

Recursion

Write a function to print array elements recursively.

- Properties:
- ① Rec should have Termination Condition } Anchor step
 - ② Repeated step (Recurrence step) } Inductive step.

Tracing

Termination

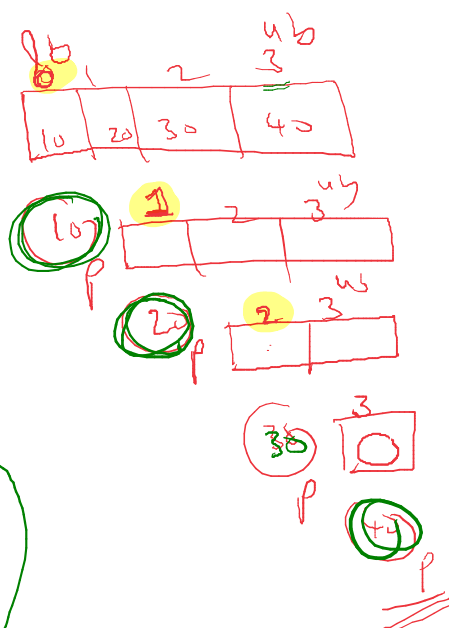
PA(a, lb, ub)

Seed call

$PA(a, 0, 3)$
 $PA(a, 1, 3)$
 $PA(a, 2, 3)$
 $PA(a, 3, 3)$

if $(lb = ub)$
 $\{$
 $\quad pf(a[lb])$
 $\}$
 else

$pf(a[lb])$
 $PA(a, lb+1, ub)$
 $\quad 0+1, 3$
 $\quad 1+1, 3$
 $\quad 2+1, 3$

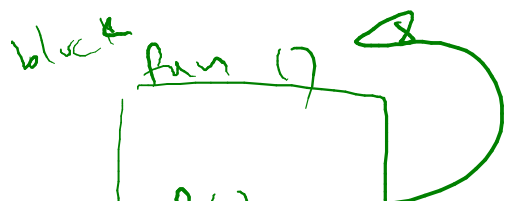


Algo

$PA(a, lb, ub)$

$\{$ $\quad pf(a[lb])$ $\}$	$(lb == ub)$ Anchor
$pf(a[lb])$ $PA(a, lb+1, ub)$	otherwise Inductive

C



tail R

void PA(int a[], int lb, int ub)

{

if (~~lb~~³ == ub)

{

~~printf("%d\n", a[³lb]);~~

}

else

{

① PA(a, lb+1, ub);

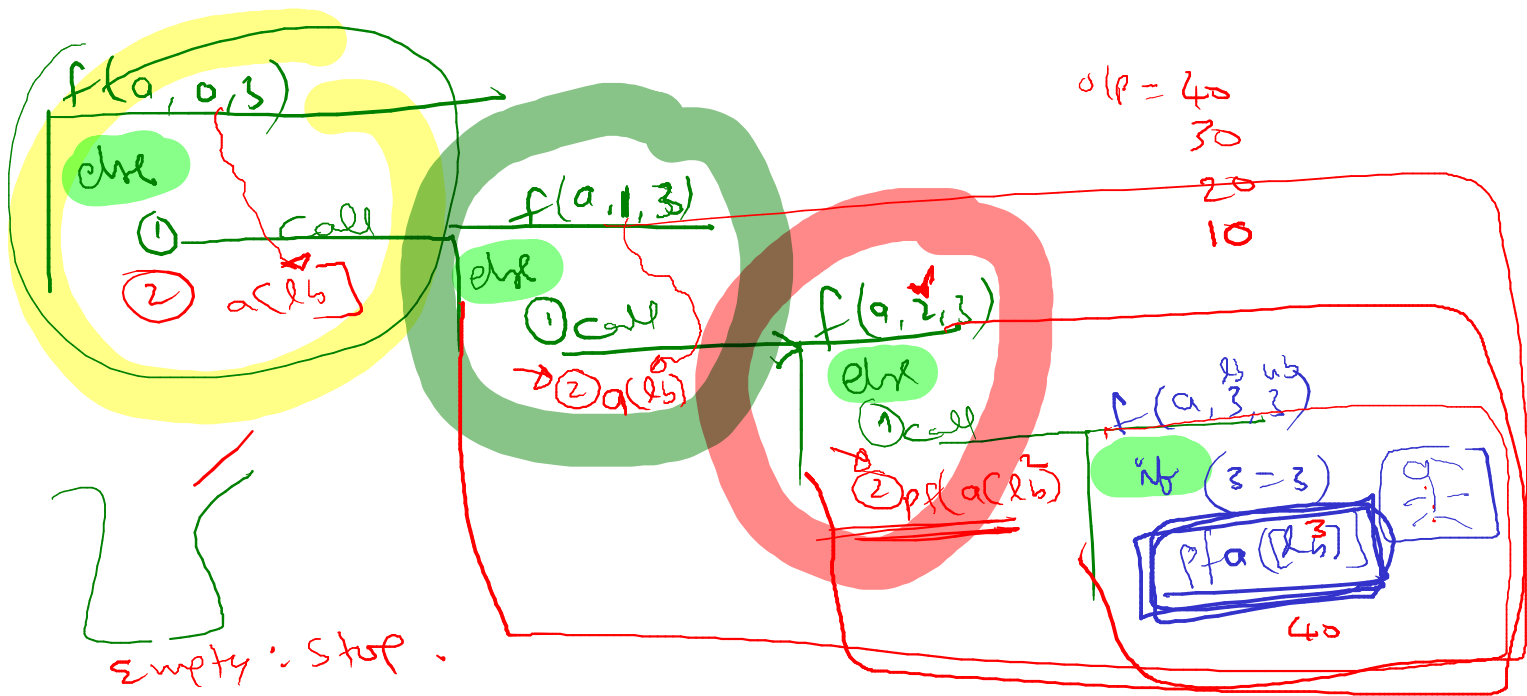
② ~~printf("%d\n", a[lb]);~~

}

}

f(a, 0, 3)
f(a, 1, 3)
f(a, 2, 3)
f(a, 3, 3) = 40

Tracing



Try: Print array elements in reverse using Tail Rec.

Tracing

0	1	2	3
10	20	30	40

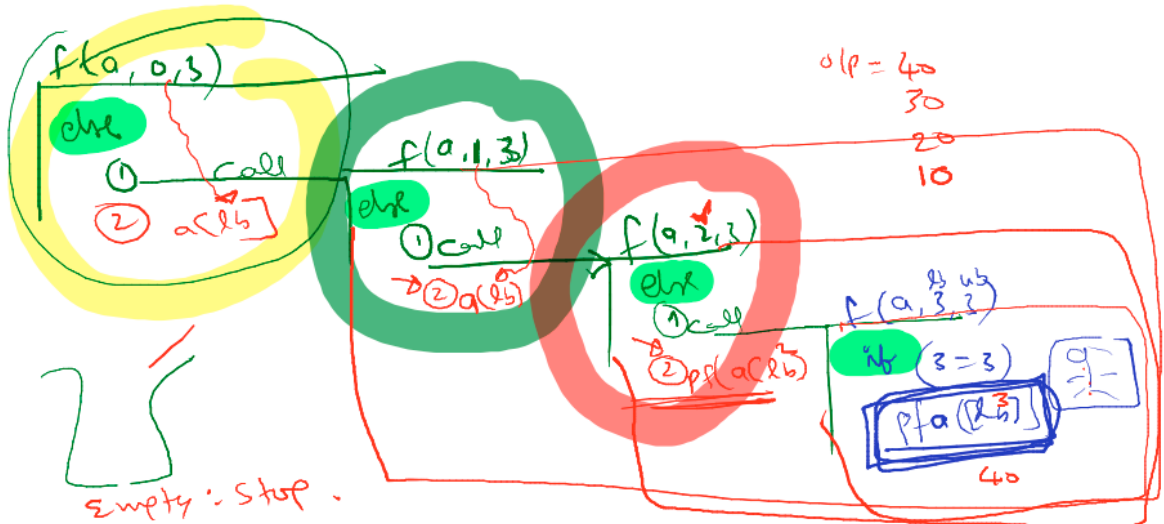
~~3~~ ub
 3
 2
 1



0 → pf: 87p - 20p

(10)p

Tracing



Try

Search an element in an array recursively

Q21 Return min element in array recursively

Exco

Multiplication

→ Repeated

$$\begin{array}{c} a \quad b \\ 2 \times 3 \end{array} = 6$$

addition

$$2 + 2 + 2 = 6$$

mul $\begin{smallmatrix} a & b \\ 2 & 3 \end{smallmatrix}$

$$\boxed{2 \times 3}$$

$$\begin{array}{c} a + \\ 2 + \end{array} \boxed{\begin{array}{c} a \quad b \\ 2 \times 2 \end{array}}$$

$$2^a + \boxed{\begin{array}{c} a \quad b \\ 2 \times 1 \end{array}}$$

Anchor

2

$pl(a)$

$(b=1)$

$$a + f(a, b-1)$$

mul → repeated +
Int \boxed{DIV} → repeated Subtraction

★ $7/2 = 3$

$$6/2 = 3$$

V. Trud

Nested Rec

Ackermann's function

$$\underline{A(x, y)} \begin{cases} A(\dots A(\underline{\underline{A}})) \end{cases}$$

Fibonacci series.

$$fib(n) \begin{cases} 0 & n=0 \\ 1 & n=1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

Q) How many calls for fib(8)
How many '+' for fib 7

n	0	1	2	3	4	5	6	7	8	9
f(n)	0	1	1	2	3	5	8	13	21	34

calls

+

Diagram showing recursive calls for fib(8):

- fib(8) calls fib(7) and fib(6)
- fib(7) calls fib(6) and fib(5)
- fib(6) calls fib(5) and fib(4)
- fib(5) calls fib(4) and fib(3)
- fib(4) calls fib(3) and fib(2)
- fib(3) calls fib(2) and fib(1)
- fib(2) calls fib(1) and fib(0)

Number of '+' operations for fib(7):

- fib(7) has 2 '+' operations (fib(6) + fib(5))
- fib(6) has 2 '+' operations (fib(5) + fib(4))
- fib(5) has 2 '+' operations (fib(4) + fib(3))
- fib(4) has 2 '+' operations (fib(3) + fib(2))
- fib(3) has 2 '+' operations (fib(2) + fib(1))
- fib(2) has 2 '+' operations (fib(1) + fib(0))

$$\begin{aligned} \text{calls} &\Rightarrow 2 \times f(n+1) - 1 \\ \text{calls}(fib(5)) &= 2 \times f(5+1) - 1 \\ &= 2 \times f(6) - 1 \\ &= 2(8) - 1 = 16 - 1 = 15 \end{aligned}$$

$$f(n) \\ \text{def} \\ f(n-1) + f(n-2)$$

$$n=0 \Rightarrow 0 \\ n=1 \Rightarrow 1$$

mit openforce

$$+ \text{ in } f(n): f(n+1) - 1$$

$$+ \text{ in } f(n) = f(n+1) - 1$$

$$= f(8) - 1$$

$$= 21 - 1 = 20$$

euclid

euclid (8, 21)

euclid (8, 5)

(5, 3)

(3, 2)

(2, 1)

(1, 0)

$$21 \overline{) 816}$$

$$\begin{array}{r} 38 \\ 21 \overline{) 816} \\ \underline{63} \\ 186 \\ \underline{147} \\ 390 \\ \underline{378} \\ 12 \end{array}$$

euclid(21, 7)

euclid(7, 0)

7

7

$$\begin{array}{r} 5 \\ 3 \overline{) 15} \\ \underline{15} \\ 0 \end{array}$$

$$\begin{array}{r} 3 \\ 2 \overline{) 6} \\ \underline{6} \\ 0 \end{array}$$

$$\begin{array}{r} 2 \\ 1 \overline{) 2} \\ \underline{2} \\ 0 \end{array}$$

$$\text{gcd}(m, n) = \begin{cases} m \\ \text{gcd}(n, m \% n) \end{cases}$$

Anchor
 $n = 0$

others

$$\text{gcd}(a, b) = \begin{cases} a \\ \text{gcd}(b, a \% b) \end{cases}$$

$b \neq 0$

GAME

Rule

only one disc

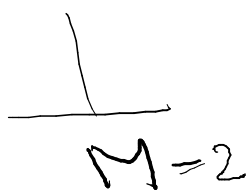


Recursively

Towers of Hanoi

200

Algo

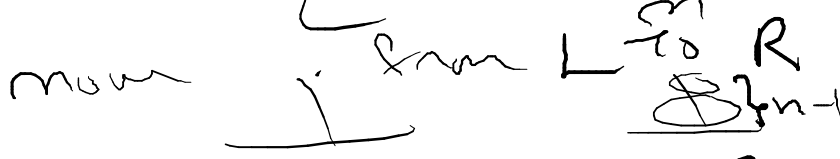


lift $(n-1)$ discs from L to M

(1)



(2)



(3)



$$\underline{fac(N)} = \begin{cases} 1 & N=0 \\ N \times \underline{fac(N-1)} & \text{otherwise} \end{cases}$$

$$f(n) = n \times f(n-1)$$

$$f(4) = 4 \times f(3)$$

$$3 \times f(2)$$

$$2 \times f(1)$$

$$1 \times f(0)$$

$$4 \times 3 \times 2 \times 1$$

Algo

$$TSH(\underline{n}, \underline{L}, \underline{R}, \underline{M})$$

$$if (n == 0)$$

$$\{$$

$$\text{① } TSH(\underline{n-1}, \underline{L}, \underline{R}, \underline{M})$$

$$\text{② move } (\underline{L}, \underline{R})$$

$$\text{③ } TSH(\underline{n-1}, \underline{M}, \underline{L}, \underline{R})$$

$$\}$$

Seed
 $T(3, 1, 2, 3)$

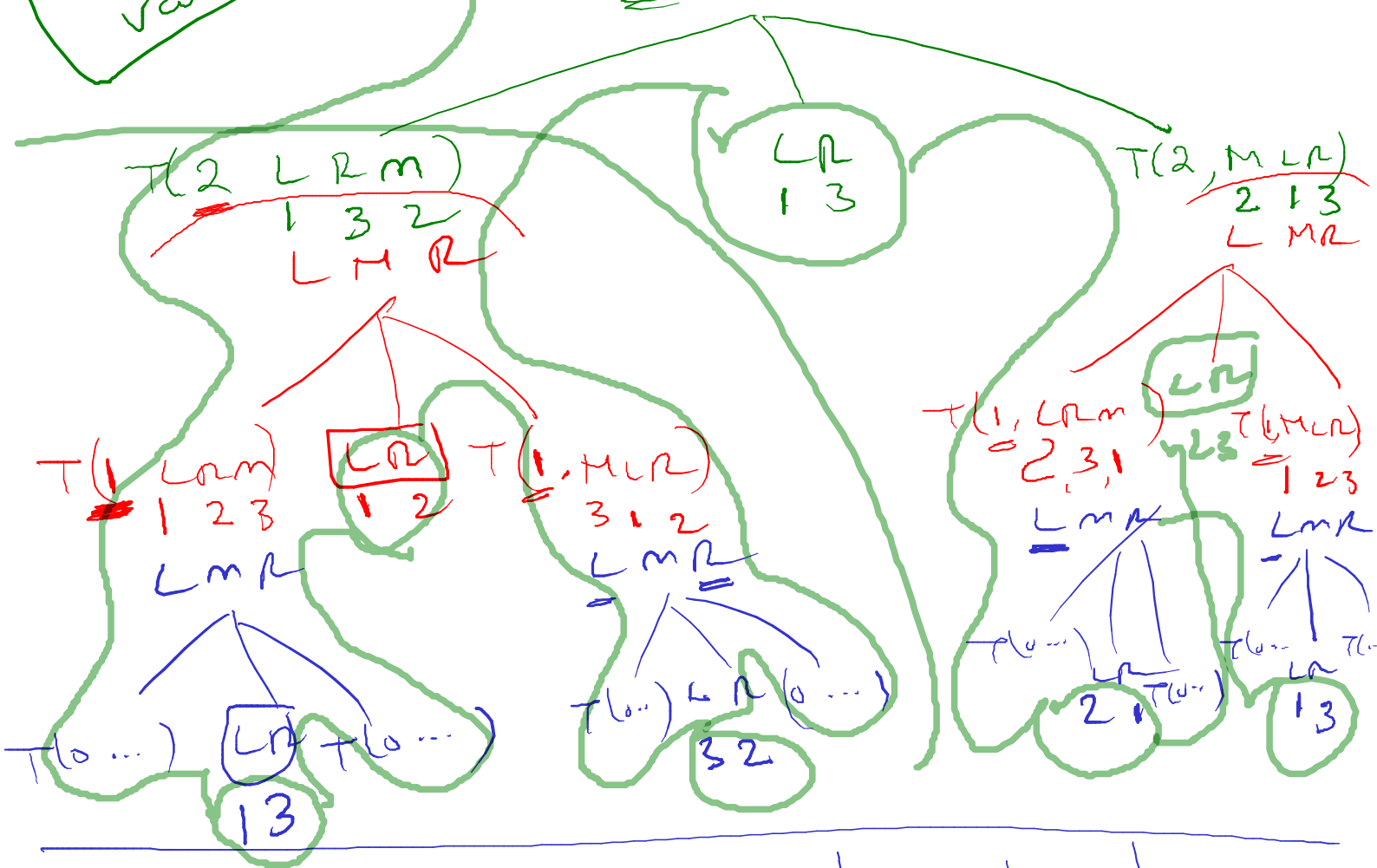
$T(n, \underline{\underline{\underline{L, M, R}}})$



Tracing

$T(\underline{3}, \underline{1}, \underline{2}, \underline{3})$

Call by value



Moves

$\frac{1}{L=1}$ $\frac{1}{M=2}$ $\frac{1}{R=3}$

- ①
- ②
- ③
- ④
- ⑤
- ⑥
- ⑦

Excessive Rec

$f(5)$

calls
 $f(5) = 15$

calls
 $f(4) = 9$

+ in $f(4) = 4$
+ in $f(5) = 7$

Initial

$L=1$

$m=2$

$R=3$

① 1,3

$L=1$ $m=2$ $R=3$

$L=1$ $m=2$ $R=3$

② 1,2

③ 3,2

$L=1$ $m=2$ $R=3$

④ 1,3

$L=1$ $m=2$ $R=3$

⑤ 2,1

$L=1$ $m=2$ $R=3$

⑥ 2,3

$L=1$ $m=2$ $R=3$

⑦ 1,2

$L=1$ $m=2$ $R=3$

$R=3$