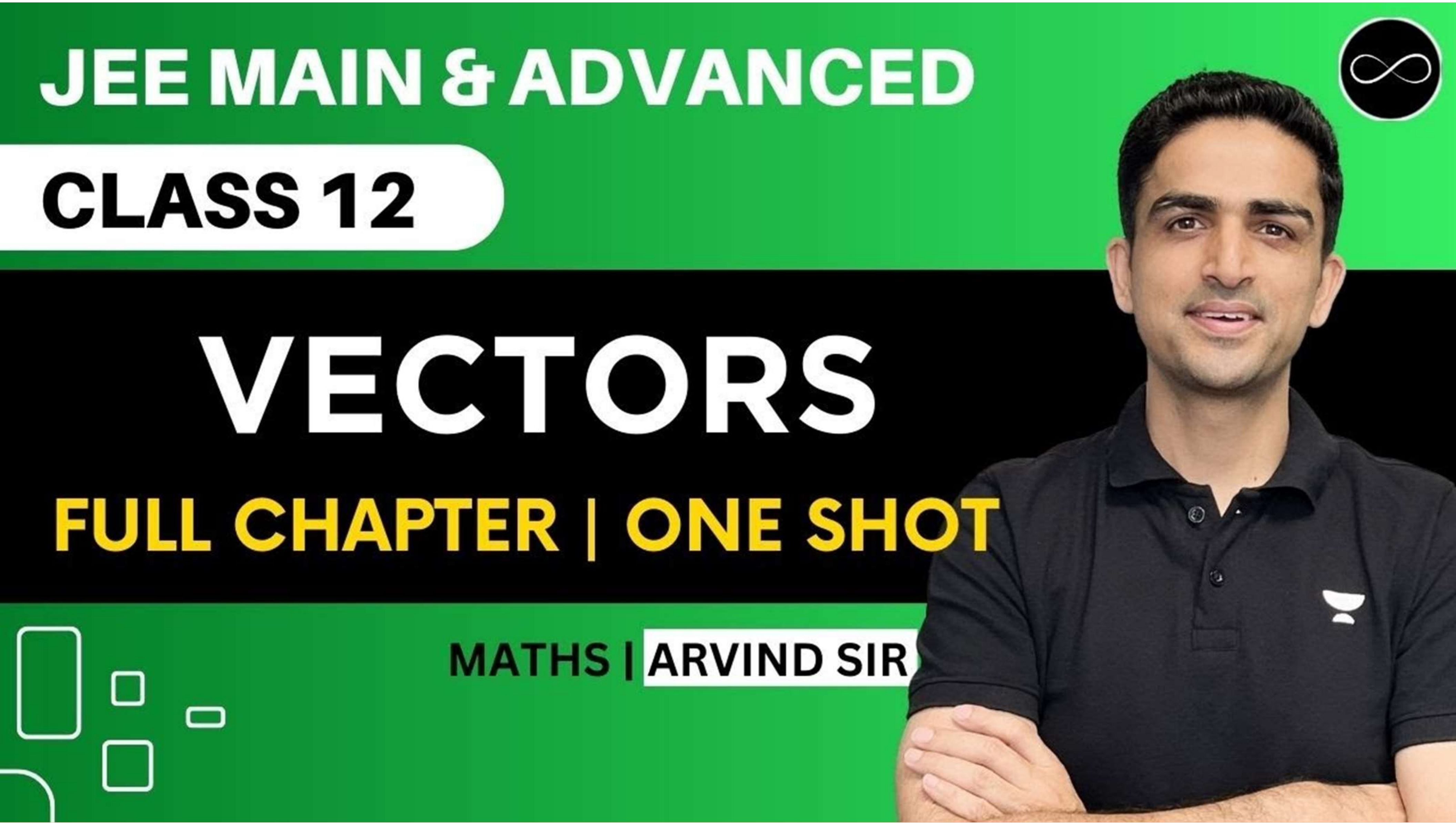

JEE MAIN & ADVANCED
CLASS 12
VECTORS
FULL CHAPTER | ONE SHOT
MATHS | ARVIND SIR

Nature of Chapter:

1. Its an independent chapter in itself, but it will be used a lot in 3D Geometry.
2. Its bit lengthy, but if you create hold on “Section formulae”, then you will be able to visualize the vector and if you create hold on “Dot” and “Cross” product then no problem will trouble you, conceptually.
3. There are some shortcuts and some observations, which will make some questions (previously asked in main and advanced) very easy, which we will be discussing, don’t miss that.
4. Students generally find questions of this topic confusing. Focussing at the right point given in the question and using some std observations (which i will be telling) is sth you need to make this topic easier for yourself.

Weightage of Vectors (Last 5 years)						
	2023	2022	2021	2020	2019	Average
JEE Main	5.8 %	4.6 %	4.5 %	3.6 %	3.5 %	4.40 %
Jee Advanced	3 %	3 %	7 %	6 %	2 %	4.20 %

Vectors

- Basics of Vectors
- Section Formulae
- Collinearity and Coplanarity
- Products of Vectors
- Triple Products
- Vector Equations

Vectors

Critical Topics:

- Dot product and Cross product
- Scalar Triple Product

Basics of Vectors

Basics of Vectors

Definition

A vector is a quantity which has magnitude as well as direction.

NOTE

In addition to magnitude and direction, two vector quantities of the same kind should be capable of being compounded according to the parallelogram law of addition.

Basics of Vectors

Types of Vectors

Zero Vector: A vector whose magnitude or length is 0 and whose direction is indeterminate is called the zero vector.

It is also known as the null vector. It is denoted by $\overline{0}$.

Basics of Vectors

Types of Vectors

Unit Vector: A unit vector is a vector whose magnitude or length is 1.

$|\bar{a}| = 1$,

$|\bar{b}| = 5$

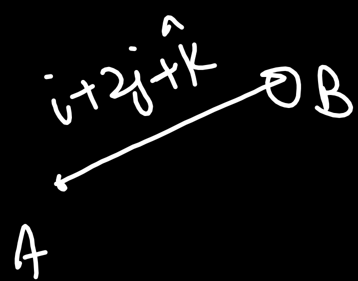
$\left|\frac{\bar{b}}{5}\right| = 1 \Rightarrow \frac{\bar{b}}{5}$ is a unit vector

Consider $\bar{p}$. Vector which parallel to $\bar{p}$ but having mag 10 $\longrightarrow \left(\frac{\bar{p}}{|\bar{p}|}\right) \times 10$

Basics of Vectors

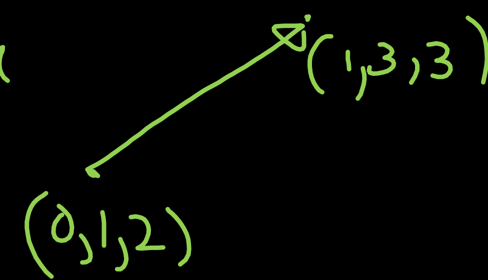
Types of Vectors

Free and Localised Vector: When there is no restriction to choose the origin of a vector, then it is called a free vector. When there is a restriction to choose a particular point, then it is called a localised vector or a sliding vector.



$\overline{AB} \rightarrow$ localised vector

$i+2j+k \rightarrow$ free vector



Basics of Vectors

Types of Vectors

Coinitial Vectors: All vectors having the same initial point are called co-initial vectors. For example, $\overline{AA}$, $\overline{AB}$, $\overline{AD}$ are all coinitial vectors.

Basics of Vectors

Types of Vectors

Reciprocal Vectors: The vector which has the same direction as $\overline{a}$ but has a magnitude reciprocal to that of $\overline{a}$ is called the reciprocal of $\overline{a}$ and is denoted by $\overline{a^{-1}}$.

$$\frac{1}{|\overline{a}|}\left(\frac{\overline{a}}{|\overline{a}|}\right) \text{ is reciprocal vector of } \overline{a}$$

-

Basics of Vectors

Types of Vectors

Collinear or Parallel Vectors: Vectors are said to be collinear or parallel if they have the same line of support or have parallel lines of support regardless of their magnitudes.

Remark

The line of which a vector is a part is called its support.

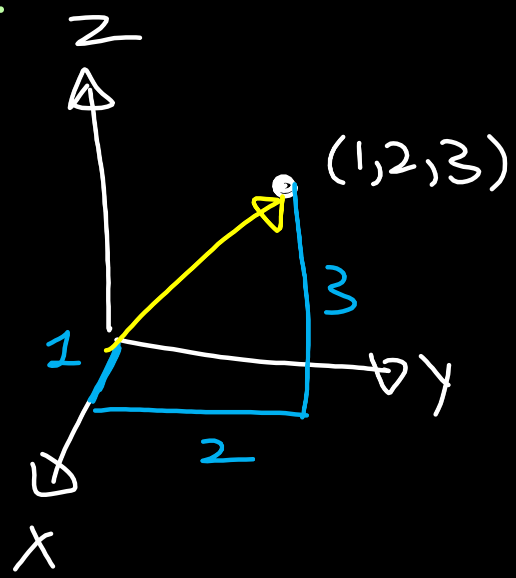
Basics of Vectors

Types of Vectors

Like and Unlike Vectors: Vectors are said to be like if they have the same direction and unlike if they have opposite directions, irrespective of their magnitudes.

Basics of Vectors

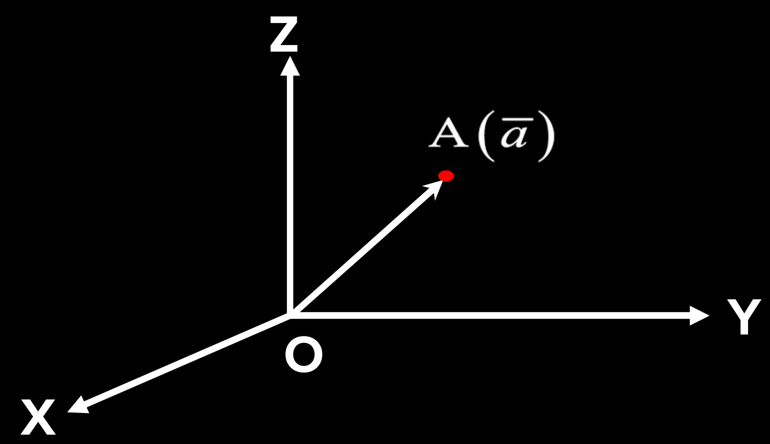
Now let's see the evolution of vectors, taking various number of points in space.



Basics of Vectors

1 point

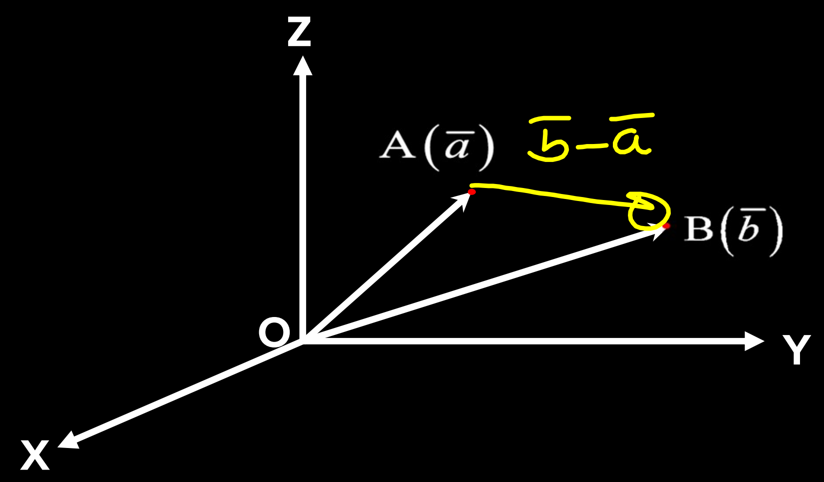
A vector formed by joining the origin to the point A (vector terminating at A) is called the position vector of point A. It is generally written as $\mathrm{A}(\overline{a})$.



PV just refers to a pt

Basics of Vectors

2 points

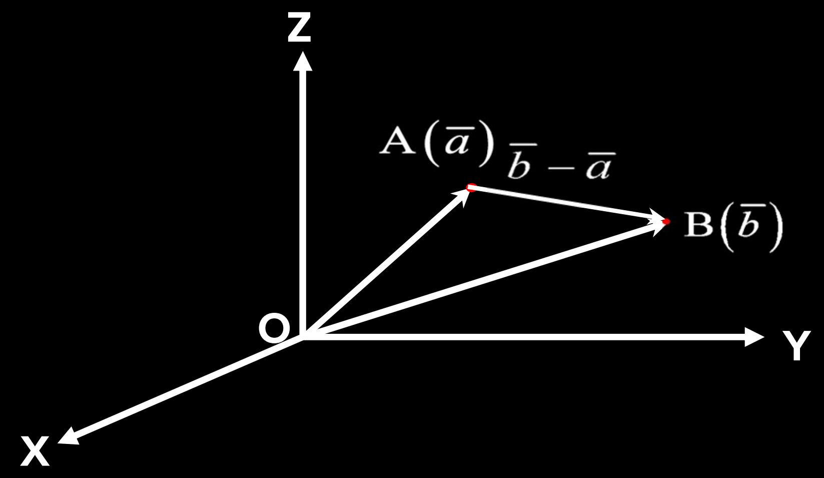


Basics of Vectors

2 points

Two points, A and B, now means two position vectors $\mathrm{A}(\overline{a})$ and $\mathrm{B}(\overline{b})$

Two position vectors $\mathrm{A}(\overline{a})$ and $\mathrm{B}(\overline{b})$ gives one free vector given by $\overline{b}-\overline{a}$



For example, the points A(1, 2, 3) and B(4, 1 ,5) means position vectors $\mathrm{A}(\hat{i}+2\hat{j}+3\hat{k})$ and $\mathrm{B}(4\hat{i}+\hat{j}+5\hat{k})$ respectively and they give a free vector $3\hat{i}-\hat{j}+2\hat{k}$

Addition of Vectors

The sum of two vectors $\overline{a}$ and $\overline{b}$ is denoted by $\overline{a}+\overline{b}$ and is also called the resultant of $\overline{a}$ and $\overline{b}$.

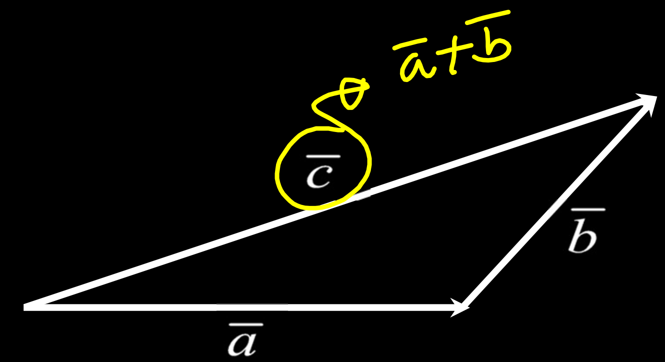
Now, let's pick the triangle law, the parallelogram law and the polygon law of addition of vectors one by one.

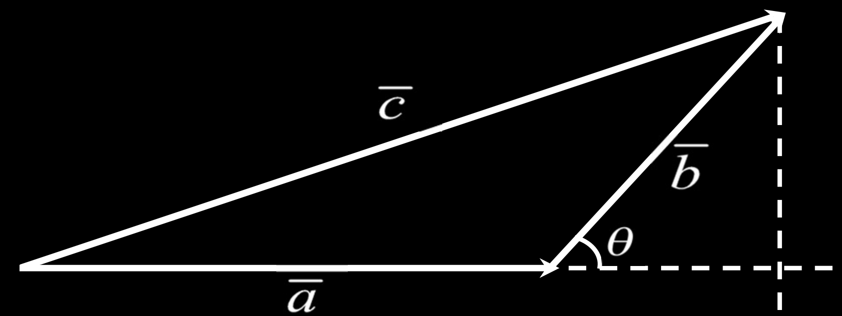
Basics of Vectors

Triangle Law of Addition of Vectors

Resultant of $\overline{a}$ and $\overline{b}$ is $\overline{c} = \overline{a} + \overline{b}$.

The length or the magnitude of the resultant can be easily found using the following relation, where θ is the angle between the vectors $\overline{a}$ and $\overline{b}$.



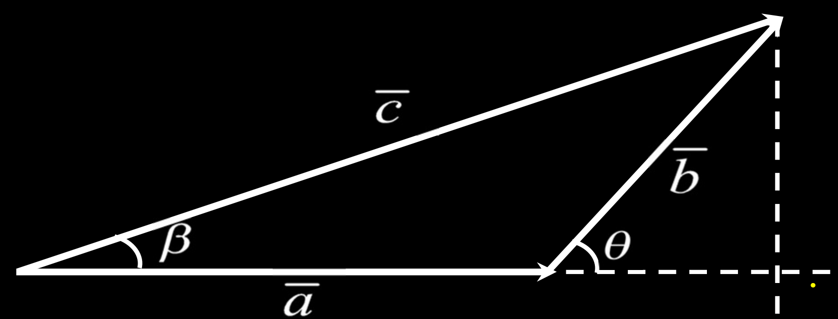


$$\left|\overline{a}+\overline{b}\right| = \sqrt{\left(\left|\overline{a}\right| + \left|\overline{b}\right|\cos\theta\right)^2 + \left(\left|\overline{b}\right|\sin\theta\right)^2} = \sqrt{\left|\overline{a}\right|^2 + \left|\overline{b}\right|^2 + 2\left|\overline{a}\right|\left|\overline{b}\right|\cos\theta}$$

Basics of Vectors

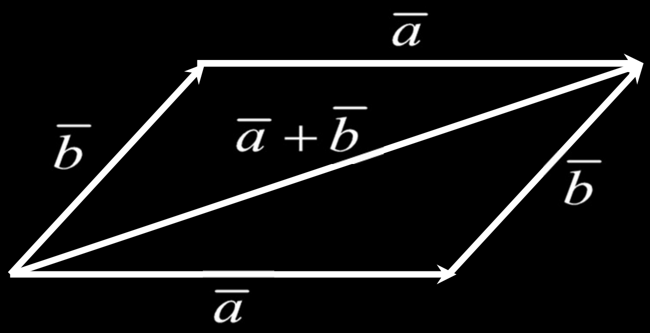
Remark

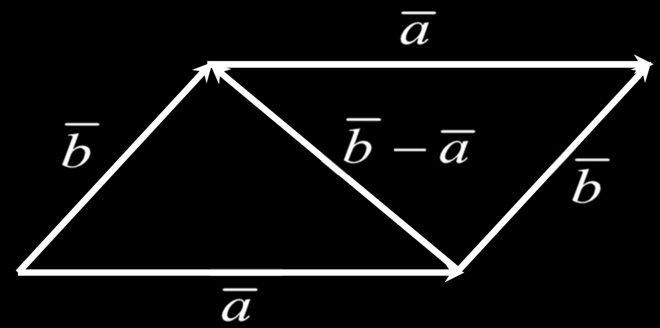
The angle which the resultant of $\overline{a}$ and $\overline{b}$ makes with $\overline{a}$ is given by $\tan\beta = \dfrac{|\overline{b}|\sin\theta}{|\overline{a}| + |\overline{b}|\cos\theta}$



Basics of Vectors

Parallelogram Law of Addition of Vectors





Q

Find the unit vector in the direction of the resultant of the vectors $\overrightarrow{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\overrightarrow{b} = 2\hat{i} + \hat{j} - 7\hat{k}$.

Sol: Ans: $\dfrac{\bar{a}+\bar{b}}{|\bar{a}+\bar{b}|}$

Find the unit vector in the direction of the resultant of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$.

Solution:

Given, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$

$$\therefore \vec{a} + \vec{b} = 2\hat{i} + 2\hat{j} - 5\hat{k} + 2\hat{i} + \hat{j} - 7\hat{k} = 4\hat{i} + 3\hat{j} - 12\hat{k}$$

Also, $\left|\vec{a} + \vec{b}\right| = \sqrt{4^2 + 3^2 + 12^2} = \sqrt{169} = 13$

Hence, the unit vector in the direction of $\vec{a} + \vec{b}$ is given by -

$$\frac{\vec{a} + \vec{b}}{\left|\vec{a} + \vec{b}\right|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13} = \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$$

Q

Find a vector of magnitude 5 units which is parallel to the resultant of the vectors $\overline{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\overline{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Ans. $\pm \left(\dfrac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|} \right) \times 5$

Find a vector of magnitude 5 units which is parallel to the resultant of the vectors $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution:

The given vectors are $\bar{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$

$$\bar{a} + \bar{b} = 2\hat{i} + 3\hat{j} - \hat{k} + \hat{i} - 2\hat{j} + \hat{k} = 3\hat{i} + \hat{j}$$

$$\therefore \ |\bar{a} + \bar{b}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{9+1} = \sqrt{10}$$

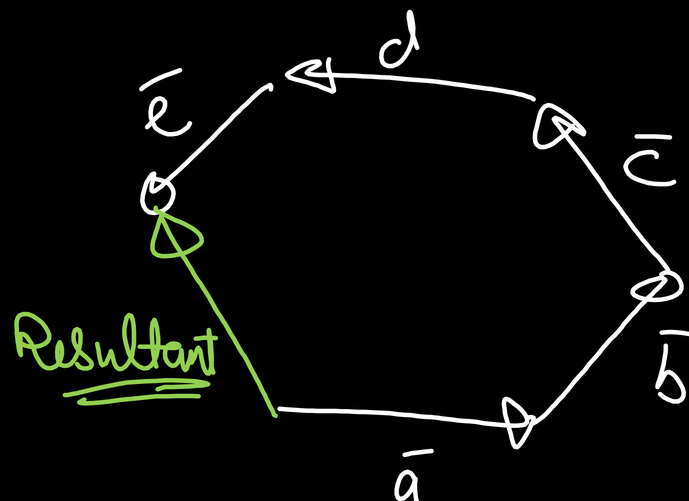
Now, if $\hat{u}$ is a unit vector in the direction of $\bar{a} + \bar{b}$,

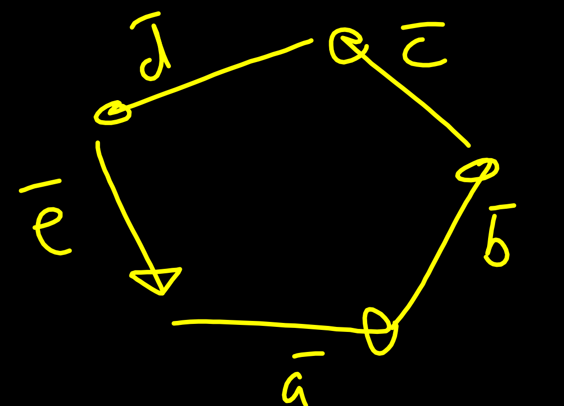
then $5\hat{u}$ will be a vector of magnitude 5 in the direction of $\bar{a} + \bar{b}$

$$\therefore \ 5\hat{u} = 5 \times \left(\frac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|} \right) = 5 \times \left(\frac{3\hat{i} + \hat{j}}{\sqrt{10}} \right) = \frac{3\sqrt{10}}{2}\hat{i} + \frac{\sqrt{10}}{2}\hat{j}$$

Basics of Vectors

Polygon Law of Addition of Vectors





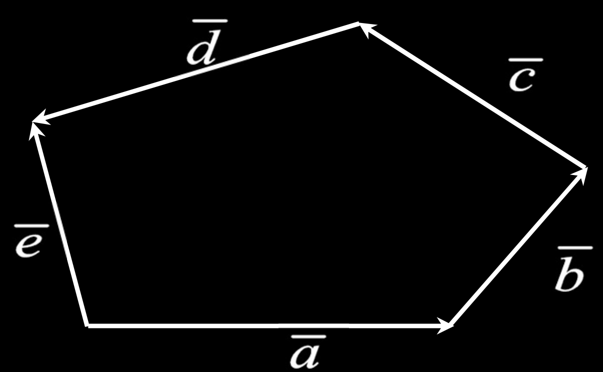
$$\bar{a}+\bar{b}+\bar{c}+d+\bar{e}=\bar{0}$$

Basics of Vectors

Polygon Law of Addition of Vectors

If n – 1 sides of a polygon represent the vectors $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_{n-1}}$, taken in order, then the n^{th} side in opposite direction gives their sum or resultant.

For example, consider the following pentagon.

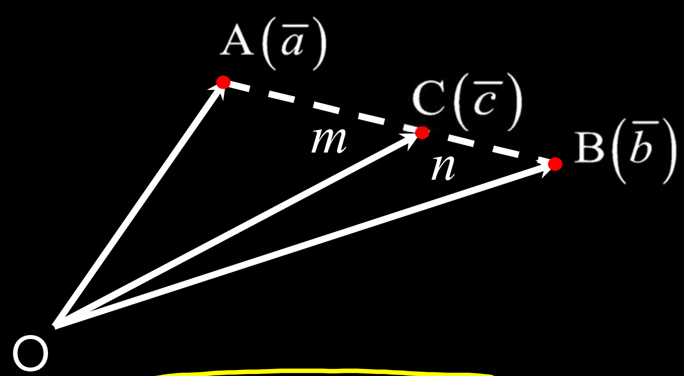


$$\overline{e} = \overline{a} + \overline{b} + \overline{c} + \overline{d}$$

Section Formulae

Section Formulae

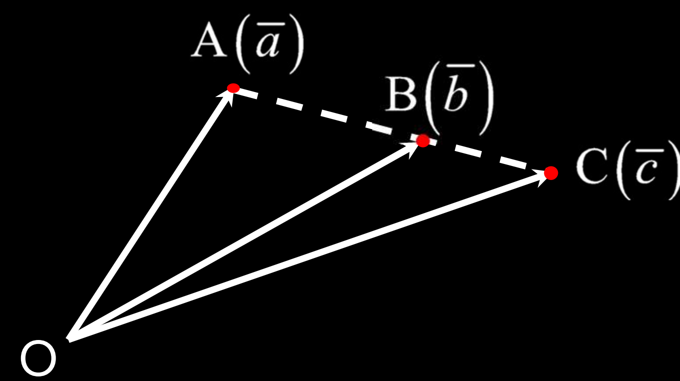
Internal Section Formula



$$\boxed{\bar{c} = \frac{m\bar{b} + n\bar{a}}{m+n}}$$

$$\overline{OC} = \frac{m\overline{OB} + n\overline{OA}}{m+n}$$

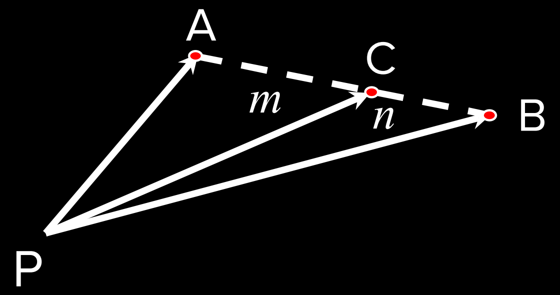
External Section Formula



$$\bar{c} = \frac{m\bar{b} - n\bar{a}}{m-n}$$

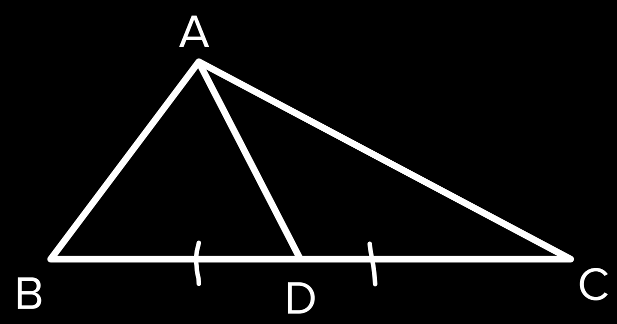
Section Formulae

NOTE



$$\overline{PC} = \frac{m\overline{PB} + n\overline{PA}}{m+n}$$

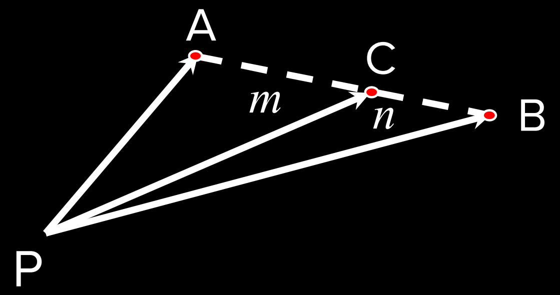
For example, in a ΔABC, if AD is the median to the side BC, then



$$\overline{AD} = \frac{\overline{AB} + \overline{AC}}{2}$$

Section Formulae

NOTE



$$\overline{PC} = \frac{m\,\overline{PB} + n\,\overline{PA}}{m+n}$$

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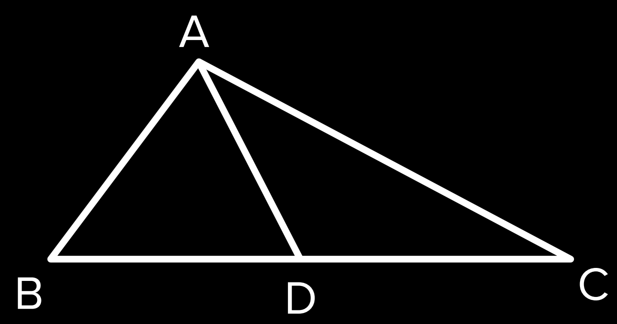


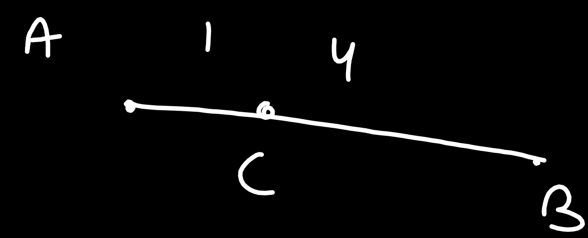
$$\overline{AD} = \frac{\overline{AB} + \overline{AC}}{2}$$

Section Formulae

Now let's do some examples on section formulae.

Q

Find the position vector of the point C on the segment AB such that AC : CB = 1 : 4. The position vectors of A and B are $3\hat{i}+2\hat{j}-\widehat{k}$ and $-2\hat{i}+7\hat{j}-\widehat{k}$ respectively.



$$\bar{c} = \frac{\bar{b}+4\bar{a}}{5}$$

Q

Find the position vector of the point C on the segment AB such that AC : CB = 1 : 4. The position vectors of A and B are $3\hat{i}+2\hat{j}-\widehat{k}$ and $-2\hat{i}+7\hat{j}-\widehat{k}$ respectively.

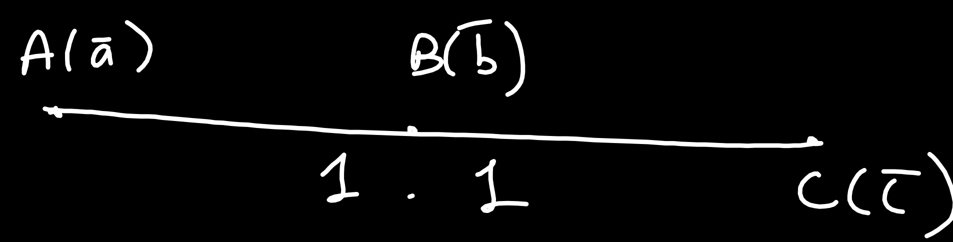
Solution:

$$\because\ AC:CB=1:4$$

$$\therefore P.V.\text{ of C}=\frac{1(-2\hat{i}+7\hat{j}-\hat{k})+4(3\hat{i}+2\hat{j}-\hat{k})}{1+4}$$

$$=\frac{1}{5}\left(10\hat{i}+15\hat{j}-5\hat{k}\right)=2\hat{i}+3\hat{j}-\hat{k}$$

Q **If $\overline{a}$, $\overline{b}$ are position vectors of A, B respectively and C is a point on AB produced such that AC = 2AB, then find the position vector of C.**



$$\bar{b} = \frac{\bar{c} + \bar{a}}{2}$$

$$\bar{c} = 2\bar{b} - a$$ Ans

If $\overline{a}, \overline{b}$ are position vectors of A, B respectively and C is a point on AB produced such that AC = 2AB, then find the position vector of C.

Solution:

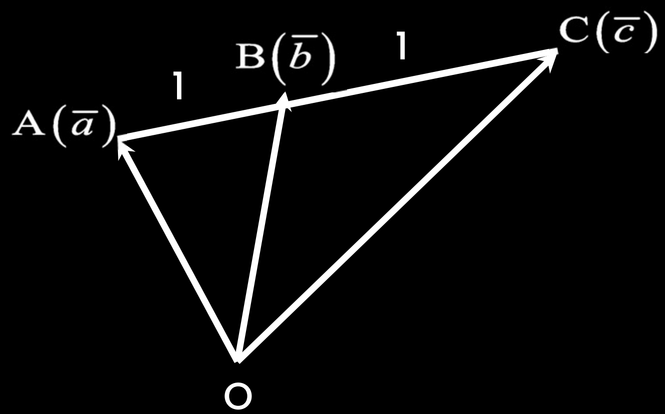
It is given that AC = 2AB

$\therefore$ AB : BC = 1 : 1

$\Rightarrow$ B divides AC in 1 : 1

$$\therefore \bar{b} = \frac{\bar{c} + \bar{a}}{2}$$

$$\Rightarrow \bar{c} = 2\bar{b} - \bar{a}$$



$$\text{Also, observe that } \overline{OB} = \frac{\overline{OC} + \overline{OA}}{2}$$

Q If A, B, C, D are any four points and E, F are the midpoints of AC and BD respectively, then prove that $\overline{AB}+\overline{CB}+\overline{CD}+\overline{AD}=4\,\overline{EF}$

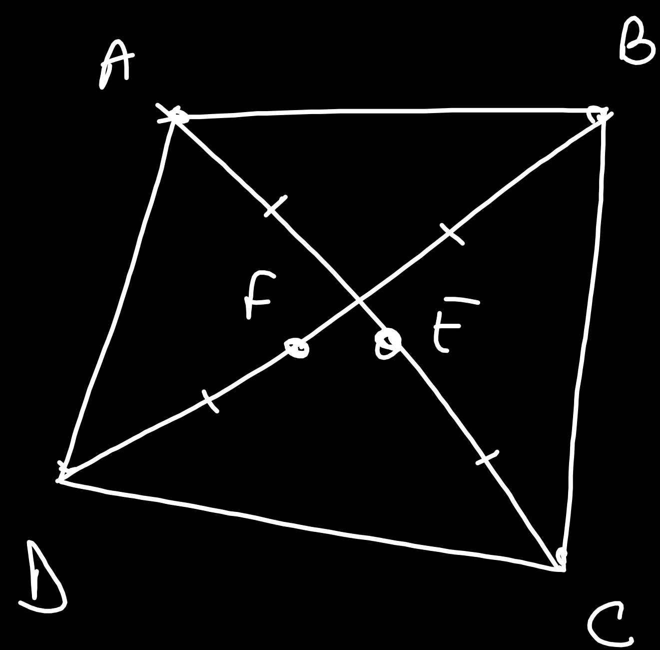
ans. $\overline{AB}+\overline{CB}+\overline{CD}+\overline{AD}$

$$2\left(\frac{\overline{AB}+\overline{AD}}{2}\right)+2\left(\frac{\overline{CB}+\overline{CD}}{2}\right)$$

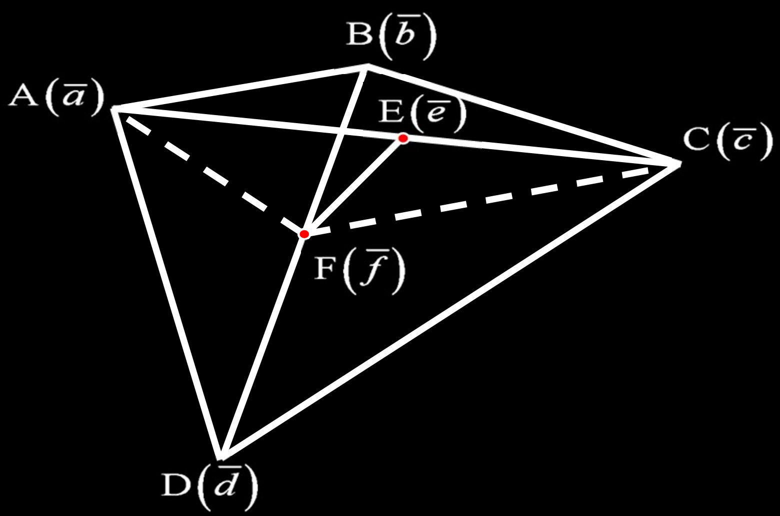
$$\Rightarrow 2\,\overline{AF}+2\,\overline{CF}$$

$$\Rightarrow -2\left(\frac{\overline{FA}+\overline{FC}}{2}\right)\times 2$$

$$\Rightarrow -4\,\overline{FE}=4\,\overline{EF}$$



Q **If A, B, C, D are any four points and E, F are the midpoints of AC and BD respectively, then prove that** $\overline{AB}+\overline{CB}+\overline{CD}+\overline{AD}=4\,\overline{EF}$



Solution:

$$\overline{AB}+\overline{AD}=2\,\overline{AF}$$

$$\overline{CB}+\overline{CD}=2\,\overline{CF}$$

$$\begin{aligned}\overline{AB}+\overline{CB}+\overline{CD}+\overline{AD}&=2\left(\overline{AF}+\overline{CF}\right)\\&=-2\left(\overline{FA}+\overline{FC}\right)\\&=-2\left(2\,\overline{FE}\right)\\&=4\,\overline{EF}\end{aligned}$$

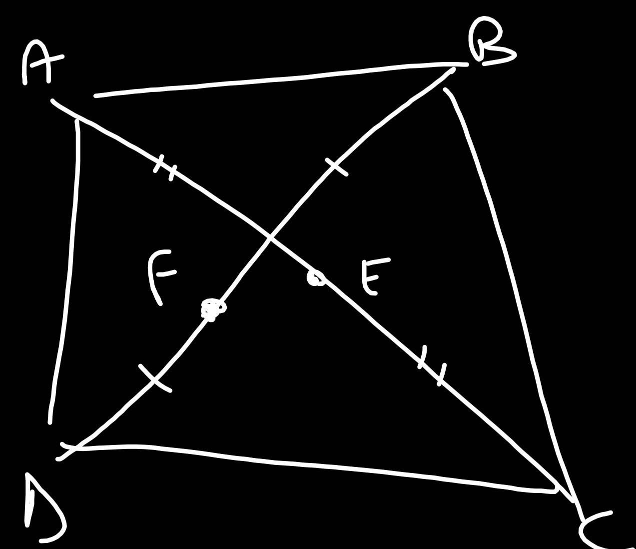
$$\text{LHS}=\text{RHS}$$

JEE Main 15th Apr, 2023

Q Let ABCD be a quadrilateral. If E and F are the midpoints of the diagonals AC and BD respectively and $\left(\overrightarrow{AB}-\overrightarrow{BC}\right)+\left(\overrightarrow{AD}-\overrightarrow{DC}\right)=k\overrightarrow{FE}$, then k is equal to

$(\overline{AB}+\overline{AD})+(\overline{CB}+\overline{CD})$

= Same as previous



(A) 4

(B) 2

(C) -2

(D) -4

JEE Main 15th Apr, 2023

Q Let ABCD be a quadrilateral. If E and F are the midpoints of the diagonals AC and BD respectively and $\left(\overrightarrow{AB}-\overrightarrow{BC}\right)+\left(\overrightarrow{AD}-\overrightarrow{DC}\right)=k\overrightarrow{FE}$, then k is equal to

(A) 4

(B) 2

(C) −2

(D) −4

Solution:

$$\overrightarrow{AB}-\overrightarrow{BC}+\overrightarrow{AB}-\overrightarrow{DC}=k\overrightarrow{FE}$$

$$\left(\vec{b}-\vec{a}\right)-\left(\vec{c}-\vec{b}\right)+\left(\vec{d}-\vec{a}\right)-\left(\vec{c}-\vec{d}\right)=k\overrightarrow{FE}$$

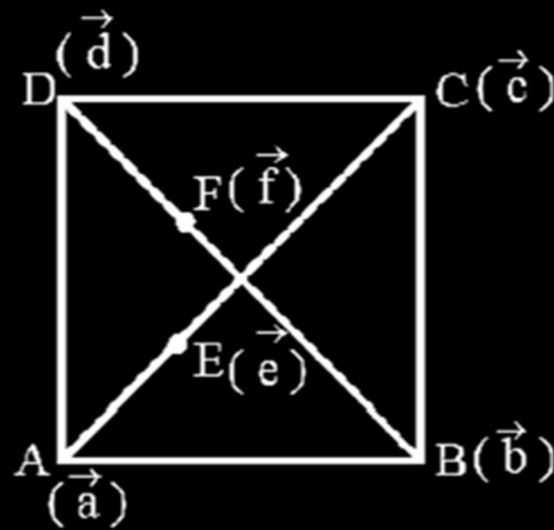
$$2\left(\vec{b}+\vec{d}\right)-2\left(\vec{a}-\vec{c}\right)=k\overrightarrow{FE}$$

$$2\left(2\vec{f}\right)-2\left(2\vec{e}\right)=k\overrightarrow{FE}$$

$$4\left(\vec{f}-\vec{e}\right)=k\overrightarrow{FE}$$

$$-4\overrightarrow{FE}=k\overrightarrow{FE}$$

$$k=-4$$



Std variety

Q Points D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 each respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.

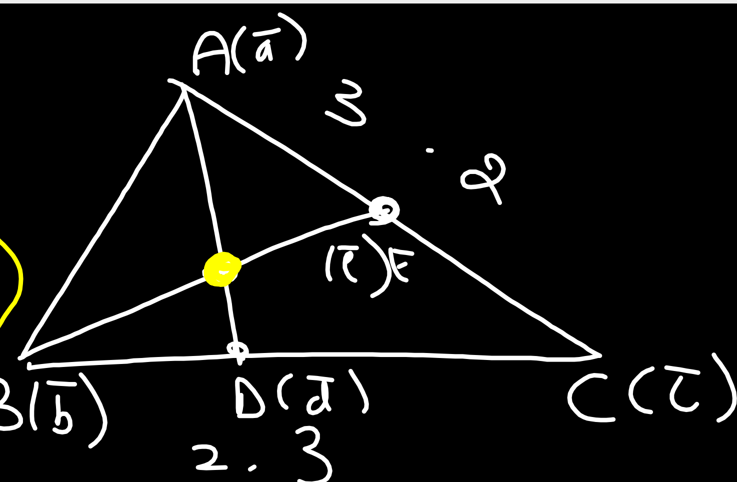
$5\bar{d} = 2\bar{c} + 3\bar{b}$

$5\bar{e} = 2\bar{a} + 3\bar{c}$

$15\bar{d} = 6\bar{c} + 9\bar{b} \Rightarrow \frac{15\bar{d}+4\bar{a}}{19} = \frac{6\bar{c}+9\bar{b}+4\bar{a}}{19}$

$10\bar{e} = 4\bar{a} + 6\bar{c} \Rightarrow \frac{10\bar{e}+9\bar{b}}{19} = \frac{4\bar{a}+6\bar{c}+9\bar{b}}{19}$

same



$\therefore \frac{15\bar{d}+4\bar{a}}{19} = \frac{10\bar{e}+9\bar{b}}{19} \longrightarrow$ pt. of inters^n

Solution:

Using section formula, we have

$$\overline{d}=\frac{2\overline{c}+3\overline{b}}{2+3}\Rightarrow 5\overline{d}=2\overline{c}+3\overline{b}$$

and $\overline{e}=\dfrac{3\overline{c}+2\overline{a}}{3+2}\Rightarrow 5\overline{e}=3\overline{c}+2\overline{a}$

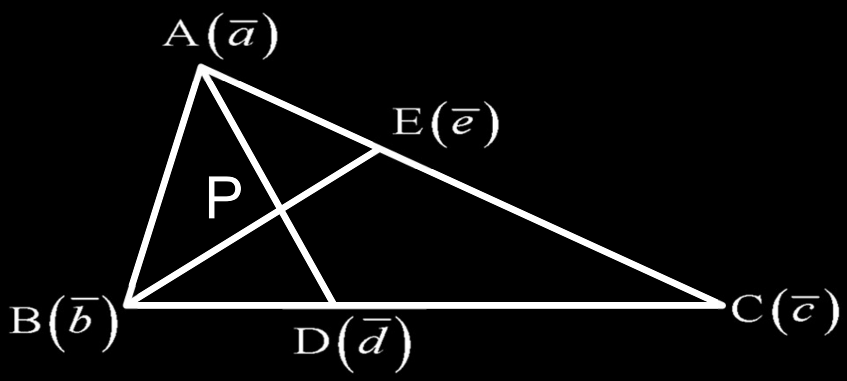
$\therefore\ 15\overline{d}=6\overline{c}+9\overline{b}$ and $10\overline{e}=6\overline{c}+4\overline{a}$

$\Rightarrow 15\overline{d}+4\overline{a}=6\overline{c}+9\overline{b}+4\overline{a}$

and $10\overline{e}+9\overline{b}=6\overline{c}+4\overline{a}+9\overline{b}$

$\therefore\ \dfrac{15\overline{d}+4\overline{a}}{19}=\dfrac{6\overline{c}+9\overline{b}+4\overline{a}}{19}$

and $\dfrac{10\overline{e}+9\overline{b}}{19}=\dfrac{6\overline{c}+4\overline{a}+9\overline{b}}{19}$

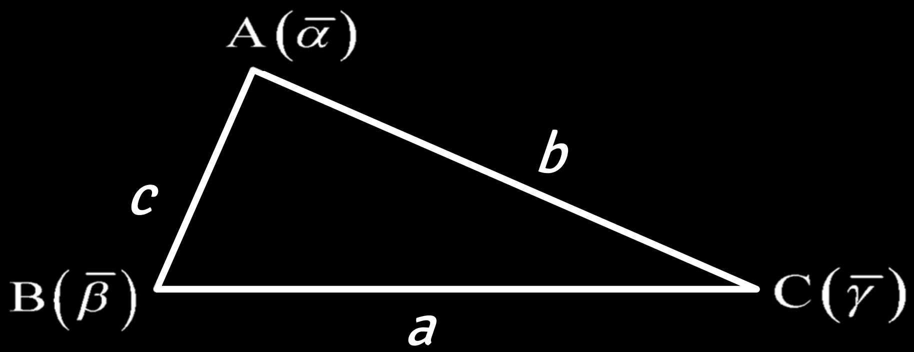


Solution:

$\therefore$ Position vector of point of concurrency $= \dfrac{6\overline{c} + 4\overline{a} + 9\overline{b}}{19}$

P divides AD in the ratio $15:4$ and BE in the ratio $10:9$

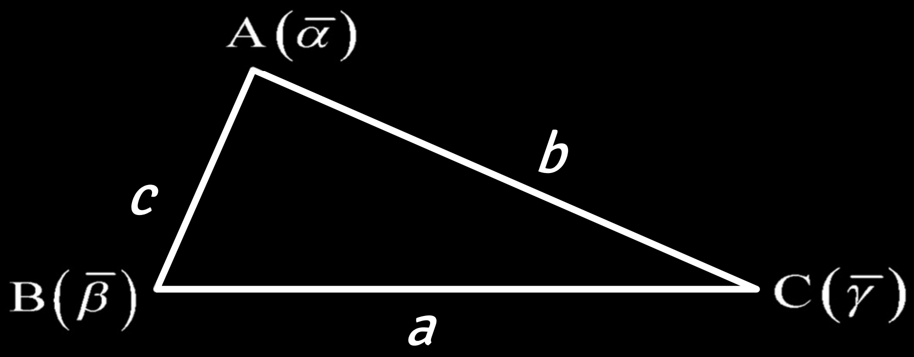
NOTE



1. Position vector of centroid is

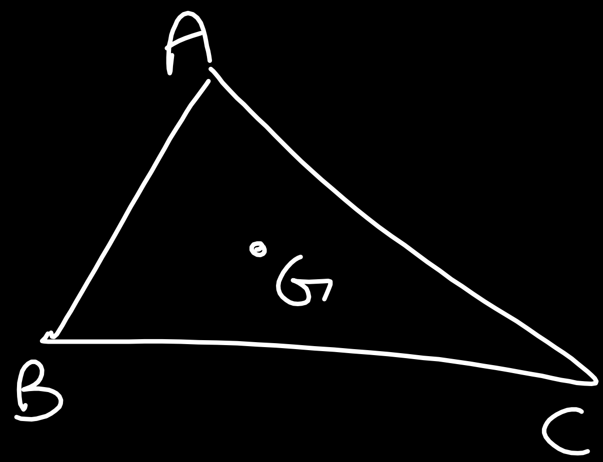
2. Position vector of incentre is

NOTE



1. Position vector of centroid is $\dfrac{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}{3}$

2. Position vector of incentre is $\dfrac{a\bar{\alpha}+b\bar{\beta}+c\bar{\gamma}}{a+b+c}$



$$\frac{\overline{PA}+\overline{PB}+\overline{PC}}{3}=\overline{PG}$$

Observation

If I is the incenter of triangle ABC, then the value of the expression $|\mathrm{BC}|\left(\overline{\mathrm{IA}}\right)+|\mathrm{CA}|\left(\overline{\mathrm{IB}}\right)+|\overline{\mathrm{AB}}|\left(\overline{\mathrm{IC}}\right)=\overline{0}$

Recall

$H \overset{2}{—} G \overset{1}{—} O$

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Q If the points P and Q are respectively the circumcenter and the orthocentre of a ΔABC, then $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is equal to:

Sol

$Q \overset{2}{—} G \overset{1}{—} P$

$$3\left(\frac{\overline{PA}+\overline{PB}+\overline{PC}}{3}\right)=3\overline{PG}$$

$$=\overline{PQ}$$

(A) $\overrightarrow{QP}$

(B) $\overrightarrow{PQ}$ ✓

(C) $2\overrightarrow{PQ}$

(D) $2\overrightarrow{QP}$

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Q

If the points P and Q are respectively the circumcenter and the orthocentre of a ΔABC, then $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is equal to:

(A) $\overrightarrow{QP}$

(B) $\overrightarrow{PQ}$

(C) $2\overrightarrow{PQ}$

(D) $2\overrightarrow{QP}$

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Q

If the points P and Q are respectively the circumcenter and the orthocentre of a ΔABC, then $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is equal to:

Solution:

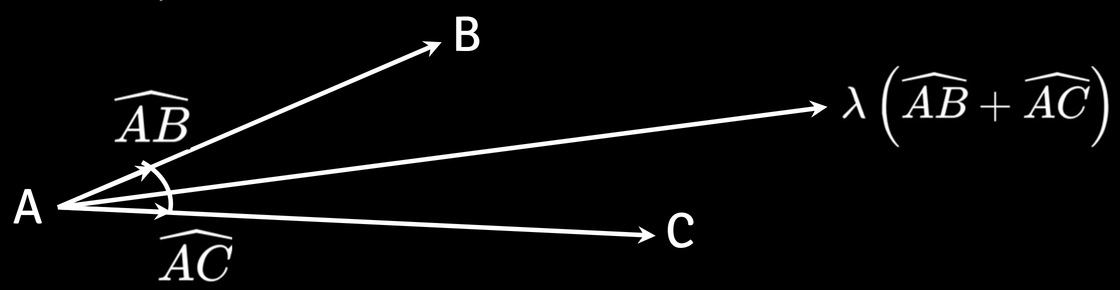
$$\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}=\vec{a}+\vec{b}+\vec{c}$$

$$\overrightarrow{PG}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\Rightarrow \vec{a}+\vec{b}+\vec{c}=3\overrightarrow{PG}=\overrightarrow{PQ}$$

Remark

A vector along the internal angle bisector of $\overline{\mathrm{AB}}$ and $\overline{\mathrm{AC}}$ is of the form $\lambda\left(\widehat{AB}+\widehat{AC}\right)$.



For external angle bisector, it is $\lambda\left(\widehat{AB}-\widehat{AC}\right)$.

Q

If $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$, determine vector $\vec{c}$ along the internal bisector of the angle between vectors $\vec{a}$ and $\vec{b}$, such that $|\vec{c}| = 5\sqrt{6}$.

complete it

Ans: $\left(\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\right) \times 5\sqrt{6}$

Solution:

$$\widehat{a} = \frac{1}{9}\left(7\hat{i} - 4\hat{j} - 4\widehat{k}\right)$$

$$\hat{b} = \frac{1}{3}\left(-2\hat{i} - \hat{j} + 2\widehat{k}\right)$$

$$\hat{c} = \lambda\left[\widehat{a} + \hat{b}\right] = \lambda\frac{1}{9}\left(\hat{i} - 7\hat{j} + 2\widehat{k}\right) \qquad \ldots(i)$$

$$\left|\vec{c}\right| = 5\sqrt{6}$$

$$\Rightarrow \frac{\lambda^2}{81}(1 + 49 + 4) = 25 \times 6$$

$$\lambda^2 = \frac{25\times6\times81}{54} = 225$$

$$\lambda = \pm15$$

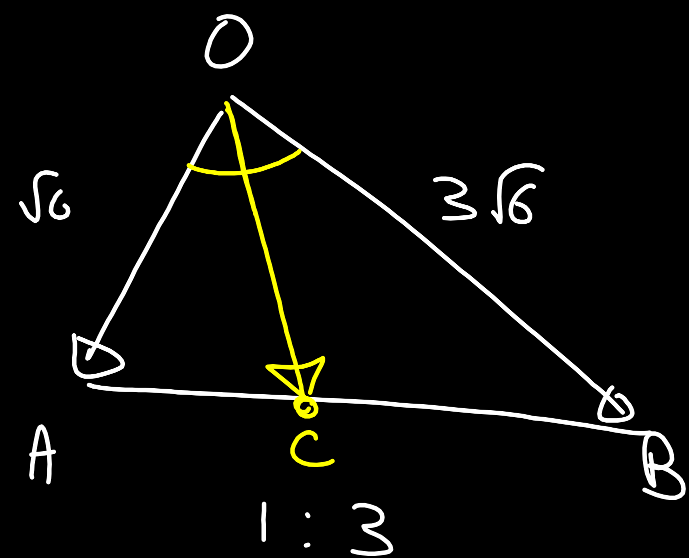
Putting the value of λ in (i), we get

$$\vec{c} = \pm\frac{5}{3}\left(\hat{i} - 7\hat{j} + 2\widehat{k}\right)$$

$|\overline{OA}| = \sqrt{6}$ $|\overline{OB}| = \sqrt{54}$

Q Let $\overline{OA} = 2\hat{i} + \hat{j} - \hat{k}$ and $\overline{OB} = 7\hat{i} + 2\hat{j} - \hat{k}$ **, find the vector** $\overline{OC}$ **bisecting ∠AOB internally such that C lies on AB.**

$$\overline{OC} = \frac{1 \times \overline{OB} + 3\overline{OA}}{4}$$



Solution:

The given vector are

$\overline{OA} = 2\hat{i} + \hat{j} - \hat{k}$ and $\overline{OB} = 7\hat{i} + 2\hat{j} - \hat{k}$

$\left|\overline{OA}\right| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$\left|\overline{OB}\right| = \sqrt{7^2 + 2^2 + 1^2} = \sqrt{54} = 3\sqrt{6}$

$\therefore \left|\overline{OB}\right| = 3\left|\overline{OA}\right| \Rightarrow \dfrac{CB}{CA} = \dfrac{3}{1}$

using section formula, we have

$$\overline{OC} = \frac{3\overline{OA} + \overline{OB}}{4} = \frac{3\left(2\hat{i} + \hat{j} - \hat{k}\right) + 7\hat{i} + 2\hat{j} - \hat{k}}{4}$$

$$= \frac{13\hat{i} + 5\hat{j} - 2\hat{k}}{4}$$

Collinearity & Coplanarity

Collinearity & Coplanarity

3 points (two vectors)

The moment we have three points, the concern is collinearity. Three points $A\left(\overline{a}\right),\ B\left(\overline{b}\right),\ C\left(\overline{c}\right)$ will be collinear if $\underbrace{\overline{AB}||\overline{BC}}_{\text{proportional}},$ that is $\overline{AB}=\lambda\overline{BC}.$

Collinearity & Coplanarity

3 points (two vectors)

The moment we have three points, the concern is collinearity. Three points $A\left(\overline{a}\right),\ B\left(\overline{b}\right),\ C\left(\overline{c}\right)$ will be collinear if $\overline{AB}||\overline{BC},$ that is $\overline{AB} = \lambda\overline{BC}$.

For example, consider the points A(1, 3, 2), B(-2, 0, 1) and C(4, 6, 3).
As $\overline{AB} = -3\hat{i} - 3\hat{j} - \hat{k}$ and $\overline{BC} = 6\hat{i} + 6\hat{j} + 2\hat{k}$ are parallel or collinear, the points A, B and C will be collinear.

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Q If the points with position vectors $\alpha\hat{i}+10\hat{j}+13\hat{k}$, $6\hat{i}+11\hat{j}+11\hat{k}$, $\frac{9}{2}\hat{i}+\beta\hat{j}-8\hat{k}$ are collinear, then $(19\alpha - 6\beta)^2$ is equal to

$$\overline{AB} = (6-\alpha)\hat{i} + \hat{j} - 2\hat{k}$$

$$\overline{BC} = -\frac{3}{2}\hat{i} + (\beta-11)\hat{j} - 19\hat{k}$$

$$\frac{6-\alpha}{-3/2} = \frac{1}{\beta-11} = \frac{-2}{-19}$$

$\alpha = \checkmark$

$\beta \checkmark$

(A) 16

(B) 49

(C) 36

(D) 25

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Q

If the points with position vectors $\alpha\hat{i} + 10\hat{j} + 13\hat{k}$, $6\hat{i} + 11\hat{j} + 11\hat{k}$, $\frac{9}{2}\hat{i} + \beta\hat{j} - 8\hat{k}$ **are collinear, then** $(19\alpha - 6\beta)^2$ **is equal to**

A 16

B 49

C 36

D 25

Solution:

Given Points with position vectors $\alpha\hat{i}+10\hat{j}+13\hat{k}$, $6\hat{i}+11\hat{j}+11\hat{k}$,

and $\frac{9}{2}\hat{i}+\beta\hat{j}-8\hat{k}$ are collinear.

So, $\dfrac{\alpha-6}{6-\frac{9}{2}}=\dfrac{10-11}{11-\beta}=\dfrac{13-11}{11+8}$

$$\Rightarrow \frac{2(\alpha-6)}{3}=\frac{-1}{11-\beta}=\frac{2}{19}$$

$$\Rightarrow \frac{2}{3}(\alpha-6)=\frac{2}{19}$$

$$\Rightarrow 19\alpha-114=3 \Rightarrow 19\alpha=117$$

$$\Rightarrow \alpha=\frac{117}{19}$$

Solution:

And $\dfrac{-1}{11-\beta} = \dfrac{2}{19}$

$\Rightarrow -19 = 22 - 2\beta$

$\Rightarrow \beta = \dfrac{41}{2}$

$\therefore (19\alpha - 6\beta)^2 = \left(19 \times \dfrac{117}{19} - \dfrac{6 \times 41}{2}\right)^2 = (117 - 123)^2 = 36$

Q Let $\overline{a}, \overline{b}, \overline{c}$ be three vectors of which every pair is non-collinear. If $\overline{a}+\overline{b}$ and $\overline{b}+\overline{c}$ are collinear with $\overline{c}$ and $\overline{a}$ respectively, then find the value of $\overline{a}+\overline{b}+\overline{c}$.

$$\left.\begin{aligned}\overline{a}+\overline{b}&=\lambda\overline{c}\\ \overline{b}+\overline{c}&=\mu\overline{a}\end{aligned}\right\}$$ Eliminate any one

$$\overline{a}-\overline{c}=\lambda\overline{c}-\mu\overline{a}$$

$$\overline{a}(1+\mu)=\overline{c}(\lambda+1)$$

$$\Rightarrow \mu+1=0=\lambda+1 \quad \lambda=-1$$

$$\overline{a}+\overline{b}=-\overline{c}$$

Solution:

Since $\overline{a}+\overline{b}$ is collinear with $\overline{c}$ and $\overline{b}+\overline{c}$ is collinear with $\overline{a}$,

$\therefore$ We have $\overline{a}+\overline{b}=\lambda\overline{c}$ and $\overline{b}+\overline{c}=\mu\overline{a}$ for some $\lambda,\mu\in\mathbb{R}$

So, $\overline{a}+\overline{b}=\lambda\overline{c} \Rightarrow \overline{c}=\dfrac{1}{\lambda}\left(\overline{a}+\overline{b}\right)$

Hence, $\overline{b}+\dfrac{1}{\lambda}\left(\overline{a}+\overline{b}\right)=\mu\overline{a} \Rightarrow \left(1+\dfrac{1}{\lambda}\right)\overline{b}=\left(\mu-\dfrac{1}{\lambda}\right)\overline{a}$

But $\overline{a}\nparallel\overline{b} \Rightarrow 1+\dfrac{1}{\lambda}=\mu-\dfrac{1}{\lambda}=0 \Rightarrow \lambda=\mu=-1$

$\therefore$ $\overline{c}=-\left(\overline{a}+\overline{b}\right) \Rightarrow \overline{a}+\overline{b}+\overline{c}=\overline{0}$

Alternate Solution:

Since $\overline{a}+\overline{b}$ is collinear with $\overline{c}$ and $\overline{b}+\overline{c}$ is collinear with $\overline{a}$,

$\therefore$ We have $\overline{a}+\overline{b}=\lambda\overline{c}$ and $\overline{b}+\overline{c}=\mu\overline{a}$ for some $\lambda,\mu\in\mathbb{R}$

So, $\overline{a}+\overline{b}=\lambda\overline{c} \Rightarrow \overline{a}+\overline{b}+\overline{c}=(\lambda+1)\overline{c}$

and $\overline{b}+\overline{c}=\mu\overline{a} \Rightarrow \overline{a}+\overline{b}+\overline{c}=(\mu+1)\overline{a}$

$\therefore (\lambda+1)\overline{c}=(\mu+1)\overline{a}$

But $\overline{a}\nparallel\overline{c} \Rightarrow \lambda+1=\mu+1=0 \Rightarrow \lambda=\mu=-1$

So, $\overline{a}+\overline{b}+\overline{c}=(-1+1)\overline{c}=\overline{0}$

AIEEE 2011

Q Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is

(A) $\vec{0}$

(B) $\vec{a} + \vec{c}$

(C) $\vec{a}$

(D) $\vec{c}$

Sol:-

$\bar{a} + 3\bar{b} = \lambda\bar{c}$

$\bar{b} + 2\bar{c} = \mu\bar{a} \quad \times 3$

$\bar{a} - 6\bar{c} = \lambda\bar{c} - 3\mu\bar{a}$

$\bar{a}(1 + 3\mu) = \bar{c}(\lambda + 6)$

$\underbrace{1+3\mu}_{0}$, $\underbrace{\lambda+6}_{0}$

$\Rightarrow \lambda = -6$

$\bar{a} + 3\bar{b} = -6\bar{c}$

AIEEE 2011

Q Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is

(A) $\vec{0}$

(B) $\vec{a} + \vec{c}$

(C) $\vec{a}$

(D) $\vec{c}$

Solution:

$$\vec{c} = \mu\left(\vec{a} + 3\vec{b}\right)$$

$$\vec{b} + 2\vec{c} = \lambda\vec{a}$$

$$\vec{b} + 2\mu\left(\vec{a} + 3\vec{b}\right) = \lambda\vec{a}$$

$$(1 + 6\mu)\vec{b} + (2\mu - \lambda)\vec{a} = 0$$

$$6\mu + 1 = 0,\ 2\mu = \lambda$$

$$\mu = -\tfrac{1}{6},\quad \lambda = -\tfrac{1}{3}$$

Now, $\vec{c} = -\frac{1}{6}\left(\vec{a} + 3\vec{b}\right) = 6\vec{c} + \vec{a} + 3\vec{b} = 0$

NOTE

$A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ are collinear if $\overline{AB} = \lambda \overline{BC}$,

which means $\bar{b} - \bar{a} = \lambda(\bar{c} - \bar{b})$

$$\Rightarrow -\bar{a} + \bar{b}(1+\lambda) - \lambda \bar{c} = 0$$

i.e. $x\bar{a} + y\bar{b} + z\bar{c} = 0$

where $x + y + z = 0$

& not all x, y, z are 0

NOTE

$A(\overline{a}), B(\overline{b}), C(\overline{c})$ are collinear if $\overline{AB} = \lambda \overline{BC}$,

which means $\overline{b} - \overline{a} = \lambda(\overline{c} - \overline{b})$

$\Rightarrow -\overline{a} + (1+\lambda)\overline{b} - \lambda\overline{c} = \overline{0}$

$\Rightarrow \quad x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}$

where x + y + z = 0 and not all of x, y, z are zero.

NOTE

$A(\overline{a}), B(\overline{b}), C(\overline{c})$ are collinear if $\overline{AB} = \lambda\, \overline{BC}$,

which means $\overline{b} - \overline{a} = \lambda(\overline{c} - \overline{b})$

$\Rightarrow\ -\overline{a} + (1+\lambda)\overline{b} - \lambda\overline{c} = \overline{0}$

$\Rightarrow\ x\overline{a} + y\overline{b} + z\overline{c} = \overline{0}$

where x + y + z = 0 and not all of x, y, z are zero.

1. This is the condition of collinearity of points A, B and C in terms of their position vectors.
2. It does not imply that $\overline{a}, \overline{b}, \overline{c}$ are collinear.

Collinearity & Coplanarity

4 points (three vectors)

The moment we have four points, the concern is coplanarity. Four points $A(\bar{a}), B(\bar{b}), C(\bar{c}), D(\bar{d})$ will be ~~collinear~~ if $\overline{AB}, \overline{BC}, \overline{CD}$ are coplanar.

coplanar

Collinearity & Coplanarity

4 points (three vectors)

The moment we have four points, the concern is coplanarity. Four points $\mathrm{A}(\bar{a}), \mathrm{B}(\bar{b}), \mathrm{C}(\bar{c}), \mathrm{D}(\bar{d})$ will be collinear if $\overline{\mathrm{AB}}, \overline{\mathrm{BC}}, \overline{\mathrm{CD}}$ are coplanar.

Condition: Three vectors $\bar{p}, \bar{q}, \bar{r}$ are coplanar if any one of them can be written as a linear combination of the other two, that is,

$$\bar{r} = \lambda\bar{p} + \mu\bar{q}$$

Q

Show that $\overline{a} = \hat{i} - \hat{j} - 2\hat{k}$, $\overline{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ **and** $\overline{c} = 7\hat{i} + 3\hat{j} - 4\hat{k}$ **are coplanar.**

Show that $\overline{a} = \hat{i} - \hat{j} - 2\hat{k}$, $\overline{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ **and** $\overline{c} = 7\hat{i} + 3\hat{j} - 4\hat{k}$ **are coplanar.**

Solution:

The given vectors are

$\overline{a} = \hat{i} - \hat{j} - 2\hat{k}$, $\overline{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\overline{c} = 7\hat{i} + 3\hat{j} - 4\hat{k}$

For coplanarity, $\overline{c} = \lambda\,\overline{a} + \mu\,\overline{b}$ for some $\lambda, \mu \in \mathbb{R}$

$\therefore\ 7\hat{i} + 3\hat{j} - 4\hat{k} = \lambda(\hat{i} - \hat{j} - 2\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

$\Rightarrow (\lambda + 2\mu)\hat{i} + (3\mu - \lambda)\hat{j} + (\mu - 2\lambda)\hat{k} = 7\hat{i} + 3\hat{j} - 4\hat{k}$

$\Rightarrow \lambda + 2\mu = 7,\ 3\mu - \lambda = 3,\ \mu - 2\lambda = -4$

Solving $\lambda + 2\mu = 7$ and $3\mu - \lambda = 3$ we get $\lambda = 3$ and $\overset{\mu}{\cancel{\lambda}} = 2$,

which also satisfies $\mu - 2\lambda = -4$

$\therefore\ \overline{a}, \overline{b}, \overline{c}$ are coplanar.

Collinearity & Coplanarity

Observation

If, out of three vectors, two are collinear, then the three vectors are coplanar.

Collinearity & Coplanarity

Remark

Four points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$, $D(\bar{d})$ are coplanar if $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ are coplanar, which means $\overline{AB} = \lambda\left(\overline{BC}\right) + \mu\left(\overline{CD}\right)$

$$\bar{b} - \bar{a} = \lambda(\bar{c} - \bar{b}) + \mu(\bar{d} - \bar{c})$$

Collinearity & Coplanarity

Remark

Four points $\mathrm{A}(\bar{a}), \mathrm{B}(\bar{b}), \mathrm{C}(\bar{c}), \mathrm{D}(\bar{d})$ are coplanar if $\overline{\mathrm{AB}}, \overline{\mathrm{BC}}, \overline{\mathrm{CD}}$ are coplanar, which means $\overline{\mathrm{AB}} = \lambda(\overline{\mathrm{BC}}) + \mu(\overline{\mathrm{CD}})$

$$\Rightarrow \bar{b} - \bar{a} = \lambda(\bar{c} - \bar{b}) + \mu(\bar{d} - \bar{c})$$

$$\Rightarrow -\bar{a} + (1+\lambda)\bar{b} + (-\lambda + \mu)\bar{c} - \mu\bar{d} = 0$$

$$\Rightarrow x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = 0$$

where $x + y + z + w = 0$ and not all of x, y, z are zero.

Collinearity & Coplanarity

Remark

Four points $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$, $D(\overline{d})$ are coplanar if $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ are coplanar, which means $\overline{AB} = \lambda(\overline{BC}) + \mu(\overline{CD})$

$\Rightarrow \overline{b} - \overline{a} = \lambda(\overline{c} - \overline{b}) + \mu(\overline{d} - \overline{c})$

$\Rightarrow -\overline{a} + (1+\lambda)\overline{b} + (-\lambda+\mu)\overline{c} - \mu\overline{d} = 0$

$\Rightarrow x\overline{a} + y\overline{b} + z\overline{c} + w\overline{d} = 0$

where $x + y + z + w = 0$ and not all of x, y, z are zero.

1. This is the condition of coplanarity of points A, B, C and D in terms of their position vectors.
2. It does not imply that the four position vectors are coplanar.

1. Fundamental Theorem in 2D

Let $\overline{a}$ and $\overline{b}$ be two given non-zero, non-collinear vectors, then any vector $\overline{r}$ coplanar with $\overline{a}$ and $\overline{b}$ can be uniquely expressed as

$\overline{r} = \underbrace{x\overline{a} + y\overline{b}}_{\text{lin combn.}}$, for some scalars *x* and *y*.

$$x_1\overline{a} + y_1\overline{b} = x_2\overline{a} + y_2\overline{b}$$

$$\Rightarrow x_1 = x_2 \; \& \; y_1 = y_2 \text{ if } \overline{a} \nparallel \overline{b}$$

2. Fundamental Theorem in 3D

Let $\overline{a},\ \overline{b},\ \overline{c}$ be three given non-zero, non-collinear vectors, then any vector $\overline{r}$ in space can be uniquely expressed as $\overline{r} = x\overline{a} + y\overline{b} + z\overline{c}$, for some scalars *x* & *y*.

(that is, $\overline{r}$ can be uniquely expressed as a linear combination of $\overline{a},\ \overline{b},\ \overline{c}$)

Product of Vectors

Product of Vectors

We define the products of two vectors in two ways.

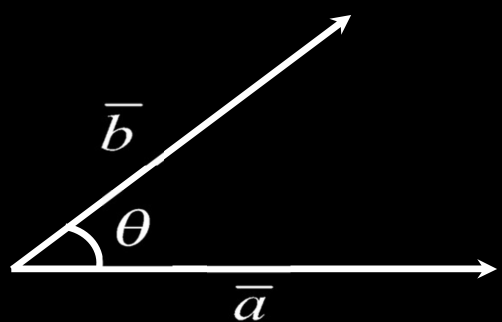
1. Scalar Product (or dot product).
2. Vector Product (or cross product).

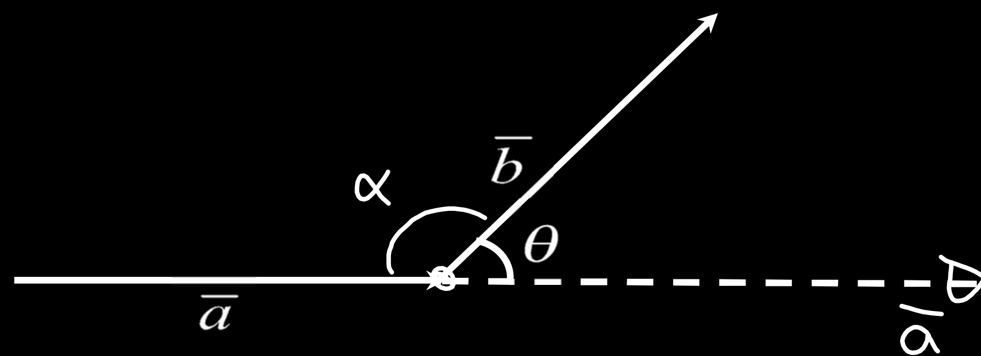
Let consider them one by one.

Dot Product

$$\overline{a}\,.\,\overline{b}$$

$$\left|\overline{a}\right|\left|\overline{b}\right|\cos\theta \qquad\qquad a_x b_x + a_y b_y + a_z b_z$$





Clearly, the angle between $\overline{a}$ and $\overline{b}$ is given by $\cos\theta = \dfrac{\overline{a}\,.\,\overline{b}}{\left|\overline{a}\right|\left|\overline{b}\right|}$.

Dot Product

Properties

1. $\overline{a}.\overline{b} = \overline{b}.\overline{a}$

2. $\overline{a}.(\overline{b} + \overline{c}) = \overline{a}.\overline{b} + \overline{a}.\overline{c}$

3. $(l\overline{a}).(m\overline{b}) = lm(\overline{a}.\overline{b})$

Dot Product

NOTE

1. $\overline{a}.\overline{b} > 0 \Rightarrow$ angle between $\overline{a}$ and $\overline{b}$ is acute.
2. $\overline{a}.\overline{b} < 0 \Rightarrow$ angle between $\overline{a}$ and $\overline{b}$ is obtuse.
3. $\overline{a}.\overline{b} = 0 \Rightarrow \overline{a}$ ad $\overline{b}$ are perpendicular to each other.
4. $\overline{a}.\overline{a} = |\overline{a}|^2$
5. $|\overline{a}+\overline{b}|^2 = |\overline{a}|^2 + |\overline{b}|^2 + 2\overline{a}.\overline{b}$
6. $|\overline{a}+\overline{b}+\overline{c}|^2 = |\overline{a}|^2 + |\overline{b}|^2 + |\overline{c}|^2 + 2\left(\overline{a}.\overline{b}+\overline{b}.\overline{c}+\overline{c}.\overline{a}\right)$

ov imp

Q **Find** $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$ if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $\left|\vec{a}\right|=8\left|\vec{b}\right|$.

$a^2 - b^2 = 8$

& $\quad a = 8b \rightarrow$ a ✓

b ✓

Find $|\vec{a}|$ and $|\vec{b}|$ if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

Solution:

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$$

$$\Rightarrow \vec{a}\,.\,\vec{a}-\vec{a}\,.\,\vec{b}+\vec{b}\,.\,\vec{a}-\vec{b}\,.\,\vec{b}=8$$

$$\Rightarrow \left|\vec{a}\right|^2-\left|\vec{b}\right|^2=8$$

$$\Rightarrow 64\left|\vec{b}\right|^2-\left|\vec{b}\right|^2=8 \quad \left\{\because \left|\vec{a}\right|=8\left|\vec{b}\right|\right\}$$

$$\Rightarrow \left|\vec{b}\right|^2=\frac{8}{63} \Rightarrow \left|\vec{b}\right|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$\Rightarrow \left|\vec{a}\right|=8\left|\vec{b}\right|=\frac{8\times 2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, then prove that $\cos\theta/2 = \frac{1}{2}\left|\hat{a}+\hat{b}\right|$.

$$|\hat{a}+\hat{b}|^2 = a^2 + b^2 + 2\hat{a}\,\hat{b}$$

$$|\hat{a}+\hat{b}|^2 = 2 + 2\cos\theta$$

$$|\hat{a}+\hat{b}|^2 = 2(2\cos^2\theta/2)$$

$$|\hat{a}+\hat{b}| = 2\cos\theta/2$$

Solution:

Let's consider $\left|\hat{a}+\hat{b}\right|^2$

$$\left|\hat{a}+\hat{b}\right|^2 = \left(\hat{a}+\hat{b}\right).\left(\hat{a}+\hat{b}\right)$$

$$= \hat{a}.\hat{a}+\hat{a}.\hat{b}+\hat{b}.\hat{a}+\hat{b}.\hat{b}$$

$$= \left|\hat{a}\right|^2 + 2\,\hat{a}.\hat{b}+\left|\hat{b}\right|^2$$

$$= 1+2\left|\hat{a}\right|\left|\hat{b}\right|\cos\theta + 1$$

$$= 2+2\cos\theta$$

$$\therefore\ 1+\cos\theta = \frac{\left|\hat{a}+\hat{b}\right|^2}{2}$$

$$\Rightarrow 2\cos^2\frac{\theta}{2} = \frac{\left|\hat{a}+\hat{b}\right|^2}{2}$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{\left|\hat{a}+\hat{b}\right|^2}{4}$$

$$\Rightarrow \cos\frac{\theta}{2} = \frac{\left|\hat{a}+\hat{b}\right|}{2}$$

JEE Main 2020

Q Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three unit vectors such that $\left|\vec{a}-\vec{b}\right|^2+\left|\vec{a}-\vec{c}\right|^2=8$. Then $\left|\vec{a}+2\vec{b}\right|^2+\left|\vec{a}+2\vec{c}\right|^2$ is equal to

Sol. $2a^2+b^2+c^2-2\bar{a}\cdot\bar{b}-2\bar{a}\cdot\bar{c}=8$

$$\boxed{\bar{a}\cdot\bar{b}+\bar{a}\cdot\bar{c}=-2}$$

Now, we need. $a^2+4b^2+4\bar{a}\cdot\bar{b}+a^2+4c^2+4\bar{a}\cdot\bar{c}$

$=10+4(\bar{a}\cdot\bar{b}+\bar{a}\cdot\bar{c})$ $\to -2$

$=2$

JEE Main 2020

Q

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three unit vectors such that $\left|\vec{a}-\vec{b}\right|^2+\left|\vec{a}-\vec{c}\right|^2=8$. Then $\left|\vec{a}+2\vec{b}\right|^2+\left|\vec{a}+2\vec{c}\right|^2$ is equal to

Ans: 2

Solution:

Given, $\left|\overrightarrow{a}\right| = \left|\overrightarrow{b}\right| = \left|\overrightarrow{c}\right| = 1$

Also $\left|\overrightarrow{a} - \overrightarrow{b}\right|^2 + \left|\overrightarrow{a} - \overrightarrow{c}\right|^2 = 8$

$\Rightarrow \left|\overrightarrow{a}\right|^2 + \left|\overrightarrow{b}\right|^2 - 2\overrightarrow{a} \cdot \overrightarrow{b} + \left|\overrightarrow{a}\right|^2 + \left|\overrightarrow{c}\right|^2 - 2\overrightarrow{a} \cdot \overrightarrow{c} = 8$

$\Rightarrow \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c} = -2$

Now, $\left|\overrightarrow{a} + 2\overrightarrow{b}\right|^2 + \left|\overrightarrow{a} + 2\overrightarrow{c}\right|^2$

$= \left|\overrightarrow{a}\right|^2 + 4\left|\overrightarrow{b}\right|^2 + 4\overrightarrow{a} \cdot \overrightarrow{b} + \left|\overrightarrow{a}\right|^2 + 4\left|\overrightarrow{c}\right|^2 + 4\overrightarrow{a} \cdot \overrightarrow{c}$

$= 10 + 4\left(\overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c}\right)$

$= 10 + 4(-2) = 2$

Q

If three unit vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol:

$$\bar{a}+\bar{b}+\bar{c}=0$$

$$|\bar{a}+\bar{b}|=|-\bar{c}|$$

$$1^2+1^2+2\bar{a}\cdot\bar{b}=1$$

$$2\times1\times1\times\cos\theta=-1$$

$$\cos\theta=-\frac{1}{2}$$

$$\theta=120$$

Solution:

$$\overrightarrow{a}+\overrightarrow{b}=-\overrightarrow{c}$$

$$\Rightarrow\left|\overrightarrow{a}+\overrightarrow{b}\right|^{2}=\left|\overrightarrow{c}\right|^{2}=1$$

$$\Rightarrow\left|\overrightarrow{a}\right|^{2}+\left|\overrightarrow{b}\right|^{2}+2\overrightarrow{a}\cdot\overrightarrow{b}=1$$

$$\Rightarrow\overrightarrow{a}\cdot\overrightarrow{b}=-\frac{1}{2}$$

$$\Rightarrow\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\cos\theta=-\frac{1}{2}$$

$$\Rightarrow\cos\theta=-\frac{1}{2}$$

$$\Rightarrow\theta=\frac{2\pi}{3}$$

Dot Product

Remark

If in some question, the value of $\overline{a}.\overline{b}+\overline{b}.\overline{c}+\overline{c}.\overline{a}$ is asked or required , then use $\left|\overline{a}+\overline{b}+\overline{c}\right|^2=\left|\overline{a}\right|^2+\left|\overline{b}\right|^2+\left|\overline{c}\right|^2+2\left(\overline{a}.\overline{b}+\overline{b}.\overline{c}+\overline{c}.\overline{a}\right)$ to create it.

Q

If $\overline{a}, \overline{b}$ and $\overline{c}$ are unit vectors then find minimum value of $\overline{a}\cdot\overline{b}+\overline{b}\cdot\overline{c}+\overline{c}\cdot\overline{a}$.

Sol. $|\overline{a}+\overline{b}+\overline{c}|^2 = 1^2+1^2+1^2+2(\overline{a}\cdot\overline{b}+\overline{b}\cdot\overline{c}+\overline{c}\cdot\overline{a})$

$$\overline{a}\cdot\overline{b}+\overline{b}\cdot\overline{c}+\overline{c}\cdot\overline{a} = \frac{|\overline{a}+\overline{b}+\overline{c}|^2-3}{2} \geq 0-\frac{3}{2}$$

Remember as a result → For unit vectors $\hat{a}, \hat{b}$ & $\hat{c}$;

$\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a}$ has min value $-3/2$

which is attained when $\hat{a}+\hat{b}+\hat{c}=0$

Q

If $\overline{a}, \overline{b}$ and $\overline{c}$ are unit vectors then find minimum value of $\overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{c} + \overline{c} \cdot \overline{a}$.

Solution:

It is given that $\overline{a}, \overline{b}, \overline{c}$ are unit vectors

$\Rightarrow |\overline{a}| = |\overline{b}| = |\overline{c}| = 1$

Consider the expression $|\overline{a} + \overline{b} + \overline{c}|$

We know that $|\overline{a} + \overline{b} + \overline{c}| \geq 0$

$\Rightarrow |\overline{a}|^2 + |\overline{b}|^2 + |\overline{c}|^2 + 2\left(\overline{a}.\overline{b} + \overline{b}.\overline{c} + \overline{c}.\overline{a}\right) \geq 0$

$\Rightarrow 1 + 1 + 1 + 2\left(\overline{a}.\overline{b} + \overline{b}.\overline{c} + \overline{c}.\overline{a}\right) \geq 0$

$\therefore \overline{a}.\overline{b} + \overline{b}.\overline{c} + \overline{c}.\overline{a} \geq -\frac{3}{2} \quad \ldots(1)$

IIT JEE 2012

Repeat

Q If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is

Sol. $2+2+2-2(\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{a}\cdot\bar{c})=9$

$\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a}=-3/2$

$\bar{a}+\bar{b}+\bar{c}=0$

$\therefore\ |2\bar{a}+5\underbrace{(\bar{b}+\bar{c})}_{-\bar{a}}|=|-3\bar{a}|=3$

IIT JEE 2012

Q

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $\left|\vec{a}-\vec{b}\right|^2+\left|\vec{b}-\vec{c}\right|^2+\left|\vec{c}-\vec{a}\right|^2=9$, then $\left|2\vec{a}+5\vec{b}+5\vec{c}\right|$ is

Ans: 3

Solution:

$$\left|\vec{a}-\vec{b}\right|^2+\left|\vec{b}-\vec{c}\right|^2+\left|\vec{c}-\vec{a}\right|^2=9$$

$$=2\left(\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{c}-\vec{c}\cdot\vec{a}\right)=9 \quad \ldots(i)$$

$$\Rightarrow 2\left(3-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{c}-\vec{c}\cdot\vec{a}\right)=9$$

$$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\tfrac{3}{2} \Rightarrow \text{angle between each pair is } 120^\circ.$$

$$\text{Now,} \left|2\vec{a}+5\vec{b}+5\vec{c}\right|^2$$

$$=4\left|\vec{a}\right|^2+25\left|\vec{b}\right|^2+25\left|\vec{c}\right|^2+20\vec{a}\cdot\vec{b}+20\vec{a}\cdot\vec{c}+50\vec{a}\cdot\vec{c}$$

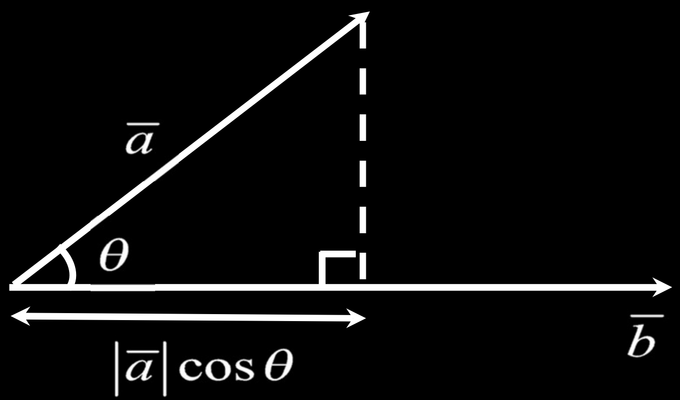
$$=4+25+25-\tfrac{20}{2}-\tfrac{20}{2}-\tfrac{50}{2}=9$$

$$\therefore \left|2\vec{a}+5\vec{b}+5\vec{c}\right|=3$$

Dot Product

Geometrical Significance of the Dot product

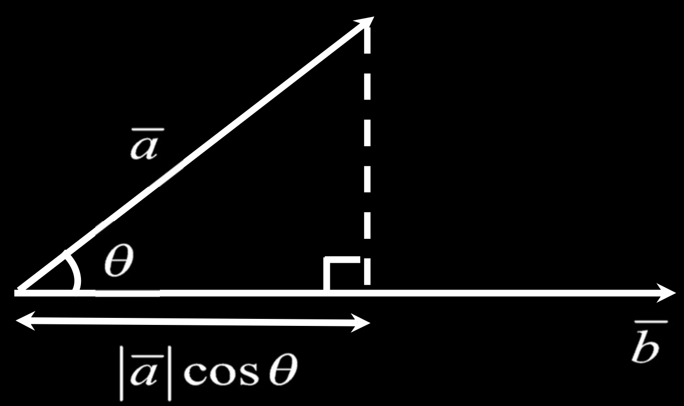
Let's look at the projection of a vector along another vector.



Dot Product

Geometrical Significance of the Dot product

Let's look at the projection of a vector along another vector.



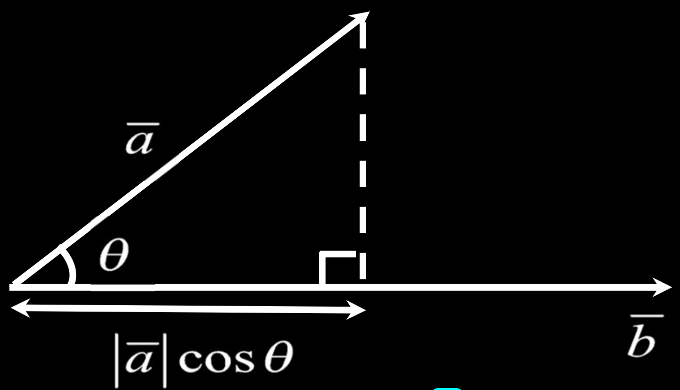
Projection of $\overline{a}$ on $\overline{b} = \overline{a}\,.\hat{b}$

Projection of $\overline{b}$ on $\overline{a} = \overline{b}\,.\hat{a}$

Dot Product

Geometrical Significance of the Dot product

Let's look at the projection of a vector along another vector.



Projection of $\overline{a}$ on $\overline{b} = \overline{a}\,.\hat{b}$

Projection of $\overline{b}$ on $\overline{a} = \overline{b}\,.\hat{a}$

NOTE

$\boxed{\left(\overline{a}\,.\hat{b}\right)\hat{b}$ is called projection vector (or component vector) of $\overline{a}$ along $\overline{b}}$

$\left(\overline{a}\,.\hat{b}\right)\hat{b}$ is called projection vector (or component vector) of $\overline{a}$ along $\overline{b}$

Q

Find the projection or the component of $\overline{a} = \hat{i} - 2\hat{j} + \hat{k}$ **on** $\overline{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Sol: $\overline{a} \cdot \hat{b}$ Ans

Q Find the projection or the component of $\bar{a} = \hat{i} - 2\hat{j} + \hat{k}$ on $\bar{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Solution:

Given $\bar{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\bar{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

$$|\bar{b}| = \sqrt{4^2 + 4^2 + 7^2} = \sqrt{16 + 16 + 49} = \sqrt{81} = 9$$

Now, $\hat{b} = \dfrac{\bar{b}}{|\bar{b}|} = \dfrac{4\hat{i} - 4\hat{j} + 7\hat{k}}{9}$

Projection of $\bar{a}$ on $\bar{b} = \bar{a}.\hat{b}$

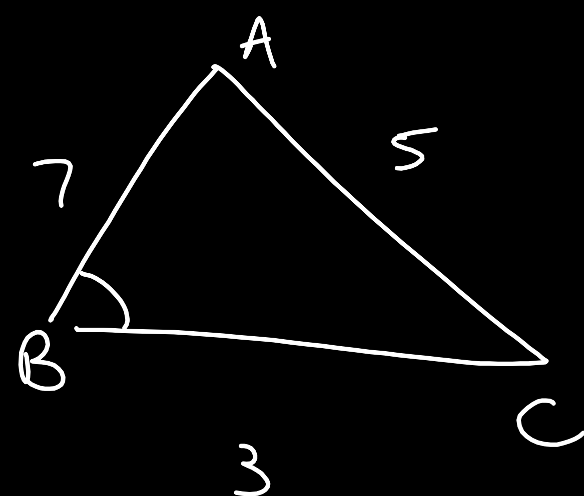
$$= \left(\hat{i} - 2\hat{j} + \hat{k}\right).\frac{\left(4\hat{i} - 4\hat{j} + 7\hat{k}\right)}{9} = \frac{1 \times 4 + 2 \times 4 + 1 \times 7}{9} = \frac{19}{9}$$

Q In a triangle ABC, if $\left|\overrightarrow{BC}\right| = 3$, $\left|\overrightarrow{CA}\right| = 5$ and $\left|\overrightarrow{BA}\right| = 7$, then projection of the vector $\overrightarrow{BA}$ on $\overrightarrow{BC}$ is equal to

Sol: $\overline{BA} \cdot \hat{BC}$

$= 7 \times 1 \times \cos B$

$\cos B = \dfrac{7^2 + 3^2 - 5^2}{2 \times 7 \times 3}$



(A) $\frac{19}{2}$

(B) $\frac{13}{2}$

(C) $\frac{11}{2}$

(D) $\frac{15}{2}$

JEE Main 20th July 2021

Q

In a triangle ABC, if $\left|\overrightarrow{BC}\right| = 3$, $\left|\overrightarrow{CA}\right| = 5$ and $\left|\overrightarrow{BA}\right| = 7$, then projection of the vector $\overrightarrow{BA}$ on $\overrightarrow{BC}$ is equal to

(A) $\frac{19}{2}$

(B) $\frac{13}{2}$

(C) $\frac{11}{2}$

(D) $\frac{15}{2}$

Q

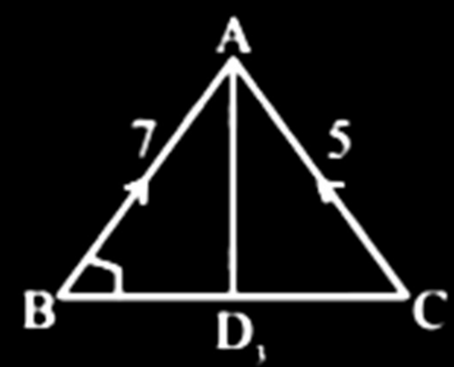
In a triangle ABC, if $\left|\overrightarrow{BC}\right| = 3$, $\left|\overrightarrow{CA}\right| = 5$ and $\left|\overrightarrow{BA}\right| = 7$, then projection of the vector $\overrightarrow{BA}$ on $\overrightarrow{BC}$ is equal to

Solution:

Projection of $\overrightarrow{BA}$ on $\overrightarrow{BC}$ is equal to

$$= \left|\overrightarrow{BA}\right| \cos \angle ABC$$

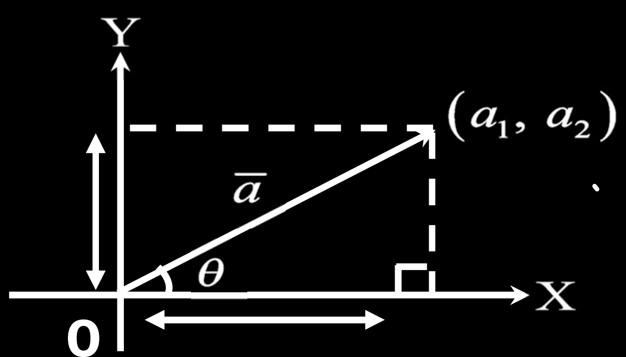
$$= 7\left|\frac{7^2+3^2-5^2}{2\times7\times3}\right| = \frac{11}{2}$$



Rectangular resolution of a Vector

Rectangular resolution of a Vector

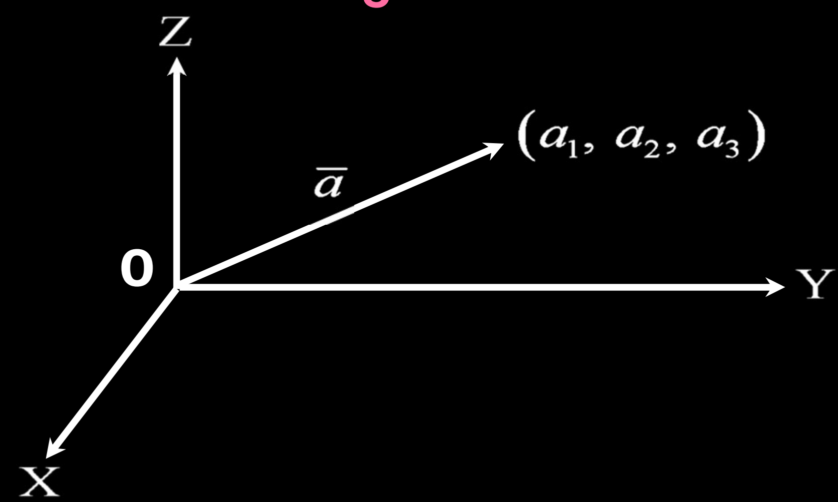
Along X - Y



$$\bar{a} = a_1\hat{i} + a_2\hat{j}$$

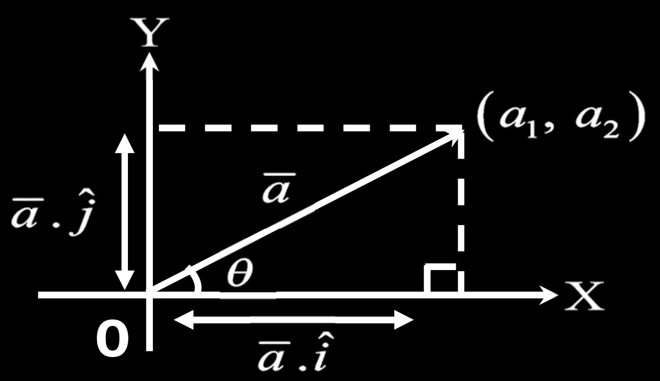
$$\bar{a} = (\bar{a}\cdot\hat{i})\hat{i} + (\bar{a}\cdot\hat{j})\hat{j}$$

Along X - Y - Z



Rectangular resolution of a Vector

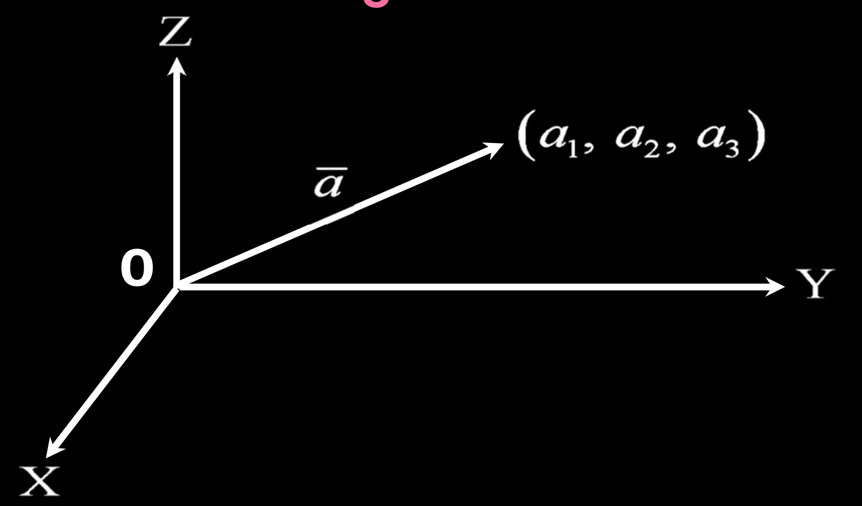
Along X - Y



Clearly, $a_1 = \overline{a}\,.\,\hat{i}$ and $\overline{a}_2 = \overline{a}\,.\,\hat{j}$

$$\therefore\ \overline{a} = \left(\overline{a}\,.\,\hat{i}\right)\hat{i} + \left(\overline{a}\,.\,\hat{j}\right)\hat{j}$$

Along X - Y - Z



Clearly, $a_1 = \overline{a}\,.\,\hat{i}$, $a_2 = \overline{a}\,.\,\hat{j}$ and $a_3 = \overline{a}\,.\,\hat{k}$

$$\therefore\ \overline{a} = \left(\overline{a}\,.\,\hat{i}\right)\hat{i} + \left(\overline{a}\,.\,\hat{j}\right)\hat{j} + \left(\overline{a}\,.\,\hat{k}\right)\hat{k}$$

Rectangular resolution of a Vector

Remark

1. If a vector $\overline{a}$ lies in a plane of two $\perp$ vectors $\overline{p}$ and $\overline{q}$ then we can resolve $\overline{a}$ as $\overline{a}=\left(\overline{a}\,.\,\hat{p}\right)\hat{p}+\left(\overline{a}\,.\,\hat{q}\right)\hat{q}$

Rectangular resolution of a Vector

Remark

1. If a vector $\overline{a}$ lies in a plane of two $\perp$ vectors $\overline{p}$ and $\overline{q}$ then we can resolve $\overline{a}$ as $\overline{a} = (\overline{a}\,.\,\hat{p})\,\hat{p} + (\overline{a}\,.\,\hat{q})\,\hat{q}$

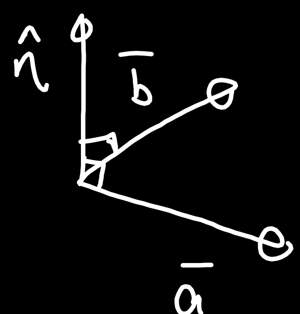
2. If $\overline{p}, \overline{q}, \overline{r}$ are any three mutually $\perp$ vectors then we can resolve $\overline{a}$ as $\overline{a} = (\overline{a}\,.\,\hat{p})\,\hat{p} + (\overline{a}\,.\,\hat{q})\,\hat{q} + (\overline{a}\,.\,\hat{r})\,\hat{r}$

Cross Product

Cross Product

$$\bar{a} \times \bar{b}$$

$$\left(|\bar{a}|\,|\bar{b}|\sin\theta\right)\hat{n}$$



$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Q If vectors $\overline{a}$ and $\overline{b}$ are such that $|\overline{a}| = 4$ and $|\overline{b}| = \frac{1}{2}$ and $|\overline{a} \times \overline{b}| = \sqrt{3}$, find angle between $\overline{a}$ and $\overline{b}$.

$$b$$

$$\cancel{4}^{2} \times \frac{1}{\cancel{2}} \sin\theta = \sqrt{3}$$

$$\theta = 60^\circ$$

Q

If vectors $\overline{a}$ and $\overline{b}$ are such that $|\overline{a}| = 4$ and $|\overline{b}| = \frac{1}{2}$ and $|\overline{a} \times \overline{b}| = \sqrt{3}$, find angle between $\overline{a}$ and $\overline{b}$.

Solution:

We know that $|\overline{a} \times \overline{b}| = |\overline{a}|\,|\overline{b}| \sin\theta$

$\Rightarrow \sqrt{3} = 4 \times \frac{1}{2} \sin\theta$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

Hence, angle between $\overline{a}$ and $\overline{b}$ is $\frac{\pi}{3} = 60^\circ$

Find a vector of magnitude 7 units, which is perpendicular to the two vectors $2\hat{i}-\hat{j}+\hat{k}$ **and** $\hat{i}+\hat{j}-\hat{k}$

$\downarrow$ $\bar{a}$ $\qquad$ $\downarrow$ $\bar{b}$

$$\pm 7\left(\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}\right)$$

Solution:

Given two vectors are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$

$\therefore$ A unit vector $\perp$ to $\overline{a}$ and $\overline{b}$ is $\hat{u}=\dfrac{\overline{a}\times\overline{b}}{\left|\overline{a}\times\overline{b}\right|}$

$$\overline{a}\times\overline{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=\hat{i}(1-1)-\hat{j}(-2-1)+\hat{k}(2+1)=3\hat{j}+3\hat{k}$$

$$\therefore\ \left|\overline{a}\times\overline{b}\right|=\sqrt{0^2+3^2+3^2}=\sqrt{0+9+9}=\sqrt{18}=3\sqrt{2}$$

$$\therefore\ \hat{u}=\frac{3\hat{j}+3\hat{k}}{3\sqrt{2}}=\frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

Hence, a vector of magnitude 7 units perpendicular to $\overline{a}$ and $\overline{b}$ is $7\hat{u}$, that is $\pm\dfrac{7}{\sqrt{2}}\left(\hat{j}+\hat{k}\right)$

JEE Main 10th April, 2023

Q Let O be the origin and the position vector of the point P be $-\hat{i}-2\hat{j}+3\widehat{k}$. If the position vectors of A, B and C are $-2\hat{i}+\hat{j}-3\widehat{k},\ 2\hat{i}+4\hat{j}-2\widehat{k}$ and $-4\hat{i}+2\hat{j}-\widehat{k}$ respectively then the projection of vector $\overrightarrow{OP}$ on a vector perpendicular to the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is :

(A) $\frac{10}{3}$

(B) $\frac{8}{3}$

(C) $\frac{7}{3}$

(D) 3

Sol.

$$\frac{\overline{OP}\cdot(\overline{AB}\times\overline{AC})}{|\overline{AB}\times\overline{AC}|} \quad \text{Ans}$$

JEE Main 10th April, 2023

Q

Let O be the origin and the position vector of the point P be $-\hat{i}-2\hat{j}+3\widehat{k}$. If the position vectors of A, B and C are $-2\hat{i}+\hat{j}-3\widehat{k},\ 2\hat{i}+4\hat{j}-2\widehat{k}$ and $-4\hat{i}+2\hat{j}-\widehat{k}$ respectively then the projection of vector $\overrightarrow{OP}$ on a vector perpendicular to the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is :

(A) $\frac{10}{3}$

(B) $\frac{8}{3}$

(C) $\frac{7}{3}$

(D) 3

Solution:

Position vector of the point $P(-1,-2,3)$, $A(-2,1,-3)$ $B(2,4,-2)$, and $C(-4,2,-1)$

Then $\overrightarrow{OP}.\dfrac{\overrightarrow{AB}\times\overrightarrow{AC}}{\left|(\overrightarrow{AB}\times\overrightarrow{AC})\right|}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 3 & 1\\ -2 & 1 & 2\end{vmatrix}$$

$$=\hat{i}(5)-\hat{j}(8+2)+\hat{k}(4+6)$$

$$=5\hat{i}-10\hat{j}+10\hat{k}$$

Now

Solution:

$$\overrightarrow{OP}.\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{\left|(\overrightarrow{AB}\times\overrightarrow{AC})\right|}=(-\hat{i}-2\hat{j}+3\hat{k}).\frac{(5\hat{i}-10\hat{j}+10\hat{k})}{\sqrt{(5)^2+(-10)^2+(10)^2}}$$

$$=\frac{-5+20+30}{\sqrt{25+100+100}}$$

$$=\frac{45}{\sqrt{225}}=\frac{45}{15}=3$$

Cross Product

Properties

1. $\overline{a} \times \overline{b} = -\overline{b} \times \overline{a}$
2. $\overline{a} \times \left(\overline{b} + \overline{c}\right) = \overline{a} \times \overline{b} + \overline{a} \times \overline{c}$
3. $\left(l\overline{a}\right) \times \left(m\overline{b}\right) = lm\left(\overline{a} \times \overline{b}\right)$

$*$

$$\bar{a} \times \bar{b} + \bar{b} \times \bar{c}$$
$$= \bar{a} \times \bar{b} - \bar{c} \times \bar{b}$$
$$= (\bar{a} - \bar{c}) \times \bar{b}$$

Cross Product

Properties

1. $\overline{a} \times \overline{b} = -\overline{b} \times \overline{a}$
2. $\overline{a} \times \left(\overline{b} + \overline{c}\right) = \overline{a} \times \overline{b} + \overline{a} \times \overline{c}$
3. $(l\overline{a}) \times \left(m\overline{b}\right) = lm\left(\overline{a} \times \overline{b}\right)$

NOTE

1. $\left(\overline{a} \times \overline{b}\right).\overline{a} = \left(\overline{a} \times \overline{b}\right).\overline{b} = 0$
2. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ & $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$
3. If $\overline{a}$ and $\overline{b}$ are non-zero vectors then $\overline{a} \times \overline{b} =$ 0 implies they are parallel

Q

Find λ if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\overline{0}$

$$\frac{2}{1}=\frac{6}{-\lambda}=\frac{14}{\lambda}$$

$$\lambda=-3$$

Q Find λ if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\overline{0}$

Solution:

We have $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right)=\overline{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix} = \overline{0}$$

$$\Rightarrow \hat{i}\left(42+14\lambda\right)-\hat{j}\left(14-14\right)+\hat{k}\left(-2\lambda-6\right)=\overline{0}$$

$$\Rightarrow 14\hat{i}\left(\lambda+3\right)-2\left(\lambda+3\right)\hat{k}=\overline{0}$$

$$\Rightarrow \lambda+3=0$$

Hence, $\lambda=-3$

IIT JEE 2012

Q

If $\overline{a}$ and $\overline{b}$ are vectors such that $|\overline{a}+\overline{b}|=\sqrt{29}$ and $\overline{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\overline{b}$, then a possible value of $(\overline{a}+\overline{b}).\left(-7\hat{i}+2\hat{j}+3\hat{k}\right)$ is

(A) 0

(B) 3

(C) 4

(D) 8

IIT JEE 2012

Q

If $\overline{a}$ and $\overline{b}$ are vectors such that $\left|\overline{a}+\overline{b}\right|=\sqrt{29}$ and $\overline{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\overline{b}$, then a possible value of $\left(\overline{a}+\overline{b}\right).\left(-7\hat{i}+2\hat{j}+3\hat{k}\right)$ is

(A) 0

(B) 3

(C) 4

(D) 8

Solution:

It is given that $\overline{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\overline{b}$

$\Rightarrow \overline{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)-\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\times\overline{b}=\overline{0}$

$\Rightarrow \overline{a}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\overline{b}\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\overline{0}$

$\Rightarrow \left(\overline{a}+\overline{b}\right)\times\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=\overline{0}$

Since $2\hat{i}+3\hat{j}+4\hat{k}\neq\overline{0}$ $\Rightarrow \overline{a}+\overline{b}$ is parallel to $2\hat{i}+3\hat{j}+4\hat{k}$ or $\overline{a}+\overline{b}=\overline{0}$

But $\overline{a}+\overline{b}\neq\overline{0}$ because it is given that $\left|\overline{a}+\overline{b}\right|=\sqrt{29}$

$\therefore\ \overline{a}+\overline{b}\ \parallel\ 2\hat{i}+3\hat{j}+4\hat{k}$ $\Rightarrow \overline{a}+\overline{b}=\lambda\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$

Solution:

$$\text{Since } \left|\bar{a}+\bar{b}\right| = \sqrt{29} \Rightarrow \sqrt{\lambda^2\left(2^2+3^2+4^2\right)} = \sqrt{29}$$

$$\Rightarrow |\lambda|\sqrt{4+9+16} = \sqrt{29}$$

$$\Rightarrow |\lambda|\sqrt{29} = \sqrt{29}$$

$$\Rightarrow |\lambda| = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

$$\therefore\ \bar{a}+\bar{b} = \pm\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$$

$$\text{Now, } \left(\bar{a}+\bar{b}\right).\left(-7\hat{i}+2\hat{j}+3\hat{k}\right) = \pm\left(2\hat{i}+3\hat{j}+4\hat{k}\right).\left(-7\hat{i}+2\hat{j}+3\hat{k}\right) = \pm 4$$

JEE Main 24^{th} June, 2022

Q Let $\hat{a}$, $\hat{b}$ be unit vectors. If $\vec{c}$ be a vector such that the angle between $\hat{a}$ and $\vec{c}$ is $\frac{\pi}{12}$ and $\hat{b} = \vec{c} + 2\left(\vec{c} \times \hat{a}\right)$, then $\left|6\vec{c}\right|^2$ is equal to

(A) $6\left(3-\sqrt{3}\right)$

(B) $6\left(3+\sqrt{3}\right)$

(C) $6\left(\sqrt{3}+1\right)$

(D) $3+\sqrt{3}$

$\hat{b} = \bar{c} + 2(\bar{c} \times \bar{a})$ $\perp$r

Taking mod & sq

$1 = c^2 + 4\boxed{(|\bar{c}\times\bar{a}|)^2}$ $c^2 \times 1 \times \sin^2 \pi/12$

$c^2 = \dfrac{1}{1+4\sin^2 \pi/12}$

JEE Main 24th June, 2022

Q

Let $\hat{a}$, $\hat{b}$ be unit vectors. If $\vec{c}$ be a vector such that the angle between $\hat{a}$ and $\vec{c}$ is $\frac{\pi}{12}$ and $\hat{b} = \vec{c} + 2\left(\vec{c} \times \hat{a}\right)$, then $\left|6\vec{c}\right|^2$ is equal to

A $6\left(3-\sqrt{3}\right)$

B $6\left(3+\sqrt{3}\right)$

C $6\left(\sqrt{3}+1\right)$

D $3+\sqrt{3}$

Solution:

$$\left|\hat{b}\right|^2=\left|\vec{c}+2\left(\vec{c}\times\hat{a}\right)\right|^2$$

$$\left|\hat{b}\right|^2=\left|\vec{c}\right|^2+4\left|\vec{c}\times\hat{a}\right|^2+4\vec{c}.\left(\vec{c}\times\hat{a}\right)$$

$$1=\left|\vec{c}\right|^2+4\left|\vec{c}\right|^2\sin^2\tfrac{\pi}{12}+0$$

$$1=\left|\vec{c}\right|^2+4\left|\vec{c}\right|^2\left(\tfrac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$\left|\vec{c}\right|^2=\tfrac{1}{3-\sqrt{3}}=\tfrac{3+\sqrt{3}}{6}$$

$$\text{So } 6^2\left|\vec{c}\right|^2=6\left(3+\sqrt{3}\right)$$

Q **Prove that:** $\left(\bar{a}\times\bar{b}\right)^2 = |\bar{a}|^2 |\bar{b}|^2 - \left(\bar{a}.\bar{b}\right)^2$ **(Lagrange's Identity)**

Sol: LHS: $a^2b^2 \sin^2\theta \rightarrow 1-\cos^2\theta$

$a^2b^2 - a^2b^2\cos^2\theta \rightarrow (\bar{a}.\bar{b})^2$

$= \text{RHS.}$

Prove that: $\left(\bar{a}\times\bar{b}\right)^2 = |\bar{a}|^2|\bar{b}|^2 - \left(\bar{a}.\bar{b}\right)^2$ **(Lagrange's Identity)**

Solution:

$$\left(\bar{a}\times\bar{b}\right)^2 = \left(|\bar{a}||\bar{b}|\sin\theta\,\hat{n}\right)^2$$

$$= |\bar{a}|^2|\bar{b}|^2\sin^2\theta\,\hat{n}^2$$

$$= |\bar{a}|^2|\bar{b}|^2\sin^2\theta$$

$$= |\bar{a}|^2|\bar{b}|^2\left(1-\cos^2\theta\right)$$

$$= |\bar{a}|^2|\bar{b}|^2 - |\bar{a}|^2|\bar{b}|^2\cos^2\theta$$

$$= |\bar{a}|^2|\bar{b}|^2 - \left(\bar{a}.\bar{b}\right)^2$$

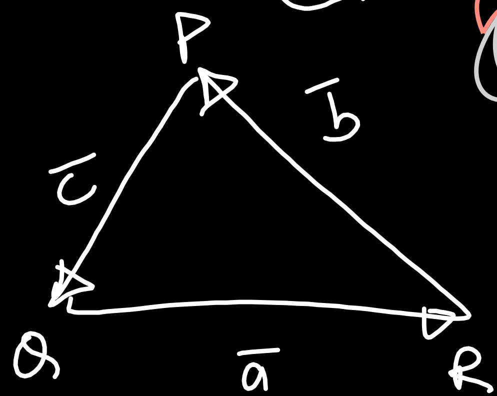
Repeat

JEE Advanced 2020

Q

In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$.

If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\dfrac{\vec{a}\cdot(\vec{c}-\vec{b})}{\vec{c}\cdot(\vec{a}-\vec{b})} = \dfrac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$ then value of $|\vec{a}\times\vec{b}|^2$ is



Sol $\boxed{\bar{a}+\bar{b}+\bar{c}=0}$

Now, $\dfrac{\bar{a}\cdot(-\bar{a}-2\bar{b})}{-(\bar{a}+\bar{b})\cdot(\bar{a}-\bar{b})} = \dfrac{3}{7}$

$\Rightarrow \dfrac{-9-2\bar{a}\cdot\bar{b}}{\cancel{-(-7)}} = \dfrac{3}{\cancel{7}}$

$\bar{a}\cdot\bar{b} = -6$

Lagrange $\rightarrow$ Ans

$|\bar{a}\times\bar{b}|^2 = 9\times16-36$.

JEE Advanced 2020

Q

In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$.

If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\dfrac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \dfrac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$ then value of $|\vec{a} \times \vec{b}|^2$ is

Ans: 108

Solution:

$$\because \vec{a}+\vec{b}+\vec{c}=0 \quad \ldots(i)$$

$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$$

$$\vec{a}\cdot\vec{c}=-9-\vec{a}\cdot\vec{b} \quad \ldots(ii) \text{ and}$$

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}=0$$

$$\Rightarrow \vec{c}\cdot\vec{b}=-16-\vec{a}\cdot\vec{b} \quad \ldots(iii)$$

$$\because \frac{\vec{a}\cdot\left(\vec{c}-\vec{b}\right)}{\vec{c}\cdot\left|\vec{a}-\vec{b}\right|}=\frac{\left|\vec{a}\right|}{\left|\vec{a}\right|+\left|\vec{b}\right|} \Rightarrow \frac{\vec{a}\cdot\vec{c}-\vec{a}\cdot\vec{b}}{\vec{c}\cdot\vec{a}-\vec{c}\cdot\vec{b}}=\frac{3}{7}$$

Solution:

$$\Rightarrow 7\vec{a}\cdot\vec{b}=4\vec{a}\cdot\vec{c}+3\,\vec{b}\cdot\vec{c}$$

$$\Rightarrow 7\vec{a}\cdot\vec{b}=-36-4\,\vec{a}\cdot\vec{b}-48-3\,\vec{a}\cdot\vec{b}$$

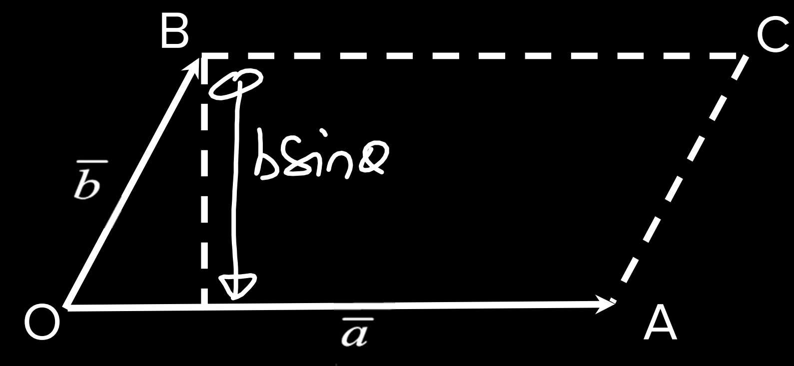
$$\Rightarrow 14\vec{a}\cdot\vec{b}=-84\Rightarrow\vec{a}\cdot\vec{b}=6$$

$$\therefore\left|\vec{a}\times\vec{b}\right|^2=\left|\vec{a}\right|^2\left|\vec{b}\right|^2-\left(\vec{a}\cdot\vec{b}\right)^2$$

$$=9\,(16)-6^2=144-36=108$$

Geometrical Significance of the Cross Product

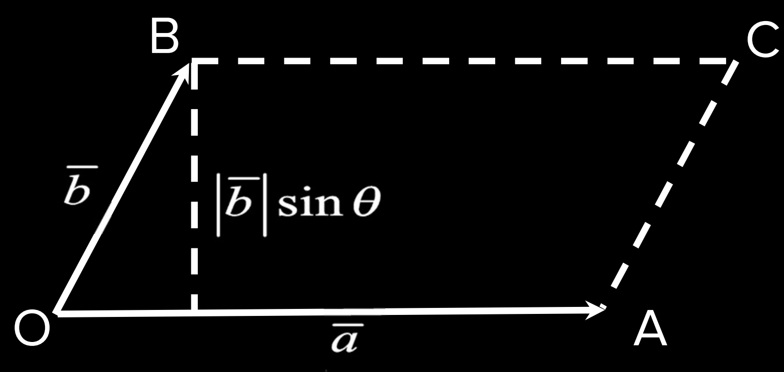
$$|\bar{a} \times \bar{b}| = ab\sin\theta$$



Geometrical Significance of the Cross Product

Area of parallelogram $\mathrm{OABC} = |\overline{a}||\overline{b}|\sin\theta$

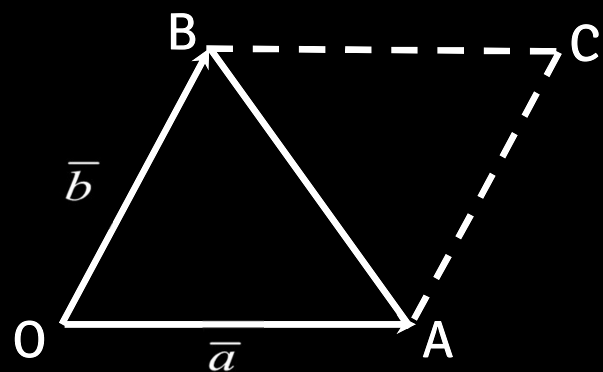
$$= |\overline{a} \times \overline{b}|$$



NOTE

$\overline{a} \times \overline{b}$ is the vector area of parallelogram OABC

Remark



Area of $\Delta\,\mathrm{OAB} = \dfrac{1}{2}\left(\text{Area of parallelogram OABC}\right)$

$$= \frac{1}{2}\left|\overline{a} \times \overline{b}\right|$$

Result

The area of a quadrilaterals is $\frac{1}{2}\left|\overline{d_1} \times \overline{d_2}\right|$ where $\overline{d_1}$ and $\overline{d_2}$ are vectors.

JEE Main 1st Feb, 2023

Q **A(2, 6, 2), B(−4, 0, λ), C(2, 3, −1) and D(4, 5, 0), $|\lambda| \leq 5$ are the vertices of a quadrilateral ABCD. If its area is 18 sq. units, then 5 − 6λ is equal to _____.**

Sol. $\frac{1}{2}\left|\bar{AC} \times \bar{BD}\right| = 18$

JEE Main 1st Feb, 2023

Q A(2, 6, 2), B(−4, 0, λ), C(2, 3, −1) and D(4, 5, 0), $|\lambda| \leq 5$ are the vertices of a quadrilateral ABCD. If its area is 18 sq. units, then $5 - 6\lambda$ is equal to _____.

Ans: 11

Solution:

$$A(2,6,2),\ B(-4,0,\lambda), C(2,3,-1),\ D(4,5,0)$$

$$\text{Area} = \frac{1}{2}|\overrightarrow{BD} \times \overrightarrow{AC}| = 18$$

$$\overrightarrow{AC} \times \overrightarrow{BD} = \begin{vmatrix} \hat{i} & j & k \\ 0 & -3 & -3 \\ 8 & 5 & -\lambda \end{vmatrix} = (3\lambda + 15)\hat{i} - \hat{j}(-24) + \hat{k}(-24)$$

$$\overrightarrow{AC} \times \overrightarrow{BD} = (3\lambda + 15)\hat{i} + 24\hat{j} - 24\hat{k}$$

$$\Rightarrow \sqrt{(3\lambda + 15)^2 + (24)^2 + (24)^2} = 36$$

$$\Rightarrow \lambda^2 + 10\lambda + 9 = 0$$

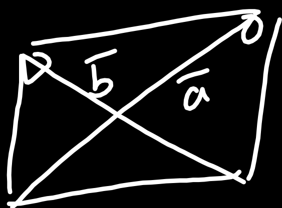
$$\Rightarrow \lambda = -1, -9$$

$$|\lambda| \leq 5 \Rightarrow \lambda = -1$$

$$5 - 6\lambda = 5 - 6(-1) = 11$$

Repeat

JEE Main 27th June, 2022



Q

Let $\vec{a}$ and $\vec{b}$ be the vectors along the diagonal of a parallelogram having area $2\sqrt{2}$. Let the angle between $\vec{a}$ and $\vec{b}$ be actue. $|\vec{a}| = 1$ and $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$. If $\vec{c} = 2\sqrt{2}\left(\vec{a}\times\vec{b}\right) - 2\vec{b}$, then an angle between $\vec{b}$ and $\vec{c}$ is:

(A) $\frac{\pi}{4}$

(B) $-\frac{\pi}{4}$

(C) $\frac{5\pi}{6}$

(D) $\frac{3\pi}{4}$

Sol

$c^2 = 8\times 32 + 4b^2$; $\frac{1}{2}|a\times\bar{b}| = 2\sqrt{2}$

$c^2 = 256 + 256 = 512$ $\quad |\bar{a}\times\bar{b}| = 4\sqrt{2} = |\bar{a}\cdot\bar{b}|$

$c = 16\sqrt{2}$

Now,

$\bar{c} = 2\sqrt{2}(\bar{a}\times\bar{b}) - 2\bar{b}$

$\Rightarrow \bar{b}\cdot\bar{c} = 0 - 2b^2$

$\Rightarrow \frac{\bar{b}\cdot\bar{c}}{bc} = -\frac{\cancel{128}}{\cancel{8\times 16}\sqrt{2}} = -\frac{1}{\sqrt{2}}$

lagrange.

$32 = a^2b^2 - 32$

$a^2b^2 = 64 \Rightarrow b^2 = 64$

JEE Main 27th June, 2022

Q

Let $\vec{a}$ and $\vec{b}$ be the vectors along the diagonal of a parallelogram having area $2\sqrt{2}$. Let the angle between $\vec{a}$ and $\vec{b}$ be actue. $|\vec{a}| = 1$ and $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$. If $\vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$, then an angle between $\vec{b}$ and $\vec{c}$ is:

A $\frac{\pi}{4}$

B $-\frac{\pi}{4}$

C $\frac{5\pi}{6}$

D $\frac{3\pi}{4}$

Solution:

$$\text{Area} = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = 2\sqrt{2} \Rightarrow \left|\vec{a} \times \vec{b}\right| = 4\sqrt{2}$$

$$|\vec{a}| = 1 \text{ and } \left|\vec{a}.\vec{b}\right| = \left|\vec{a} \times \vec{b}\right|$$

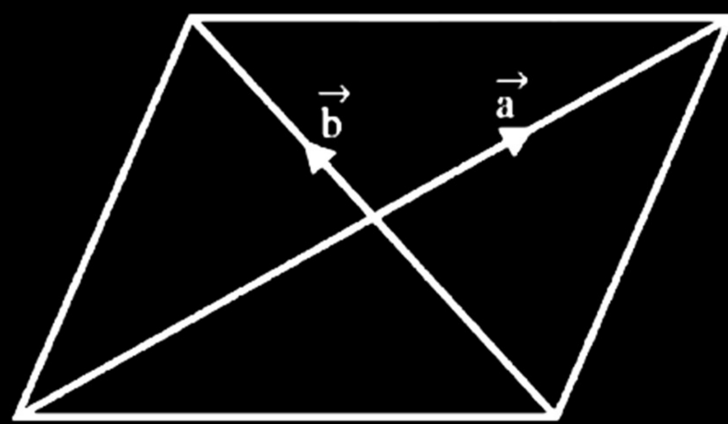
$$\Rightarrow \cos\theta = \sin\theta$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = 4\sqrt{2} \Rightarrow |\vec{a}|\left|\vec{b}\right| \sin\frac{\pi}{4} = 4\sqrt{2}$$

$$\Rightarrow \left|\vec{b}\right| = 8$$

$$\text{Now, } \vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$$



Solution:

$$|\vec{c}| = \sqrt{\left(2\sqrt{2}\right)^2 \left|\vec{a} \times \vec{b}\right|^2 + \left(2\left|\vec{b}\right|\right)^2} = 16\sqrt{2}$$

$$\text{Now, } \vec{b}.\vec{c} = -2\left|\vec{b}\right|^2$$

$$\Rightarrow 8 \times 16\sqrt{2} \times \cos\alpha = -2.64$$

$$\Rightarrow \cos\alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{3\pi}{4}$$

Triple Products

Triple Products

There are two kinds of triple products, namely

1. Scalar triple product. → (or) Box product
2. Vector triple product

Let's consider them one by one.

Scalar Triple Product

Scalar Triple Product (Box Product)

$\overline{a}.\left(\overline{b}\times\overline{c}\right)$ or $\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ is called scalar triple product of $\overline{a}, \overline{b}, \overline{c}$.

Clearly, it's a scalar value.

Scalar Triple Product

Scalar Triple Product (Box Product)

$\overline{a}.\left(\overline{b}\times\overline{c}\right)$ or $\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ is called scalar triple product of $\overline{a}, \overline{b}, \overline{c}$.

Clearly, it's a scalar value.

$$\overline{a}.\left(\overline{b}\times\overline{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Scalar Triple Product

Scalar Triple Product (Box Product)

$\overline{a}.\left(\overline{b}\times\overline{c}\right)$ or $\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ is called scalar triple product of $\overline{a}, \overline{b}, \overline{c}$.

Clearly, it's a scalar value.

$$\overline{a}.\left(\overline{b}\times\overline{c}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

In Cartesian form, $\overline{a}.\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}\times\overline{b}\right).\overline{c}$

Try to observe that

That is, we can interchange the . and the x operations, changing the brackets appropriately.

Scalar Triple Product

Properties of Box Product

(a) $[\, l\overline{a} \quad m\overline{b} \quad n\overline{c}\,] = lmn\,[\,\overline{a} \quad \overline{b} \quad \overline{c}\,]$

(b) $[\overline{a}+\overline{b} \quad \overline{c} \quad \overline{d}] = [\,\overline{a} \quad \overline{c} \quad \overline{d}\,] + [\,\overline{b} \quad \overline{c} \quad \overline{d}\,]$

$\left[\overline{a}+\overline{b} \quad \overline{c}+\overline{d} \quad \overline{e}\right]$ and $\left[\overline{a}+\overline{b} \quad \overline{c}+\overline{d} \quad \overline{e}+\overline{f}\right]$ will break into 4 and 8 scalar triple products respectively.

$$\{\bar a+\bar b \quad \bar c+\bar d \quad \bar e+\bar f\}$$

$$\{\bar a\ \bar c\ \bar e\}+\{\bar a\ \bar c\ \bar f\}+\{\bar a\ \bar d\ \bar e\}+\{\bar a\ \bar d\ \bar f\}$$

$$+\{\bar b\ \text{''}\}+\{\bar b\ \text{''}\}+\{\bar b\ \text{''}\}+\{\bar b\ \text{''}\}$$

Scalar Triple Product

Properties of Box Product

(a) $[\, l\overline{a} \quad m\overline{b} \quad n\overline{c} \,] = lmn\,[\overline{a} \quad \overline{b} \quad \overline{c}]$

(b) $\left[\overline{a}+\overline{b} \;\; \overline{c} \;\; \overline{d}\right] = [\overline{a} \quad \overline{c} \quad \overline{d}\,] + [\,\overline{b} \quad \overline{c} \quad \overline{d}\,]$

$\left[\overline{a}+\overline{b} \quad \overline{c}+\overline{d} \quad \overline{e}\right]$ and $\left[\overline{a}+\overline{b} \quad \overline{c}+\overline{d} \quad \overline{e}+\overline{f}\right]$ will break into 4 and 8 scalar triple products respectively.

(c) $\left[\overline{a} \quad \overline{b} \quad \overline{c}\right] = \left[\overline{b} \quad \overline{c} \quad \overline{a}\right] = \left[\overline{c} \quad \overline{a} \quad \overline{b}\right] = -\left[\overline{b} \quad \overline{a} \quad \overline{c}\right]$

Q If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors having magnitude 1, 2, 3 respectively, then $\left[\vec{a}+\vec{b}+\vec{c} \quad \vec{b}-\vec{a} \quad \vec{c}\right] =$

(A) 0

(B) 6

(C) 12

(D) 18

Sol.

M1

$[\bar{a}\ \bar{b}\ \bar{c}] + \cancel{[\bar{a}\ -\bar{a}\ \bar{c}]}^{0}$

$+ \cancel{[\bar{b}\ \bar{b}\ \bar{c}]}^{0} + [\bar{b}\ -\bar{a}\ \bar{c}]$

$+ \cancel{[\bar{c}\ \bar{b}\ \bar{c}]}^{0} + \cancel{[\bar{c}\ -\bar{a}\ \bar{c}]}^{0}$

$= [\bar{a}\ \bar{b}\ \bar{c}] - [\bar{b}\ \bar{a}\ \bar{c}]$

$= 2[\bar{a}\ \bar{b}\ \bar{c}] = 2 \times a \times b \times c$

M2

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times [\bar{a}\ \bar{b}\ \bar{c}]$$

$$= 2[\bar{a}\ \bar{b}\ \bar{c}]$$

NOTE

If $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors then $\left[\vec{p}\ \vec{q}\ \vec{r}\right] = |\vec{p}|.|\vec{q}|.|\vec{r}|$

Q

If $\vec{a},\ \vec{b},\ \vec{c}$ are mutually perpendicular vectors having magnitude 1, 2, 3 respectively, then $\left[\vec{a}+\vec{b}+\vec{c}\quad \vec{b}-\vec{a}\quad \vec{c}\right]=$

A 0

B 6

C 12

D 18

Solution:

Given, $\left|\vec{a}\right| = 1,\ \left|\vec{b}\right| = 2,\ \left|\vec{c}\right| = 3$ and

$\vec{a}\,.\,\vec{b} = 0 = \vec{b}\,.\,\vec{c} = \vec{c}\,.\vec{a}$

(as the three vectors are mutually perpendicular)

So,

$$\left[\left(\vec{a}+\vec{b}+\vec{c}\right) \times \left(\vec{b}-\vec{a}\right)\right] \cdot \vec{c}$$

$$= \left[\vec{a} \times \vec{b} - 0 + 0 - \vec{b} \times \vec{a} + \vec{c} \times \vec{b} - \vec{c} \times \vec{a}\right] \cdot \vec{c}$$

$$= \left[2\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} + \left(\vec{c} \times \vec{b}\right) \cdot \vec{c} - \left(\vec{c} \times \vec{a}\right) \cdot \vec{c}\right]$$

$$= 2\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} + 0 - 0 = 2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = 2 \cdot 1 \cdot 2 \cdot 3 = 12$$

Scalar Triple Product

Remark

Three vectors $\overline{a}, \overline{b}, \overline{c}$ are coplanar if $\begin{bmatrix}\overline{a} & \overline{b} & \overline{c}\end{bmatrix} = 0$

Scalar Triple Product

Remark

Three vectors $\overline{a}, \overline{b}, \overline{c}$ are coplanar if $\left[\overline{a} \quad \overline{b} \quad \overline{c}\right] = 0$

Proof

$\overline{a}.\left(\overline{b} \times \overline{c}\right) = 0$ implies that $\overline{a}$ is perpendicular to $\overline{b} \times \overline{c}$, which means that $\overline{a}$ lies in the plane of $\overline{b}$ and $\overline{c}$.

NOTE

If any two of $\overline{a}, \overline{b}, \overline{c}$ are collinear, then $\left[\overline{a} \quad \overline{b} \quad \overline{c}\right] = 0$

JEE Main 11th Apr, 20223

Q If the four distinct points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}]$ is equal to

(A) $[\vec{d}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{b}\ \vec{c}]$

(B) $[\vec{a}\ \vec{d}\ \vec{b}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{b}\ \vec{c}]$

(C) $[\vec{d}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{d}\ \vec{a}] + [\vec{c}\ \vec{d}\ \vec{b}]$

(D) $[\vec{b}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{a}\ \vec{c}] + [\vec{d}\ \vec{b}\ \vec{a}]$

$\overline{AB}, \overline{BC}, \overline{CD}$ are coplanar

$[\overline{AB}\ \ \overline{BC}\ \ \overline{CD}] = 0$

$\Rightarrow [\bar{b}-\bar{a}\ \ \bar{c}-\bar{b}\ \ \bar{d}-\bar{c}] = 0$

$\Rightarrow [\bar{b}\ \bar{c}\ \bar{d}] + 0 + 0 + 0$

$[-\bar{a}\ \bar{c}\ \bar{d}] + 0 + [-\bar{a}\ -\bar{b}\ \bar{d}] + [-\bar{a}\ -\bar{b}\ -\bar{c}] = 0$

$= [\bar{a}\ \bar{b}\ \bar{c}]$

JEE Main 11th Apr, 20223

Q

If the four distinct points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}]$ is equal to

A $[\vec{d}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{b}\ \vec{c}]$

B $[\vec{a}\ \vec{d}\ \vec{b}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{b}\ \vec{c}]$

C $[\vec{d}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{d}\ \vec{a}] + [\vec{c}\ \vec{d}\ \vec{b}]$

D $[\vec{b}\ \vec{c}\ \vec{d}] + [\vec{d}\ \vec{a}\ \vec{c}] + [\vec{d}\ \vec{b}\ \vec{a}]$

Solution:

$$[\vec{b}-\vec{a}\quad \vec{c}-\vec{a}\quad \vec{d}-\vec{a}]=0$$

$$(\vec{b}-\vec{a})\cdot[(\vec{c}-\vec{a})\times(\vec{d}-\vec{a})]=0$$

$$(\vec{b}-\vec{a})\cdot(\vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{a}\times\vec{d})=0$$

$$[\vec{b}\quad \vec{c}\quad \vec{d}]-[\vec{b}\quad \vec{c}\quad \vec{a}]-[\vec{b}\quad \vec{a}\quad \vec{d}]-[\vec{a}\quad \vec{c}\quad \vec{d}]=0$$

$$\therefore [\vec{a}\quad \vec{b}\quad \vec{c}]=[\vec{b}\quad \vec{c}\quad \vec{d}]+[\vec{a}\quad \vec{b}\quad \vec{d}]+[\vec{a}\quad \vec{d}\quad \vec{c}]$$

$$=[\vec{d}\quad \vec{c}\quad \vec{a}]+[\vec{b}\quad \vec{d}\quad \vec{a}]+[\vec{c}\quad \vec{d}\quad \vec{b}]$$

Q

If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\bar{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent and $|\bar{c}| = \sqrt{3}$, then find α, β.

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

$$\sqrt{1+\alpha^2+\beta^2} = \sqrt{3}$$

Solution:

It is given that $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$, $\bar{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent, which means that they are coplanar.

$$\therefore \begin{bmatrix} \bar{a} & \bar{b} & \bar{c} \end{bmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0 \qquad \dots (1)$$

$$\Rightarrow 3\beta - 4\alpha + 4 - 4\beta + 4\alpha - 3 = 0 \Rightarrow -\beta + 1 = 0 \Rightarrow \beta = 1$$

It is also given that $|\bar{c}| = \sqrt{3}$

$$\therefore |\bar{c}|^2 = 1^2 + \alpha^2 + \beta^2 = \left(\sqrt{3}\right)^2 \Rightarrow \alpha^2 + \beta^2 = 2 \qquad \dots (2)$$

Solving equations 1 and 2, we get $\alpha = \pm 1$, $\beta = 1$

linearly dep.

$\bar{a}, \bar{b}, \bar{c}$

$x\bar{a} + y\bar{b} + z\bar{c} = 0$

$\bar{a} = -\frac{y}{x}\bar{b} - \frac{z}{x}\bar{c}$

lin ind

$\bar{a}, \bar{b}, \bar{c}$

$x\bar{a} + y\bar{b} + z\bar{c} = 0$

$\Rightarrow x = y = z = 0$

JEE Main 28th July, 2022

Q

Let the vectors $\vec{a} = (1+t)\hat{i} + (1-t)\hat{j} + \hat{k}$, $\vec{b} = (1-t)\hat{i} + (1+t)\hat{j} + 2\hat{k}$ $\vec{c} = t\hat{i} - t\hat{j} + \hat{k}, t \in R$ be such that for $\alpha, \beta, \gamma \in R$, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ $\Rightarrow \alpha = \beta = \gamma = 0$. Then, the set of all values of t is:

$\bar{a}, \bar{b}, \bar{c}$ are lin ind

$$\Rightarrow \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ t & -t & 1 \end{vmatrix} \neq 0$$

(A) A non-empty finite set

(B) Equal to N

(C) Equal to R − {0}

(D) Equal to R

JEE Main 28th July, 2022

Q Let the vectors $\vec{a} = (1+t)\hat{i} + (1-t)\hat{j} + \hat{k}$, $\vec{b} = (1-t)\hat{i} + (1+t)\hat{j} + 2\hat{k}$ $\vec{c} = t\hat{i} - t\hat{j} + \hat{k}, t \in R$ be such that for $\alpha, \beta, \gamma \in R, \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ $\Rightarrow \alpha = \beta = \gamma = 0$. Then, the set of all values of t is:

(A) A non-empty finite set

(B) Equal to N

(C) Equal to R - {0}

(D) Equal to R

Solution:

By its given condition:

$\vec{a},\ \vec{b},\ \vec{c}$ are linearly independent vectors

$$\Rightarrow \left[\ \bar{a}\ \bar{b}\ \bar{c}\right] \neq 0 \quad \ldots(i)$$

Now,

$$\left[\bar{a}\ \bar{b}\ \bar{c}\ \right] = \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ t & -t & 1 \end{vmatrix} C_2 \to C_1 + C_2$$

$$\begin{vmatrix} 1+t & 2 & 1 \\ 1-t & 2 & 2 \\ t & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1+t & 1 & 1 \\ 1-t & 1 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2[(1+t) - (1-t) + t]$$

$$= 2[3t] = 6t$$

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] \neq 0 \Rightarrow t \neq 0$$

Scalar Triple Product

Result

1. $\left[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}\right]=2\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

2. $\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]\left(\bar{l}\times\bar{m}\right)=\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}.\bar{l} & \bar{b}.\bar{l} & \bar{c}.\bar{l} \\ \bar{a}.\bar{m} & \bar{b}.\bar{m} & \bar{c}.\bar{m} \end{vmatrix}$

3. $\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]^2=\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$

$$\underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}}_{\to 2} [\bar{a} \ \bar{b} \ \bar{c}]$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} i & j & k \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$$

If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar, then prove that $\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} = \bar{0}$

$\bar{0}$ 0

$[\bar{a}\ \bar{b}\ \bar{c}](\bar{a}\times\bar{b})$

$= 0$

Solution:

$$\begin{vmatrix} \overline{a} & \overline{b} & \overline{c} \\ \overline{a}.\overline{a} & \overline{a}.\overline{b} & \overline{a}.\overline{c} \\ \overline{b}.\overline{a} & \overline{b}.\overline{b} & \overline{b}.\overline{c} \end{vmatrix} = \begin{bmatrix} \overline{a} & \overline{b} & \overline{c} \end{bmatrix} \left(\overline{a} \times \overline{b}\right) = (0)\left(\overline{a} \times \overline{b}\right) = \overline{0}$$

Q

If $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ such that $|\vec{a}| = 2$, $|\vec{b}| = 3$, $|\vec{c}| = 4$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{2\pi}{3}$, then $\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|$ is equal to

(A) $4\sqrt{3}$

(B) $6\sqrt{3}$

(C) $12\sqrt{3}$

(D) $18\sqrt{3}$

Sol:

$$[\bar{a}\ \bar{b}\ \bar{c}]^2 = \begin{vmatrix} \bar{a}\,\bar{a} & \bar{b}\,\bar{a} & \bar{c}\,\bar{a} \\ \bar{a}\,\bar{b} & \bar{b}\,\bar{b} & \bar{c}\,\bar{b} \\ \bar{a}\cdot\bar{c} & \bar{b}\,\bar{c} & \bar{c}\,\bar{c} \end{vmatrix}$$

$$[\bar{a}\ \bar{b}\ \bar{c}]^2 = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 16 \end{vmatrix}$$

Q

If $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ such that $\left|\vec{a}\right| = 2$, $\left|\vec{b}\right| = 3$, $\left|\vec{c}\right| = 4$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{2\pi}{3}$, then $\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|$ is equal to

A $4\sqrt{3}$

B $6\sqrt{3}$

C $12\sqrt{3}$

D $18\sqrt{3}$

Solution:

$$\vec{a}\,.\,\vec{b}=\vec{a}\,.\,\vec{c}=0,\ \ \vec{b}\,.\,\vec{c}=\left|\vec{b}\right|\left|\vec{c}\right|\cos\tfrac{2\pi}{3}=3\times4\times\tfrac{-1}{2}=-6$$

$$\text{So,}\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2=\begin{vmatrix}\vec{a}\,.\,\vec{a} & \vec{a}\,.\,\vec{b} & \vec{a}\,.\,\vec{c}\\ \vec{b}\,.\,\vec{a} & \vec{b}\,.\,\vec{b} & \vec{b}\,.\,\vec{c}\\ \vec{c}\,.\,\vec{a} & \vec{c}\,.\,\vec{b} & \vec{c}\,.\,\vec{c}\end{vmatrix}=\begin{vmatrix}4 & 0 & 0\\ 0 & 9 & -6\\ 0 & -6 & 16\end{vmatrix}$$

$$=4\,(144-36)=4\times108=432$$

$$\Rightarrow\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|=2\times2\times3\sqrt{3}=12\sqrt{3}$$

Q

If $\left[\left(\vec{a}+2\vec{b}+3\vec{c}\right)\times\left(\vec{b}+2\vec{c}+3\vec{a}\right)\right]\cdot\left(\vec{c}+2\vec{a}+3\vec{b}\right)\Big] = 54$ where $\vec{a},\vec{b}$ and c are 3 non-coplanar vectors, then the value of $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$ is equal to

$\to [\bar{a}\ \bar{b}\ \bar{c}]^2$

(A) 9

(B) 3

(C) 6

(D) 12

Sol. Given. $\to 18$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} [\bar{a}\ \bar{b}\ \bar{c}] = 54$$

$$[\bar{a}\ \bar{b}\ \bar{c}] = 3$$

Q

If $\left[\left(\vec{a}+2\vec{b}+3\vec{c}\right)\times\left(\vec{b}+2\vec{c}+3\vec{a}\right)\right]\cdot\left(\vec{c}+2\vec{a}+3\vec{b}\right)\right]=54$ where $\vec{a},\vec{b}$ and c are 3 non-coplanar vectors, then the value of $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$ is equal to

A 9

B 3

C 6

D 12

Solution:

$$\left[\vec{a}+2\vec{b}+3\vec{c}\vec{b}+2\vec{c}+3\vec{a}\vec{c}+2\vec{a}+3\vec{b}\right]=54$$

$$\begin{vmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{vmatrix}\left[\vec{a}\,\vec{b}\,\vec{c}\right]=54$$

$$\left(1\left(-5\right)-2\left(-1\right)+3\left(7\right)\right)\left[\vec{a}\,\vec{b}\,\vec{c}\right]=54$$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=3$$

$$\text{Also}\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2=9$$

Scalar Triple Product

NOTE

Reciprocal System of Vectors

Let $\bar{a}, \bar{b}, \bar{c}$ be three non-coplanar vectors such that

$\left[\bar{a} \quad \bar{b} \quad \bar{c}\right] \neq 0$.

Then the vectors $\overline{a'} = \dfrac{\bar{b} \times \bar{c}}{\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]}, \overline{b'} = \dfrac{\bar{c} \times \bar{a}}{\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]}, \overline{c'} = \dfrac{\bar{a} \times \bar{b}}{\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]}$

are said to form a reciprocal system of vectors for the vectors $\bar{a}, \bar{b}, \bar{c}$

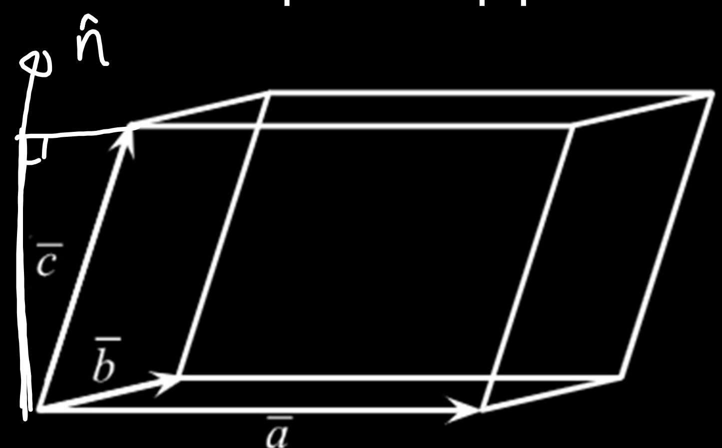
Scalar Triple Product

Geometrical Significance of the Box Product

The magnitude of $\begin{bmatrix}\overline{a} & \overline{b} & \overline{c}\end{bmatrix}$ gives the volume of the parallelepiped whose coinitial edges are $\overline{a},\ \overline{b}$ and $\overline{c}$.

$$\left|\bar{c}\cdot(\bar{a}\times\bar{b})\right|$$

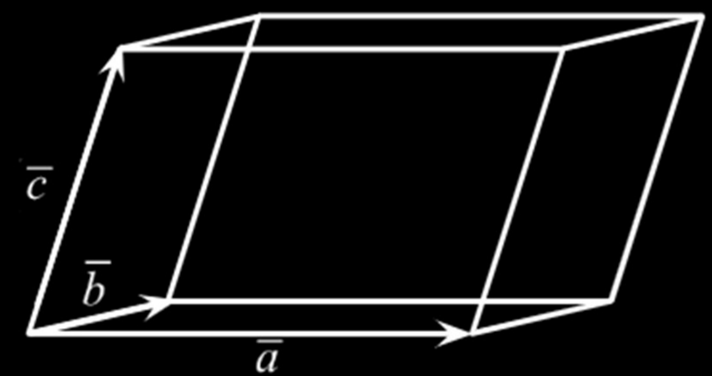
$$\bar{c}\cdot\hat{n}\,|\bar{a}\times\bar{b}|$$



Scalar Triple Product

Geometrical Significance of the Box Product

The magnitude of $\left[\overline{a} \quad \overline{b} \quad \overline{c}\right]$ gives the volume of the parallelepiped whose coinitial edges are $\overline{a},\ \overline{b}$ and $\overline{c}$.



Remark

The magnitude of $\dfrac{1}{6}\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ gives the volume of the tetrahedron whose coinitial edges are $\overline{a},\ \overline{b}$ and $\overline{c}$.

Vector Triple Product

Vector Triple Product

For three vectors $\overline{a},\ \overline{b}$ and $\overline{c}$: $\left(\overline{a}\times\overline{b}\right)\times\overline{c}$ and $\overline{a}\times\left(\overline{b}\times\overline{c}\right)$ are called vector triple products.

Vector Triple Product

For three vectors $\overline{a},\ \overline{b}$ and $\overline{c}$: $\left(\overline{a}\times\overline{b}\right)\times\overline{c}$ and $\overline{a}\times\left(\overline{b}\times\overline{c}\right)$ are called vector triple products.

Theorem

$$\overline{a}\times\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}.\overline{c}\right)\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{c}$$

NOTE

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$$

Recall

If $\bar{a}$ and $\bar{b}$ are non-collinear vectors such that $x_1\,\bar{a} + y_1\,\bar{b} = x_2\,\bar{a} + y_2\,\bar{b}$ then $x_1 = x_2$ and $y_1 = y_2$.

JEE Main 29th July, 2022

Q

Let $\vec{a} = 3\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. Let $\vec{c}$ be a vector satisfying $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b} + \lambda\vec{c}$. If $\vec{b}$ and $\vec{c}$ are non-parallel, then the value of λ is:

$$(\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c} = \bar{b} + \lambda\bar{c}$$

$$\Rightarrow \bar{a}\cdot\bar{c} = 1 \quad \& \quad -\bar{a}\cdot\bar{b} = \lambda$$

$$\therefore \lambda = -(3+2)$$

(A) -5

(B) 5

(C) 1

(D) -1

JEE Main 29th July, 2022

Q

Let $\vec{a} = 3\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. Let $\vec{c}$ be a vector satisfying $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b} + \lambda\vec{c}$. If $\vec{b}$ and $\vec{c}$ are non-parallel, then the value of λ is:

A -5

B 5

C 1

D -1

Solution:

$$\overrightarrow{a} = 3\hat{i} + \hat{j}, \quad \overrightarrow{b} = \hat{i} + 2\hat{j} + \widehat{k}$$

$$\text{As } \overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c}\right) = \overrightarrow{b} + \lambda\overrightarrow{c}$$

$$\Rightarrow \vec{a} \cdot \vec{c}\left(\vec{b}\right) - \left(\vec{a}.\vec{b}\right)\vec{c} = \vec{b} + \lambda\vec{c}$$

$$\Rightarrow \overrightarrow{a} \cdot \overrightarrow{c} = 1, \ \overrightarrow{a} \cdot \overrightarrow{b} = -\lambda$$

$$\Rightarrow \left(3\hat{i} + \hat{j}\right).\left(\hat{i} + 2\hat{j} + \widehat{k}\right) = -\lambda$$

$$\Rightarrow \lambda = -5$$

JEE Main 26th June, 2022

Q If $\vec{a}\cdot\vec{b}=1,\ \ \vec{b}\cdot\vec{c}=2$ and $\vec{c}\cdot\vec{a}=3,$ then the value of $\left[\vec{a}\times\left(\vec{b}\times\vec{c}\right)\ \vec{b}\times\left(\vec{c}\times\vec{a}\right)\ \vec{c}\times\left(\vec{b}\times\vec{a}\right)\right]$ is:

$$\left[3\bar{b}-\bar{c}\quad \bar{c}-2\bar{a}\quad 3\bar{b}-2\bar{a}\right]$$

$$=\begin{vmatrix} 0 & 3 & -1 \\ -2 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} \rightarrow 0 \ \left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$= $ Ans

(A) 0

(B) $-6\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

(C) $12\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$

(D) $-12\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$

JEE Main 26th June, 2022

Q

If $\vec{a} \cdot \vec{b} = 1,\ \ \vec{b} \cdot \vec{c} = 2$ and $\vec{c} \cdot \vec{a} = 3,$ then the value of $\left[\vec{a} \times \left(\vec{b} \times \vec{c}\right) \vec{b} \times \left(\vec{c} \times \vec{a}\right) \vec{c} \times \left(\vec{b} \times \vec{a}\right)\right]$ is:

A 0

B $-6\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$

C $12\vec{c} \cdot \left(\vec{a} \times \vec{b}\right)$

D $-12\vec{b} \cdot \left(\vec{c} \times \vec{a}\right)$

Solution:

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}.\vec{c})\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}=3\vec{b}-\vec{c}$$

$$\vec{b}\times(\vec{c}\times\vec{a})=\left(\vec{b}.\vec{a}\right)\vec{c}-\left(\vec{b}.\vec{c}\right)\vec{a}=\vec{c}-2\vec{a}$$

$$\vec{c}\times\left(\vec{b}\times\vec{a}\right)=(\vec{c}.\vec{a})\vec{b}-\left(\vec{c}.\vec{b}\right)\vec{a}=3\vec{b}-2\vec{a}$$

$$\left[3\vec{b}-\vec{c}\ \ \vec{c}-2\vec{a}\ \ 3\vec{b}-2\vec{a}\right]$$

$$\left(3\vec{b}-\vec{c}\right).\left[(\vec{c}-2\vec{a})\times\left(3\vec{b}-2\vec{a}\right)\right]$$

$$\left(3\vec{b}-\vec{c}\right).\left[3\left(\vec{c}\times\vec{b}\right)-2\left(\vec{c}\times\vec{a}\right)-6\left(\vec{a}\times\vec{b}\right)\right]$$

$$-6\left[\vec{b}\ \ \vec{c}\ \ \vec{a}\right]+6\left[\vec{c}\ \ \vec{a}\ \ \vec{b}\right]$$

Vector Triple Product

Observation

$$\overline{a} \times \left(\overline{a} \times \overline{b}\right) = \left(\overline{a} \,.\, \overline{b}\right)\overline{a} - \left(\overline{a} \,.\, \overline{a}\right)\overline{b} = \left(\overline{a} \,.\, \overline{b}\right)\overline{a} - \left|\overline{a}\right|^2 \overline{b}$$

Q If $\overline{a} = \hat{i} + \hat{j} + \widehat{k}$, $\overline{a}.\overline{b} = 1$ and $\overline{a} \times \overline{b} = \hat{j} - \widehat{k}$ then $\overline{b} =$____

(A) $\hat{i} - \hat{j} - \hat{k}$

(B) $2\hat{j} - \hat{k}$

(C) $\hat{i}$

(D) $2\hat{i}$

$$\bar{a} \times \bar{b} = j - k$$

$$\bar{a} \times (\bar{a} \times \bar{b}) = \bar{a} \times (j - k)$$

$$(\bar{a} \cdot \bar{b})\,\bar{a} - (\bar{a} \cdot \bar{a})\,\bar{b} = (i + j + k) \times (j - k)$$

1

$i + j + k$

3

Q If $\overline{a} = \hat{i} + \hat{j} + \widehat{k}$, $\overline{a}.\overline{b} = 1$ and $\overline{a} \times \overline{b} = \hat{j} - \widehat{k}$ then $\overline{b} =$____

A) $\hat{i} - \hat{j} - \hat{k}$

B) $2\hat{j} - \hat{k}$

C) $\hat{i}$

D) $2\hat{i}$

Solution:

It is given that $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{a}.\bar{b} = 1$ and $\bar{a} \times \bar{b} = \hat{j} - \hat{k}$

So, $\bar{a} \times \left(\bar{a} \times \bar{b}\right) = \bar{a} \times \left(\hat{j} - \hat{k}\right)$

$\Rightarrow \left(\bar{a}.\bar{b}\right)\bar{a} - |\bar{a}|^2 \bar{b} = \bar{a} \times \left(\hat{j} - \hat{k}\right)$

$\Rightarrow (1)\left(\hat{i} + \hat{j} + \hat{k}\right) - \left(\sqrt{3}\right)^2 \bar{b} = \left(\hat{i} + \hat{j} + \hat{k}\right) \times \left(\hat{j} - \hat{k}\right)$

$\Rightarrow 3\bar{b} = \hat{i} + \hat{j} + \hat{k} + \left(\hat{j} - \hat{k}\right) \times \left(\hat{i} + \hat{j} + \hat{k}\right)$

$\Rightarrow 3\bar{b} = \hat{i} + \hat{j} + \hat{k} - \hat{k} + \bar{0} + \hat{i} - \hat{j} + \hat{i} - \bar{0}$

$\Rightarrow 3\bar{b} = 3\hat{i}$

$\Rightarrow \bar{b} = \hat{i}$

Repeat

Q Let $\overline{a}, \overline{b}$ be non-collinear vectors of which $\overline{a}$ is a unit vector. Find the angles of a triangle, two of whose sides are represented by $\overline{b} - \left(\overline{a}.\overline{b}\right)\overline{a}$ and $\sqrt{3}\left(\overline{a} \times \overline{b}\right)$.

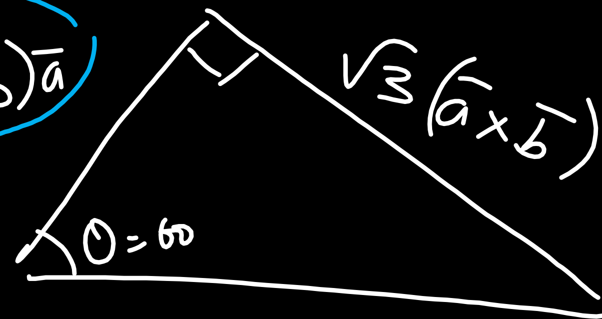
Sol.

$$\tan\theta = \frac{\left|\sqrt{3}\,\cancel{(\bar{a}\times\bar{b})}\right|}{\left|\bar{a}\times(\bar{b}\times\bar{a})\right|} = \sqrt{3}$$

$\cancel{1 \times |\bar{b}\times\bar{a}| \sin 90}$

$\bar{a}^2\bar{b} - (\bar{a}\cdot\bar{b})\bar{a}$

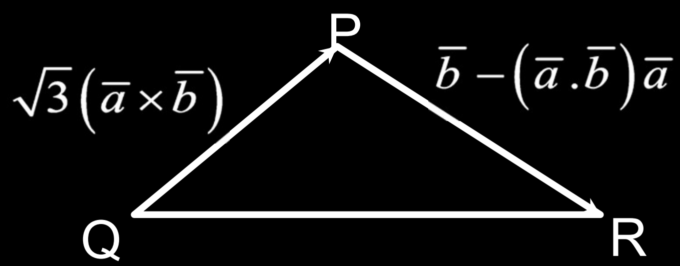
$\bar{a}\times(\bar{b}\times\bar{a})$



$\bar{a}\times(\bar{a}\times\bar{b})$

$= (\bar{a}\cdot\bar{b})\bar{a} - a^2\bar{b}$

Solution:



It is given that $\overline{a}$ and $\overline{b}$ are non-collinear vectors and $\overline{a}$ is a unit vector $\left(\text{that is, } |\overline{a}|=1\right)$.

Now, in triangle PQR,

$\overline{\mathrm{QP}}=\sqrt{3}\left(\overline{a}\times\overline{b}\right)$ and $\overline{\mathrm{PR}}=\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{a}$

Clearly $\angle \mathrm{P}=90^\circ$ $\quad \left(\because\ \overline{\mathrm{QP}}.\overline{\mathrm{PR}}=0\right)$

Solution:

Also, $\bar{b}-(\bar{a}.\bar{b})\bar{a}=|\bar{a}|^2\,\bar{b}-(\bar{a}.\bar{b})\bar{a}=\bar{a}\times(\bar{b}\times\bar{a})$

Now, $\dfrac{|\overline{\mathrm{PR}}|}{|\overline{\mathrm{QP}}|}=\tan\mathrm{Q} \Rightarrow \dfrac{|\bar{a}||\bar{b}\times\bar{a}|}{\sqrt{3}|\bar{a}\times\bar{b}|}=\tan\mathrm{Q}$

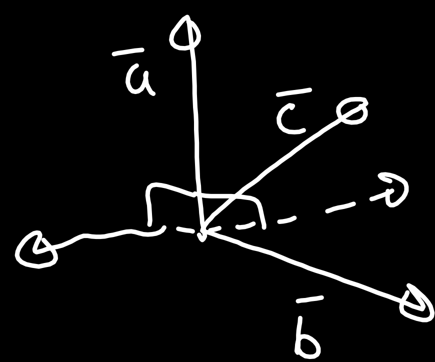
$(\because$ angle between $\bar{a}$ and $\bar{a}\times\bar{b}=90^\circ)$

$\therefore\ \tan\mathrm{Q}=\dfrac{1}{\sqrt{3}} \Rightarrow \angle\mathrm{B}=30^\circ \Rightarrow \angle\mathrm{C}=60^\circ$

Vector Triple Product

NOTE

Geometrically, $\overline{a}\times\left(\overline{b}\times\overline{c}\right)$ is a vector which is perpendicular to $\cancel{\overline{b}\text{ and }\overline{c}}$ $\overline{a}$ lies in the plane of $\cancel{\text{and }\overline{a}}$. $\overline{b}$ & $\overline{c}$



Find a vector of magnitude 5 units which is coplanar with $3\hat{i}-\hat{j}-\hat{k}$ and $\hat{i}+\hat{j}-2\hat{k}$, and is perpendicular to $2\hat{i}+2\hat{j}+\hat{k}$

$\bar{b}$ $\bar{c}$ $\bar{a}$

$(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$

$$\pm 5\left(\frac{\bar{a}\times(\bar{b}\times\bar{c})}{|\bar{a}\times(\bar{b}\times\bar{c})|}\right)$$

Solution:

Let $\overline{a}=2\hat{i}+2\hat{j}+\hat{k}$, $\overline{b}=3\hat{i}-\hat{j}-\hat{k}$ and $\overline{c}=\hat{i}+\hat{j}-2\hat{k}$

Let $\overline{p}=\overline{a}\times\left(\overline{b}\times\overline{c}\right)=(\overline{a}.\overline{c})\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{c}$

Clearly, $\overline{p}\perp\overline{a}$ and $\overline{p}$ is coplanar with $\overline{b}$ and $\overline{c}$.

$\therefore$ The required vector is $\pm 5\hat{p}$, that is $\pm\dfrac{5\overline{p}}{\left|\overline{p}\right|}$

Now, $\overline{p}=(\overline{a}.\overline{c})\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{c}$

$$=(2+2-2)\left(3\hat{i}-\hat{j}-\hat{k}\right)-(6-3-1)\left(\hat{i}+\hat{j}-2\hat{k}\right)$$

$$=4\hat{i}-4\hat{j}+2\hat{k}$$

And $\left|\overline{p}\right|=\sqrt{4^2+4^2+2^2}=\sqrt{16+16+4}=\sqrt{36}=6$

$$\therefore\ \pm 5\hat{p}=\pm\frac{5\left(4\hat{i}-4\hat{j}+2\hat{k}\right)}{6}=\pm\left(\frac{10}{3}\hat{i}-\frac{10}{3}\hat{j}+\frac{5}{3}\hat{k}\right)$$

Now, lets learn how do we tackle the following.

1. $\left(\overline{a} \times \overline{b}\right).\left(\overline{c} \times \overline{d}\right)$

2. $\left(\overline{a} \times \overline{b}\right) \times \left(\overline{c} \times \overline{d}\right)$

Now, lets learn how do we tackle the following.

Result

1. $\left(\bar{a}\times\bar{b}\right).\left(\bar{c}\times\bar{d}\right)=\begin{vmatrix}\bar{a}.\bar{c} & \bar{a}.\bar{d}\\ \bar{b}.\bar{c} & \bar{b}.\bar{d}\end{vmatrix}$

2. $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$

$$=\left[\bar{a}\quad\bar{b}\quad\bar{d}\right]\bar{c}-\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]\bar{d}$$

$$=\left[\bar{c}\quad\bar{d}\quad\bar{a}\right]\bar{b}-\left[\bar{c}\quad\bar{d}\quad\bar{b}\right]\bar{a}$$

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Q Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors such that $\boxed{\vec{b} \cdot \vec{c} = 0}$ and $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} - \vec{c}}{2}$. If $\vec{d}$ be a vector such that $\underbrace{\vec{b} \cdot \vec{d} = \vec{a} \cdot \vec{b}}$, then $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is equal to

$(\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{c}$

$\bar{a} \cdot \bar{c} = \frac{1}{2}$ & $\bar{a} \cdot \bar{b} = \frac{1}{2} = \bar{b} \cdot \bar{d}$

$\begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}$

$= \begin{vmatrix} 1/2 & \bar{a} \cdot \bar{d} \\ 0 & 1/2 \end{vmatrix}$

$= 1/4$

(A) $\frac{3}{4}$

(B) $\frac{1}{2}$

(C) $-\frac{1}{4}$

(D) $\frac{1}{4}$

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Q

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors such that $\vec{b} \cdot \vec{c} = 0$ and $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \dfrac{\vec{b} - \vec{c}}{2}$. If $\vec{d}$ be a vector such that $\vec{b} \cdot \vec{d} = \vec{a} \cdot \vec{b}$, then $\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right)$ is equal to

A $\dfrac{3}{4}$

B $\dfrac{1}{2}$

C $-\dfrac{1}{4}$

D $\dfrac{1}{4}$

Solution:

$$(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}=\frac{\vec{b}-\vec{c}}{2}$$

$$\vec{a}\cdot\vec{c}=\frac{1}{2},\ \vec{a}\cdot\vec{b}=\frac{1}{2}$$

$$\therefore \vec{b}\cdot\vec{d}=\frac{1}{2}$$

$$\begin{aligned}\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)&=\vec{a}\cdot\left(\vec{b}\times\left(\vec{c}\times\vec{d}\right)\right)\\&=\vec{a}\cdot\left(\left(\vec{b}\cdot\vec{d}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{d}\right)\\&=(\vec{a}\cdot\vec{c})\left(\vec{b}\cdot\vec{d}\right)=\frac{1}{4}\end{aligned}$$

Result

$[a \times b \;\; b \times c \;\; c \times a] = [a \;\; b \;\; c]^2$

$$[\bar{a}+\bar{b} \;\; \bar{b}+\bar{c} \;\; \bar{c}+\bar{a}] = 2[\bar{a} \;\; \bar{b} \;\; \bar{c}]$$

Q Find the value of [(2a × 3b) (3b × 4c) (4c × 2a)] if [a b c] = 2

$$\Rightarrow$$

$$[2\bar{a}\ \ 3\bar{b}\ \ 4\bar{c}]^2$$

$$= 24^2 \times [\bar{a}\ \ \bar{b}\ \ \bar{c}]^2$$

$$= 24^2 \times 2^2$$ Ans

Q Find the value of [(2a × 3b) (3b × 4c) (4c × 2a)] if [a b c] = 2

Solution:

We have,

$$\left[(2a \times 3b)\ \ (3b \times 4c)\ \ (4c \times 2a)\right]$$

$$= \left[x \times y \quad y \times z \quad z \times x\right]$$

$$\text{where } 2a = x\ 3b = y\ 4c = z$$

$$= \left[x\ y\ z\right]^2$$

$$= \left[2a\ 3b\ 4c\right]^2$$

$$= 4.9.16\left[a\ b\ c\right]^2$$

$$= 576 \times 4 = 2304$$

Vector Equations

Q Solve $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$ and $\bar{r} \cdot \bar{c} = 0$, where $\bar{c}$ is not perpendicular to $\bar{b}$.

for $\bar{r}$

$$\bar{r} \times \bar{b} = \bar{a} \times \bar{b} \longrightarrow \bar{r} \times \bar{b} - \bar{a} \times \bar{b} = 0$$

$$(\bar{r} - \bar{a}) \times \bar{b} = 0$$

$$\bar{r} - \bar{a} = \lambda \bar{b}$$

$$\bar{r} = \bar{a} + \lambda \bar{b}$$

$$\bar{r} \cdot \bar{c} = 0$$

Taking dot with $\bar{c}$

$$0 = \bar{a} \cdot \bar{c} + \lambda \bar{b} \cdot \bar{c}$$

$$\lambda = -\frac{\bar{a} \cdot \bar{c}}{\bar{b} \cdot \bar{c}}$$

$$\bar{r} = \bar{a} - \left(\frac{\bar{a} \cdot \bar{c}}{\bar{b} \cdot \bar{c}}\right) \bar{b} \quad \text{Ans}$$

Q Solve r × b = a × b and r . c = 0, where c is not perpendicular to b.

Solution:

Given,

$r \times b = a \times b$

$\Rightarrow (r - a) \times b = 0$

$\Rightarrow (r - a) = \lambda b, \lambda \in R$

$\Rightarrow r = a + \lambda b$

$\Rightarrow r.c = (a.c) + \lambda (b.c)$

$\Rightarrow \lambda = \frac{(a.c)}{(b.c)}$

Thus, $r = a - \left(\frac{(a.c)}{(b.c)}\right) b.$

Vector Equations

To solve vector equations, we generally express the unknown vector as a linear combination of three non-coplanar vectors and then find the coefficients.

Q If $\overline{r}$ satisfies $\overline{r} \times \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - \hat{k}$, then prove that $\overline{r} = \hat{j} + t\left(\hat{i} + 2\hat{j} + \hat{k}\right)$ for some scalar *t*.

Sol: Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

As given

$$(xi + yj + zk) \times (i + 2j + k) = i - k$$

Solution:

It is given that $\overline{r} \times \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - \hat{k}$ ...(1)

Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Substituting the value of $\overline{r}$ in equation 1, we get

$\therefore \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \times \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - \hat{k}$

$$\Rightarrow \hat{i} - \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 2 & 1 \end{vmatrix} \Rightarrow \hat{i} - \hat{k} = \hat{i}\left(y - 2z\right) - \hat{j}\left(x - z\right) + \hat{k}\left(2x - y\right)$$

Comparing the coefficients, we get $y - 2z = 1,\ x - z = 0,\ 2x - y = -1$

$x = z = \frac{1+y}{2} = t$

$x = t,\ y = 2t - 1,\ z = t$

$\overline{r} = ti + (2t-1)j + tK$

From which we get, $x = z = \dfrac{y-1}{2} = t$

$\therefore\ \overline{r} = t\hat{i} + \left(2t + 1\right)\hat{j} + t\hat{k} \Rightarrow \overline{r} = \hat{j} + t\left(\hat{i} + 2\hat{j} + \hat{k}\right)$

Q

If non-zero vectors $\overline{a}$ and $\overline{b}$ are perpendicular to each other and $\overline{r}$ satisfies $\overline{r} \times \overline{b} = \overline{a}$, then express $\overline{r}$ in terms of $\overline{a}$ and $\overline{b}$

Sol. $\bar{r} \times \bar{b} = \bar{a}$

$\bar{r} = x\bar{a} + y\bar{b} + z\,\bar{a}\times\bar{b}$

Std.

$$(x\bar{a} + y\bar{b} + z\,\bar{a}\times\bar{b}) \times \bar{b} = \bar{a}$$

$$x(\bar{a}\times\bar{b}) + y\times 0 + z\left((\bar{b}\cdot\bar{a})\bar{b} - b^2\bar{a}\right) = \bar{a} + 0\bar{b} + 0\cdot\bar{a}\times\bar{b}$$

$\bar{a} \to -b^2 z = 1$ $\quad$ $\bar{a}\times\bar{b} \to x = 0$ $\quad$ $\bar{b} \to z(\bar{b}\cdot\bar{a}) = 0$ also 0

$z = -\frac{1}{b^2}$ $\quad$ $x = 0$

Ans: $\bar{r} = y\bar{b} - \frac{1}{b^2}(\bar{a}\times\bar{b}),\ y \in R$

Solution:

It is given that $\overline{r}\times\overline{b}=\overline{a}$ ... (1)

Consider the vectors $\overline{a},\overline{b}$ and $\overline{a}\times\overline{b}$.

$\overline{a},\overline{b}$ should be non-collinear and $\overline{a}\times\overline{b}$ should be non-coplanar.

$\therefore\ \overline{r}=x\overline{a}+y\overline{b}+z\left(\overline{a}\times\overline{b}\right)$

Substituting the value of $\overline{r}$ in equation 1, we get

$\left\{x\overline{a}+y\overline{b}+z\left(\overline{a}\times\overline{b}\right)\right\}\times\overline{b}=\overline{a}$

$\Rightarrow x\left(\overline{a}\times\overline{b}\right)-z\overline{b}\times\left(\overline{a}\times\overline{b}\right)=\overline{a}$

$\Rightarrow x\left(\overline{a}\times\overline{b}\right)-z\left(\left|\overline{b}\right|^{2}\right)\overline{a}=\overline{a}$

$\Rightarrow x=0$ and $z=-\dfrac{1}{\left|\overline{b}\right|^{2}}$

Q **Express $\bar{a} \times \bar{b}$ in terms of the non-coplanar vectors $\bar{a}, \bar{b}, \bar{c}$**

Std.

Sol:
$$\bar{a} \times \bar{b} = x\bar{a} + y\bar{b} + z\bar{c}$$

① $(\bar{a} \times \bar{b})\,(\bar{b} \times \bar{c}) = x[\bar{a}\,\bar{b}\,\bar{c}] + 0 + 0$

$$x = \frac{(\bar{a} \times \bar{b})\,(\bar{b} \times \bar{c})}{[\bar{a}\,\bar{b}\,\bar{c}]}$$

② $(\bar{a} \times \bar{b})\,(\bar{c} \times \bar{a}) = 0 + y[\bar{b}\,\bar{c}\,\bar{a}] + 0$

$y = \checkmark$

lly $z = \checkmark$

Solution:

Let $\bar{a} \times \bar{b} = l\bar{a} + m\bar{b} + n\bar{c}$

Taking dot product on both sides with $\bar{b} \times \bar{c}$, we get

$$(\bar{a} \times \bar{b}).(\bar{b} \times \bar{c}) = l\bar{a}.(\bar{b} \times \bar{c}) + m\bar{b}.(\bar{b} \times \bar{c}) + n\bar{c}.(\bar{b} \times \bar{c})$$

$$\Rightarrow (\bar{a} \times \bar{b}).(\bar{b} \times \bar{c}) = l\bar{a}.(\bar{b} \times \bar{c}) \Rightarrow l = \frac{(\bar{a} \times \bar{b}).(\bar{b} \times \bar{c})}{\bar{a}.(\bar{b} \times \bar{c})}$$

Similarly, taking dot product on both sides with $\bar{a} \times \bar{b}$ and $\bar{c} \times \bar{a}$, we get

$$n = \frac{(\bar{a} \times \bar{b}).(\bar{a} \times \bar{b})}{\bar{c}.(\bar{a} \times \bar{b})} \text{ and } m = \frac{(\bar{a} \times \bar{b}).(\bar{c} \times \bar{a})}{\bar{b}.(\bar{c} \times \bar{a})} \text{ respectively.}$$

Solution:

$$\therefore\ \overline{a}\times\overline{b}=\frac{\left(\overline{a}\times\overline{b}\right).\left(\overline{b}\times\overline{c}\right)}{\overline{a}.\left(\overline{b}\times\overline{c}\right)}\overline{a}+\frac{\left(\overline{a}\times\overline{b}\right).\left(\overline{c}\times\overline{a}\right)}{\overline{b}.\left(\overline{c}\times\overline{a}\right)}\overline{b}+\frac{\left(\overline{a}\times\overline{b}\right).\left(\overline{a}\times\overline{b}\right)}{\overline{c}.\left(\overline{a}\times\overline{b}\right)}\overline{c}$$

Q

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors and $\vec{a} = \alpha\left(\vec{a} \times \vec{b}\right) + \beta\left(\vec{b} \times \vec{c}\right) + \gamma\left(\vec{c} \times \vec{a}\right)$ and $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 1$, then find the value of $\alpha + \beta + \gamma$.

(a) a^2 ✓

(b) $-a^2$

(c) $3a^2$

(d) $a+b+c$

Sol. (1) $\bar{a}\cdot\bar{c} = \alpha[\bar{a}\,\bar{b}\,\bar{c}] + 0 + 0$

$\alpha = \bar{a}\cdot\bar{c} = 0$

(2) $\bar{a}\cdot\bar{b} = \gamma[\bar{c}\,\bar{a}\,\bar{b}]$

$\gamma = \bar{a}\cdot\bar{b} = 0$

(3) $\bar{a}\cdot\bar{a} = \beta[\bar{a}\,\bar{b}\,\bar{c}]$

$\beta = a^2$

Solution:

Taking dot product with $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$, respectively, we get

$$\left|\overrightarrow{a}\right|^2 = \beta \cdot \left[\overrightarrow{a}\ \overrightarrow{b}\ \overrightarrow{c}\right] = \beta,$$

$$0 = \gamma \cdot \left[\overrightarrow{a}\ \ \overrightarrow{b}\ \ \overrightarrow{c}\right] = \gamma \text{ and}$$

$$0 = \alpha \cdot \left[\overrightarrow{a}\ \overrightarrow{b}\ \overrightarrow{c}\right] = \alpha$$

$$\therefore \alpha + \beta + \gamma = \left|\overrightarrow{a}\right|^2$$

Q If C is a given non-zero scalar and $\bar{A}$ and $\bar{B}$ be given non-zero vectors such that $\bar{A} \perp \bar{B}$, then find $\bar{X}$ which satisfies $\bar{A}.\bar{X} = C$ and $\bar{A} \times \bar{X} = \bar{B}$

$\bar{A} \cdot \bar{X} = C$ & $\bar{A} \times \bar{X} = \bar{B}$

$$\bar{A} \times (\bar{A} \times \bar{X}) = \bar{A} \times \bar{B}$$

$$C\bar{A} - A^2\bar{X} = \bar{A} \times \bar{B}$$

$$\bar{X} = \frac{C\bar{A} - \bar{A} \times \bar{B}}{A^2}$$ Ans

Solution:

It is given that $\bar{A}$ and $\bar{B}$ are non-zero perpendicular vectors.

And $\bar{X}$ is such that $\bar{A}.\bar{X} = C$ and $\bar{A} \times \bar{X} = \bar{B}$

Consider $\bar{A} \times \bar{X} = \bar{B}$

Taking cross product with $\bar{A}$ on both sides, we get

$$\bar{A} \times \left(\bar{A} \times \bar{X}\right) = \bar{A} \times \bar{B}$$

$$\Rightarrow \left(\bar{A}.\bar{X}\right)\bar{A} - \left(\bar{A}.\bar{A}\right)\bar{X} = \bar{A} \times \bar{B}$$

$$\Rightarrow C\bar{A} - \left|\bar{A}\right|^2 \bar{X} = \bar{A} \times \bar{B}$$

$$\therefore \ \bar{X} = \frac{C\bar{A} - \bar{A} \times \bar{B}}{\left|A\right|^2}$$