I.9 Partial Product 2

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Programming for Problem Solving

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The divergence and convergence of:

$$\prod_{n=1}^{\infty}\left(1+\frac{f(n)}{(g(n))}\right)$$

Where $f(n)$ and $g(n)$ are polynomials (i.e. $f(n) = n^3 + n^2 + n + 1$)

For the above product, we can determine whether it will converge, or diverge based solely upon the value of the largest exponent found in the functions $f(n)$ and $g(n)$.

If the largest exponent for each equation is the same, then the product will diverge exponentially as the previous value will always be multiplied by 2. We can predict what the final value of the product will be as 2^n for this case.

If the numerators largest exponent is greater than the denominators largest exponent by 2 or more, then the product will always diverge exponentially without regard to any follow on values. As an example:

$$\prod_{n=1}^{\infty}\left(1+\frac{n^7+n^3+n+17}{(n^5+n^3+n^2+n+200)}\right)$$

While the denominator has more terms and initially is larger than the numerator, it quickly begins to be surpassed by the value that the first exponent in $f(n)$ and diverges.

Interestingly when the largest exponent of the denominator is only 1 greater than the numerator, the product still diverges but it does so at a linear rate. as shown by a graph of $\prod_{n=1}^{\infty}\left(1+\frac{n^5+20}{(n^4+n^3+2)}\right)$:

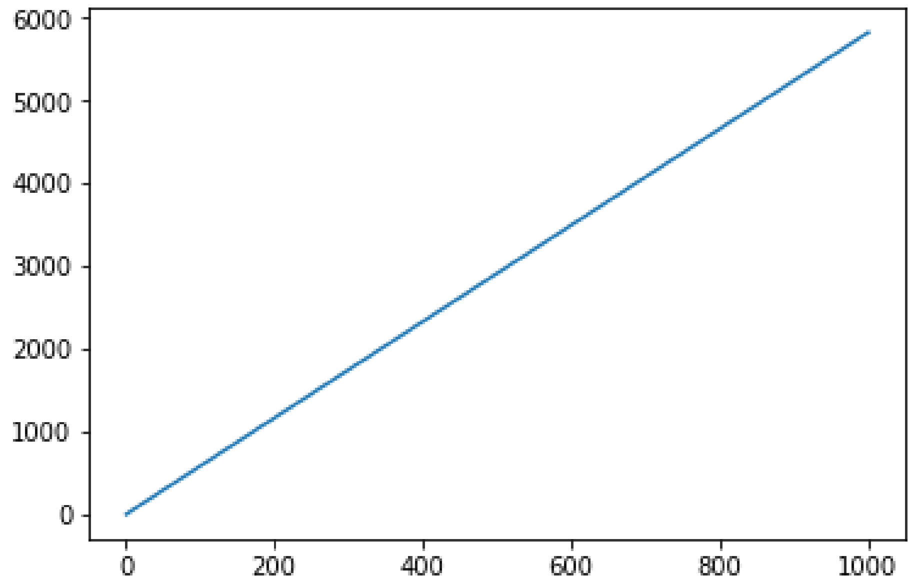

Figure 1: $\prod_{n=1}^{1000}\left(1+\frac{n^5+20}{(n^4+n^3+2}\right)$

The product begins converging once the largest exponent of the denominator was at least 2 greater than the largest exponent of the denominator. This convergence occurred regardless of any follow-on values. If we check the product of $\prod_{n=1}^{20}\left(1+\frac{n^3+n^2+n+135}{(n^6}\right)$ as an example, we see that the after 20 terms the product is converging towards 2149.27 with variation towards a higher number, and after 1000 terms it has converge to 2257.59 and has much less variation. Convergence occurs quicker as the difference between the values of the exponents grows.

The divergence and convergence of:

$$\prod_{n=1}^{\infty}(1+b^n)$$

Where $b > 0$ and is a constant

For this product, anytime the constant b is equal to or greater than 1, the product diverges exponentially upwards. As an example:

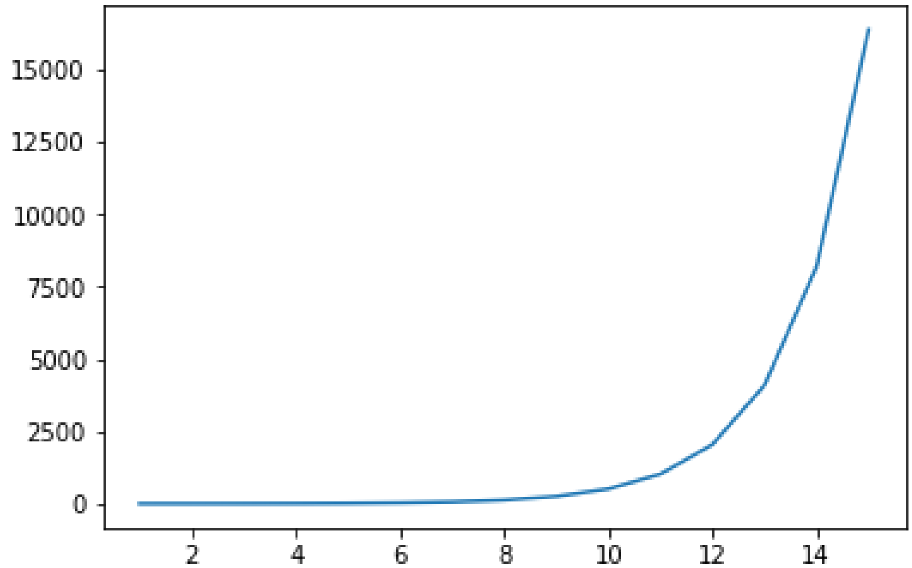

Figure 2: $\prod_{n=1}^{15}(1+1^n)$

Here we have the graph of $\prod_{n=1}^{15}(1+1^n)$ and we can easily see that it diverges rapidly within the first 15 terms. this divergence continues for as many terms as the computer was able to handle.

For the product to converge, a constant value of less than 1 but greater than 0 was substituted for b. When $\prod_{n=1}^{15}(1+0.5^n)$ is tested, convergence is evident within the first six terms, and after 15 terms comes to 2.384 and little variance. Any constant between 1 and zero converge with values closer to 0 converging much faster.

While it was stated that $b > 0$, we did test for values equal to and less than zero and some interesting results came about. At zero, the product is a constant 1 from the beginning, but once we get into the negatives, the predictable negative divergence occurs for values equal to or less than 1, but oscillation then convergence occurs for values between 1 and 0.

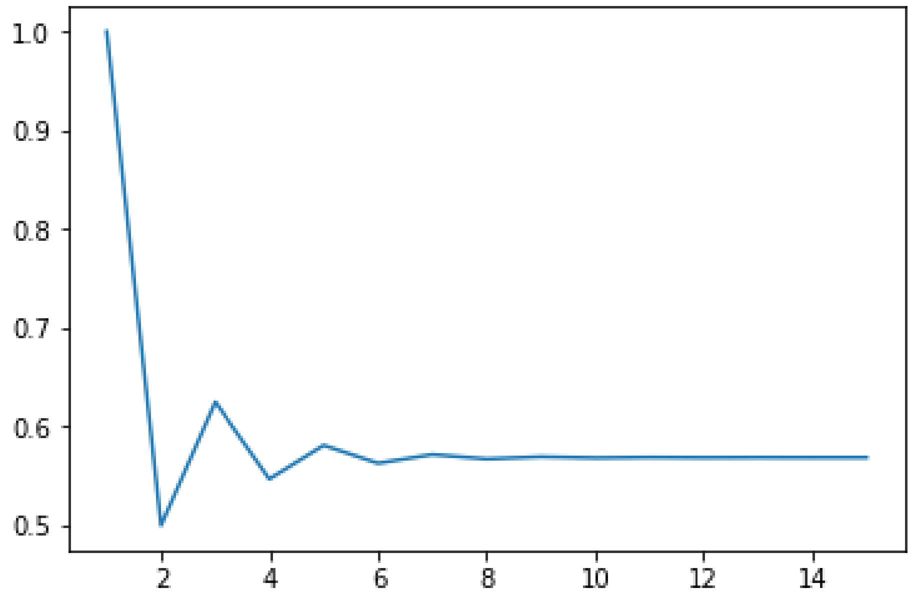

Figure 3: $\prod_{n=1}^{15}(1+-.5^n)$