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SDS 385 HW. 1

A]

$$\text{Min} : Z = \sum_{i=1}^{N} \frac{w_i}{2} (Y_i - x_i^T \beta)^2, \quad \ldots\ldots (1)$$

S.T. $\beta \in R^P$

W be a diagonal matrix such that

$$W = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & 0 & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & w_n \end{bmatrix}$$

$\Rightarrow$ Writing Equation (1) in matrix notations:

$$\text{Min} : Z = \frac{1}{2} \left[(Y - X\beta)^T W (Y - X\beta) \right]$$

$\Downarrow$

PROOF

$$Z = \frac{1}{2} \left[(Y_1 - x_1^T \beta), (Y_2 - x_2^T \beta), \cdots (Y_n - x_n^T \beta) \right] \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & 0 \\ 0 & & \ddots & \\ 0 & 0 & \cdots & w_n \end{bmatrix} \begin{bmatrix} Y_1 - x_1^T \beta \\ Y_2 - x_2^T \beta \\ \vdots \\ Y_n - x_n^T \beta \end{bmatrix}$$

$$Z = \frac{1}{2} \left[w_1 (Y_1 - x_1^T \beta)^2 + w_2 (Y_2 - x_2^T \beta)^2 + \cdots + w_n (Y_n - x_n^T \beta)^2 \right]$$

$$Z = \frac{1}{2}(Y - X\beta)^T W (Y - X\beta)$$

$$= \frac{1}{2}\left[Y^T W Y - Y^T W X \beta - \beta^T X^T W Y + \beta^T X^T W X \beta \right]$$

Using properties

① $(A \pm B)^T = A^T \pm B^T$

$(AB)^T = B^T A^T$

To minimize with respect to β, we will differentiate with respect to β.

$$\frac{dz}{d\beta} = \frac{1}{2}\left[-X^T W Y + X^T W X \beta \right]$$

$$\frac{d}{d\beta} = \begin{bmatrix} \frac{d}{d\beta_1} \\ \frac{d}{d\beta_2} \\ \vdots \\ \frac{d}{d\beta_n} \end{bmatrix} \Rightarrow \frac{dz}{d\beta} = \begin{bmatrix} \frac{dz_1}{d\beta} \\ \frac{dz_2}{d\beta} \\ \frac{dz_3}{d\beta} \\ \vdots \end{bmatrix}$$

[$z_1, z_2, z_3 \cdots z_n$ are Rows of z]

$$\Rightarrow \quad \frac{d^2}{d\beta} = 0 \quad , \quad \Rightarrow \quad X^T W Y = X^T W X \beta$$

$$\Rightarrow \quad (X^T W X)^{-1} X^T W Y = (X^T W X)^{-1} X^T W X \beta$$

$$= \beta$$

$$\Rightarrow \quad \boxed{\beta = (X^T W X)^{-1} X^T W Y}$$