Attitude Dynamics and Stability of Space Vehicles | AER 4063

Ahmed Salah Hammad Ahmed

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Contents

Abstract

Please note that this summary is totally made by students, so it is so criticizable.

Part I

Dynamics

Chapter 1

Linearized Modelling of Satellite Dynamics

1.1 Conservation of Angular Momentum

Let ω be the vector of angular velocity of the satellite about the three axes, and $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ be the angular velocities about the body axes.
Then the torque angular momentum general equation:

$$T = \dot{h}_I = \dot{h}_b + \omega \times h_b \tag{1.1}$$

Where I indicates inertial frame and b indicates body frame, now write equation in vector form:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \dot{h}_{b1} \\ \dot{h}_{b2} \\ \dot{h}_{b3} \end{bmatrix} + \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ h_{b1} & h_{b2} & h_{b3} \end{vmatrix} \tag{1.2}$$

Expand the cross product and let go of the b subscript:

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \dot{h}_x \\ \dot{h}_y \\ \dot{h}_z \end{bmatrix} + \begin{bmatrix} \omega_y h_z - \omega_z h_y \\ \omega_z h_x - \omega_x h_z \\ \omega_x h_y - \omega_y h_x \end{bmatrix} \tag{1.3}$$

We only need to find expressions for $\dot{h}_b$, ω, and h_b for each of the axes x, y and z.

1.2 Transformation of angular velocity components to body axes

We can use euler rotation matrices (rotation using euler angles) to deduce a formula for components of the vector $\vec{\omega}$, after neglecting nonlinear terms and linearizing for small angles:

$$\begin{aligned}
\omega_x &= \dot{\phi} - \omega_0 \psi \\
\omega_y &= \dot{\theta} - \omega_0 \\
\omega_z &= \dot{\psi} + \omega_0 \phi
\end{aligned} \tag{1.4}$$

1.3 Expressions for angular momentum and its rate

Before we shall continue to find an expression for the vector $\dot{h_b}$, assume (just for the sake of clarification) the existence of a momentum bias reaction wheel along the $\hat{J}$ axis, where h_{yw} is its angular momentum. Now, keep in mind that momentum about each of the axes is given by:

$$\begin{aligned}
h_x &= I_x \omega_x &&= I_x \dot{\phi} - I_x \omega_0 \psi \\
h_y &= I_y \omega_y + h_{yw} &&= I_y \dot{\theta} - I_y \omega_0 + h_{yw} \\
h_z &= I_z \omega_z &&= I_z \dot{\psi} + I_z \omega_0 \phi
\end{aligned} \tag{1.5}$$

Now, let's find $\dot{h_b}$:

$$\begin{bmatrix} \dot{h}_x \\ \dot{h}_y \\ \dot{h}_z \end{bmatrix} = \begin{bmatrix} I_x \dot{\omega}_x \\ I_y \dot{\omega}_y + \dot{h}_{yw} \\ I_z \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} I_x \frac{d}{dt}\left(\dot{\phi} - \omega_0 \psi\right) \\ I_y \frac{d}{dt}\left(\dot{\theta} - \omega_0\right) \\ I_z \frac{d}{dt}\left(\dot{\psi} + \omega_0 \phi\right) \end{bmatrix} \tag{1.6}$$

$$\begin{bmatrix} \dot{h}_x \\ \dot{h}_y \\ \dot{h}_z \end{bmatrix} = \begin{bmatrix} I_x \ddot{\phi} - I_x \omega_0 \dot{\psi} \\ I_y \ddot{\theta} \\ I_z \ddot{\psi} + I_z \omega_0 \dot{\phi} \end{bmatrix} \tag{1.7}$$

We are now ready to substitute equations 1.4, 1.5 and 1.7 into equation 1.3, and neglect all nonlinear terms:

$$\begin{aligned}
T_x &= I_x \ddot{\phi} - \left[\left(I_x + I_z - I_y\right)\omega_0 + h_{yw}\right]\dot{\psi} + \left[\left(I_y - I_z\right)\omega_0^2 - h_{yw}\omega_0\right]\phi \\
T_y &= I_y \ddot{\theta} \\
T_z &= I_z \ddot{\psi} + \left[\left(I_x + I_z - I_y\right)\omega_0 + h_{yw}\right]\dot{\phi} + \left[\left(I_y - I_x\right)\omega_0^2 - h_{yw}\omega_0\right]\psi
\end{aligned} \tag{1.8}$$

1.4 Introducing Gravity Gradient Moments as an external torque

$$\begin{aligned}
G_x &= \tfrac{3\mu}{2R_0^3}\left(I_z - I_y\right)\sin\left(2\phi\right)\cos^2\left(\nu\right) &= \tfrac{3\mu}{2R_0^3}\left(I_z - I_y\right)a_{23}a_{33} \\
G_y &= \tfrac{3\mu}{2R_0^3}\left(I_z - I_x\right)\sin\left(2\nu\right)\cos\left(\phi\right) &= \tfrac{3\mu}{2R_0^3}\left(I_z - I_x\right)a_{13}a_{33} \\
G_z &= \tfrac{3\mu}{2R_0^3}\left(I_x - I_y\right)\sin\left(2\nu\right)\sin\left(\phi\right) &= \tfrac{3\mu}{2R_0^3}\left(I_x - I_y\right)a_{13}a_{23}
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
G_x &= 3\omega_0^2\left(I_z - I_y\right)\phi \\
G_y &= 3\omega_0^2\left(I_z - I_x\right)\theta \\
G_z &= 0
\end{aligned} \tag{1.10}$$

Linearizing for small angles ϕ and ψ and using equation:

$$\omega_0 = \frac{V}{R_0} = \sqrt{\frac{\mu}{R_0^3}} \tag{1.11}$$

We can reach now to the governing equations for the dynamics of satellite written in the next section.

1.5 Governing Equations

$$\begin{aligned}
T_x &= I_x\ddot{\phi} - \left[\left(I_x + I_z - I_y\right)\omega_0 + h_{yw}\right]\dot{\psi} + \left[4\left(I_y - I_z\right)\omega_0^2 - h_{yw}\omega_0\right]\phi \\
T_y &= I_y\ddot{\theta} + 3\omega_0^2\left(I_x - I_z\right)\theta \\
T_z &= I_z\ddot{\psi} + \left[\left(I_x + I_z - I_y\right)\omega_0 + h_{yw}\right]\dot{\phi} + \left[\left(I_y - I_x\right)\omega_0^2 - h_{yw}\omega_0\right]\psi
\end{aligned} \tag{1.12}$$

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Part II

Stability

Chapter 2

Motion & Disturbances

2.1 Types of motion

1. Uncontrolled
   (a) Tumbling
   (b) Stand by
2. Controlled
   (a) Pointing Maneuver
   (b) Disturbance Rejection

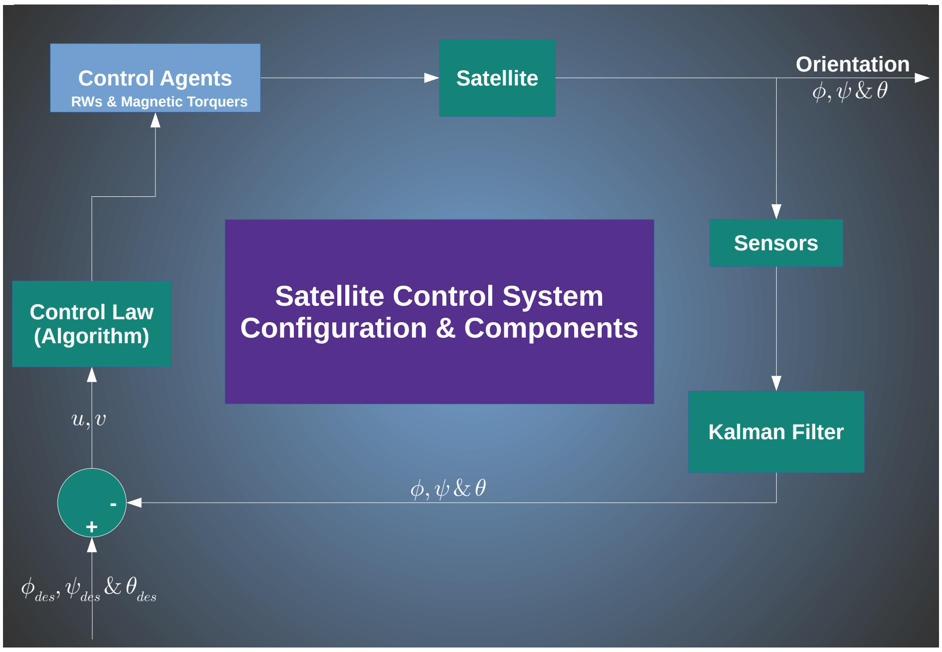

Figure 2.1: Satellite Control System Configuration & Components

2.2 Control System Functions

1. Disturbance Rejection
2. Pointing Maneuver
3. Detumbling
4. Stability during pointing

2.3 Components

2.3.1 Reaction Wheel

1. Slew Maneuver
2. Disturbance Rejection
3. Stabilization

2.3.2 Megnetic Torquer

1. Detumbling
2. Desaturation
3. Can assist Reaction Wheels in all their functions

2.4 Types of Disturbance

This section discusses external torque that might act on a satellite due to different disturbances, taking this satellite as an example:

Universat: $m = 60\,Kg$, $I_{xx} = 6.684942\,Kg.m^2$, $I_{yy} = 6.916894\,Kg.m^2$, $I_{zz} = 4.915737\,Kg.m^2$, $h = 619\,Km$, orbit period $T = 97$ minutes, $v = 7.5465\,Km/s$, dimensions: $0.55*0.55*0.85\ m^3$, $R_E = 6367\,Km$.
Full examples are in lecture notes.

2.4.1 Gravity Gradient Torque

$$T_g = \frac{3\mu}{2R^3} \times |I_z - I_y| \times \sin{(2\theta)} \tag{2.1}$$

Where $\mu \equiv$ Earth's Gravity Constant $= 3.986 \times 10^{14}\ (m^3/s^2)$, $R \equiv$ Orbit Radius $= R_E + h = 6986\,Km$, $\theta \equiv$ Maximum deviation of the $z - axis$ from local vertical.

Example: let $\theta = 30^{\circ} \Longrightarrow T_g = 3.0391 \times 10^{-6} N.m.$

2.4.2 Solar Radiation Torque

$$T_s = F\left(C_{ps} - C_g\right) \tag{2.2}$$

$$F = \left(F_s/C\right) \times A_s \left(1+q\right) \cos\left(i\right) \tag{2.3}$$

Where $F_s \equiv$ Solar Constant $=$ 1367 $\left(W/m^2\right)$, $C \equiv$ Speed of Light $=$ 3×10^8 $\left(m/s\right)$, $A_s \equiv$ Surface Area, $C_g \equiv$ Centre of Gravity, $C_{ps} \equiv$ Centre of Solar Pressure, $q \equiv$ Reflectance factor $= 1$ for blackbody $(q = 0.6$ average$)$, $i \equiv$ Incidence Angle ($i = 0$ worst case).
You can assume that: $C_{ps} - C_g = 0.3$ for roll/pitch, and $C_{ps} - C_g = 0.05$ for yaw.

Example to try:

Roll/Pitch:

$A_s = .55 * .85 = 0.4675\, m^2 \Longrightarrow F = \frac{F_s}{C} \times A_s \times (1+.6) \times \cos\left(0^{\circ}\right) = 3.4084 \times 10^{-6} \Longrightarrow T_s = 1.02252 \times 10^{-6} N.m$

Yaw:

$A_s = .55 * .55 = 0.3025\, m^2 \Longrightarrow F = \frac{F_s}{C} \times A_s \times (1+.6) \times \cos\left(0^{\circ}\right) = 2.20543 \times 10^{-6} \Longrightarrow T_s = 1.10271 \times 10^{-7} N.m$

2.4.3 Magnetic Torque

$$T_m = M \times B \tag{2.4}$$

Where $M \equiv$ Residual Magnetic Dipole Moment $\left(Am^2\right)$, $B \equiv$ Earth's Magnetic Field (T).

$$B = \frac{2D}{R^3} \tag{2.5}$$

Considering: $D \equiv$ Magnetic Moment of Earth $= 7.96 \times 10^{15}\, T/m^3$.

Example: $M = 1\ Am^2 \Longrightarrow T_m = 47 \times 10^{-6} N.m$

2.4.4 Aerodynamic Torque

$$T_a = F\left(C_{pa} - C_g\right) \tag{2.6}$$

$$F = \frac{1}{2}\rho V^2 A C_d \tag{2.7}$$

Where $C_d \equiv$ Drag Coefficient $C_d = 2 \sim 2.5$, $\rho \equiv$ Atmospheric Density, $A \equiv$ Surface Area normal to the direction of satellite motion, $V \equiv$ Satellite Velocity, $C_{pa} \equiv$ Centre of Aerodynamic Pressure, $C_g \equiv$ Centre of Gravity.
You can assume that: $C_{pa} - C_g = 0.3$ for roll/pitch, and $C_{pa} - C_g = 0.05$ for yaw.

Example: let $C_d = 2.5$ **and** $\rho_{atm} = 10^{-13}\, Kg.m^{-3}$

Roll/Pitch:

$A = 0.4675\, m^2 \Longrightarrow F = \frac{1}{2} \times 10^{-13} \times (7546.5)^2 \times A \times 2.5 = 3.328 \times 10^{-6} N \Longrightarrow T_a = F * 0.3 = 9.984 \times 10^{-7} N.m.$

Yaw:

$A = 0.4675\, m^2 \Longrightarrow F = \frac{1}{2} \times 10^{-13} \times (7546.5)^2 \times A \times 2.5 = 3.328 \times 10^{-6} N \Longrightarrow T_a = F * 0.3 = 9.984 \times 10^{-7} N.m.$
Worst Torque can be the summation of the 4 torques:

$$T_{max} = T_g + T_s + T_m + T_a \tag{2.8}$$

Chapter 3

Pitch Mode without Damping

Let's define:

$$\sigma_x = \frac{I_y - I_z}{I_x},\ \sigma_y = \frac{I_x - I_z}{I_y},\ \sigma_z = \frac{I_y - I_x}{I_z} \tag{3.1}$$

We have seen that the equation for the pitch mode without damping is given by:

$$T_y = I_y \ddot{\theta} + 3\omega_0^2 \left(I_x - I_z\right) \theta \tag{3.2}$$

3.1 Laplace Domain

$$T_y = \left[I_y s^2 + 3\omega_0^2 \left(I_x - I_z\right)\right] \theta \tag{3.3}$$

$$G(s) = \frac{\theta}{T_y} = \frac{1}{I_y s^2 + 3\omega_0^2 \left(I_x - I_z\right)} \tag{3.4}$$

Characteristic Equation:

$$I_y s^2 + +3\omega_0^2 \left(I_x - I_z\right) = 0 \tag{3.5}$$

$$s^2 + 3\omega_0^2 \sigma_y = 0 \tag{3.6}$$

You shoud now find the stability conditions:

$$I_x > I_z \tag{3.7}$$

$$\frac{I_y}{I_z} > \frac{3}{4} \tag{3.8}$$

$$\frac{I_x}{I_z} > \frac{3}{4} \tag{3.9}$$

3.2 State Space Approach

From equation 4.2:

$$\ddot{\theta} = -3\omega_0^2\sigma_y\theta + \frac{T_y}{I_y} \tag{3.10}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3\omega_0^2\sigma_y & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_y} \end{bmatrix} \left[T_y\right] \tag{3.11}$$

You can find the same characteristic equation as before by calculating $det\,(sI - A)$, and then you can check the stability as before as well.

Chapter 4

Pitch Mode with Damping

Define the damping equation:

$$I_w \dot{\omega}_w = D_y \left(\dot{\theta} - \omega_w \right) \tag{4.1}$$

Where D is the damping coefficient. Now introduce damping into the equation:

$$T_y = I_y \ddot{\theta} + I_w \dot{\omega}_w + 3\omega_0^2 \left(I_x - I_z \right) \theta \tag{4.2}$$

4.1 Laplace Domain

Take laplace for equation (4.1):

$$I_w s \omega_w = D s \theta - D \omega_w \tag{4.3}$$

Then:

$$\omega_{w,\theta} = \frac{Ds}{I_w s + D} \theta \tag{4.4}$$

Take laplace for equation 4.2:

$$T_y = \left[I_y s^2 + 3\omega_0^2 \left(I_x - I_z \right) \right] \theta + I_w s \omega_w \tag{4.5}$$

Substitute for ω_w:

$$T_y = \left[I_y s^2 + I_w \frac{Ds^2}{I_w s + D} + 3\omega_0^2 \left(I_x - I_z \right) \right] \theta \tag{4.6}$$

$$G(s) = \frac{\theta}{T_y} = \frac{1}{I_y s^2 + I_w \frac{Ds^2}{I_w s + D} + 3\omega_0^2 \left(I_x - I_z \right)} \tag{4.7}$$

$$G(s) = \frac{\theta}{T_y} = \frac{\frac{1}{I_y} \left(I_w s + D \right)}{s^2 \left(I_w s + D \right) + \frac{I_w}{I_y} D s^2 + 3\omega_0^2 \sigma_y \left(I_w s + D \right) = 0} \tag{4.8}$$

$$G(s) = \frac{\theta}{T_y} = \frac{\frac{1}{I_y} \left(I_w s + D \right)}{I_w s^3 + D \left(\frac{I_w}{I_y} + 1 \right) s^2 + 3\omega_0^2 \sigma_y I_w s + 3\omega_0^2 \sigma_y D = 0} \tag{4.9}$$

$$G(s) = \frac{\theta}{T_y} = \frac{\frac{1}{I_y}s + \frac{D}{I_w I_y}}{s^3 + D\left(\frac{1}{I_y} + \frac{1}{I_w}\right)s^2 + 3\omega_0^2\sigma_y s + 3\omega_0^2\sigma_y \frac{D}{I_w}} \tag{4.10}$$

Characteristic Equation:

$$s^3 + D\left(\frac{1}{I_y} + \frac{1}{I_w}\right)s^2 + 3\omega_0^2\sigma_y s + 3\omega_0^2\sigma_y \frac{D}{I_w} = 0 \tag{4.11}$$

$$s^3 + a_2 s^2 + a_1 s + a_0 = 0 \tag{4.12}$$

Now, let's check stability using Hermite-Beihler, substitute for $s = j\omega$:

$$-\omega^3 j - a_2\omega^2 + a_1\omega j + a_0 = 0 \tag{4.13}$$

Real Part $P(\omega)$:

$$a_0 - a_2\omega^2 = 0 \tag{4.14}$$

$$\omega_{p1}^2 = \frac{a_0}{a_2} \tag{4.15}$$

Imaginary Part $Q(\omega)$:

$$-\omega^3 + a_1\omega \tag{4.16}$$

$$\omega_{q1} = 0,\ \omega_{q2}^2 = a_1 \tag{4.17}$$

For the satellite to be stable, it must be that:

$$\omega_{q1}^2 < \omega_{p1}^2 < \omega_{q2}^2 \tag{4.18}$$

Then:

$$0 < \frac{a_0}{a_2} < a1 \tag{4.19}$$

4.2 State Space Approach

From equation 4.2:

$$\ddot{\theta} = -3\omega_0^2\sigma_y\theta - \frac{D}{I_y}\dot{\theta} + \frac{D}{I_y}\omega_w + \frac{T_y}{I_y} \tag{4.20}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\omega}_w \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3\omega_0^2\sigma_y & \frac{-D}{I_y} & \frac{D}{I_y} \\ 0 & \frac{D}{I_w} & -\frac{D}{I_w} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \omega_w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_y} \\ 0 \end{bmatrix} [T_y] \tag{4.21}$$

You can find the same characteristic equation as before by calculating $det\,(sI - A)$, and then you can check the stability as before as well.

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Chapter 5

Roll-Yaw Modes without Damping

The equations in Chapter 1 for the coupled modes have been introduced, but as said before, the existence of a momentum bias reaction wheel was just for illustration of how the equation would look like if you were to consider it, however, it's going to be neglected for the sake of simplicity in this chapter, remember the equations are:

$$\begin{aligned} T_x &= I_x\ddot{\phi} - \left(I_x + I_z - I_y\right)\omega_0\dot{\psi} + 4\left(I_y - I_z\right)\omega_0^2\phi \\ T_z &= I_z\ddot{\psi} + \left(I_z + I_x - I_y\right)\omega_0\dot{\phi} + \left(I_y - I_x\right)\omega_0^2\psi \end{aligned} \tag{5.1}$$

5.1 Laplace Domain

$$\begin{aligned} T_x &= \left[I_x s^2 + 4\left(I_y - I_z\right)\omega_0^2\right]\phi - \left(I_x + I_z - I_y\right)\omega_0 s\psi \\ T_z &= \left[I_z s^2 + \left(I_y - I_x\right)\omega_0^2\right]\psi + \left(I_x + I_z - I_y\right)\omega_0 s\phi \end{aligned} \tag{5.2}$$

Divide by I_x in first equation and I_z in second equation:

$$\begin{aligned} \tfrac{T_x}{I_x} &= \left[I_x s^2 + 4\omega_0^2\sigma_x\right]\phi - \omega_0\left(1 - \sigma_x\right)s\psi \\ \tfrac{T_z}{I_z} &= \left[s^2 + \omega_0^2\sigma_z\right]\psi + \omega_0\left(1 - \sigma_z\right)s\phi \end{aligned} \tag{5.3}$$

Write in matrix form:

$$\begin{bmatrix} s^2 + 4\omega_0^2\sigma_x & -\omega_0\left(1 - \sigma_x\right)s \\ \omega_0\left(1 - \sigma_z\right)s & s^2 + \omega_0^2\sigma_z \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \frac{T_x}{I_x} \\ \frac{T_z}{I_z} \end{bmatrix} \tag{5.4}$$

The characteristic equation can be found by taking the determinant of the matrix:

$$\left(s^2 + 4\omega_0^2\sigma_x\right)\left(s^2 + \omega_0^2\sigma_z\right) + \omega_0^2\left(1 - \sigma_z\right)\left(1 - \sigma_x\right)s^2 = 0 \tag{5.5}$$

$$s^4 + \omega_0^2 \left(\sigma_z + 4\sigma_x\right) s^2 + 4\omega_0^4 \sigma_x \sigma_z + \omega_0^2 \left(1 - \sigma_x - \sigma_z + \sigma_x \sigma_z\right) s^2 = 0 \tag{5.6}$$

$$s^4 + \omega_0^2 \left(1 + 3\sigma_x + \sigma_x \sigma_z\right) s^2 + 4\omega_0^4 \sigma_x \sigma_z = 0 \tag{5.7}$$

You should now achieve stability conditions.

5.2 State Space Approach

From equations 5.1:

$$\begin{aligned} \ddot{\phi} &= \omega_0 \left(1 - \sigma_x\right) \dot{\psi} - 4\omega_0^2 \sigma_x \phi + \tfrac{T_x}{I_x} \\ \ddot{\psi} &= -\omega_0 \left(1 - \sigma_z\right) \dot{\phi} - \omega_0^2 \sigma_z \psi + \tfrac{T_z}{I_z} \end{aligned} \tag{5.8}$$

State Space Equation:

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4\omega_0^2 \sigma_x & 0 & 0 & \omega_0 \left(1 - \sigma_x\right) \\ 0 & 0 & 0 & 1 \\ 0 & -\omega_0 \left(1 - \sigma_z\right) & -\omega_0^2 \sigma_z & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{I_x} & 0 \\ 0 & 0 \\ 0 & \frac{1}{I_z} \end{bmatrix} \begin{bmatrix} T_x \\ T_z \end{bmatrix} \tag{5.9}$$

Quick Reminder: Characteristic Equation $= det\left(sI - A\right)$

Chapter 6

Roll-Yaw Modes with Damping

6.1 State Space Approach

From equations 5.1:

$$\begin{aligned}
\ddot{\phi} &= \omega_0 \left(1-\sigma_x\right) \dot{\psi} + \tfrac{D}{I_x}\omega_{w,\phi} - \tfrac{D}{I_x}\dot{\phi} - 4\omega_0^2\sigma_x\phi + \tfrac{T_x}{I_x} \\
\ddot{\psi} &= -\omega_0 \left(1-\sigma_z\right) \dot{\phi} + \tfrac{D}{I_z}\omega_{w,\psi} - \tfrac{D}{I_z}\dot{\psi} - \omega_0^2\sigma_z\psi + \tfrac{T_z}{I_z}
\end{aligned} \tag{6.1}$$

Damping equations:

$$\begin{aligned}
I_{w,\phi}\dot{\omega}_{w,\phi} &= D_x\left(\dot{\phi} - \omega_{w,\phi}\right) \\
I_{w,\psi}\dot{\omega}_{w,\psi} &= D_z\left(\dot{\psi} - \omega_{w,\psi}\right)
\end{aligned} \tag{6.2}$$

State Space Equation:

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\omega}_{w,\phi} \\ \dot{\psi} \\ \ddot{\psi} \\ \dot{\omega}_{w,\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -4\omega_0^2\sigma_x & \frac{-D_x}{I_x} & \frac{D_x}{I_x} & 0 & \omega_0\left(1-\sigma_x\right) & 0 \\ 0 & \frac{D_x}{I_{w,\phi}} & \frac{-D_x}{I_{w,\phi}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\omega_0\left(1-\sigma_z\right) & 0 & -\omega_0^2\sigma_z & \frac{-D_z}{I_z} & \frac{D_z}{I_z} \\ 0 & 0 & 0 & 0 & \frac{D_z}{I_{w,\psi}} & \frac{-D_z}{I_{w,\psi}} \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \omega_{w,\phi} \\ \psi \\ \dot{\psi} \\ \omega_{w,\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{I_x} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{I_z} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_x \\ T_z \end{bmatrix} \tag{6.3}$$

Usually, we assume that $I_{w,\phi} = I_{w,\psi}$, and $D_x = D_z = D_y$.
You can find the characteristic equation of this 6×6 matrix $[sI - A]$ by using matlab, or by rearranging the rows of the matrix such that the first two rows are the ones with many 0 and only 1, then you'll have to find the determinant of a 4×4 matrix by yourself.

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Chapter 7

Total System (Roll-Pitch-Yaw)

$$
\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\omega}_{w,\phi} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{\omega}_{w,\theta} \\ \dot{\psi} \\ \ddot{\psi} \\ \dot{\omega}_{w,\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4\omega_0^2\sigma_x & \frac{-D_x}{I_x} & \frac{D_x}{I_x} & 0 & 0 & 0 & 0 & \omega_0\left(1-\sigma_x\right) & 0 \\ 0 & \frac{D_x}{I_{w,\phi}} & \frac{-D_x}{I_{w,\phi}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\omega_0^2\sigma_y & \frac{-D_y}{I_y} & \frac{D_y}{I_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{D_y}{I_{w,\theta}} & \frac{-D_y}{I_{w,\theta}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\omega_0\left(1-\sigma_z\right) & 0 & 0 & 0 & 0 & -\omega_0^2\sigma_z & \frac{-D_z}{I_z} & \frac{D_z}{I_z} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{D_z}{I_{w,\psi}} & \frac{-D_z}{I_{w,\psi}} \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \omega_{w,\phi} \\ \theta \\ \dot{\theta} \\ \omega_{w,\theta} \\ \psi \\ \dot{\psi} \\ \omega_{w,\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{I_x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_z} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \tag{7.1}
$$

You can find the characteristic equation of the total system by **simply multiplying the characteristic equation of the pitch with that of the coupled modes**.

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Part III

Control

Chapter 8

Decoupling Control using Diagonal Dominance

For coupled 2x2 systems it is difficult to design the system controller using the classical SISO methods. For the design of a controller for this system we need first to decouple the system using a decoupling controller and then design the SISO controller for each channel. To do this we use inverse design and the Diagonal Dominance concept:
Let the open loop transfer function of the total coupled system be:

$$T(s) = \begin{bmatrix} t_{11}(s) & t_{12}(s) \\ t_{21}(s) & t_{22}(s) \end{bmatrix} \tag{8.1}$$

8.1 Theorem 1

A system given by the previous transfer function is said to be diagonally dominant if

$$||t_{ii}(j\omega)|| \geq ||t_{ij}(j\omega)|| \ \forall \, \omega \tag{8.2}$$

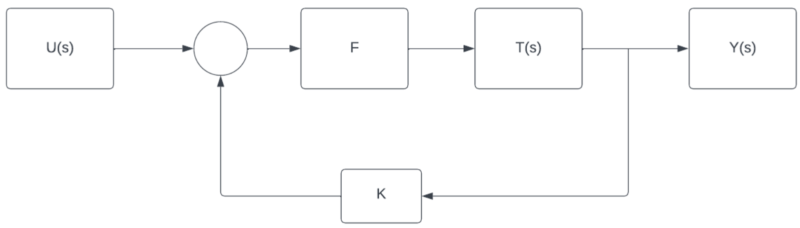

Figure 8.1: Control Loop

8.2 Theorem 2

The 2×2 system $T(s)$ is stable if the inverse of the system $T^{-1}(s)$ is diagonally dominant and the diagonal elements are stable polynomials.

$$Y(s) = T(s)\,F\,[U(s) - KY(s)] \tag{8.3}$$

$$[I + T(s)FK]\,Y(s) = T(s)\,F\,U(s) \tag{8.4}$$

$$Y(s) = T_c(s)\,U(s) = [1 + T(s)FK]^{-1}\,T(s)\,F\,U(s) \tag{8.5}$$

$$T_c(s) = [1 + T(s)FK]^{-1}\,T(s)\,F \tag{8.6}$$

$$T_c^{-1}(s) = F^{-1}T^{-1}(s)\,[1 + T(s)FK] \tag{8.7}$$

$$T_c^{-1}(s) = F^{-1}T^{-1}(s) + K \tag{8.8}$$

F & K are two constant compensators which are to be designed. F is called the decoupling compensator while K is the feedback compensator.

8.3 How do we measure diagonal dominance?

Diagonal dominance is measured by plotting the two sides of the inequality (2). That is easily done by plotting the magnitude of the polynomial in a manner similar to plotting the Magnitude Part of Bode diagram. We don't need the phase part here. To plot the magnitude of the polynomial with elements of F in it we need to operate as if we are designing using the root locus method.

8.4 Design of F and K

Usually F will achieve diagonal dominance for the high frequencies in the Bode diagram. To achieve dominance in the low frequencies we will use K. The off diagonal elements of K will be used to lower the off diagonal elements, while the diagonal elements of K will either be zeros or are used to achieve some feature of the control system (Stability or Steady State).

Bibliography

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