Physics

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Part I.

Classical Mechanics

1. General concepts

1.1. Generalised coordinates

A set $\{q_i\}$ of **generalised coordinates** is a set of values that identify the state of a physical system. It can be useful to treat the set of generalised coordinates as a vector $\mathbf{q}$. It should be kept in mind that this is not a vector in the physical sense, so it does not necessarily behave as expected under spatial transformations, it is merely represented as such for the sake of mathematical convenience.

The least number of generalised coordinates that are necessary to uniquely identify the state of a system is the number of **degrees of freedom** of the system. A physical system composed of N particles in 3 dimensions has, in the most general case, $3N$ degrees of freedom, as it is necessary to give the position $\mathbf{r}_s$ of each particle as a set of cartesian coordinates. However, it is often the case that particles are not independent of each other; consider the following examples:

- The state of a rigid body can be described by simply giving the position of one of its points (3 vector components), and its orientation in space (3 angles), resulting in a total of only 6 degrees of freedom.

- The state of a thermodynamical system at equilibrium is described by its pressure and temperature, resulting in a total of only 2 degrees of freedom.

In a majority of cases, for each particle there exists an invertible map $\mathbf{r}_s = \mathbf{r}_s(\mathbf{q}, t)$, that allows us to find the position of each constituent of the system once we know the generalised coordinates. This allows us to express **kinematic displacements** as

$$\mathrm{d}\mathbf{r}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial \mathbf{r}_s}{\partial t}\,\mathrm{d}t \tag{1.1}$$

these are to be contrasted with **virtual displacements**

$$\delta\mathbf{r}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} \tag{1.2}$$

The term *virtual* comes from the fact that they happen at a fixed time (they are obtained from real displacements by setting $\mathrm{d}t = 0$), but there can obviously be no motion without passage of time, so the displacement is simply a mathematical construct.

The time derivatives of generalised coordinates are called **generalised velocities** ($\dot{\mathbf{q}}$), and **generalised accelerations** ($\ddot{\mathbf{q}}$). As with generalised coordinates, it is possible to find the positions and accelerations of each particle. This is complicated by the fact that, while the cartesian coordinates only depend on the generalised coordinates, it is not generally true that the cartesian velocities and acceleration depend solely on their generalised equivalent, but rather $\dot{\mathbf{r}}_s = \dot{\mathbf{r}}_s(\dot{\mathbf{q}}, \mathbf{q}, t)$ and $\ddot{\mathbf{r}}_s = \ddot{\mathbf{r}}_s(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t)$.

We can compute the cartesian velocities $\dot{\mathbf{r}}_s$ by taking Eq. 1.1.

$$\dot{\mathbf{r}}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial \mathbf{r}_s}{\partial t} \tag{1.3}$$

From this expression we observe that the cartesian velocities are linear in the generalised velocities (while this is not generally true for the coordinates).

As with displacement, we can define a **kinematic differential of velocity**

$$\mathrm{d}\dot{\mathbf{r}}_s = \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{\mathbf{q}}} \cdot \mathrm{d}\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{r}}_s}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} + \frac{\partial \dot{\mathbf{r}}_s}{\partial t}\, \mathrm{d}t \tag{1.4}$$

By observing Eq. 1.3 we can compute the various terms in $\mathrm{d}\dot{\mathbf{r}}_s$, that will be useful in computing the equation for the accelerations.

$$\frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \qquad \frac{\partial \dot{\mathbf{r}}_s}{\partial \mathbf{q}} = \frac{\partial^2 \mathbf{r}}{\partial \mathbf{q}^2} \cdot \dot{\mathbf{q}} + \frac{\partial^2 \mathbf{r}}{\partial \mathbf{q}\, \partial t} \qquad \frac{\partial \dot{\mathbf{r}}_s}{\partial t} = \frac{\partial^2 \mathbf{r}}{\partial \mathbf{q}\, \partial t} \cdot \dot{\mathbf{q}} + \frac{\partial^2 \mathbf{r}}{\partial t^2}$$

We also define a **virtual differential of velocity**

$$\delta \dot{\mathbf{r}}_s = \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{\mathbf{q}}} \cdot \mathrm{d}\dot{\mathbf{q}} = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \mathrm{d}\dot{\mathbf{q}} \tag{1.5}$$

that is obtained from the former by setting $\mathrm{d}\mathbf{q} = 0$ and $\mathrm{d}t = 0$.

We now take Eq. 1.4 and use it to compute the cartesian accelerations $\ddot{\mathbf{r}}_s$.

$$\ddot{\mathbf{r}}_s = \frac{\partial \dot{\mathbf{r}}_s}{\partial \dot{\mathbf{q}}} \cdot \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{r}}_s}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{r}}_s}{\partial t}$$
$$\ddot{\mathbf{r}}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \ddot{\mathbf{q}} + \frac{\partial^2 \mathbf{r}_s}{\partial \mathbf{q}^2} \cdot \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + 2\frac{\partial^2 \mathbf{r}_s}{\partial \mathbf{q}\, \partial t} \cdot \dot{\mathbf{q}} + \frac{\partial^2 \mathbf{r}_s}{\partial t^2} \tag{1.6}$$

Here we see that the cartesian accelerations depend linearly on generalised accelerations and may have both a quadratic and a linear term in the generalised velocities.

In the most general case:

$$\mathrm{d}\mathbf{r}^{(n)} = \sum_{k=0}^{n} \frac{\partial \mathbf{r}^{(n)}}{\partial \mathbf{q}^{(k)}} \mathrm{d}\mathbf{q}^{(k)} + \frac{\partial \mathbf{r}^{(n)}}{\partial t} \mathrm{d}t$$

$$\mathbf{r}^{(n)} = \frac{\mathrm{d}\mathbf{r}^{(n-1)}}{\mathrm{d}t} = \sum_{k=0}^{n-1} \frac{\partial \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(k)}} \mathbf{q}^{(k+1)} + \frac{\partial \mathbf{r}^{(n-1)}}{\partial t}$$

$$\frac{\partial \mathbf{r}^{(n)}}{\partial \mathbf{q}^{(m)}} = \sum_{k=0}^{n-1} \left(\frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial \mathbf{q}^{(k)}} \mathbf{q}^{(k+1)} + \frac{\partial \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(k)}} \frac{\partial \mathbf{q}^{(k+1)}}{\partial \mathbf{q}^{(m)}} \right) + \frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial t}$$

$$\frac{\partial \mathbf{r}^{(n)}}{\partial \mathbf{q}^{(m)}} = \sum_{k=0}^{n-1} \left(\frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial \mathbf{q}^{(k)}} \mathbf{q}^{(k+1)} + \frac{\partial \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(k)}} \delta_{m,k+1} \right) + \frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial t}$$

$$\frac{\partial \mathbf{r}^{(n)}}{\partial \mathbf{q}^{(m)}} = \frac{\partial \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m-1)}} + \sum_{k=0}^{n-1} \frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial \mathbf{q}^{(k)}} \mathbf{q}^{(k+1)} + \frac{\partial^2 \mathbf{r}^{(n-1)}}{\partial \mathbf{q}^{(m)} \partial t}$$

I would like to find an expression of this in terms solely of partial derivatives by q and t. Moreover, it should be possible to prove that

$$\frac{\partial r^{(n)}}{\partial q^{(m)}} = \begin{cases} 0 & \text{if } m > n \\ \frac{\partial r}{\partial q} & \text{if } m = n \end{cases}$$

$$\frac{\partial r^{(n)}}{\partial q^{(n)}} = \sum_{k=0}^{n-1} \left(\frac{\partial^2 r^{(n-1)}}{\partial q^{(n)} \partial q^{(k)}} q^{(k+1)} + \frac{\partial r^{(n-1)}}{\partial q^{(k)}} \delta_{n,k+1} \right) + \frac{\partial^2 r^{(n-1)}}{\partial q^{(n)} \partial t}$$

$$\frac{\partial r^{(n)}}{\partial q^{(n)}} = \sum_{k=0}^{n-1} \frac{\partial r^{(n-1)}}{\partial q^{(k)}} \delta_{n,k+1} = \frac{\partial r^{(n-1)}}{\partial q^{(n-1)}}$$

Applying this recursively we get what we want. For proof that the higher-order derivatives are zero, use induction by assuming it work for $n-1$. You have to change the first formula to account for higher orders in principle, though.

1.2. Constraints

A **constraint** is a relation between coordinates and velocities. Using a set $\mathbf{q}$ of generalised coordinates, a constrant can be represented as an equality or inequality involving a **constraint function** $f(\mathbf{q}, \dot{\mathbf{q}}, t)$.

There is several ways to classify constraints.

- Directionality:
  - **Unilateral:** the constraint may prevent displacements in one direction, but not necessarily in the opposite direction. These constraints generally represent bodies that cannot be penetrated. Mathematically, they are represented by inequalities: $f(\mathbf{q}, \dot{\mathbf{q}}, t) \geq 0$
  - **Bilateral:** if the constraint prevents displacements in one direction, it must prevent an equivalent displacement in the opposite direction, as well. These constraints generally represent bodies moving along a rail or a similar object. Mathematically, they are represented by equalities: $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$
- Time-dependence:
  - **Scleronomic:** the constraint does not depend explicity on time and can therefore be written as $f(\mathbf{q}, \dot{\mathbf{q}})$.
  - **Rheonomic:** the constraint depends explicitly on time.
- Velocity-dependence:
  - **Geometric:** the constraint does not depend explicity on velocity and can therefore be written as $f(\mathbf{q}, t)$.
  - **Kinetic:** the constraint depends explicitly on velocity.
  - **Linear:** the constraint is kinetic and depends linearly on velocity.
  - **Integrable:** the constraint is kinetic, but it is *equivalent* to a geometric constraint. That is to say, there exists a geometric constraint that identifies the same subspace of allowed configurations.
  - **Holonomic:** the constraint is geometric or integrable.

- **Anholonomic (nonholonomic):** the constraint is neither geometric, nor integrable. Unilateral constraints are generally considered anholonomic.

A single bilateral holonomic constraint reduces the number of degrees of freedom of a physical system by one. Bilateral holonomic constraints can always be expressed as $f(\mathbf{q}, t) = 0$, we now pick a specific q_k and denote as $\boldsymbol{\varrho}_k$ the set of generalised coordinates *except* q_k, so that we can write the constraint as $f(q_k, \boldsymbol{\varrho}_k, t) = 0$. Invoking Dini's implicit function theorem, there exists a function $g(\boldsymbol{\varrho}_k, t)$ such that $f(g(\boldsymbol{\varrho}_k, t), \boldsymbol{\varrho}_k, t) = 0$. Since the constraint must be satisfied, this forces us to choose $q_k = g(\boldsymbol{\varrho}_k, t)$. The $\boldsymbol{\varrho}_k$ are therefore the generalised coordinates for a new, unconstrained system, with one fewer degree of freedom than the original.

2. Alternative formulations (to be moved)

2.1. Udwadia-Kalaba equation

Where $\mathbf{Q}$ is the generalised force and $\mathcal{Q}$ is the generalised constraint force. The matrix M is symmetric and positive semi-definite.

$$M(\mathbf{q}, t)\,\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathcal{Q}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

2.2. Gauss's principle of least constraint

The **Gauss's principle of least constraint** states that the function C defined as

$$C = \frac{1}{2} \sum_{s=1}^{N} m_s \left\| \ddot{\mathbf{r}}_s - \frac{\mathbf{F}_s}{m_s} \right\|^2 \tag{2.1}$$

must be minimised by the equations of motion. Alternative ways to write C include

$$C = \frac{1}{2} \sum_{s=1}^{N} m_s \left(\ddot{\mathbf{r}}_s - \frac{\mathbf{F}_s}{m_s} \right) \cdot \left(\ddot{\mathbf{r}}_s - \frac{\mathbf{F}_s}{m_s} \right) \tag{2.2a}$$

$$C = \frac{1}{2} \sum_{s=1}^{N} \left(m_s \ddot{\mathbf{r}}_s \cdot \ddot{\mathbf{r}}_s - 2\mathbf{F}_s \cdot \ddot{\mathbf{r}}_s + \frac{\mathbf{F}_s \cdot \mathbf{F}_s}{m_s} \right) \tag{2.2b}$$

$$C = \frac{1}{2} \sum_{s=1}^{N} \left(m_s \left\| \ddot{\mathbf{r}}_s \right\|^2 - 2\mathbf{F}_s \cdot \ddot{\mathbf{r}}_s + \frac{\left\| \mathbf{F}_s \right\|^2}{m_s} \right) \tag{2.2c}$$

2.3. Jourdain's principle

D'Alembert principle only applies to holonomic systems, a more general principle that can also be applied to anholonomic system can be obtained by working with the virtual velocity differentials from 1.5. As it was the case with d'Alembert's principle, if the boundaries are smooth, then the constraint forces must be normal to the constraint surface. Much like the virtual displacements, the virtual velocity differential must also lie on the constraint surface, so that $\mathbf{\Phi}_s \cdot \delta\dot{\mathbf{r}}_s = 0$. By writing $\mathbf{\Phi}_s$ out as $m_s\ddot{\mathbf{r}}_s - \mathbf{F}_s$ we obtain the statement of **Jourdain's principle**.

$$\sum_{s=1}^{N} \left(m_s\ddot{\mathbf{r}}_s - \mathbf{F}_s\right) \cdot \delta\dot{\mathbf{r}}_s = 0$$

2.4. Gauss-Gibbs principle

Another principle can be obtained by working with virtual changes in acceleration, this is known as the **Gauss-Gibbs principle**.

$$\sum_{s=1}^{N} \left(m_s\ddot{\mathbf{r}}_s - \mathbf{F}_s\right) \cdot \delta\ddot{\mathbf{r}}_s = 0$$

2.5. Mangeron-Deleanu principle

The most general case is known as **Mangeron-Deleanu principle** and is stated in terms of a virtual variation of the *n*-th time derivative of the cartesian positions.

$$\sum_{s=1}^{N} \left(m_s\ddot{\mathbf{r}}_s - \mathbf{F}_s\right) \cdot \delta\mathbf{r}_s^{(n)} = 0$$

2.6. Appell's equation

We take the principle of least constraint (Eq. 2.2c) and we define the **Appellian** $\mathcal{A}$ as

$$\mathcal{A} = \frac{1}{2}\sum_{s=1}^{N} m_s \left\|\ddot{\mathbf{r}}_s\right\|^2$$

which we recognise as the first term in $\mathcal{C}$.

$$\mathcal{C} = \mathcal{A} + \frac{1}{2}\sum_{s=1}^{N}\left(\frac{\left\|\mathbf{F}_s\right\|^2}{m_s} - 2\mathbf{F}_s \cdot \ddot{\mathbf{r}}_s\right)$$

Since $\mathcal{C}$ must be minimal, its derivative with regards to the $\ddot{\mathbf{q}}$ must be zero. Assuming the forces are adequately well-behaved (it is sufficient that they do not depend on the accelerations), their derivative with regards to the $\ddot{\mathbf{q}}$ will also be zero. We also recall Eq. 1.6 to find the derivative of the cartesian accelerations.

$$0 = \frac{\partial \mathcal{C}}{\partial \ddot{\mathbf{q}}} = \frac{\partial \mathcal{A}}{\partial \ddot{\mathbf{q}}} - \sum_{s=1}^{N}\frac{\partial \ddot{\mathbf{r}}_s}{\partial \ddot{\mathbf{q}}} \cdot \mathbf{F}_s = \frac{\partial \mathcal{A}}{\partial \ddot{\mathbf{q}}} - \sum_{s=1}^{N}\frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \cdot \mathbf{F}_s$$

The second term gives the generalised forces $\mathbf{Q}$, we now have **Appell's equation**.

$$\mathbf{Q} = \frac{\partial \mathcal{A}}{\partial \ddot{\mathbf{q}}}$$

3. Newtonian Mechanics

3.1. Constraints

In Newtonian Mechanics, constraints manifest in terms of forces that prevent the system from violating the constraint. These **constraint forces** are usually denoted with $\mathbf{\Phi}$. The value of such forces depends not only upon the constraint function, but also on other forces present in the system. Consider, as an example, a book lying on a table: the amount of force required to prevent the book from falling through the table depends not only on the shape of the table, but also on the weight of the book and other forces that might be pushing the book towards the table.

This gives us a new way to categorise constraints, based on their reaction forces. Here we assume to have a system composed by several objects, and we denote with $\mathbf{\Phi}_s$ the force acting on the s-th object, the coordinates of which are given by $\mathbf{r}_s$.

- **Smooth:** there are no components of the reaction force along the unconstrained directions, such forces are therefore unable to perform any work, being perpendicular to all allowed displacements. Mathematically, this means that the constraint force is in the same direction as the gradient of the constraint function:

$$\mathbf{\Phi}_s \times \frac{\partial f}{\partial \mathbf{r}_s} = 0$$

- **Rough:** the reaction force may have components along the unconstrained directions. This is usually the case in systems where dry friction is present.

$$\mathbf{\Phi}_s \times \frac{\partial f}{\partial \mathbf{r}_s} \neq 0$$