Physics

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Part I.

Classical Mechanics

1. General concepts

1.1. Constraints

A **constraint** is a relation between coordinates and velocities, reducing the number of degrees of freedom of a system. Using a set $\mathbf{q}$ of generalised coordinates, it can be represented as an equality or inequality involving a **constraint function** $f(\mathbf{q}, \dot{\mathbf{q}}, t)$.

There is several ways to classify constraints.

- Directionality:
  - **Unilateral:** the constraint may prevent displacements in one direction, but not necessarily in the opposite direction. These constraints generally represent bodies that cannot be penetrated. Mathematically, they are represented by inequalities: $f(\mathbf{q}, \dot{\mathbf{q}}, t) \geq 0$
  - **Bilateral:** if the constraint prevents displacements in one direction, it must prevent an equivalent displacement in the opposite direction, as well. These constraints generally represent bodies moving along a rail or a similar object. Mathematically, they are represented by equalities: $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$
- Time-dependence:
  - **Scleronomic:** the constraint does not depend explicity on time and can therefore be written as $f(\mathbf{q}, \dot{\mathbf{q}})$.
  - **Rheonomic:** the constraint depends explicitly on time.
- Velocity-dependence:
  - **Holonomic:** the constraint does not depend explicity on velocity, or depends on velocity in a way that can be integrated and reduced to a dependence on time and coordinates. This means there exists a constraint function

$g(\mathbf{q}, t)$ that is equivalent to the original, that is it allows or forbids exactly the same configurations as the original constraint.

- **Anholonomic (nonholonomic):** the constraint depends on velocities in a manner that cannot be integrated and reduced to a dependence on time and coordinates.

1.2. Udwadia-Kalaba equation (to be moved)

Where $\mathbf{Q}$ is the generalised force and $\mathcal{Q}$ is the generalised constraint force. The matrix M is symmetric and positive semi-definite.

$$M(\mathbf{q}, t)\, \ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathcal{Q}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

2. Newtonian Mechanics

2.1. Constraints

In Newtonian Mechanics, constraints manifest in terms of forces that prevent the system from violating the constraint. These **constraint forces** are usually denoted with $\mathbf{\Phi}$. The value of such forces depends not only upon the constraint function, but also on other forces present in the system. Consider, as an example, a book lying on a table: the amount of force required to prevent the book from falling through the table depends not only on the shape of the table, but also on the weight of the book and other forces that might be pushing the book towards the table.

This gives us a new way to categorise constraints, based on their reaction forces. Here we assume to have a system composed by several objects, and we denote with $\mathbf{\Phi}_s$ the force acting on the s-th object, the coordinates of which are given by $\mathbf{r}_s$.

- **Smooth:** there are no components of the reaction force along the unconstrained directions, such forces are therefore unable to perform any work, being perpendicular to all allowed displacements. Mathematically, this means that the constraint force is in the same direction as the gradient of the constraint function:

$$\mathbf{\Phi}_s \times \frac{\partial f}{\partial \mathbf{r}_s} = 0$$

- **Rough:** the reaction force may have components along the unconstrained directions. This is usually the case in systems where dry friction is present.

$$\mathbf{\Phi}_s \times \frac{\partial f}{\partial \mathbf{r}_s} \neq 0$$