y}{3}+i\sin\frac{y}{3}\right)$$

$$\xi=e^{\frac{x}{3}}\cos\frac{y}{3},\quad \eta=e^{\frac{x}{3}}\sin\frac{y}{3}$$

$$u(\xi_0,\eta_0)=\frac{\eta_0}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)}{(\xi-\xi_0)^2+\eta_0^2}\,d\xi=\frac{2u_0\eta_0}{\pi}\int_{-\infty}^{0}\frac{d\xi}{(\xi-\xi_0)^2+\eta_0^2}=$$

$$=\frac{2u_0}{\pi}\int_{-\infty}^{\frac{\xi_0}{\eta_0}}\frac{d\left(\frac{\xi-\xi_0}{\eta_0}\right)}{\left(\frac{\xi-\xi_0}{\eta_0}\right)^2+1}=\frac{2u_0}{\pi}\operatorname{arctg}\frac{\xi-\xi_0}{\eta_0}\Bigg|_{\xi=-\infty}^{\xi=0}=$$

$$=\frac{2u_0}{\pi}\left(-\operatorname{arctg}\frac{\xi_0}{\eta_0}+\frac{\pi}{2}\right)=u_0-\frac{2u_0}{\pi}\operatorname{arctg}\left(\operatorname{ctg}\frac{y_0}{3}\right)=$$

$$=u_0-\frac{2u_0}{\pi}\left(\frac{\pi}{2}-\operatorname{arcctg}\operatorname{ctg}\frac{y}{3}\right)=\frac{2y_0}{3\pi}u_0=u(x_0,y_0)$$

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$$\begin{cases} u_{tt} = u_{xx} & x \in (0, \pi) \\ u(0,t) = 0 & t > 0 \\ u_x(\pi, t) = t \\ u(x,0) = 2 \\ u_t(x,0) = x \end{cases}$$

$$u(x,t) = v(x,t) + tx$$

$$\begin{cases} v_{tt} = v_{xx} \\ v(0,t) = v_x(\pi,t) = 0 \\ v(x,0) = 2 \\ v_t(x,0) = 0 \end{cases}$$

$$v(x,t) = X(x) \cdot T(t)$$

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X'(\pi) = 0 \end{cases}, \quad \lambda_n = \left(\frac{(2n-1)\pi}{2\pi}\right)^2 = \left(\frac{2n-1}{2}\right)^2$$

$$X_n = \sin \frac{2n-1}{2} x$$

$$\begin{cases} T_n'' + \lambda_n T_n = 0 \\ T(0) = 2 \\ T'(0) = 0 \end{cases} \qquad T_n(t) = A_n \cos \frac{2n-1}{2} t + B_n \sin \frac{2n-1}{2} t$$

$$v(x,t=0)=\sum_{n=1}^{\infty} A_n \sin\frac{2n-1}{2}x = 2$$

$$A_n=\frac{2}{\pi}\int_0^{\pi} 2\sin\frac{2n-1}{2}x\,dx = -\frac{8}{\pi(2n-1)}\cos\frac{2n-1}{2}x\Big|_0^{\pi}$$

$$=\frac{8}{(2n-1)\pi}$$

$$v_t(x,t=0)=\sum_{n=1}^{\infty} B_n\ldots = 0 \Longrightarrow B_n=0$$

$$u(x,t)=tx+\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos t\left(n-\frac{1}{2}\right)\cdot\sin x\left(n-\frac{1}{2}\right)$$

5. Решить задачу:
$u_{tt}=u_{xx}$, $0<x<+\infty$, $t>0$,
$u(x,t=0)=u_0\sin kx$, $u_t(x,t=0)=u_0 k\cos kx$.
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$u_x(x=0,t)=0$, $t>0$. Построить график функции $u\left(x,t=\dfrac{3\pi}{2k}\right)$ в зависимости от

Чётное продолжение:

$$\Phi(x,t)=\begin{cases}u_0\sin kx & x\geq 0\\ -u_0\sin kx & x<0\end{cases}$$

$$\Psi(x,t)=u_0k\cos kx$$

$x+t>0$

$$u(x,t)=\begin{cases}\frac{1}{2}\left[u_0\sin k(x+t)+u_0\sin k(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}u_0k\cos k\alpha\,d\alpha\\[2ex] \frac{1}{2}\left[u_0\sin k(x+t)-u_0\sin k(x-t)\right]+\frac{1}{2}\int_0 u_0k\cos k\alpha\,d\alpha + \int_0^{at-x}\end{cases}$$

$$= \frac{u_0}{2}\begin{cases} \sin k(x+t) + \sin k(x-t) + \sin k\alpha \Big|_{\alpha=x-t}^{\alpha=x+t} \\ \sin k(x+t) - \sin k(x-t) + \sin k\alpha \Big|_{\alpha=x-t}^{\alpha=x+t} \end{cases} =$$

$$= \begin{cases} u_0 \sin(x+t)k & x-t>0 \\ u_0\left(\sin k(x+t) + \sin k(x-t)\right) & x-t<0 \end{cases}$$

$$u(x,t_0) = \begin{cases} u_0 \sin\left(kx + \frac{3\pi}{2}\right) = -u_0 \cos kx & kx > \frac{3\pi}{2} \\ \underbrace{u_0\left[\sin\left(kx+\frac{3\pi}{2}\right) + \sin\left(kx - \frac{3\pi}{2}\right)\right]}_{-2u_0 \cos kx} & kx < \frac{3\pi}{2} \end{cases}$$