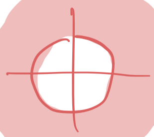
Вариант 2.

1. Решить задачу:

$\Delta u(r,\varphi)=1,\ a<r<\infty,\ 0\le\varphi<2\pi$,

$u(r=a,\varphi)=a^2\cdot\cos^2\varphi = \frac{a^2}{2}+\frac{a^2}{2}\cos 2\varphi$



$\cos 2\varphi = 2\cos^2\varphi - 1$

$\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$

в полярных коорд:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}=1$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial\varphi^2}=r^2$$

$$u_{част.}=Ar^2 \longrightarrow \frac{\partial u_{част}}{\partial r}=2Ar$$

$$4Ar^2=r^2 \longrightarrow A=\frac{1}{4} \longrightarrow u_{част}=\frac{r^2}{4}$$

$$u(r,\varphi)=v(r,\varphi)+\frac{r^2}{4}$$

$$\begin{cases}\Delta v=0\\ v(r=a,\varphi)=a^2\cos^2\varphi\end{cases}$$

неогр. при $r\to\infty \Rightarrow A_n, B_n=0$

$$v(r,\varphi)=A_0+B_0\ln\frac{1}{r}+\sum_{n=1}^{\infty}r^n(A_n\cos n\varphi+B_n\sin n\varphi)$$
$$+\sum_{n=1}^{\infty}r^{-n}(C_n\cos n\varphi+D_n\sin n\varphi)$$

$$v(a,\varphi)=A_0+\sum_{n=1}^{\infty}\frac{1}{a^n}(C_n\cos n\varphi+D_n\sin n\varphi)=\frac{a^2}{2}+\frac{a^2\cos 2\varphi}{2}$$

$$A_0=\frac{a^2}{2}\qquad C_2=\frac{a^4}{2};\quad C_{n\neq 2}=D_n=0$$

$$u(r,\varphi)=\frac{a^2}{2}+\frac{a^4r^{-2}}{2}\cdot\cos 2\varphi+\frac{r^2}{4}$$

2. Решить задачу:

$\Delta u(x,y)=0,\ 0<x<a,\ 0<y<b$,

$u(x,y=0)=u_1,\ u(x,y=b)=u_2$,

$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0$.

$$\begin{cases}\Delta V(x,y)=0\\ V|_{y=0}=u_1\\ V|_{y=b}=0\\ V_x|_{x=0}=V_x|_{x=a}=0\end{cases}\qquad\begin{cases}\Delta w=0\\ w|_{y=0}=0\\ w|_{y=b}=u_2\\ w_x|_{x=0}=w_x|_{x=a}=\end{cases}$$

1) $v = Y(y)$

$$\begin{cases} Y''(y) = 0 \\ Y(b) = 0 \\ Y(0) = u_1 \end{cases} \longrightarrow \quad Y = \alpha y + \beta$$

$$Y(0) = \beta = u_1 \longrightarrow v(x,y) = \left(-\frac{u_1}{b}y + u_1\right)$$

$$Y(b) = \alpha b + u_1 = 0$$

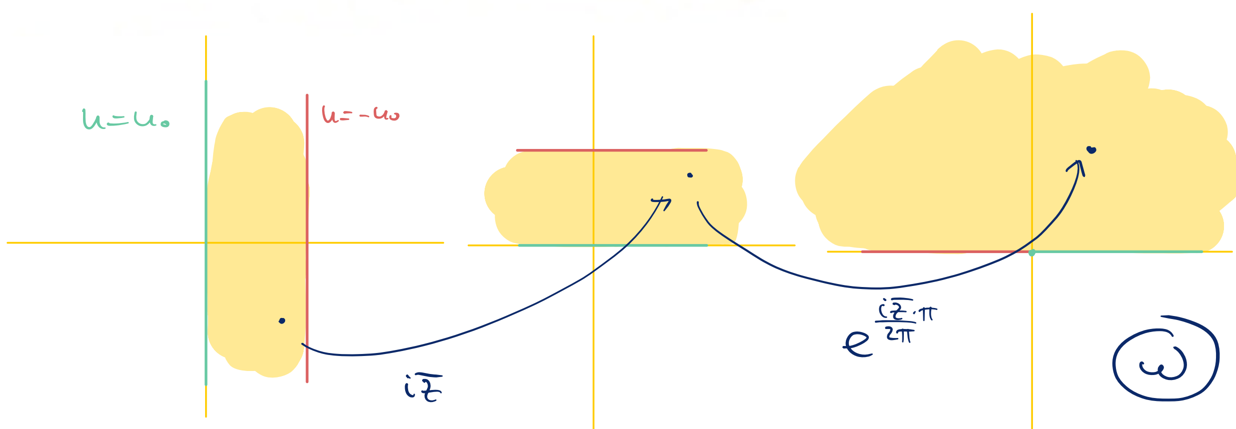
$$\alpha = -\frac{u_1}{b}$$

2) $w = Y(y)$

$$\begin{cases} Y''(y) = 0 \\ Y(0) = 0 \\ Y(b) = u_2 \end{cases} \qquad \begin{aligned} &Y(y) = \alpha y + \beta \\ &Y(0) = \beta = 0 \\ &Y(b) = \alpha b = u_2 \end{aligned} \Rightarrow w = \frac{u_2}{b}$$

$$u(x,y) = \left(-\frac{u_1}{b}y + u_1\right) + \frac{u_2}{b}$$

3. Решить методом конформного отображения:
$\Delta u(x,y) = 0$, $0 < x < 2\pi$, $-\infty < y < \infty$,
$u(x=0,y) = u_0$, $u(x=2\pi,y) = -u_0$.



$$w = e^{\frac{iz}{2}} = e^{\frac{y+ix}{2}} = e^{\frac{y}{2}}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right) = \xi + i\eta$$

$\xi = \frac{y}{2}\cos\frac{x}{2}$; $\eta = \frac{y}{2}\sin\frac{x}{2}$

$$u(\xi_0, \eta_0) = \frac{\eta_0}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)\,d\xi}{(\xi-\xi_0)^2+\eta_0^2} = \frac{-u_0}{\pi}\int_{-\infty}^{-\frac{\xi_0}{\eta_0}}\frac{d\left(\frac{\xi-\xi_0}{\eta_0}\right)}{\left(\frac{\xi-\xi_0}{\eta_0}\right)^2+1} +$$

$$+\frac{u_0}{\pi}\int_{-\frac{\xi_0}{\eta_0}}^{+\infty}\frac{dt}{t^2+1} = \frac{u_0}{\pi}\left[-\left(-\operatorname{arctg}\frac{\xi_0}{\eta_0}+\frac{\pi}{2}\right)+\frac{\pi}{2}+\operatorname{arctg}\frac{\xi_0}{\eta_0}\right] =$$

$= \dots$

4. Решить задачу:

$$u_{tt} = u_{xx} + \sin\frac{7x}{2},\quad 0 < x < \pi,\quad t > 0,$$

$$u(0,t) = t,\quad u_x(\pi,t) = 1,\quad t > 0,$$

$$u(x,0) = x + \frac{3}{49}\sin\frac{7x}{2},\quad u_t(x,0) = 0,\quad 0 < x < \pi.$$

$V(x,t) = u(x,t) - t - x$

$$\begin{cases} V_{tt} = u_{xx} + \sin\frac{7x}{2} \\ V(0,t) = 0 \\ V_x(\pi,t) = 0 \\ V(x,0) = \frac{3}{49}\sin\frac{7x}{2} \\ V_t(x,0) = -1 \end{cases}$$

$V = w + h$