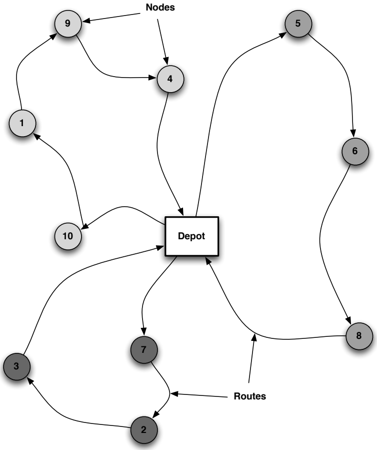
Capacitated Vehicle Routing Problem

Topic 5 - Scheduling Problems

Capacitated Vehicle Routing Problem

- The CVRP was introduced for the first time by [2], is a complex combinatorial optimisation problem, which can be seen as a merge of two well-known problems: the TSP and the Bin Packing Problem (BPP).
- The CVRP is often described as the problem in which vehicles based on a central depot are required to visit geographically dispersed customers in order to fulfill known customer demands [3].
- A simple schematic is given in the following, where the *nodes* represent the customers, and the *route* is the tour which is undertaken by a specific vehicle. The depot is where each vehicle starts and ends the tour.

Directed Graph



Capacitated Vehicle Routing Problem

Let $G = (V, E)$ be a graph where $V = \{i_1, i_2, \dots, i_n\}$ is the vertex set (i_1 refers to the depot and the customers are indexed $i_2, \dots, i_n$) and $E = \{(i_l, i_m) : i_l, i_m \in V\}$ is the edge set. Each customer must be assigned to exactly one of the k vehicles and the total size of deliveries for customers assigned to each vehicle must not exceed the vehicle capacity (Q_k).
If the vehicles are homogeneous, the capacity for all vehicles is equal and denoted by Q. A demand q_l and a service time st_l are associated with each customer node i_l. The demand q_1 and the service time st_1 which are referred to the demand and service time of the depot are set equal to zero.

Capacitated Vehicle Routing Problem

The travel cost and the travel time between customers i_l and i_m is $c_{l,m}$ and $tt^k_{l,m}$, respectively, and T_k is the maximum time allowed for a route of vehicle k.

The problem is to construct a low cost, feasible set of routes - one for each vehicle. A route is a sequence of locations that a vehicle must visit along with the indication of the service it provides, where the variable $x^k_{l,m}$ is equal to 1 if the arc (i_l, i_m) is traversed by vehicle k and 0 otherwise. The vehicle must start and finish its tour at the depot.

Formulation

- The mathematical formulation of the CVRP can be given as the following (as outlined in [1] and [3]):

$$J = \min \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{k=1}^{K} c_{l,m} x_{l,m}^{k} \tag{1}$$

$$\sum_{i_l=1}^{n} \sum_{k=1}^{K} x_{l,m}^{k} = 1, \qquad i_m = 2, \ldots, n \tag{2}$$

$$\sum_{i_m=1}^{n} \sum_{k=1}^{K} x_{l,m}^{k} = 1, \qquad i_l = 2, \ldots, n \tag{3}$$

Formulation

$$\sum_{i_l=1}^{n} x_{l,f}^{k} - \sum_{i_m=1}^{n} x_{f,m}^{k} = 0, \qquad \mathrm{k} = 1, \ldots, K;\ i_f = 1, \ldots, n \tag{4}$$

$$\sum_{i_l=1}^{n} q_l \sum_{i_{m=\mathbf{1}}}^{n} x_{l,m}^{k} \leq Q_k, \qquad k = 1, \ldots, K \tag{5}$$

$$\sum_{i_l=1}^{n} st_l^k \sum_{i_m=1}^{n} x_{l,m}^{k} + \sum_{i_l=1}^{n} \sum_{i_m=1}^{n} tt_{l,m}^{k} x_{l,m}^{k} \leq T_k, \quad k = 1, \ldots, K \tag{6}$$

Formulation

$$\sum_{i_m=2}^{n} x_{1,m}^{k} \leq 1, \qquad k = 1, \dots, K \tag{7}$$

$$\sum_{i_l=2}^{n} x_{l,1}^{k} \leq 1, \qquad k = 1, \dots, K \tag{8}$$

$$X \in S \tag{9}$$

$$x_{l,m}^{k} = 0 \text{ or } 1, \quad \text{for all } i_l, i_m, k \tag{10}$$

Formulation

Objective function (1) states that the total distance is to be minimised. Equations (2) and (3) ensure that each demand node is served by exactly one vehicle. Route continuity is represented by (4), i.e. if a vehicle enters in a demand node, it must exit from that node. Equation (5) represents the vehicle capacity constraints and (6) are the total elapsed route time constraints. Equations (7) and (8) guarantee that vehicle availability is not exceeded [3].

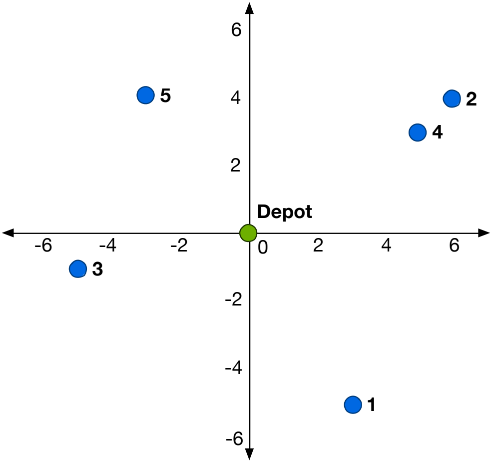
CVRP Example

- Consider a 5 tour with the following x-y coordinations. The total capacity of a vehicle is **10** and the depot coordinates is $\{0,\ 0\}$.

Table: Sample coordinates

Tour	x	y	Demand
1	3	-5	2
2	6	4	6
3	-5	-1	8
4	5	3	4
5	-3	4	6

Customers



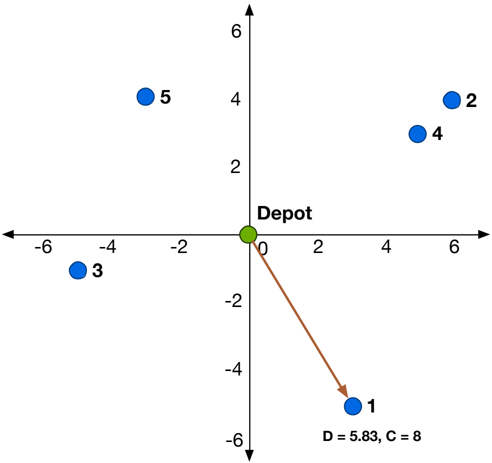
CVRP Example

- Consider a tour of $\prod = \{1,\ 2,\ 3,\ 4,\ 5\}$.
- For the first customer, the distance form the depot can be calculated using the Pythagorean theorem as the distance from the Deport to the first customer, provided that there is enough capacity in the vehicle.
- The distance is from $\{0,\ 0\}$ to $\{3,\ \text{-}\ 5\}$, therefore the travelled distance is $\sqrt{3^2 + -5^2}$, which is 5.83. The capacity of the vehicle can decrease by 2 to (10 - 2) = 8.

Table: First Customer

Tour	x	y	Demand	Distance	Capacity
1	3	-5	2	5.83	8

First Customer



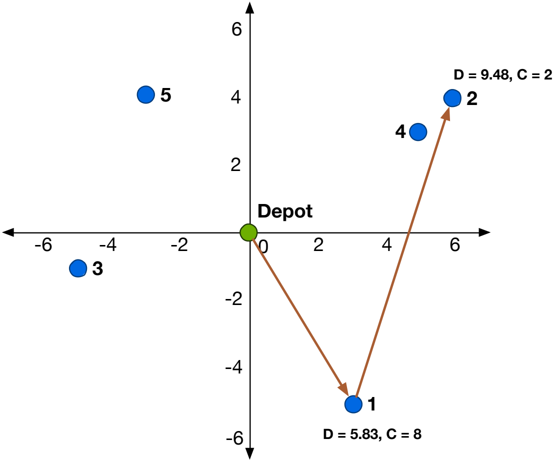
CVRP Example

- For the second customer, we first have to ascertain if the vehicle has the sufficient capacity. The capacity of the second customer is 6, and the current capacity of the vehicle is 8, therefore the vehicle can service the customer.
- The distance is from $\{3, \ -5\}$ to $\{6, \ 4\}$, therefore the difference is $x = (6 - 3) = 3$ and $y = (4 - -5) = 9$ travelled distance is $\sqrt{3^2 + 9^2}$, which is 9.48. The capacity of the vehicle can decrease by 6 to (8 - 6) = 2.

Table: Second Customer

Tour	x	y	Demand	Distance	Capacity
1	3	-5	2	5.83	8
2	6	4	6	9.48	2

Second Customer



CVRP Example

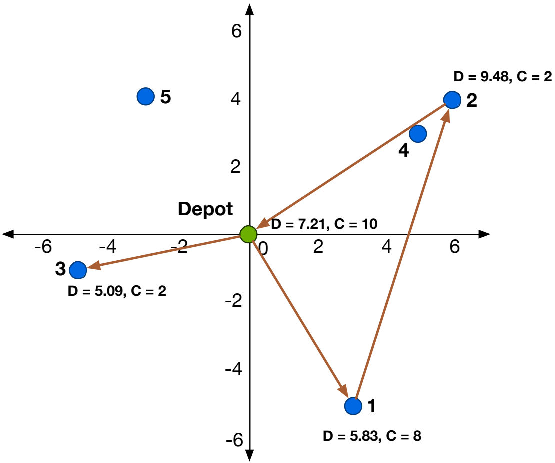
- For the third customer, the capacity is 8, however the current capacity of the vehicle is 2, therefore the vehicle cant service the customer.
- Therefore, we first need to visit the depot.
- The distance from the second customer to the depot is $\{6,\ 4\}$ to $\{0, 0\}$, therefore the travelled distance is $\sqrt{6^2 + 4^2}$, which is 7.21. The capacity of the vehicle is now reset to 10.
- As the capacity of the vehicle is now 10, it can service the third customer. The vechicle need to travel from the depot to the third client as $\{0,\ 0\}$ to $\{-5,$-$1\}$, therefore the travelled distance is $\sqrt{-5^2 + -1^2}$, which is 5.09. The capacity of the vehicle is now reset to (10 - 8) $=$ 2.
- The total distance travelled by the vehicle from the second customer to the third customer is $7.21 + 5.09 = 12.3$.

CVRP Example

Table: Third Customer

Tour	x	y	Demand	Distance	Capacity
1	3	-5	2	5.83	8
2	6	4	6	9.48	2
3	-5	-1	8	12.3	2

Third Customer



CVRP Example

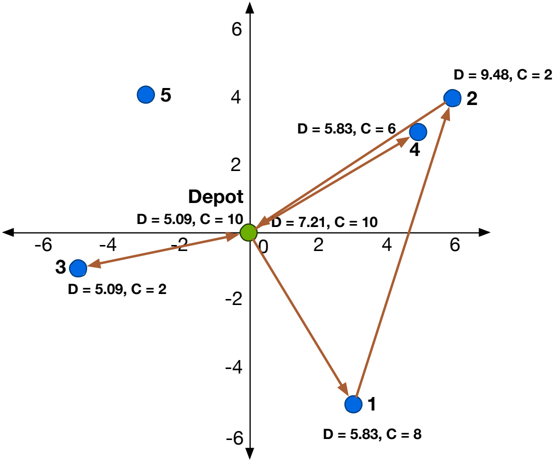
- Likewise, the fourth and fifth customers are served.
- The vehicle ends up at the Depot after serving the fifth customer
- The distance travelled is the total distance covered by the vehicle.

CVRP Example

Table: Fourth Customer

Tour	x	y	Demand	Distance	Capacity
1	3	-5	2	5.83	8
2	6	4	6	9.48	2
3	-5	-1	8	12.3	2
4	5	3	4	10.92	6

Fourth Customer



CVRP Example

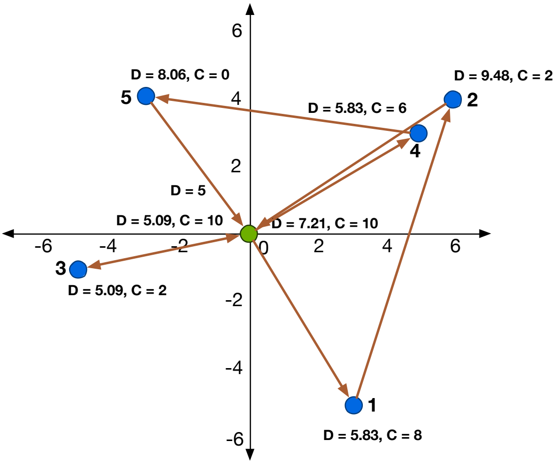
Table: Fifth Customer

Tour	x	y	Demand	Distance	Capacity
1	3	-5	2	5.83	8
2	6	4	6	9.48	2
3	-5	-1	8	12.3	2
4	5	3	4	10.92	6
5	-3	4	6	8.06	0
D	0	0	-	5	-

Total distance travelled

The total distance travelled is (D - 1) 5.83 + (1 - 2) 9.48 + (2 - D) 7.21 + (D - 3) 5.09 + (3 - D) 5.09 + (D - 4) 5.83 + (4 - 5) 8.06 + (5 - D) 5 = **51.59**.

Depot



6
4
2
-2
-4
-6
-6
-4
-2
0
2
4
6
D = 8.06, C = 0
5
D = 9.48, C = 2
2
D = 5.83, C = 6
4
D = 5
D = 5.09, C = 10
D = 7.21, C = 10
3
D = 5.09, C = 2
1
D = 5.83, C = 8

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