CS528 Homework Three

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1. Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $\langle 5, 10, 3, 12, 5, 50, 6 \rangle$

   (a) The answer and all work is included below, but for brevity, the final answer is also included here:

$$m = \begin{bmatrix} 0 & 150 & 330 & 405 & 1655 & 2010 \\ - & 0 & 360 & 330 & 2430 & 1950 \\ - & - & 0 & 180 & 930 & 1770 \\ - & - & - & 0 & 3000 & 1860 \\ - & - & - & - & 0 & 1500 \\ - & - & - & - & - & 0 \end{bmatrix}$$

   and an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $\langle 5, 10, 3, 12, 5, 50, 6 \rangle$ is $((A_1 A_2)((A_3 A_4)(A_5 A_6)))$.

   (b) We have a matrix-chain product whose sequence of dimensions is: $\langle 5, 10, 3, 12, 5, 50, 6 \rangle$, therefore we have the following matrices and p-values:

$A_1 = 5 \times 10$	$p_0 = 5$
$A_2 = 10 \times 3$	$p_1 = 10$
$A_3 = 3 \times 12$	$p_2 = 3$
$A_4 = 12 \times 5$	$p_3 = 12$
$A_5 = 5 \times 50$	$p_4 = 5$
$A_6 = 50 \times 6$	$p_5 = 50$
	$p_6 = 6$

   We will need a 6×6 matrix m such that, for all i rows and j columns of matrix m where $i \leq j$, we have the following:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \leq k < j} \{m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j\} & \text{if } i < j \end{cases}$$

In order to fill the upper diagonal matrix m, we need to check all combinations where $i < j$, since the diagonal of the matrix will be all zeros (since $i = j$ on the diagonal). At this stage in the problem, we have:

$$m = \begin{bmatrix} 0 & & & & & \\ - & 0 & & & & \\ - & - & 0 & & & \\ - & - & - & 0 & & \\ - & - & - & - & 0 & \\ - & - & - & - & - & 0 \end{bmatrix}$$

To assist in calculation, we will calculate all of the entries where $j = i + 1$ first, since these will only rely on trivial solutions to fully calculate the entry in table m.

- For $m[1, 2]$, we know that $k = 1$ is the only possible value. Thus:

$$\begin{aligned} m[1,2] &= \min_{1 \le k < 2} \{m[1,1] + m[2,2] + p_0 \cdot p_1 \cdot p_2 \\ &= 0 + 0 + 5 \cdot 10 \cdot 3 \\ &= 150 \end{aligned} \tag{1}$$

- For $m[2, 3]$, we know that $k = 2$ is the only possible value. Thus:

$$\begin{aligned} m[2,3] &= \min_{2 \le k < 3} \{m[2,2] + m[3,3] + p_1 \cdot p_2 \cdot p_3 \\ &= 0 + 0 + 10 \cdot 3 \cdot 12 \\ &= 360 \end{aligned} \tag{2}$$

- For $m[3, 4]$, we know that $k = 3$ is the only possible value. Thus:

$$\begin{aligned} m[3,4] &= \min_{3 \le k < 4} \{m[3,3] + m[4,4] + p_2 \cdot p_3 \cdot p_4 \\ &= 0 + 0 + 3 \cdot 12 \cdot 5 \\ &= 180 \end{aligned} \tag{3}$$

- For $m[4, 5]$, we know that $k = 4$ is the only possible value. Thus:

$$\begin{aligned} m[4,5] &= \min_{4 \le k < 5} \{m[4,4] + m[5,5] + p_3 \cdot p_4 \cdot p_5 \\ &= 0 + 0 + 12 \cdot 5 \cdot 50 \\ &= 3000 \end{aligned} \tag{4}$$

- For $m[5, 6]$, we know that $k = 5$ is the only possible value. Thus:

$$\begin{aligned} m[5,6] &= \min_{5 \le k < 6} \{m[5,5] + m[6,6] + p_4 \cdot p_5 \cdot p_6 \\ &= 0 + 0 + 5 \cdot 50 \cdot 6 \\ &= 1500 \end{aligned} \tag{5}$$

At this stage in the problem, we have:

$$m = \begin{bmatrix} 0 & 150 & & & & \\ - & 0 & 360 & & & \\ - & - & 0 & 180 & & \\ - & - & - & 0 & 3000 & \\ - & - & - & - & 0 & 1500 \\ - & - & - & - & - & 0 \end{bmatrix}$$

Now to start solving for cases where $j = i + 2$:

- For $m[1,3]$, we know that $k = 1, 2$. Thus:

$$\begin{aligned} m[1,3] &= \min_{1 \le k < 3} \{m[1,k] + m[k+1,3] + p_0 \cdot p_k \cdot p_3 \\ &= \min_{1 \le k < 3} \begin{cases} m[1,1] + m[2,3] + p_0 \cdot p_1 \cdot p_3, & k = 1 \\ m[1,2] + m[3,3] + p_0 \cdot p_2 \cdot p_3, & k = 2 \end{cases} \\ &= \min_{1 \le k < 3} \begin{cases} 0 + 360 + 5 \cdot 10 \cdot 12 \\ 150 + 0 + 5 \cdot 3 \cdot 12 \end{cases} \\ &= \min_{1 \le k < 3} \begin{cases} 960 \\ 330 \end{cases} \\ &= 330 \end{aligned} \tag{6}$$

- For $m[2,4]$, we know that $k = 2, 3$. Thus:

$$\begin{aligned} m[2,4] &= \min_{2 \le k < 4} \{m[2,k] + m[k+1,4] + p_1 \cdot p_k \cdot p_4 \\ &= \min_{2 \le k < 4} \begin{cases} m[2,2] + m[3,4] + p_1 \cdot p_2 \cdot p_4, & k = 2 \\ m[2,3] + m[4,4] + p_1 \cdot p_3 \cdot p_4, & k = 3 \end{cases} \\ &= \min_{2 \le k < 4} \begin{cases} 0 + 180 + 10 \cdot 3 \cdot 5 \\ 360 + 0 + 10 \cdot 12 \cdot 5 \end{cases} \\ &= \min_{2 \le k < 4} \begin{cases} 330 \\ 960 \end{cases} \\ &= 330 \end{aligned} \tag{7}$$

- For $m[3,5]$, we know that $k = 3, 4$. Thus:

$$\begin{aligned} m[3,5] &= \min_{3 \le k < 5} \{m[3,k] + m[k+1,5] + p_2 \cdot p_k \cdot p_5 \\ &= \min_{3 \le k < 5} \begin{cases} m[3,3] + m[4,5] + p_2 \cdot p_3 \cdot p_5, & k = 3 \\ m[3,4] + m[5,5] + p_2 \cdot p_4 \cdot p_5, & k = 4 \end{cases} \\ &= \min_{3 \le k < 5} \begin{cases} 0 + 3000 + 3 \cdot 12 \cdot 50 \\ 180 + 0 + 3 \cdot 5 \cdot 50 \end{cases} \\ &= \min_{3 \le k < 5} \begin{cases} 4800 \\ 930 \end{cases} \\ &= 930 \end{aligned} \tag{8}$$

- For $m[4,6]$, we know that $k = 4, 5$. Thus:

$$
\begin{aligned}
m[4,6] &= \min_{4 \le k < 6} \{m[4,k] + m[k+1,6] + p_3 \cdot p_k \cdot p_6 \\
&= \min_{4 \le k < 6} \begin{cases} m[4,4] + m[5,6] + p_3 \cdot p_4 \cdot p_6, & k = 4 \\ m[4,5] + m[6,6] + p_3 \cdot p_5 \cdot p_6, & k = 5 \end{cases} \\
&= \min_{4 \le k < 6} \begin{cases} 0 + 1500 + 12 \cdot 5 \cdot 6 \\ 3000 + 0 + 12 \cdot 50 \cdot 6 \end{cases} \\
&= \min_{4 \le k < 6} \begin{cases} 1860 \\ 6600 \end{cases} \\
&= 1860
\end{aligned}
\tag{9}
$$

At this stage in the problem, we have:

$$
m = \begin{bmatrix}
0 & 150 & 330 & & & \\
- & 0 & 360 & 330 & & \\
- & - & 0 & 180 & 930 & \\
- & - & - & 0 & 3000 & 1860 \\
- & - & - & - & 0 & 1500 \\
- & - & - & - & - & 0
\end{bmatrix}
$$

Now to start solving for cases where $j = i + 3$:

- For $m[1,4]$, we know that $k = 1, 2, 3$. Thus:

$$
\begin{aligned}
m[1,4] &= \min_{1 \le k < 4} \{m[1,k] + m[k+1,4] + p_0 \cdot p_k \cdot p_4 \\
&= \min_{1 \le k < 4} \begin{cases} m[1,1] + m[2,4] + p_0 \cdot p_1 \cdot p_4, & k = 1 \\ m[1,2] + m[3,4] + p_0 \cdot p_2 \cdot p_4, & k = 2 \\ m[1,3] + m[4,4] + p_0 \cdot p_3 \cdot p_4, & k = 3 \end{cases} \\
&= \min_{1 \le k < 4} \begin{cases} 0 + 330 + 5 \cdot 10 \cdot 5 \\ 150 + 180 + 5 \cdot 3 \cdot 5 \\ 330 + 0 + 5 \cdot 12 \cdot 5 \end{cases} \\
&= \min_{1 \le k < 4} \begin{cases} 580 \\ 405 \\ 630 \end{cases} \\
&= 405
\end{aligned}
\tag{10}
$$

- For $m[2,5]$, we know that $k = 2,3,4$. Thus:

$$
\begin{aligned}
m[2,5] &= \min_{2\leq k<5} \{m[2,k] + m[k+1,5] + p_1 \cdot p_k \cdot p_5 \\
&= \min_{2\leq k<5} \begin{cases} m[2,2] + m[3,5] + p_1 \cdot p_2 \cdot p_5, & k=2 \\ m[2,3] + m[4,5] + p_1 \cdot p_3 \cdot p_5, & k=3 \\ m[2,4] + m[5,5] + p_1 \cdot p_4 \cdot p_5, & k=4 \end{cases} \\
&= \min_{2\leq k<5} \begin{cases} 0 + 930 + 10 \cdot 3 \cdot 50 \\ 360 + 3000 + 10 \cdot 12 \cdot 50 \\ 330 + 0 + 10 \cdot 5 \cdot 50 \end{cases} \\
&= \min_{2\leq k<5} \begin{cases} 2430 \\ 9360 \\ 2830 \end{cases} \\
&= 2430
\end{aligned} \tag{11}
$$

- For $m[3,6]$, we know that $k = 3,4,5$. Thus:

$$
\begin{aligned}
m[3,6] &= \min_{3\leq k<6} \{m[3,k] + m[k+1,6] + p_2 \cdot p_k \cdot p_6 \\
&= \min_{3\leq k<6} \begin{cases} m[3,3] + m[4,6] + p_2 \cdot p_3 \cdot p_6, & k=3 \\ m[3,4] + m[5,6] + p_2 \cdot p_4 \cdot p_6, & k=4 \\ m[3,5] + m[6,6] + p_2 \cdot p_5 \cdot p_6, & k=5 \end{cases} \\
&= \min_{3\leq k<6} \begin{cases} 0 + 1860 + 3 \cdot 12 \cdot 6 \\ 180 + 1500 + 3 \cdot 5 \cdot 6 \\ 930 + 0 + 3 \cdot 50 \cdot 6 \end{cases} \\
&= \min_{3\leq k<6} \begin{cases} 2076 \\ 1770 \\ 1830 \end{cases} \\
&= 1770
\end{aligned} \tag{12}
$$

At this stage in the problem, we have:

$$
m = \begin{bmatrix} 0 & 150 & 330 & 405 & & \\ - & 0 & 360 & 330 & 2430 & \\ - & - & 0 & 180 & 930 & 1770 \\ - & - & - & 0 & 3000 & 1860 \\ - & - & - & - & 0 & 1500 \\ - & - & - & - & - & 0 \end{bmatrix}
$$

Now to start solving for cases where $j = i + 4$:

- For $m[1,5]$, we know that $k = 1,2,3,4$. Thus:

$$
\begin{aligned}
m[1,5] &= \min_{1\leq k<5} \{m[1,k] + m[k+1,5] + p_0 \cdot p_k \cdot p_5 \\
&= \min_{1\leq k<5} \begin{cases} m[1,1] + m[2,5] + p_0 \cdot p_1 \cdot p_5, & k=1 \\ m[1,2] + m[3,5] + p_0 \cdot p_2 \cdot p_5, & k=2 \\ m[1,3] + m[4,5] + p_0 \cdot p_3 \cdot p_5, & k=3 \\ m[1,4] + m[5,5] + p_0 \cdot p_4 \cdot p_5, & k=4 \end{cases} \\
&= \min_{1\leq k<5} \begin{cases} 0 + 2430 + 5 \cdot 10 \cdot 50 \\ 150 + 930 + 5 \cdot 3 \cdot 50 \\ 330 + 3000 + 5 \cdot 12 \cdot 50 \\ 405 + 0 + 5 \cdot 5 \cdot 50 \end{cases} \\
&= \min_{1\leq k<5} \begin{cases} 4930 \\ 1830 \\ 6330 \\ 1655 \end{cases} \\
&= 1655
\end{aligned} \tag{13}
$$

- For $m[2,6]$, we know that $k = 2,3,4,5$. Thus:

$$
\begin{aligned}
m[2,6] &= \min_{2\leq k<6} \{m[2,k] + m[k+1,6] + p_1 \cdot p_k \cdot p_6 \\
&= \min_{2\leq k<6} \begin{cases} m[2,2] + m[3,6] + p_1 \cdot p_2 \cdot p_6, & k=2 \\ m[2,3] + m[4,6] + p_1 \cdot p_3 \cdot p_6, & k=3 \\ m[2,4] + m[5,6] + p_1 \cdot p_4 \cdot p_6, & k=4 \\ m[2,5] + m[6,6] + p_1 \cdot p_5 \cdot p_6, & k=5 \end{cases} \\
&= \min_{2\leq k<6} \begin{cases} 0 + 1770 + 10 \cdot 3 \cdot 6 \\ 360 + 1860 + 10 \cdot 12 \cdot 6 \\ 330 + 1500 + 10 \cdot 5 \cdot 6 \\ 2430 + 0 + 10 \cdot 50 \cdot 6 \end{cases} \\
&= \min_{2\leq k<6} \begin{cases} 1950 \\ 2940 \\ 2130 \\ 5430 \end{cases} \\
&= 1950
\end{aligned} \tag{14}
$$

At this stage in the problem, we have:

$$
m = \begin{bmatrix} 0 & 150 & 330 & 405 & 1655 & \\ - & 0 & 360 & 330 & 2430 & 1950 \\ - & - & 0 & 180 & 930 & 1770 \\ - & - & - & 0 & 3000 & 1860 \\ - & - & - & - & 0 & 1500 \\ - & - & - & - & - & 0 \end{bmatrix}
$$

Now all we must solve is for the case when $j = i + 5$:

- For $m[1,6]$, we know that $k = 1, 2, 3, 4, 5$. Thus:

$$
\begin{aligned}
m[1,6] &= \min_{1 \le k < 6} \{m[1,k] + m[k+1,6] + p_0 \cdot p_k \cdot p_6 \\
&= \min_{1 \le k < 6} \begin{cases} m[1,1] + m[2,6] + p_0 \cdot p_1 \cdot p_6, & k = 1 \\ m[1,2] + m[3,6] + p_0 \cdot p_2 \cdot p_6, & k = 2 \\ m[1,3] + m[4,6] + p_0 \cdot p_3 \cdot p_6, & k = 3 \\ m[1,4] + m[5,6] + p_0 \cdot p_4 \cdot p_6, & k = 4 \\ m[1,5] + m[6,6] + p_0 \cdot p_5 \cdot p_6, & k = 5 \end{cases} \\
&= \min_{1 \le k < 6} \begin{cases} 0 + 1950 + 5 \cdot 10 \cdot 6 \\ 150 + 1770 + 5 \cdot 3 \cdot 6 \\ 330 + 1860 + 5 \cdot 12 \cdot 6 \\ 405 + 1500 + 5 \cdot 5 \cdot 6 \\ 1655 + 0 + 5 \cdot 50 \cdot 6 \end{cases} \\
&= \min_{1 \le k < 6} \begin{cases} 2250 \\ 2010 \\ 2550 \\ 2055 \\ 3155 \end{cases} \\
&= 2010
\end{aligned} \tag{15}
$$

At this stage in the problem, we have:

$$
\text{m=} \begin{bmatrix} 0 & 150 & 330 & 405 & 1655 & 2010 \\ - & 0 & 360 & 330 & 2430 & 1950 \\ - & - & 0 & 180 & 930 & 1770 \\ - & - & - & 0 & 3000 & 1860 \\ - & - & - & - & 0 & 1500 \\ - & - & - & - & - & 0 \end{bmatrix}
$$

Now that we have found the matrix m, we can construct matrix s from the k-value of each entry in m that provided the minimum value. Thus, we have:

$$
\text{s=} \begin{bmatrix} - & 1 & 2 & 2 & 4 & 2 \\ - & - & 2 & 2 & 2 & 2 \\ - & - & - & 3 & 4 & 4 \\ - & - & - & - & 4 & 4 \\ - & - & - & - & - & 5 \\ - & - & - & - & - & - \end{bmatrix}
$$

Each entry of $s[i,j]$ records a value of k such that an optimal parenthesization of $A_1 A_{i+1} \ldots A_j$ splits the product between A_k and A_{k+1}, and thus the final matrix multiplication for computing $A_{1..n}$ optimally is $A_{1..s[1,n]} A_{s[1,n]+1..n}$. Since, in our case, $n = 6$, we know $A_{1..s[1,6]} A_{s[1,6]+1..6} = A_{1..2} A_{2+1..6} = A_{1..2} A_{3..6}$. Thus, the first parenthesization is $(A_1 A_2)(A_{3..6})$. Now for the parenthesization for $A_{3..6}$. We have $A_{3..s[3,6]} A_{s[3,6]+1..6} = A_{3..4} A_{4+1..6} = (A_3 A_4)(A_5 A_6)$. Therefore, the optimal parenthesization for the matrix-chain whose sequence of dimensions is $\langle 5, 10, 3, 12, 5, 50, 6 \rangle$ is $((A_1 A_2)((A_3 A_4)(A_5 A_6)))$