GA Schema Theorem

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Schema Theorem

- Schema theorem serves as the analysis tool for the GA process
- Explain why GAs work by showing the expectation of schema survival
- Applicable to a canonical GA
  - binary representation
  - fixed length individuals
  - fitness proportional selection
  - single point crossover
  - gene-wise mutation

Schema

- A **schema** is a set of binary strings that match the template for schema ***H***
- A template is made up of 1s, 0s, and *s where * is the 'don't care' symbol that matches either 0 or 1

Schema Examples

- The schema $H = 10{*}1{*}$ represents the set of binary strings

$$10010,\ \ 10011,\ \ 10110,\ \ 10111$$

- The string '10' of length $l = 2$ belongs to $2^l = 2^2$ different schemas

$$**,\ \ *0,\ \ 1*,\ \ 10$$

Schema: Order o(*H*)

- The order of a schema is the number of its fixed bits, i.e. the number of bits that are not '*' in the schema H

- Example: if H = 10*1* then o(H) = 3

Schema: Defining Length $\delta(H)$

- The defining length is the distance between its first and the last fixed bits

- Example: if H = *1*01 then $\delta(H) = 5 - 2 = 3$

- Example: if H = 0**** then $\delta(H) = 1 - 1 = 0$

Schema: Count

- Suppose x is an individual that belongs to the schema H, then we say that ***x* is an instance of *H*** ($\boldsymbol{x \in H}$)

- $m(H, k)$ denotes the number of instances of H in the k th generation

Schema: Fitness

- $f(x)$ denotes fitness value of x

- $\boldsymbol{f(H,k)}$ denotes **average fitness of *H*** in the k-th generation

$$f(H,k) = \frac{\sum_{x \in H} f(x)}{m(H,k)}$$

Effect of GA On A Schema

- Effect of Selection
- Effect of Crossover
- Effect of Mutation
- = Schema Theorem

Effect of Selection on Schema

- Assumption: fitness proportional selection

- Selection probability for the individual x

$$p_s(x) = \frac{f(x)}{\sum_{i=1}^{N} f(x_i)}$$

where the N is the total number of individuals

Net Effect of Selection

- The expected number of instances of *H* in the mating pool $M(H,k)$ is

$$M(H,k) = \frac{\sum_{x \in H} f(x)}{\bar{f}} = m(H,k)\frac{f(H,k)}{\bar{f}}$$

Schemas with fitness greater than the population average are likely to appear more in the next generation

Effect of Crossover on Schema

- Assumption: single-point crossover

- Schema *H* survives crossover operation if
  - one of the parents is an instance of the schema *H* **AND**
  - one of the offspring is an instance of the schema *H*

Crossover Survival Examples

Consider $H = *10**$

$P_1 = 1\ 1\ 0\ \vdots\ 1\ 0 \in H$ → $S_1 = 1\ 1\ 0\ 1\ 1 \in H$ **Schema *H***

$P_2 = 1\ 0\ 1\ \vdots\ 1\ 1 \notin H$ → $S_2 = 1\ 0\ 1\ 1\ 0 \notin H$ **survived**

$P_1 = 1\ 1\ \vdots\ 0\ 1\ 0 \in H$ → $S_1 = 1\ 1\ 1\ 1\ 1 \notin H$ **Schema *H***

$P_2 = 1\ 0\ \vdots\ 1\ 1\ 1 \notin H$ → $S_2 = 1\ 0\ 0\ 1\ 0 \notin H$ **destroyed**

Crossover Operation

- Suppose a parent is an instance of a schema *H*. When the crossover is occurred within the bits of the defining length, it is destroyed unless the other parent repairs the destroyed portion

- Given a string with length l and a schema H with the defining length $\delta(H)$, the probability that the crossover occurs within the bits of the defining length is $\delta(H)/(l - 1)$

Crossover Probability Example

- Suppose H = *1**0
- We gave
  - $l = 5$
  - $\delta(H) = 5 - 2 = 3$
- Thus, the probability that the crossover occurs within the defining length is 3/4

Crossover Operation

- The upper bound of the probability that the schema H being destroyed is

$$D_c(H) \leq p_c \frac{\delta(H)}{l-1}$$

where p_c is the crossover probability

Net Effect of Crossover

- The lower bound on the probability $S_c(H)$ that H survives is

$$S_c(H) = 1 - D_c(H) \geq 1 - p_c \frac{\delta(H)}{l-1}$$

➡ **Schemas with low order are more likely to survive**

Mutation Operation

- Assumption: mutation is applied gene by gene

- For a schema H to survive, all fixed bits must remain unchanged

- Probability of a gene not being changed is

$$(1 - p_m)$$

where p_m is the mutation probability of a gene

Net Effect of Mutation

- The probability a schema H survives under mutation

$$S_m(H) = (1 - p_m)^{o(H)}$$

Schemas with low order are more likely to survive

Schema Theorem

Exp. # of Schema H in Next Generation >

Exp. # in Mating Pool ($M(H,k) = m(H,k)\dfrac{f(H,k)}{\bar{f}}$)

Prob. of Surviving Crossover ($S_c(H) \geq 1 - p_c \dfrac{\delta(H)}{l-1}$)

Prob. of Surviving Mutation ($S_m(H) = (1 - p_m)^{o(H)}$)

Schema Theorem

- Mathematically

$$E[m(H,k+1)] \geq m(H,k)\frac{f(H,k)}{\bar{f}}\left(1-p_c\frac{\delta(H)}{l-1}\right)(1-p_m)^{o(H)}$$

The schema theorem states that the schema with *above average fitness*, *short defining length* and *lower order* is more likely to survive.

The theorem concentrates on the *number of schema surviving* **not** *which schema survive***.**