THE Winter School

Causal Machine Learning

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References

- Belloni, Chernozhukov, and Hansen (2014): ”High-Dimensional Methods and Inference on Structural and Treatment Effects”, Journal of Economic Perspectives, 28 (2), pp. 29-50, download.
- Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey (2017): ”Double/Debiased/Neyman Machine Learning of Treatment Effects”, American Economic Review, 107 (5), pp. 261-265, download.

Estimation Target

- Multivariate Linear Regression Model:

$$
\begin{aligned}
Y_i &= D_i\delta + X_i\beta_g + U_i && \text{(structural model)} \\
D_i &= X_i\beta_m + V_i && \text{(selection model)}
\end{aligned}
$$

with $E[U_i|D_i, X_i] = 0$ and $E[V_i|X_i] = 0$.

- Parameter of interest: δ
- Nuisance parameters: β_g and β_m

- X_i contains $p \gg N$ covariates.
- We assume controlling for $K \ll N$ covariates is sufficient to identify δ.
- Controlling for too many irrelevant covariates may reduce the efficiency of OLS.

Types of Covariates

Relation between covariates and outcome (for some $s_g > 0$):

- $|\beta_{gj}| > s_g$: covariate X_j has a **strong association** with Y_i
- $0 < |\beta_{gj}| \leq s_g$: covariate X_j has a **weak association** with Y_i
- $\beta_{gj} = 0$: covariate X_j has a **no association** with Y_i

Relation between covariates and treatment (for some $s_m > 0$):

- $|\beta_{mj}| > s_m$: covariate X_j has a **strong association** with D_i
- $0 < |\beta_{mj}| \leq s_m$: covariate X_j has a **weak association** with D_i
- $\beta_{mj} = 0$: covariate X_j has a **no association** with D_i

$\rightarrow$ All covariates are standardised

Naive Approach I: Structural Model

Apply Lasso to the structural model

$$\min_{\beta_g}\{E[(Y_i - D_i\delta - X_i\beta_g)^2] + \lambda\|\beta_g\|_1\}$$

without a penalty on δ and estimate a Post-Lasso model using all covariates with non-zero β_g coefficients.

Covariates that are weakly associated with Y_i could be dropped.

$\rightarrow$ Potentially we drop "weak" confounders with $0 < |\beta_{gj}| \leq s_g$ and $|\beta_{mj}| > 0$.

Covariates that are strongly associated with D_i could be dropped.

$\rightarrow$ Potentially we drop "strong" confounders with $|\beta_{gj}| > s_g$ and $|\beta_{mj}| > s_m$.

Naive Approach II: Selection Model

Apply Lasso to the selection model

$$\min_{\beta_m}\{E[(D_i - X_i\beta_m)^2] + \lambda\|\beta_m\|_1\}$$

and estimate a Post-Lasso structural model using all covariates with non-zero β_m coefficients.

Covariates that are weakly associated with D_i could be dropped.

→ Potentially we drop “weak” confounders with $0 < |\beta_{mj}| \leq s_m$ and $|\beta_{gj}| > 0$.

Double Selection Procedure

1. Apply Lasso to the reduced form models

$$\min_{\tilde{\beta}_g}\{E[(Y_i - X_i\tilde{\beta}_g)^2] + \lambda\|\tilde{\beta}_g\|_1\}, \tag{1}$$

$$\min_{\beta_m}\{E[(D_i - X_i\beta_m)^2] + \lambda\|\beta_m\|_1\}, \tag{2}$$

   with $\tilde{\beta}_g \approx \delta\beta_m + \beta_g$.

2. Take the union of all covariates $\tilde{X}_i$ with either non-zero β_m or $\tilde{\beta}_g$ coefficients and estimate the Post-Lasso structural model

$$Y_i = D_i\delta + \tilde{X}_i\beta_g^* + u_i.$$

Double Selection Procedure (cont.)

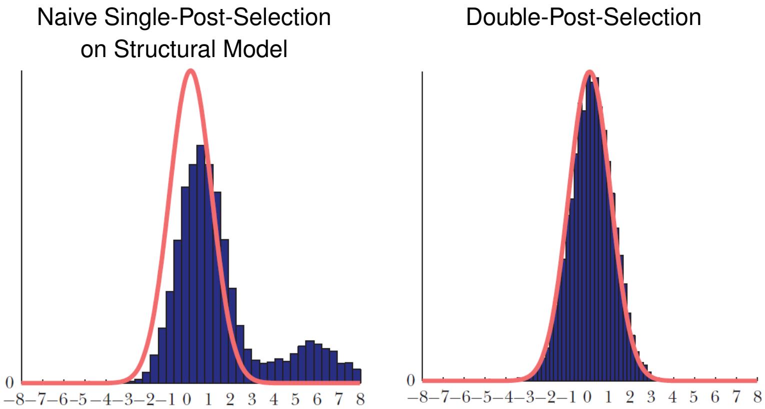
Potentially (2) misses “weak” confounders with $0 < |\beta_{mj}| \leq s_m$ and $|\beta_{gj}| > 0$.

$\tilde{\beta}_{gj} \approx \beta_g$ when $0 < |\beta_{mj}| \leq s_m$, such that the missing “weak” confounders with $|\beta_{gj}| > s_g$ are likely selected in (1).

→ Potentially we omit “very weak” confounders with $0 < |\beta_{gj}| \leq s_g$ and $0 < |\beta_{mj}| \leq s_g$.

Simulation Exercise

Distribution of Estimators



Naive Single-Post-Selection on Structural Model

Double-Post-Selection

Source: Belloni, Chernozhukov, and Hansen (2014)

Asymptotic Results

- Consistency and asymptotic normality

$$\sqrt{N}(\hat{\delta} - \delta) \overset{d}{\to} N(0, \sigma).$$

- Model selection step is asymptotically negligible for building confidence intervals.
- Optimal penalty parameter $\lambda^* = 2c \cdot \Phi^{-1}(1 - \gamma/2p)/\sqrt{N}$ (e.g., $c = 1.1$ and $\gamma \leq 0.05$) for ”Feasible LASSO”

$$\min_{\beta} E[(Y_i - X_i\beta)^2] + \lambda^* \|\beta\|_1.$$

Reference: Belloni, Chernozhukov, and Hansen (2014)

Summary Double Selection Procedure

Advantages:

- Standard inference
- Computationally fast

Disadvantages:

- Effect homogeneity
- Potentially too many covariates selected
- Sparsity assumptions required

Potential Outcome Framework

Notation:

- D_i binary treatment dummy (e.g., assignment to training program)
- $Y_i(1)$ potential outcome under treatment (e.g., earnings under participation in training)
- $Y_i(0)$ potential outcome under non-treatment (e.g., earnings under non-participation in training)

Infeasible parameter:

- Individual causal effect: $\delta_i = Y_i(1) - Y_i(0)$

Feasible parameters:

- Average Treatment Effect (ATE): $\delta = E[Y_i(1) - Y_i(0)] = E[\delta_i]$
- Average Treatment Effect on the Treated (ATET): $\rho = E[\delta_i | D_i = 1]$

Identifying Assumptions for ATE

- **Stable Unit Treatment Value Assumption (SUTVA):**

$$Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$$

- **Exogeneity of Covariates:**

$$X_i(1) = X_i(0)$$

- **No Support Problems:**

$$\varepsilon < Pr(D_i = 1 | X_i = x) = p(x) < 1 - \varepsilon$$

  for some small $\varepsilon > 0$ and all x in the support of X_i

- **Conditional Independence Assumption (CIA):**

$$Y_i(1), Y_i(0) \perp\!\!\!\perp D_i | X_i = x$$

  for all x in the support of X_i

Modified Outcome Method for ATE

$$Y^*_{i,IPW} = W_i Y_i$$

with the Inverse Probability Weights (IPW)

$$W_i = \frac{D_i - p(x)}{p(x)(1 - p(x))}$$

with $p(x) = Pr(D_i = 1 | X_i = x)$.

ATE: $\delta = E[Y^*_{i,IPW}]$ and $\hat{\delta} = \frac{1}{N} \sum_{i=1}^{N} \hat{Y}^*_{i,IPW}$

We can use standard ML methods to estimate $\hat{p}(x)$ (see, e.g., Goller, Lechner, Moczall, Wolff, 2019).

Proof of Identification

$$
\begin{aligned}
\delta = E[Y_i(1)] - E[Y_i(0)] &\overset{LIE}{=} \int E[Y_i(1)|X_i = x] - E[Y_i(0)|X_i = x]\, f_X(x)dx \\
&\overset{CIA}{=} \int E[Y_i(1)|D_i = 1, X_i = x] - E[Y_i(0)|D_i = 0, X_i = x]\, f_X(x)dx \\
&= \int E[Y_i|D_i = 1, X_i = x] - E[Y_i|D_i = 0, X_i = x]\, f_X(x)dx \\
&= \int E[D_iY_i|D_i = 1, X_i = x] - E[(1 - D_i)Y_i|D_i = 0, X_i = x]\, f_X(x)dx \\
&\overset{LIE}{=} \int E\left[\frac{D_iY_i}{p(x)}\middle| X_i = x\right] - E\left[\frac{(1 - D_i)Y_i}{1 - p(x)}\middle| X_i = x\right] f_X(x)dx \\
&= \int E\left[\frac{D_iY_i}{p(x)} - \frac{(1 - D_i)Y_i}{1 - p(x)}\middle| X_i = x\right] f_X(x)dx \\
&= \int E\left[\frac{D_i - p(x)}{p(x)(1 - p(x))} Y_i\middle| X_i = x\right] f_X(x)dx \overset{LIE}{=} E\left[\frac{D_i - p(x)}{p(x)(1 - p(x))} Y_i\right]
\end{aligned}
$$

Reference: Horvitz and Thompson (1952)

Modified Outcome Method with IPW

Advantages:

- Generic approach
- Heterogeneous treatment effects

Disadvantages:

- Potentially omitting “weak outcome confounders”

Double/Debiased Machine Learning (DML)

$$Y^*_{i,DML} = \mu_1(X_i) - \mu_0(X_i) + \frac{D_i(Y_i - \mu_1(X_i))}{p(X_i)} - \frac{(1-D_i)(Y_i - \mu_0(X_i))}{1-p(X_i)}$$

with $\mu_1 = E[Y_i(1)|X_i = x]$ and $\mu_0 = E[Y_i(0)|X_i = x]$.

ATE: $\delta = E[Y^*_{i,DML}]$ and $\hat{\delta} = \frac{1}{N}\sum_{i=1}^{N} \hat{Y}^*_{i,DML}$

We can use standard ML methods to estimate $\hat{\mu}_1(x)$, $\hat{\mu}_0(x)$, and $\hat{p}(x)$.

Additional Advantages compared to IPW:

- Treatment and outcome equations are modelled explicitly
- Double robustness property

Reference: Chernozhukov et al., 2017

Proof of Identification

$$
\begin{aligned}
\delta =& E\left[\mu_1(x)-\mu_0(x)+\frac{D_i(Y_i-\mu_1(x))}{p(x)}-\frac{(1-D_i)(Y_i-\mu_0(x))}{1-p(x)}\right] \\
=& E\left[\frac{D_i-p(x)}{p(x)(1-p(x))}Y_i+\frac{(p(x)-D_i)\mu_1(x)}{p(x)}-\frac{(D_i-p(x))\mu_0(x)}{1-p(x)}\right] \\
=& \int E\left[\frac{D_i-p(x)}{p(x)(1-p(x))}Y_i+\frac{(p(x)-D_i)\mu_1(x)}{p(x)}-\frac{(D_i-p(x))\mu_0(x)}{1-p(x)}\middle| X_i=x\right] f_X(x)dx \\
=& \int\left(E\left[\frac{D_i-p(x)}{p(x)(1-p(x))}Y_i\middle| X_i=x\right]+\frac{E[p(x)-D_i|X_i=x]}{p(x)}\mu_1(x)\right. \\
& \left.-\frac{E[D_i-p(x)|X_i=x]}{1-p(x)}\mu_0(x)\right) f_X(x)dx \\
=& \int E\left[\frac{D_i-p(x)}{p(x)(1-p(x))}Y_i\middle| X_i=x\right] f_X(x)dx = E\left[Y_i(1)-Y_i(0)\right]
\end{aligned}
$$

Reference: Robins and Rotnitzki (1995)

DML Cross-Fitting Algorithm

1. Partition the data randomly in samples S^A and S^B
2. Estimate the nuisance parameters $\hat{\mu}_1^A(x)$,$\hat{\mu}_0^A(x)$, and $\hat{p}^A(x)$ in S^A; and $\hat{\mu}_1^B(x)$,$\hat{\mu}_0^B(x)$, and $\hat{p}^B(x)$ in S^B with ML
3. Calculate the efficient scores in samples S^A and S^B, respectively:

$$\hat{Y}_{i,DML}^{A*} = \hat{\mu}_1^B(X_i^A) - \hat{\mu}_0^B(X_i^A) + \frac{D_i^A(Y_i^A - \hat{\mu}_1^B(X_i^A))}{\hat{p}^B(X_i^A)} - \frac{(1-D_i^A)(Y_i^A - \hat{\mu}_0^B(X_i^A))}{1-\hat{p}^B(X_i^A)}$$

$$\hat{Y}_{i,DML}^{B*} = \hat{\mu}_1^A(X_i^B) - \hat{\mu}_0^A(X_i^B) + \frac{D_i^B(Y_i^B - \hat{\mu}_1^A(X_i^B))}{\hat{p}^A(X_i^B)} - \frac{(1-D_i^B)(Y_i^B - \hat{\mu}_0^A(X_i^B))}{1-\hat{p}^A(X_i^B)}$$

4. Calculate ATE,

$$\hat{\delta} = \frac{1}{2}(\underbrace{\hat{E}[\hat{Y}_{i,DML}^{A*}|S^A]}_{=\hat{\delta}_A} + \underbrace{\hat{E}[\hat{Y}_{i,DML}^{B*}|S^B]}_{=\hat{\delta}_B}),$$

Asymptotic Results for ATE

- Main Regularity Condition: Convergence rate of nuisance parameters is at least $\sqrt[4]{N}$.
- ATE can be estimated $\sqrt{N}$-consistently

$$\sqrt{N}(\hat{\delta}-\delta) \xrightarrow{d} N(0,\sigma)$$

with $\sigma^2 = Var(Y^*_{i,DML})$ and $Var(\hat{\delta}) = \sigma^2/N$

- Split sample estimator of σ^2

$$\hat{\sigma}^2 = \frac{1}{2}\left(\hat{\sigma}^2_A + (\hat{\delta}_A - \hat{\delta})^2\right) + \frac{1}{2}\left(\hat{\sigma}^2_B + (\hat{\delta}_B - \hat{\delta})^2\right)$$

for $\hat{\delta} = 1/2(\hat{\delta}_A + \hat{\delta}_B)$

Orthogonal Score for ATET

$$Y^*_{i,ATET} = \frac{D_i(Y_i - \mu_0(x))}{p} - \frac{p(x)(1-D_i)(Y_i - \mu_0(x))}{p(1-p(x))}$$

with $p = Pr(D_i = 1)$.

ATET: $\rho = E[Y^*_{i,ATET}]$ and $\hat{\rho} = \frac{1}{N}\sum_{i=1}^{N} \hat{Y}^*_{i,ATET}$

Asymptotic Variance:

$$\sigma^2 = E\left[\left(\frac{D_i(Y_i - \mu_0(x))}{p} - \frac{p(x)(1-D_i)(Y_i - \mu_0(x))}{p(1-p(x))} - \rho\frac{D}{p}\right)^2\right]$$

and $Var(\hat{\rho}) = \sigma^2/N$

Reference: Chernozhukov et al., 2017

Proof of Identification for ATET

$$
\begin{aligned}
\rho =&E\left[\frac{D_i(Y_i-\mu_0(x))}{p}-\frac{p(x)(1-D_i)(Y_i-\mu_0(x))}{p(1-p(x))}\right]\\
=&\int E\left[\frac{D_iY_i}{p}-\frac{p(x)(1-D_i)Y_i}{p(1-p(x))}-\frac{(D_i-p(x))\mu_0(x)}{p(1-p(x))}\middle|X_i=x\right]f_X(x)dx\\
=&\int\left(\frac{E[D_iY_i|X_i=x]}{p}-\frac{p(x)E[(1-D_i)Y_i|X_i=x]}{p(1-p(x))}\right.\\
&\left.-\frac{E[D_i-p(x)|X_i=x]}{p(1-p(x))}\mu_0(x)\right)f_X(x)dx\\
=&\int\left(\frac{E[D_iY_i|X_i=x]}{p}-\frac{p(x)E[(1-D_i)Y_i|X_i=x]}{p(1-p(x))}\right)f_X(x)dx\\
=&\int\frac{p(x)}{p}\left(E[D_iY_i|D_i=1,X_i=x]-E[(1-D_i)Y_i|D_i=0,X_i=x]\right)f_X(x)dx\\
=&\int\left(E[Y_i(1)|D_i=1,X_i=x]-E[Y_i(0)|D_i=0,X_i=x]\right)f_{X|D=1}(x)dx\\
=&\int\left(E[Y_i(1)|D_i=1,X_i=x]-E[Y_i(0)|D_i=1,X_i=x]\right)f_{X|D=1}(x)dx\\
=&E[Y_i(1)-Y_i(0)|D_i=1]
\end{aligned}
$$

Other Orthogonal Scores

- LATE (see Chernozhukov et al., 2018).
- Difference-in-differences (see, e.g., Chen, Nie, and Wager, 2018, Zimmert, 2019).
- Multiple treatments (see, e.g., Farrell, 2015).
- Continuous treatments (see, e.g., Graham and Pinto, 2018).
- Mediation analysis (see Tchetgen Tchetgen and Shpitser, 2012).
- Synthetic control group method (see, e.g., Arkhangelsky et al., 2018).