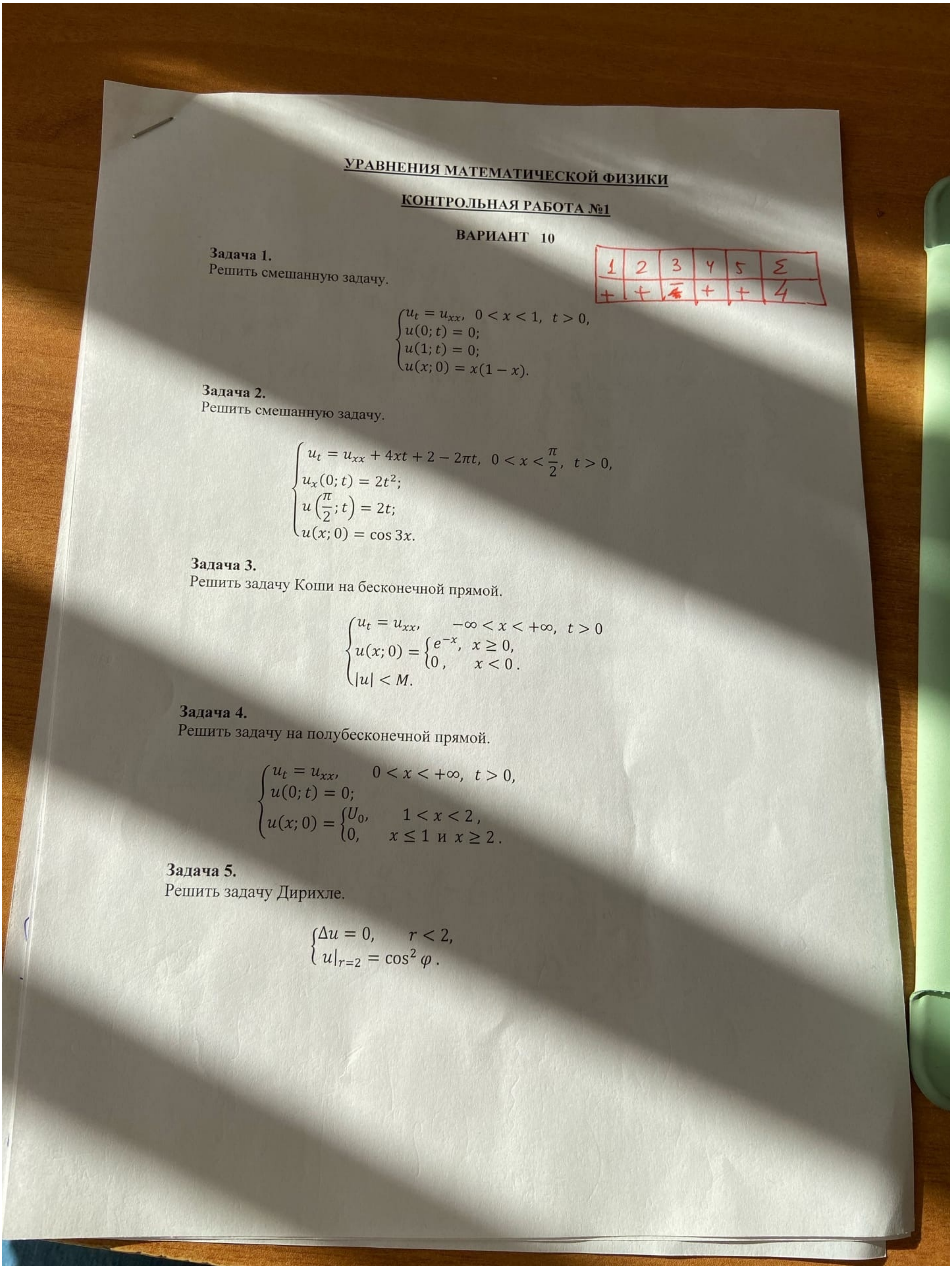
Фотография 1



УРАВНЕНИЯ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

КОНТРОЛЬНАЯ РАБОТА №1

ВАРИАНТ 10

1	2	3	4	5	Σ
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Задача 1.
Решить смешанную задачу.

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \ t > 0, \\ u(0;t) = 0; \\ u(1;t) = 0; \\ u(x;0) = x(1-x). \end{cases}$$

Задача 2.
Решить смешанную задачу.

$$\begin{cases} u_t = u_{xx} + 4xt + 2 - 2\pi t, \quad 0 < x < \dfrac{\pi}{2}, \ t > 0, \\ u_x(0;t) = 2t^2; \\ u\left(\dfrac{\pi}{2};t\right) = 2t; \\ u(x;0) = \cos 3x. \end{cases}$$

Задача 3.
Решить задачу Коши на бесконечной прямой.

$$\begin{cases} u_t = u_{xx}, \quad -\infty < x < +\infty, \ t > 0 \\ u(x;0) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \\ |u| < M. \end{cases}$$

Задача 4.
Решить задачу на полубесконечной прямой.

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < +\infty, \ t > 0, \\ u(0;t) = 0; \\ u(x;0) = \begin{cases} U_0, & 1 < x < 2, \\ 0, & x \leq 1 \text{ и } x \geq 2. \end{cases} \end{cases}$$

Задача 5.
Решить задачу Дирихле.

$$\begin{cases} \Delta u = 0, \quad r < 2, \\ u|_{r=2} = \cos^2 \varphi. \end{cases}$$

Фотография 2

...а Анастасия Вариант 10.

$$\begin{cases} u_t = u_{xx}, & 0<x<1,\ t>0 \\ u(0,t)=0 \\ u(1,t)=0 \\ u(x,0)=x(1-x) \end{cases}$$

$u = X(x)T(t)$ $\quad \dfrac{X''}{X} = \dfrac{T'}{T} = -\lambda$

$X = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

$X\big|_{(0,t)} = c_1 = 0$ $\quad X\big|_{(1,t)} = c_2 \sin\sqrt{\lambda} = 0$

$\Rightarrow \sqrt{\lambda} = \pi n$ $\quad \lambda_n = \pi^2 n^2$ $\quad X_n = \sin \pi n x$

$T_n = c_n e^{-\lambda_n t}$

$$c_n = 2\int_0^1 x(1-x)\sin \pi n x\,dx = \left\{\begin{array}{l} f=(1-x)x \\ g' = \sin\pi n x \\ f' = 1-2x \\ g = -\dfrac{\cos\pi n x}{\pi n} \end{array}\right\} =$$

$$\cancel{= 2\int_0^1 (1-2x)\frac{\cos\pi n x}{\pi n}}$$

$$= 2\left(\frac{(x-1)x\cos\pi n x}{\pi n}\Big|_0^1 + \int_0^1 \frac{(1-2x)\cos\pi n x}{\pi n}\,dx\right) =$$

$$= 2\left(0 + \int_0^1 \frac{(1-2x)\cos\pi n x}{\pi n}\,dx\right) = \left\{\begin{array}{l} f = 1-2x \\ g' = \cos\pi n x \\ f' = -2 \\ g = \dfrac{\sin\pi n x}{\pi n} \end{array}\right\} =$$

$$= \frac{2}{\pi n}\left(\frac{(1-2x)\sin\pi n x}{\pi n}\Big|_0^1 + \int_0^1 \frac{2\sin\pi n x\,dx}{\pi n}\right) = \frac{4}{\pi^2 n^2}\int_0^1 \sin\pi n x\,dx =$$

$$= \frac{4}{\pi^2 n^2}\left(-\cos\pi n x\right)\Big|_0^1 = \frac{4}{\pi^3 n^3}\left(-(-1)^n + 1\right) = \frac{4\left(1-(-1)^n\right)}{(\pi n)^3}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\pi n x \cdot e^{-\pi^2 n^2 t} \cdot \frac{4\left(1-(-1)^n\right)}{(\pi n)^3} \ //.$$

+

Фотография 3

...ва Анастасия Вариант 10

$$\begin{cases} u_t = u_{xx} + 4xt + 2 - 2\pi t, & 0 < x < \frac{\pi}{2}, t > 0 \\ u_x(0,t) = 2t^2 \\ u(\frac{\pi}{2}, t) = 2t \\ u(x,0) = \cos 3x \end{cases}$$

$u(x,t) = v(x,t) + ax + b.$

$\underbrace{v_x(0,t)}_{0} + a = 2t^2 \Rightarrow a = 2t^2$

$\underbrace{v(\frac{\pi}{2},t)}_{0} + \frac{a\pi}{2} + b = 0 + t^2\pi + b = 2t$

$b = 2t - t^2\pi$

$u(x,t) = v(x,t) + 2t^2 x + 2t - t^2\pi$

$v_t + \cancel{4tx} + \cancel{2} - \cancel{2t\pi} = v_{xx} + \cancel{4xt} + \cancel{2} - \cancel{2\pi t}$

$$\begin{cases} v_t = v_{xx} \\ v_x(0,t) = 0 \\ v(\frac{\pi}{2}, t) = 0 \\ v(x,0) = \cos 3x \end{cases}$$

$v = X(x)T(t)$

$X = c_1 \cos\sqrt{\lambda} x + c_2 \sin\sqrt{\lambda} x$

$X' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos x\sqrt{\lambda}$

$X'(0) = c_2\sqrt{\lambda} = 0 \qquad c_2 = 0.$

$X = c_1\cos\sqrt{\lambda}x$

$X(\frac{\pi}{2}) = c_1\cos\frac{\pi\sqrt{\lambda}}{2} = 0 \Rightarrow \frac{\cancel{\pi}\sqrt{\lambda}}{2} = \frac{\cancel{\pi}}{2} + \cancel{\pi}n$

$\sqrt{\lambda_n} = 2n+1$

$X_n(x) = \cos(2n+1)x$

$T_n = c_n e^{-\lambda_n t}$

$v(x,0) = \cos 3x = \sum X_n T_n = \sum_{n=1}^{\infty} \cos(2n+1)x\, c_n$

$\Rightarrow C_n = 0, n \neq 1 \qquad c_1 = 1.$

$v(x,t) = \cos 3x\, e^{-9t}$ // +

Проверка: $-9e^{-9t}\cos 3x = e^{-9t}\cdot(-9)\cos 3x$ – верно.

$u(x,t) = \cos 3x\, e^{-9t} + 2t^2 x + 2t - t^2\pi$ // – Ответ.

Проверка 2:

$\cancel{-9e^{-9t}\cos 3x} + \cancel{4tx} + \cancel{2} - \cancel{2t\pi} = \cancel{-9\cos 3x\, e^{-9t}} + \cancel{4xt} + \cancel{2} - \cancel{2\pi t}$ верно.

$u(\frac{\pi}{2},t) = \cancel{2}\cdot t^2\frac{\pi}{\cancel{2}} + 2t - t^2\pi = 2t$ – верно.

$u(x,0) = \cos 3x + 0$ – верно.

$u'_x = -3\sin 3x\, e^{-9t} + 2t^2$

$u'_x(0,t) = 2t^2$ – верно.

Фотография 4

№5. $\begin{cases} \Delta u = 0, & r<2 \\ u|_{r=2} = \cos^2\varphi \end{cases}$ $\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1$

$\cos^2\varphi = \frac{\cos 2\varphi + 1}{2}$

$$u(r,\varphi) = c + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (A_n \cos n\varphi + B_n \sin n\varphi), \quad R = 2$$

$$u(2,\varphi) = c + \sum_{n=1}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi) = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$$

$c = \frac{1}{2}$, $A_2 = \frac{1}{2}$, $B_n = 0$, $A_k = 0, k \neq 2$

$$u(r,\varphi) = \frac{1}{2} + \frac{1}{2}\cdot\frac{r^2}{2^2}\cos 2\varphi = \frac{1}{2} + \frac{r^2}{8}\cos 2\varphi \,//$$

+

№4. $\begin{cases} u_t = u_{xx}, & 0<x<+\infty, t>0 \\ u(0,t) = 0 \\ u(x,0) = \begin{cases} u_0, & 1<x<2 \\ 0, & x\le 1 \cup x \ge 2 \end{cases} \end{cases}$ $G(x,\xi,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4t}}$

$$u(x,t) = \int_{-\infty}^{+\infty} G(x,\xi,t)\varphi(\xi)d\xi = \int_1^2 u_0 G(x,\xi,t)d\xi =$$

$$= \frac{u_0}{2\sqrt{\pi t}} \int_1^2 e^{-\frac{(x-\xi)^2}{4t}} d\xi = \left\{ \frac{\xi - x}{2\sqrt{t}} = p, \quad \xi = 2\sqrt{t}p + x, \quad d\xi = 2\sqrt{t}dp \right\} =$$

$$= \frac{u_0}{2\sqrt{\pi t}} \int_a^b e^{-p^2} 2\sqrt{t}\, dp = \frac{u_0}{\cancel{2}\sqrt{\pi}} \int_a^b e^{-p^2} dp = \begin{cases} a: \xi = 1 \Rightarrow p = \frac{1-x}{2\sqrt{t}} \\ b: \xi = 2 \Rightarrow p = \frac{2-x}{2\sqrt{t}} \end{cases}$$

$$= \frac{u_0}{\sqrt{\pi}}\left(\int_0^b e^{-p^2}dp - \int_0^a e^{-p^2}dp \right) = \frac{u_0}{\sqrt{\pi}}\left(\Phi(b) - \Phi(a)\right)\frac{\sqrt{\pi}}{2} =$$

$$= \frac{u_0}{2}\left(\Phi(b) - \Phi(a)\right) // = \frac{u_0}{2}\left(\Phi\left(\frac{2-x}{2\sqrt{t}}\right) - \Phi\left(\frac{1-x}{2\sqrt{t}}\right)\right) //$$

Фотография 5

№3. $\begin{cases} u_t = u_{xx}, & -\infty < x < +\infty, t > 0 \\ u(x,0) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \\ |u| < M \end{cases}$ $\quad G(x,\xi,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4t}}$

$$u(x,t) = \int_{-\infty}^{+\infty} G(x,\xi,t)\varphi(\xi)d\xi = \int_0^{+\infty} e^{-x} G(x,\xi,t)d\xi =$$

$$= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} e^{-\frac{(x-\xi)^2}{4t} \cancel{-x}\ \xi} d\xi = \left\{ \begin{array}{l} \frac{\xi - x}{2\sqrt{t}} = p \quad d\xi = 2\sqrt{t}dp \\ \xi = 2\sqrt{t}p + x \end{array} \right\} =$$

$$\int_{-\infty}^{+\infty} G(x,\xi,t)\varphi(\xi)d\xi$$

$$= \frac{e^{-x}}{2\sqrt{\pi t}} \int_a^b e^{-p^2} 2\sqrt{t}dp \ominus$$

$$\left\{ \begin{array}{l} a: \xi = 0 \Rightarrow p = -\frac{x}{2\sqrt{t}} \\ b: \xi \to +\infty \Rightarrow p \to +\infty \end{array} \right\} \ominus \frac{e^{-x}}{2\sqrt{\pi t}} \cdot 2\sqrt{t} \int_{-\frac{x}{2\sqrt{t}}}^{+\infty} e^{-p^2} dp =$$

$$= \frac{e^{-x}}{\sqrt{\pi}} \left(\int_0^{+\infty} e^{-p^2} dp + \int_0^{\frac{x}{2\sqrt{t}}} e^{-p^2} dp \right) = \quad \ominus$$

$$= \frac{e^{-x}}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \cdot \Phi\left(\frac{x}{2\sqrt{t}}\right) \right) = \frac{e^{-x}}{2}\left(1 + \Phi\left(\frac{x}{2\sqrt{t}}\right)\right) //.$$

№4 (~~[illegible]~~) Продолжим нечетным образом. ~~У Вас задача на всей прямой~~

$$u(x,t) = \int_0^{+\infty} \left(e^{-\frac{(x-\xi)^2}{4t}} \frac{1}{2\sqrt{\pi t}} - e^{-\frac{(x+\xi)^2}{4t}} \frac{1}{2\sqrt{\pi t}} \right) \varphi(\xi) d\xi =$$

$$= \frac{u_0}{2\sqrt{\pi t}} \int_1^2 \left(e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right) d\xi = u_0 \left(\frac{\Phi\left(\frac{x-1}{2\sqrt{t}}\right) - \Phi\left(\frac{x-2}{2\sqrt{t}}\right)}{2} - \right.$$

$$\left. - \left(\frac{\Phi\left(\frac{x+2}{2\sqrt{t}}\right) - \Phi\left(\frac{x+1}{2\sqrt{t}}\right)}{2} \right) \right) =$$

$$= \frac{u_0}{2} \left(\Phi\left(\frac{x-1}{2\sqrt{t}}\right) + \Phi\left(\frac{x+1}{2\sqrt{t}}\right) - \Phi\left(\frac{x-2}{2\sqrt{t}}\right) - \Phi\left(\frac{x+2}{2\sqrt{t}}\right) \right) // \quad \oplus$$