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A) Since in this case we are dealing with gaussian distributions: For mean $\bar{\phi}$ & variance σ^2

$$\text{p.d.f} = f(y|\bar{\phi},\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(y-\bar{\phi})^2}{2\sigma^2}\right)$$

For given y,

Likelihood of ϕ given y is $L(\phi/y,\sigma^2)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(y-\phi)^2}{2\sigma^2}\right]$$

we only consider one value of y & thus the variance is also constant.

$\Rightarrow$ negative log likelihood $= -\log(L(\phi/y,\sigma^2))$

$$= -\log\left(\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(y-\phi)^2}{2\sigma^2}\right]\right]$$

$$= -\left(-\frac{1}{2}\log(2\pi\sigma^2) - \frac{(y-\phi)^2}{2\sigma^2}\right)$$

$$= C + \frac{(y-\phi)^2}{2\sigma^2}$$

$\because$ C & σ are constants, so to minimize negative of the loglikelihood, we need to minimize $(y-\phi)^2$

For the convenience of it, we will keep the $\frac{1}{2}$

$$\Rightarrow \boxed{\min : \frac{1}{2}(y-\phi)^2 \quad \forall\, y \in Y.}$$

To introduce a penalty function, we add the absolute value of ϕ multiplied by a multiplier,

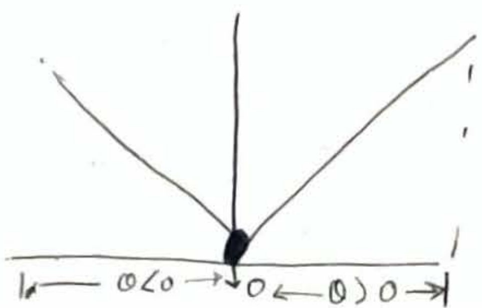
$$\forall \quad \boxed{\min : \frac{1}{2}(y-\phi)^2 + \lambda|\phi|}$$

$$\Rightarrow \boxed{\delta_\lambda(y) = \arg\min_{\theta} \frac{1}{2}(y-\theta)^2 + \lambda|\phi|} \quad \text{---} \; (1)$$

MINIMIZING

To minimize, we differentiate equation (1) with respect to ϕ & set it to zero,

clearly $|\phi|$ is not differentiable, so we divide it in differentiable zones.



$$\frac{d\, S_\lambda(y)}{d\Phi} = \frac{d}{d\Phi}\left(\tfrac{1}{2}(y-\Phi)^2 + \lambda|\Phi|\right)$$

$$= -(y-\Phi) + \lambda \cdot \text{sign}(\Phi)$$

(1) $\boxed{\Phi > 0}$

$\Rightarrow \frac{d S_\lambda}{d\phi} = -(y-\Phi) + \lambda = 0 \Rightarrow \Phi = y - \lambda$

& $\Theta > 0$

$\Rightarrow y - \lambda > 0 \Rightarrow y > \lambda$

(2) $\boxed{\Phi < 0}$ $\Rightarrow \frac{d S_\lambda}{d\phi} = -y + \Phi - \lambda = 0 \Rightarrow \Phi = y + \lambda$

& $\Phi < 0$

$\Rightarrow y < -\lambda$

(3) $\Phi = 0$ $\Rightarrow -\lambda < y < \lambda$

from (1) & (2)

$y > \lambda$

& $y < -\lambda$

$\Rightarrow |y| > \lambda$

& for $|y| < \lambda$ for $\phi = 0$.

$$\Rightarrow \boxed{S_\lambda(y) = \text{sign}(y) \cdot (|y| - \lambda)_+}$$