

# LECTURE 4: RANDOM VARIABLES

ECON 480 - ECONOMETRICS - FALL 2018

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Random Variables

Expected Value and Variance

Normal Distribution

## RANDOM VARIABLES

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## EXPERIMENTS

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### Example

Flip a coin 10 times



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## Example

The number of Heads from 10 coin flips





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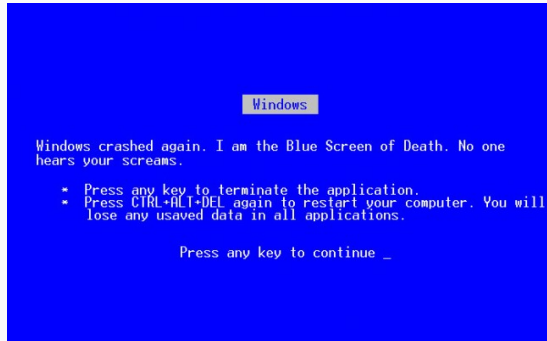
### Example

Let  $X$  be the number of Heads from 10 coin flips.  $x_i \in \{0, 1, 2, \dots, 10\}$

- A **discrete random variable** takes on a finite/countable set of possible values

### Example

Let  $X$  be the number of times your computer crashes this semester,  $x_i \in \{0, 1, 2, 3, 4\}$



- The **probability distribution** of a R.V. fully lists all the possible values of  $X$  and their associated probabilities

### Example

$x_i$	$P(X = x_i)$
0	0.80
1	0.10
2	0.06
3	0.03
4	0.01

- The **probability distribution function (pdf)** summarizes the possible outcomes of  $X$  and their probabilities

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- For any real number  $x_i$ ,  $f(x_i)$  is the probability that  $X = x_i$



### Example

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- What is  $f(0)$ ?

### Example

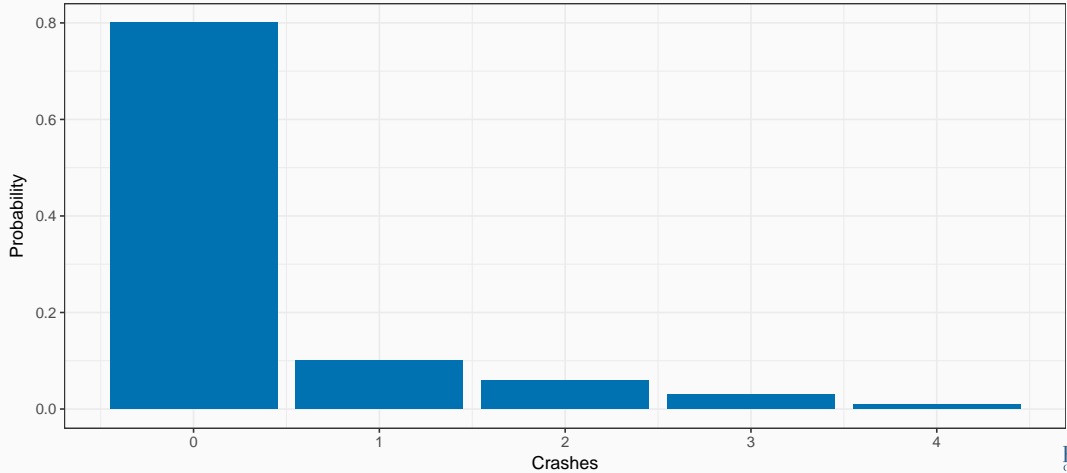
$x_i$	$P(X = x_i)$
0	0.80
1	0.10
2	0.06
3	0.03
4	0.01

- What is  $f(0)$ ?
- What is  $f(3)$ ?

```
crashes<-c(0,1,2,3,4)
prob<-c(0.8,0.1,0.06,0.03,0.01)
crashes<-data.frame(crashes, prob)

crashes.pdf<-ggplot(crashes, aes(x=crashes, y=prob))+
  geom_bar(stat="identity", fill="#0072B2")+xlab("Crashes")+ylab("Probability")
```

## DISCRETE RANDOM VARIABLES: PDF GRAPH II



- The *cumulative density function (cdf)* lists probability  $X$  will be *at most* (less than or equal to) a given value  $x_i$

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### Example

$x_i$	$f(X)$	$F(X)$
0	0.80	0.80
1	0.10	0.90
2	0.06	0.96
3	0.03	0.99
4	0.01	1.00



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- What is the probability your computer will crash at most once

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$x_i$	$f(X)$	$F(X)$
0	0.80	0.80
1	0.10	0.90
2	0.06	0.96
3	0.03	0.99
4	0.01	1.00

- What is the probability your computer will crash at most once,  $F(1)$ ? three times,  $F(3)$ ?

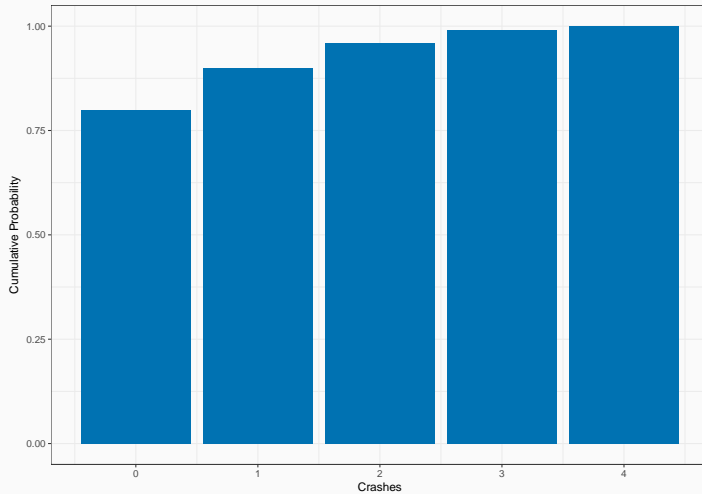
```
crashes$cprob<-(cumsum(crashes$prob))
```

```
crashes #show table
```

```
##   crashes prob cprob
## 1         0 0.80  0.80
## 2         1 0.10  0.90
## 3         2 0.06  0.96
## 4         3 0.03  0.99
## 5         4 0.01  1.00
```

```
crashes.cdf<-ggplot(crashes, aes(x=crashes, y=cprob))+  
  geom_bar(stat="identity", fill="#0072B2")+xlab("Crashes")+  
  ylab("Cumulative Probability")
```

## DISCRETE RANDOM VARIABLES: CDF GRAPH II



## EXPECTED VALUE AND VARIANCE

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- The **expected value** of a random variable  $X$ , written  $E(X)$ , is the long-run average value of  $X$  “expected” after many repetitions

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$$E(X) = p_1x_1 + p_2x_2 + \dots + p_kx_k$$



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$$E(X) = p_1x_1 + p_2x_2 + \dots + p_kx_k = \sum_{i=1}^k p_ix_i$$

- A probability-weighted average of  $X$ , with each  $x_i$  weighted by its associated probability  $p_i$
- Also called the **mean** or **expectation** of  $X$ , also denoted  $\mu_X$

### Example

Suppose you lend your friend \$100 at 10% interest. If the loan is repaid, you receive \$110 . Your friend is 99% likely to repay, but there is a default risk of 1% where you get nothing. What is the expected value of repayment?

### Example

Let  $X$  be a random variable that is described by the following pdf:

$x_i$	$P(X = x_i)$
1	0.50
2	0.25
3	0.15
4	0.10

- What is  $E(X)$ ?

```
x<-c(1,2,3,4) #x values  
p.x<-c(0.50,0.25,0.15,0.10) #probabilities  
rv<-data.frame(x,p.x) #combine into data frame  
sum(rv$x*rv$p.x) #sum of each value times its probability
```

```
## [1] 1.85
```

- The **variance** of a random variable  $X$ , denoted  $\text{var}(X)$  or  $\sigma_X^2$  is:

$$\sigma_X^2 = \sum_{i=1}^n (x_i - \mu_X)^2 p_i$$

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$$\sigma_X^2 = \sum_{i=1}^n (x_i - \mu_X)^2 p_i = E[(X - \mu_X)^2]$$

- This is the expected value of the squared deviations from the mean
  - i.e. the probability-weighted average of the squared deviations

- The **standard deviation** of a random variable  $X$ , denoted  $sd(X)$  or  $\sigma_X$  is:

$$\sigma_X = \sqrt{\sigma_X^2}$$



### Example

What is the standard deviation of computer crashes?

	Number of Computer Crashes				
$x_i$	0	1	2	3	4
$P(X = x_i)$	0.80	0.10	0.06	0.03	0.01

### Example

What is the standard deviation of computer crashes?

	Number of Computer Crashes				
$x_i$	0	1	2	3	4
$P(X = x_i)$	0.80	0.10	0.06	0.03	0.01

- First, calculate the mean:

$$\begin{aligned} E(X) &= 0(0.80) + 1(0.10) + 2(0.06) + 3(0.03) + 4(0.01) \\ &= 0 + 0.1 + 0.12 + 0.09 + 0.04 \\ &= 0.35 \end{aligned}$$

### Example

What is the standard deviation of computer crashes?

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- Next, find the deviations from the mean and square them:

	Number of Crashes ( $\mu = 0.35$ )				
$x_i$	0	1	2	3	4
$P(X = x_i)$	0.80	0.10	0.06	0.03	0.01
$(x_i - \mu)$	-0.35	0.65	1.65	2.65	3.65
$(x_i - \mu)^2$	0.1225	0.4225	2.7225	7.0225	13.3225

## Example

What is the standard deviation of computer crashes?

- Next, find the deviations from the mean and square them:

	Number of Crashes ( $\mu = 0.35$ )				
$x_i$	0	1	2	3	4
$P(X = x_i)$	0.80	0.10	0.06	0.03	0.01
$(x_i - \mu)$	-0.35	0.65	1.65	2.65	3.65
$(x_i - \mu)^2$	0.1225	0.4225	2.7225	7.0225	13.3225

- Take the expectation of the squared deviations to get variance

$$\begin{aligned}
 \sigma_x^2 &= 0.1225(0.80) + 0.4225(0.10) + 2.7225(0.06) + 7.0225(0.03) + 13.3225(0.01) \\
 &= 0.098 + 0.04225 + 0.16335 + 0.210675 + 0.133225 \\
 &= 0.6475
 \end{aligned}$$

### Example

What is the standard deviation of computer crashes?

- To get standard deviation,  $\sigma$ , take the square root:

$$\begin{aligned}\sigma &= \sqrt{\sigma^2} \\ &= \sqrt{0.6475} \\ &= 0.8047\end{aligned}$$

### Example

For the random variable we saw before, what is its standard deviation?

$x_i$	$P(X = x_i)$
1	0.50
2	0.25
3	0.15
4	0.10

```
rv$deviations<-(rv$x-1.85) #take deviations  
rv$sqdeviations<-(rv$deviations^2) #square deviations
```

```
rv #show table
```

```
##   x  p.x deviations sqdeviations  
## 1 1 0.50      -0.85         0.7225  
## 2 2 0.25       0.15         0.0225  
## 3 3 0.15       1.15         1.3225  
## 4 4 0.10       2.15         4.6225
```

```
sum(rv$sqdeviations*rv$p.x) #add squared deviations times probabilities
```

```
## [1] 1.0275
```



```
sqrt(sum(rv$sqdeviations*rv$p.x)) #take square root of variance for sd
```

```
## [1] 1.013657
```

- *Continuous random variables* can take on an uncountable (infinite) number of values



- *Continuous random variables* can take on an uncountable (infinite) number of values
- So many values that the probability of any specific value is infinitely small  $\rightarrow 0$ .



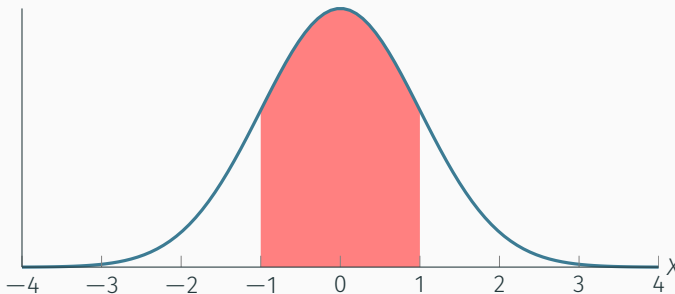
## CONTINUOUS RANDOM VARIABLES

- *Continuous random variables* can take on an uncountable (infinite) number of values
- So many values that the probability of any specific value is infinitely small  $\rightarrow 0$ .
- Instead, we focus on a *range* of values it might take on



## CONTINUOUS RANDOM VARIABLE PDF

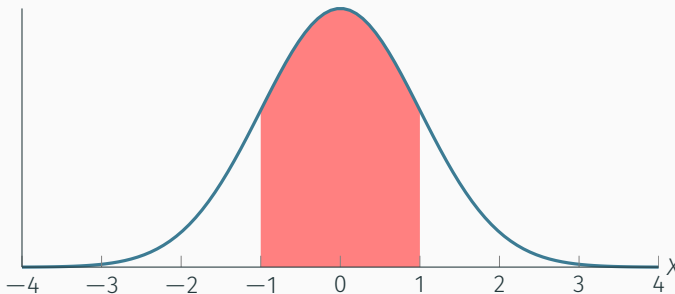
- The **probability density function (pdf)** of a continuous variable represents the probability between two values as the area under a curve



$P(-1 \leq X \leq 1)$ : area under the curve between -1 and 1.

## CONTINUOUS RANDOM VARIABLE PDF

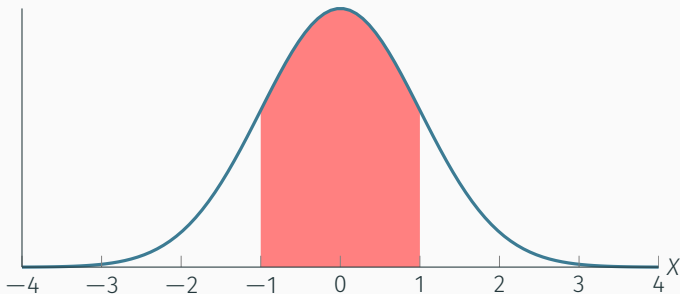
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## CONTINUOUS RANDOM VARIABLE PDF

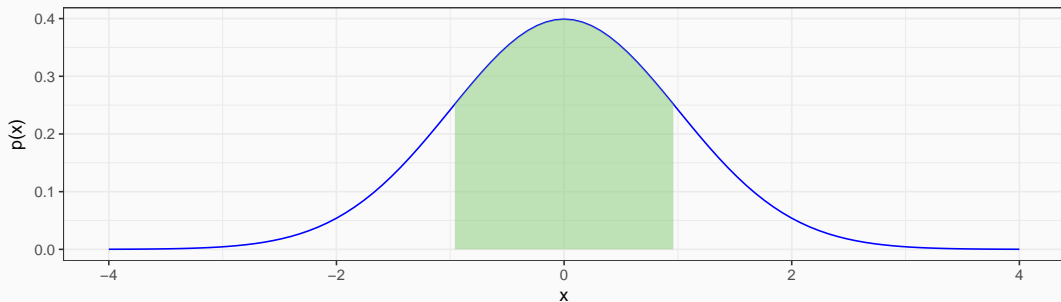
- The **probability density function (pdf)** of a continuous variable represents the probability between two values as the area under a curve
- The total area under the curve is 1
- Since  $P(a) = 0$  and  $P(b) = 0$ ,  $P(a < X < b) = P(a \leq X \leq b)$



$P(-1 \leq X \leq 1)$ : area under the curve between -1 and 1.

- FYI using calculus:

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

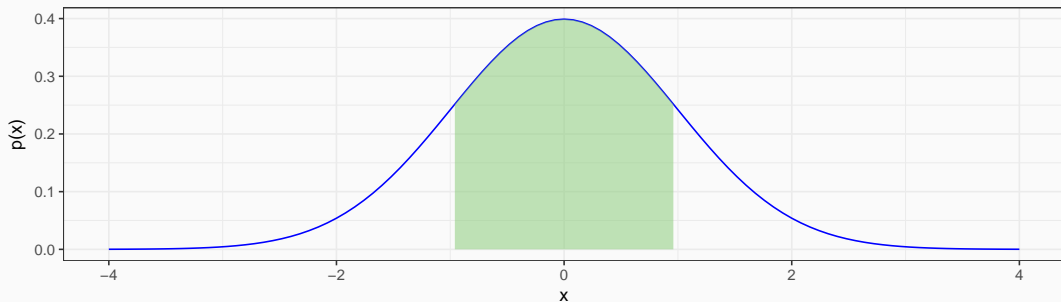




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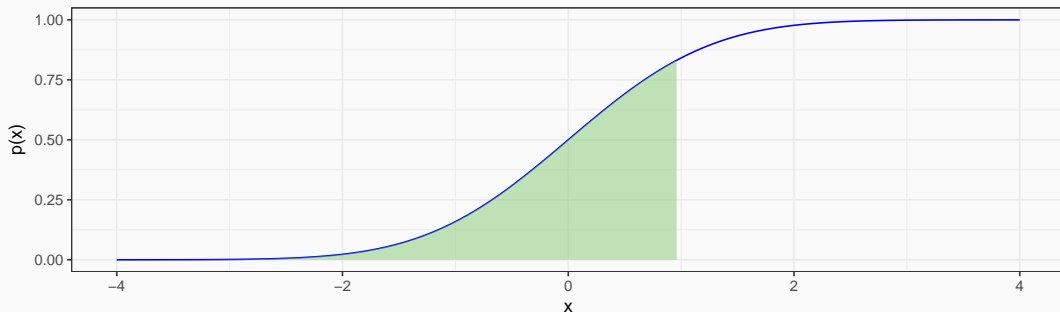
- Complicated: software or (old fashioned!) probability tables to calculate



## CONTINUOUS RANDOM VARIABLE CDF

- The *cumulative density function (cdf)* describes the area under the pdf for all values less than or equal to (i.e. to the left of) a given value,  $k$

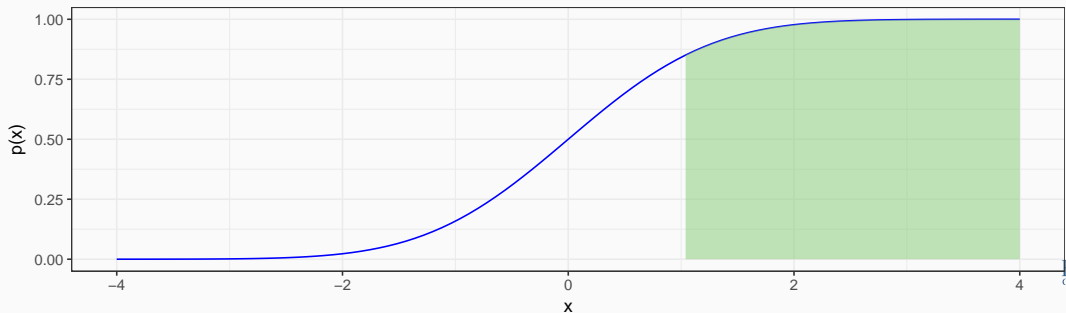
$$P(X \leq k)$$



## CONTINUOUS RANDOM VARIABLE CDF II

- Note: to find the probability of values *greater* than or equal to (to the right of) a given value  $k$ :

$$P(X \geq k) = 1 - P(X \leq k)$$



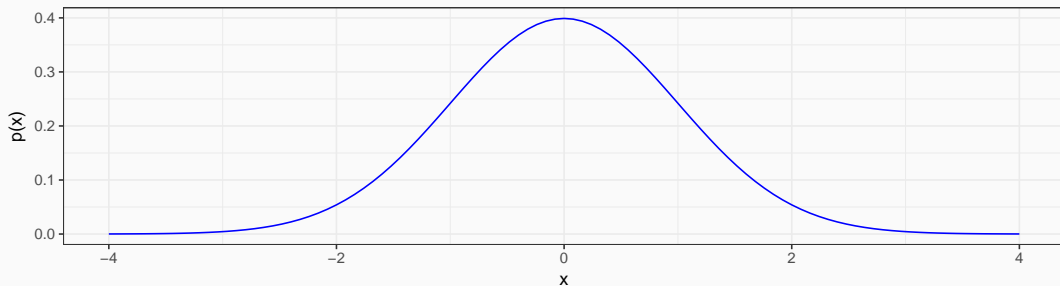
## NORMAL DISTRIBUTION

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# THE NORMAL DISTRIBUTION

- The **Gaussian** or **normal distribution** is the most useful type of probability distribution

$$X \sim N(\mu, \sigma)$$

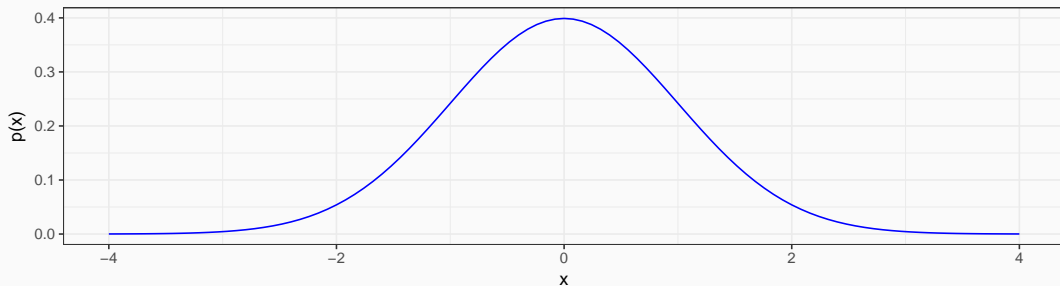


# THE NORMAL DISTRIBUTION

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$$X \sim N(\mu, \sigma)$$

- Continuous, symmetric, unimodal, with mean  $\mu$  and standard deviation  $\sigma$



- The pdf of  $X \sim N(\mu, \sigma)$  is

$$P(X = k) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{(k-\mu)}{\sigma}\right)^2}$$

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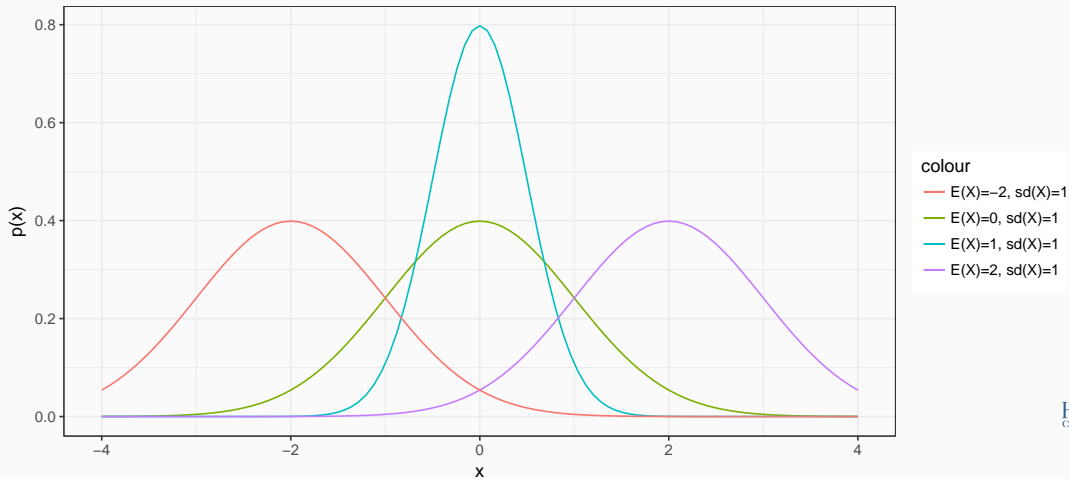
$$P(X = k) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{(k-\mu)}{\sigma}\right)^2}$$

- **Do not try and learn this**, we have software and (previously tables) to calculate pdfs and cdfs



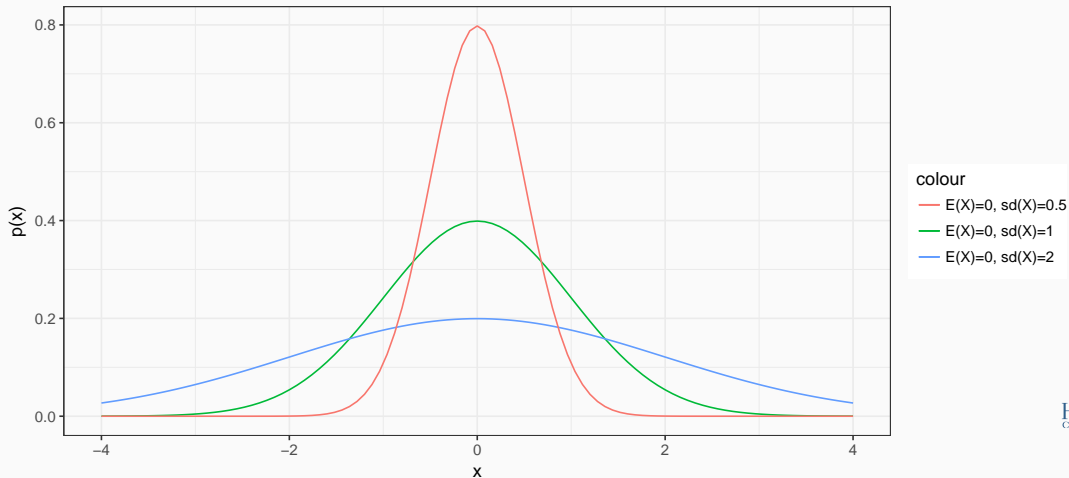
# THE EFFECTS OF PARAMETER CHANGES I

- The pdf moves left/right based on  $\mu$



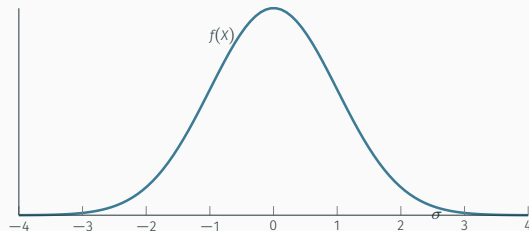
## THE EFFECTS OF PARAMETER CHANGES II

- The pdf gets fatter/skinnier based on  $\sigma$



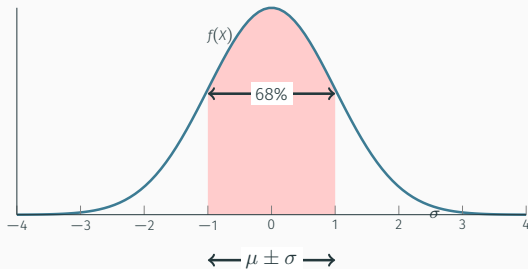
## THE 68-95-99.7% RULE

- The 68-95-99.7% (empirical) rule: for a normal distribution:



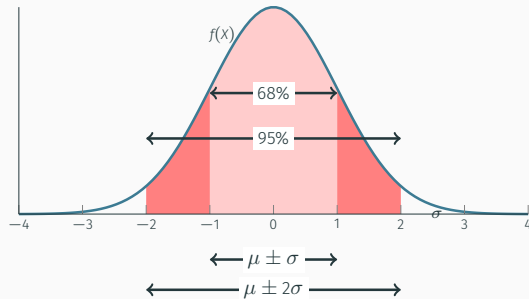
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- The 68-95-99.7% (empirical) rule: for a normal distribution:
  - $P(\mu - 1\sigma \leq X \leq \mu + 1\sigma) \approx 68\%$



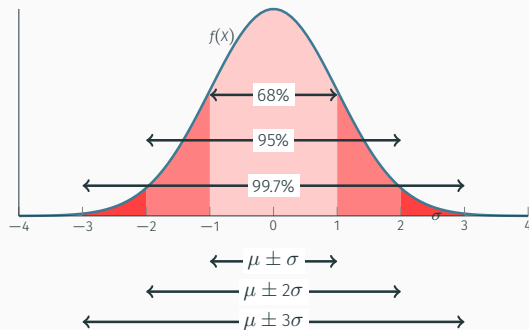
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  - $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 95\%$



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  - $P(\mu - 1\sigma \leq X \leq \mu + 1\sigma) \approx 68\%$
  - $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 95\%$
  - $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 99.7\%$



- We **standardize** a variable by calculating its **Z-score** and converting to the **standard normal distribution**

$$Z = \frac{x - \mu}{\sigma}$$

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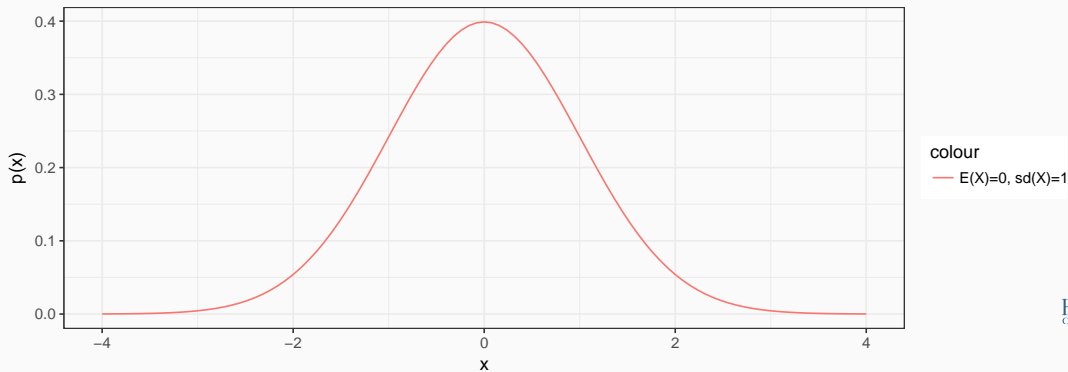
$$Z = \frac{x - \mu}{\sigma}$$

- Z is the number of standard deviations a value is above/below of its mean
- Can compare distributions of variables with very different units!

## THE STANDARD NORMAL DISTRIBUTION II

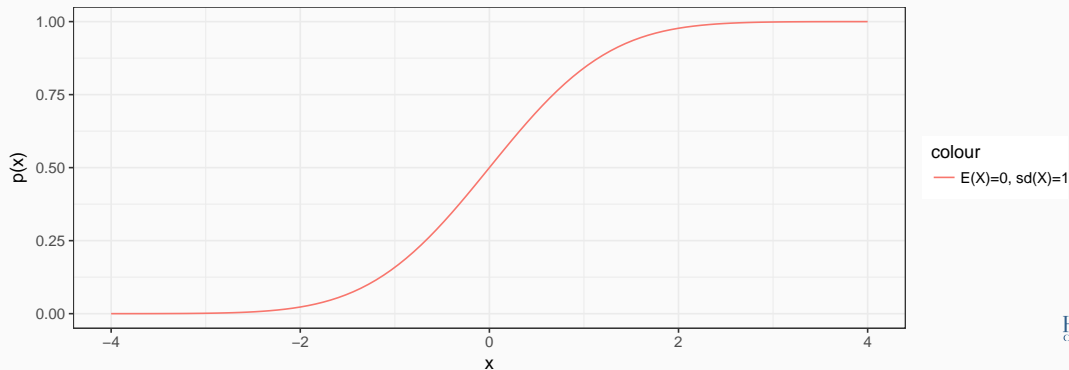
- The **standard normal distribution** has mean 0 and standard deviation 1

$$Z \sim N(0, 1)$$



- Standard normal cdf:

$$\Phi(k) = P(Z \leq k)$$



### Example

On August 8, 2011, the Dow dropped 634.8 points, sending shock waves through the financial community. Assume that during mid-2011 to mid-2012 the daily change for the Dow is normally distributed, with the mean daily change of 1.87 points and a standard deviation of 155.28 points. What is the Z-score?

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- Convert to Z-score:

$$Z = \frac{X - \mu}{\sigma} = \frac{634.8 - 1.87}{155.28}$$

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- Convert to Z-score:

$$Z = \frac{X - \mu}{\sigma} = \frac{634.8 - 1.87}{155.28} = -4.1$$

- This is 4.1 standard deviations ( $\sigma$ ) beneath the mean.

### Example

In the last quarter of 2015, a group of 64 mutual funds had a mean return of 2.4% with a standard deviation of 5.6%. These returns can be approximated by a normal distribution.



### Example

In the last quarter of 2015, a group of 64 mutual funds had a mean return of 2.4% with a standard deviation of 5.6%. These returns can be approximated by a normal distribution.

- What percent of the funds would you expect to be earning between -3.2% and 8.0% returns?

- Convert to standard normal to find Z-scores for 8 and -3.2.

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$$P(-3.2 < X < 8)$$

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$$P(-3.2 < X < 8)$$

$$P\left(\frac{-3.2 - 2.4}{5.6} < \frac{X - 2.4}{5.6} < \frac{8 - 2.4}{5.6}\right)$$

- Convert to standard normal to find Z-scores for 8 and -3.2.

$$P(-3.2 < X < 8)$$

$$P\left(\frac{-3.2 - 2.4}{5.6} < \frac{X - 2.4}{5.6} < \frac{8 - 2.4}{5.6}\right)$$

$$P(-1 < Z < 1)$$

- Convert to standard normal to find Z-scores for 8 and -3.2.

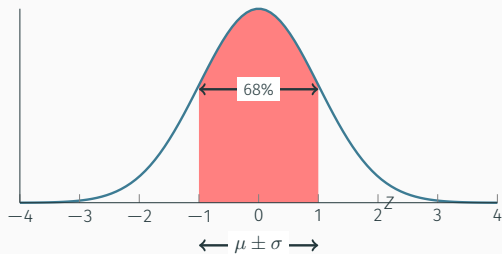
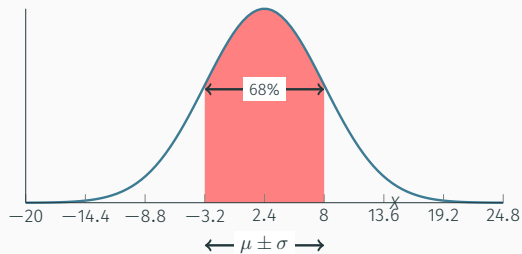
$$P(-3.2 < X < 8)$$

$$P\left(\frac{-3.2 - 2.4}{5.6} < \frac{X - 2.4}{5.6} < \frac{8 - 2.4}{5.6}\right)$$

$$P(-1 < Z < 1)$$

- $P(X \pm 1\sigma) = 0.68$

## STANDARDIZING VARIABLES: EXAMPLE IV



### Example

In the last quarter of 2015, a group of 64 mutual funds had a mean return of 2.4% with a standard deviation of 5.6%. These returns can be approximated by a normal distribution.

- What percent of the funds would you expect to be earning between -3.2% and 8.0% returns?



### Example

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### Example

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- What percent of the funds would you expect to be earning between -3.2% and 8.0% returns?
- What percent of the funds would you expect to be earning 2.4% or less?

### Example

In the last quarter of 2015, a group of 64 mutual funds had a mean return of 2.4% with a standard deviation of 5.6%. These returns can be approximated by a normal distribution.

- What percent of the funds would you expect to be earning between -3.2% and 8.0% returns?
- What percent of the funds would you expect to be earning 2.4% or less?
- What percent of the funds would you expect to be earning between -8.8% and 13.6%?

### Example


In the last quarter of 2015, a group of 64 mutual funds had a mean return of 2.4% with a standard deviation of 5.6%. These returns can be approximated by a normal distribution.

- What percent of the funds would you expect to be earning between -3.2% and 8.0% returns?
- What percent of the funds would you expect to be earning 2.4% or less?
- What percent of the funds would you expect to be earning between -8.8% and 13.6%?
- What percent of the funds would you expect to be earning returns greater than 13.6%?

# FINDING Z-SCORE PROBABILITIES

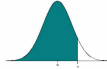
- How do we actually find the probabilities for Z-scores?

**Table of Standard Normal Probabilities for Negative Z-scores**



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0010	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0400	0.0392	0.0384	0.0377	0.0369
-1.6	0.0548	0.0537	0.0526	0.0514	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0664	0.0652	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2063	0.2033	0.2005	0.1977	0.1949	0.1923	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3631	0.3593	0.3555	0.3517	0.3479
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

**Table of Standard Normal Probabilities for Positive Z-scores**

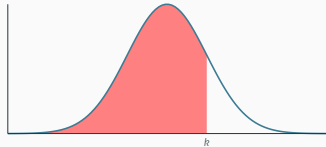


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5949	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6369	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6629	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8645	0.8668	0.8689	0.8708	0.8727	0.8746	0.8764	0.8781	0.8798	0.8810
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9494	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9895	0.9898	0.9899	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9923	0.9925	0.9926	0.9928	0.9929	0.9930	0.9931
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9992	0.9992	0.9993	0.9993	0.9993	0.9994	0.9994
3.2	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
3.3	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Note that the probabilities given in this table represent the area to the LEFT of the z-score.  
The area to the RIGHT of a z-score = 1 – the area to the LEFT of the z-score

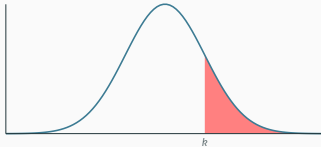
## FINDING Z-SCORE PROBABILITIES II

Probability to Left of  $k$



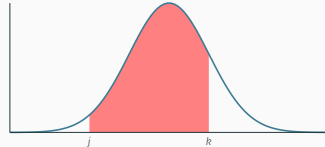
$$P(Z \leq k) = \Phi(k)$$

Probability to the Right of  $k$



$$P(Z \geq k) = 1 - \Phi(k)$$

Probability Between  $j$  and  $k$



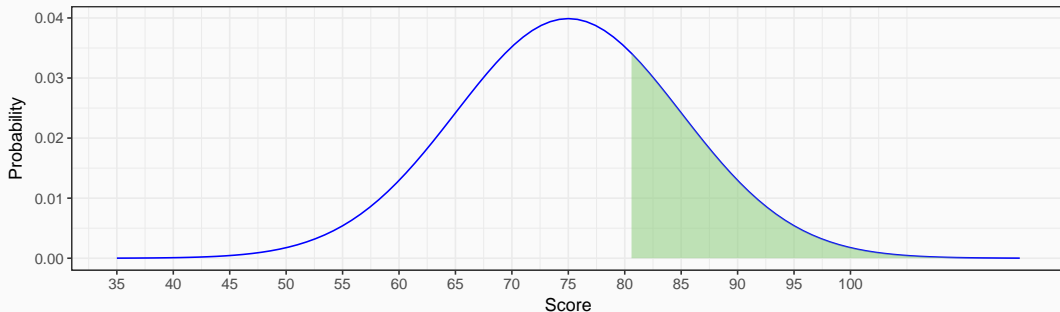
$$P(j \leq Z \leq k) = \Phi(k) - \Phi(j)$$

## FINDING Z-SCORE PROBABILITIES WITH R

- Let the distribution of grades be normal, with mean 75 and standard deviation 10.

```
pnorm(80, mean=75, sd=10, lower.tail=FALSE)
```

```
## [1] 0.3085375
```

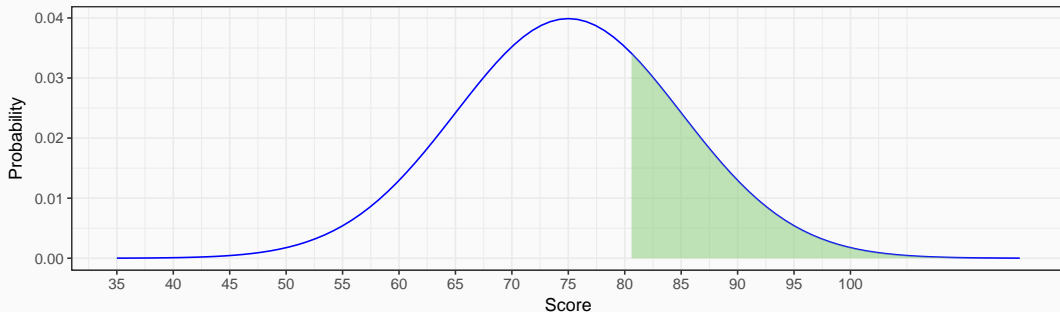


## FINDING Z-SCORE PROBABILITIES WITH R

- Let the distribution of grades be normal, with mean 75 and standard deviation 10.
- Probability a student gets at least an 80:

```
pnorm(80, mean=75, sd=10, lower.tail=FALSE)
```

```
## [1] 0.3085375
```



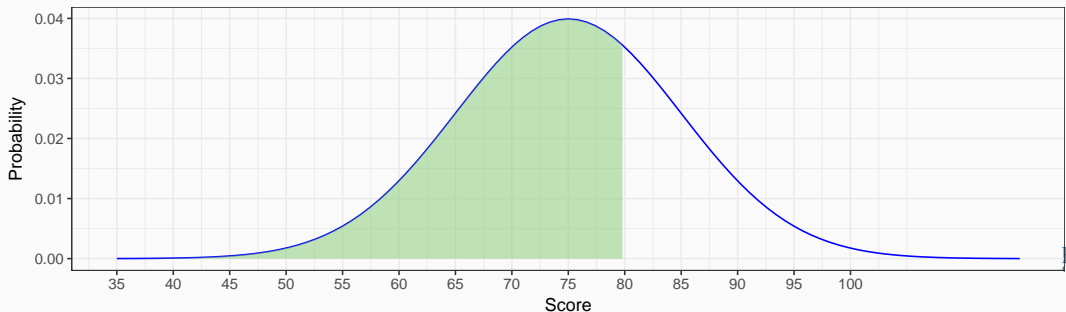


## FINDING Z-SCORE PROBABILITIES WITH R II

- Probability a student gets at most an 80:

```
pnorm(80, mean=75, sd=10, lower.tail=TRUE)
```

```
## [1] 0.6914625
```



## FINDING Z-SCORE PROBABILITIES WITH R III

- Probability a student gets between a 65 and 85:

```
pnorm(85, mean=75, sd=10, lower.tail=TRUE)-pnorm(
  65, mean=75, sd=10, lower.tail=TRUE)
```

```
## [1] 0.6826895
```

