1. Определить тип уравнения и привести к каноническому виду:

$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0$$

Хар. ур-е: $(dy)^2 - 2\,dx\,dy + 2(dx)^2 = 0$

$D = 4 - 8 = -4 \Rightarrow$ эллиптич. тип.

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 2 = 0$$

$$\frac{dy}{dx} = 1 \pm i \Rightarrow y = (1 \pm i)x + C$$

$$C = y - (1 \pm i)x$$

$$\begin{cases} \xi = \mathrm{Re}(C) = y - x \\ \eta = \mathrm{Im}(C) = x \end{cases}$$

$u_x = u_\eta - u_\xi$

$u_y = u_\xi$

$u_{xy} = u_{\eta\xi} - u_{\xi\xi}$

$u_{xx} = -2u_{\eta\xi} + u_{\eta\eta} + u_{\xi\xi}$

$u_{yy} = u_{\xi\xi}$

$\Rightarrow -2u_{\xi\eta} + u_{\eta\eta} + u_{\xi\xi} + 2u_{\eta\xi} - 2u_{\xi\xi} + 2u_{\xi\xi} + u_\xi = 0$

$u_{\eta\eta} + u_{\xi\xi} + u_\xi = 0$

$u = v e^{\alpha\xi + \beta\eta}$

$u_\xi = (v_\xi + \alpha v)e^{\dots}$

$u_{\xi\xi} = (v_{\xi\xi} + 2\alpha v_\xi + \alpha^2 v)e^{\dots}$

$u_{\eta\eta} = (v_{\eta\eta} + 2\beta v_\eta + \beta^2 v)e^{\dots}$

$$v_{\eta\eta} + 2\beta v_\eta + \beta^2 v + v_{\xi\xi} + 2\alpha v_\xi + \alpha^2 v + v_\xi + \alpha v = 0$$

$$V_{\xi\xi} + V_{\eta\eta} + \cancel{2\beta V_\eta} + \cancel{(2\alpha+1)}V_\xi + \underbrace{(\beta^2+\alpha+\alpha^2)}_{-1/4}V = 0$$

$\beta := 0$, $\alpha := -\frac{1}{2}$

$$\Rightarrow \quad V_{\xi\xi} + V_{\eta\eta} = \frac{V}{4}$$

2.

$$u_t = u_{xx} + x + 2t + \sin x \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

$$u(0,t) = t^2, \quad u(\pi,t) = t^2 + \pi t \quad , \quad t \geq 0 \quad ,$$

$$u(x,0) = x \quad , \quad 0 \leq x \leq \pi .$$

1) Обнуление кр. усл: $v(x,t) = u(x,t) - t^2 - xt$

$u(x,t) = v(x,t) + t^2 + xt$

$v_t = u_t - 2t - x$

$v_x = u_x + t \quad \Rightarrow \quad v_t + 2t + x = v_{xx} + x + 2t + \sin x$

$v_{xx} = u_{xx}$

$$\begin{cases} v_t = v_{xx} + \sin x & 0 < x < \pi,\ t > 0 \\ v(0,t) = 0 & \\ v(\pi,t) = 0 & t \geq 0 \\ v(x,0) = x & 0 \leq x \leq \pi \end{cases}$$

$\lambda_n = \left(\frac{\pi n}{\pi}\right)^2 = n^2$

$X_n(x) = \sin nx \Rightarrow \sin x$ – соб. ф-ия

Разобьём на две задачи:

$$\begin{cases} g_t = g_{xx} \\ g(0,t) = g(\pi,t) = 0 \\ g(x,0) = x \end{cases}$$

$g_n(x,t) = C_n e^{-n^2 t} \sin nx$

$g(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin nx$

$x = \sum_{n=1}^{\infty} C_n \sin nx$

$C_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$

$$\begin{cases} h_t = h_{xx} + \sin x \\ h(0,t) = h(\pi,t) = 0 \\ h(x,0) = 0 \end{cases}$$

$h(x,t) = T(t) \sin t$

$$\begin{cases} T' = -T + 1 \\ T(0) = 0 \end{cases}$$

$T_0 = Ce^{-t}$

$T_r = 1$

$T = Ce^{-t} + 1$

$T(0) = C + 1 = 0$

$u(x,t) = \left(-e^{-t} + 1\right)\sin x$

$$\int_0^\pi x \sin nx \, dx = \underbrace{-\frac{x\cos nx}{n}\Bigg|_0^\pi}_{-\frac{\pi\cos\pi n}{n} = \frac{(-1)^{n+1}}{n}\cdot\pi} + \frac{1}{n}\underbrace{\int_0^\pi \cos nx \, dx}_{0} = \frac{\pi\,(-1)^{n+1}}{n}$$

$$u(x,t) = t^2 + xt + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 t} \sin nx + \left(-e^{-t} + 1\right)\sin x$$

3. $u_t = u_{xx} + tx^2 + \cos(2\pi x) + e^t \quad , \quad 0 < x < 1 \quad , \quad t > 0 \quad ,$

$u_x(0,t) = 0 \quad , \quad u_x(1,t) = t^2 \quad , \quad t \geq 0 \quad ,$

$u(x,0) = 0 \quad , \quad 0 \leq x \leq 1.$

Обнуляем: $v(x,t) = u(x,t) - \frac{x^2 t^2}{2}$

$v_x(x,t) = u_x(x,t) - xt^2$

$u_t = v_t + tx^2$

$u_{xx} = v_{xx} + t^2$

$$\begin{cases} v_t = v_{xx} + t^2 + \cos 2\pi x + e^t \\ v_x(0,t) = 0 \;, \; v_x(1,t) = 0 \\ v(x,0) = 0 \end{cases}$$

1. $$\begin{cases} g_t = g_{xx} + \cos 2\pi x \\ g_x(0,t) = g_x(1,t) = 0 \\ g(x,0) = 0 \end{cases}$$

$\lambda = (\pi n)^2 \rightarrow X_n(x) = \cos \pi n x$ соб. ф-я

$g(x,t) = T(t) \cos 2\pi x$

$T' = -4\pi^2 T + 1 \quad \rightarrow \quad T_0 = Ce^{-4\pi t} \quad \rightarrow T = Ce^{-4\pi^2 t} + \frac{1}{4\pi^2}$

$\{T(0)=0$ $\quad T_2 = \frac{1}{4\pi^2}$

$g(x,t) = \frac{1}{4\pi^2}\left(1-e^{-4\pi^2 t}\right)\cos 2\pi x$

$T(0) = c + \frac{1}{4\pi^2}$

$c = -\frac{1}{4\pi^2}$

② $\begin{cases} h_t = h_{xx} + e^t + t^2 \\ h_x(0,t) = h_x(1,t) = 0 \\ h(x,0) = 0 \end{cases}$

$\lambda_n = (\pi n)^2, \quad X_n(x) = \cos \pi n x$

$X_0 = 1$ - с.ф.

$h(x,t) = 1 \cdot T(t)$

$\begin{cases} T' = e^t + t^2 \\ T(0) = 0 \end{cases} \Rightarrow$

$T = e^t + \frac{t^3}{3} + c$

$T(0) = 1 + c = 0$

$T = e^t + \frac{t^3}{3} - 1$

Ответ: $u(x,t) = \frac{1}{4\pi^2}\left(1-e^{-4\pi^2 t}\right)\cos 2\pi x + e^t + \frac{t^3}{3} - 1 + \frac{t^2 x^2}{2}$

4.

$$u_t = u_{xx} + e^{-t}\cos(2x) \quad , \quad -\infty < x < +\infty \quad , \quad t > 0 \quad ,$$
$$u(x,0) = \cos(4x) + \cos(2x) \quad , \quad -\infty < x < +\infty.$$

$$\begin{cases} v_t = v_{xx} \\ v(x,0) = \cos 4x \end{cases}$$

$v(x,t) = T(t)\cos 4x$

$$\begin{cases} T' = -16T \\ T(0) = 1 \end{cases}$$

$T = Ce^{-16t}$

$T(0) = C = 1 \Rightarrow \quad v(x,t) = e^{-16t}\cos 4x$

$$\begin{cases} w_t = w_{xx} + e^{-t}\cos 2x \\ w(x,0) = \cos 2x \end{cases}$$

$w(x,t) = T(t)\cos 2x$

$$\begin{cases} T' = -4T + e^{-t} \\ T(0) = 1 \end{cases}$$

$T = Ce^{-4t} + \alpha e^{-t}$

$T' = \cancel{-4Ce^{-4t}} - \alpha e^{-t} =$

$= \cancel{-4Ce^{-4t}} - 4\alpha e^{-t} + e^{-t}$

$\alpha = \frac{1}{3} \Rightarrow \quad T = Ce^{-4t} + \frac{1}{3}e^{-t}$

$T(0) = C + 1/3 = 1 = C = \frac{2}{3}$

$w(x,t) = \left[\frac{2}{3}e^{-4t} + \frac{1}{3}e^{-t}\right]\cos 2x$

$$u(x,t) = e^{-16t}\cos 4x + \left[\frac{2}{3}e^{-4t} + \frac{1}{3}e^{-t}\right]\cos 2x$$

5. $$u_t = a^2 u_{xx} - hu \quad , \quad 0 < x < +\infty \quad , \quad t > 0 \quad ,$$

$$u(x,0) = \begin{cases} 0 & ,0 < x < c \quad , \\ u_0 \,, & c < x < +\infty, \end{cases}$$

$$u(0,t) = 0.$$

Нечётное продолж-е: $\Phi(x) = \begin{cases} -u_0, & x < -C \\ 0, & |x| < C \\ u_0, & x > C \end{cases}$

$$\begin{cases} U_t = a^2 U_{xx} - hU \\ U(x,0) = \Phi(x) \\ U(0,t) = 0 \end{cases} \qquad U(x,t) = w(x,t)e^{-ht}$$

$$(w_t - wh)e^{-ht} = a^2 w_{xx} e^{-ht} - hwe^{-ht}$$

$$\begin{cases} w_t = a^2 w_{xx} \\ w(0,t) = 0 \\ w(x,0) = \begin{cases} -u_0 & x < -C \\ 0 & |x| < C \\ u_0 & x > C \end{cases} \end{cases}$$

$$w(x,t) = \int_{-\infty}^{+\infty} \Phi(\xi) \cdot e^{-\frac{(x-\xi)^2}{4a^2t}} \cdot \frac{1}{2\sqrt{\pi a^2 t}}\, d\xi =$$

$$= \frac{u_0}{2\sqrt{\pi a^2 t}} \left[-\int_{-\infty}^{-c} e^{-\frac{(x-\xi)^2}{4a^2t}}\, d\xi + \int_{c}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2t}}\, d\xi \right] =$$

$$= \left\{ \begin{array}{l} z = \dfrac{x-\xi}{2a\sqrt{t}} \\ d\xi = -2a\sqrt{t}\, dz \end{array} \right\} = \frac{-u_0}{\sqrt{\pi}} \left[-\int_{+\infty}^{\frac{x+c}{2a\sqrt{t}}} e^{-z^2}\, dz + \int_{\frac{x-c}{2a\sqrt{t}}}^{-\infty} e^{-z^2}\, dz \right] =$$

$$= -\frac{u_0}{2} \left[-\Phi\left(\frac{x+c}{2a\sqrt{t}}\right) + 1 + (-1) - \Phi\left(\frac{x-c}{2a\sqrt{t}}\right) \right] =$$

$$= \frac{u_0}{2} \left[\Phi\left(\frac{x-c}{2a\sqrt{t}}\right) + \Phi\left(\frac{x+c}{2a\sqrt{t}}\right) \right] e^{-ht}$$