CS528 Homework Zero

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Solve the following recurrence relations:

1. $x(n) = x(n-1) + 5$ for $n > 1$, $x(1) = 0$

   (a) Characteristic Equation Method:

   To solve this relation, first we separate the equation into the homogeneous part and the non-homogeneous part. First we will get the characteristic equation of the homogeneous part. Observe:

$$\begin{aligned} x(n) &= x(n-1) + 5 && \text{Relation} \\ x(n) - x(n-1) &= 5 \\ x(n) - x(n-1) &= 0 && \text{Homogeneous Part} \\ x - 1 &= 0 && \text{Homogeneous Characteristic Equation} \end{aligned} \tag{1}$$

   Therefore the homogeneous characteristic equation is $(x-1) = 0$. Now we will determine the characteristic equation for the hon-homogeneous part. We must get the right side of the equation in the form of $b^n p(n)$ where b is a constant and $p(n)$ is a polynomial in n of degree d. Since the right side of the equation is 5, we have $1^n 5$, where $p(n) = 5$ is a polynomial of degree 0. We can turn the right hand side to $(x-b)^{d+1}$, so in our case we have $(x-1)^1$, which is the non-homogeneous characteristic equation. Multiplying the homogeneous characteristic equation with the non-homogeneous characteristic equation, and get:

$$(x-1)(x-1) = 0$$

   The roots of this equation are $r_1 = r_2 = 1$, so our solution is in the form of:

$$x(n) = c_1(r_1)^n + c_2 n(r_2)^n = c_1 + c_2 n$$

   We know $x(1) = 0 = c_1 + c_2$, and from $x(1) = 0$, we can use the initial definition to calculate $x(2) = x(2-1) + 5 = x(1) + 5 = 0 + 5 = 5$. Thus we have the following relationship:

$$\begin{aligned} 0 &= c_1 + c_2 \\ 5 &= c_1 + 2c_2 \end{aligned} \tag{2}$$

Solving the relationship:

$$\begin{aligned} c_1 &= -c_2 \\ 5 &= -c_2 + 2c_2 \\ 5 &= c_2 \\ c_1 &= -5 \end{aligned} \tag{3}$$

Thus we have the solution:

$$x(n) = -5 + 5n = 5(n-1)$$

(b) Backward Substitution Method:

We have the following relationship:

$$\begin{aligned} x(1) &= 0 \\ x(n) &= x(n-1) + 5 \end{aligned} \tag{4}$$

We can rewrite this relationship to the following:

$$\begin{aligned} x(n-1) &= x(n-2) + 5 \\ x(n-2) &= x(n-3) + 5 \\ ... &= ... \end{aligned} \tag{5}$$

Now, using backward substitution, we get the following:

$$\begin{aligned} x(n) &= x(n-1) + 5 \\ &= x(n-2) + 5 + 5 \\ &= x(n-3) + 5 + 5 + 5 \\ &= ... \\ &= x(n-(n-1)) + (n-1)5 \\ &= x(1) + (n-1)5 \\ &= 0 + 5(n-1) \\ &= 5(n-1) \end{aligned} \tag{6}$$

Thus we have the general solution that solves the recurrence relation is:

$$x(n) = 5(n-1)$$

2. $x(n) = 3x(n-1)$ for $n > 1$, $x(1) = 4$

(a) Characteristic Equation Method:

This equation is a linear homogeneous equation, so we will solve the relation using the linear characteristic method.

$$\begin{aligned} x(n) &= 3x(n-1) && \text{Relation} \\ x(n) - 3x(n-1) &= 0 && \text{Homogeneous Part} \\ x - 3 &= 0 && \text{Homogeneous Characteristic Equation} \end{aligned} \tag{7}$$

Therefore the homogeneous characteristic equation is $(x-3) = 0$. The root of this equation is $r_1 = 3$, so our solution is in the form of:

$$x(n) = c_1(r_1)^n = c_1 3^n$$

We know $x(1) = 4 = c_1 3^1 = c_1 3$. Using simple arithmetic, we know that $c_1 = \frac{4}{3}$. Thus we have the solution:

$$x(n) = \frac{4}{3}3^n = 4(3)^{n-1}$$

(b) Backward Substitution Method:

We have the following relationship:

$$\begin{aligned} x(1) &= 4 \\ x(n) &= 3x(n-1) \end{aligned} \tag{8}$$

We can rewrite this relationship to the following:

$$\begin{aligned} x(n-1) &= 3x(n-2) \\ x(n-2) &= 3^2x(n-3) \\ x(n-3) &= 3^3x(n-4) \\ ... &= ... \end{aligned} \tag{9}$$

Now, using backward substitution, we get the following:

$$\begin{aligned} x(n) &= 3x(n-1) \\ &= 3^2x(n-2) \\ &= 3^3x(n-3) \\ &= ... \\ &= 3^{n-1}x(n-(n-1)) \\ &= 3^{n-1}x(1) \\ &= 3^{n-1}4 \end{aligned} \tag{10}$$

Thus we have the general solution that solves the recurrence relation is:

$$x(n) = 4(3)^{n-1}$$

3. $x(n) = x(n-1) + n$ for $n > 0$, $x(0) = 0$

(a) Characteristic Equation Method:

To solve this relation, first we separate the equation into the homogeneous part and the non-homogeneous part. First we will get the characteristic equation of the homogeneous part. Observe:

$$\begin{aligned} x(n) = x(n-1) + n & \qquad \text{Relation} \\ x(n) - x(n-1) = n & \\ x(n) - x(n-1) = 0 & \qquad \text{Homogeneous Part} \\ (x-1) = 0 & \qquad \text{Homogeneous Characteristic Equation} \end{aligned} \tag{11}$$

Therefore the homogeneous characteristic equation is $(x-1) = 0$. Now we will determine the characteristic equation for the hon-homogeneous part. We must get the right side of the equation in the form of $b^n p(n)$ where b is a constant and $p(n)$ is a polynomial in n of degree d. Since the right side of the equation is n, we have $1^n n$, where $p(n) = n$ is a polynomial of degree 1. We can turn the right hand side to $(x-b)^{d+1}$, so in our case we have $(x-1)^2$, which is the non-homogeneous characteristic equation. Multiplying the homogeneous characteristic equation with the non-homogeneous characteristic equation, and get:

$$(x-1)(x-1)^2 = 0$$

The roots of this equation are $r_1 = r_2 = r_3 = 1$, so our solution is in the form of:

$$x(n) = c_1(r_1)^n + c_2 n(r_2)^n + c_3 n^2 (r_3)^n = c_1 + c_2 n + c_3 n^2$$

We know $x(0) = 0 = c_1$, and from $x(0) = 0$, we can use the initial definition to calculate $x(1) = x(1-1) + n = x(0) + 1 = 0 + 1 = 1$ and $x(2) = x(2-1) + x(1) + 2 = 1 + 2 = 3$ Thus we have the following relationship:

$$\begin{aligned} 0 &= c_1 \\ 1 &= c_2 + c_3 \\ 3 &= 2c_2 + 4c_3 \end{aligned} \tag{12}$$

Solving the relationship:

$$\begin{aligned} c_1 &= 0 \\ c_3 &= 1 - c_2 \\ 3 &= 2c_2 + 4(1 - c_2) \\ 3 &= 2c_2 + 4 - 4c_2 \\ -1 &= -2c_2 \\ c_2 &= \frac{1}{2} \\ 1 &= \frac{1}{2} + c_3 \\ c_3 &= \frac{1}{2} \end{aligned} \tag{13}$$

Thus we have the solution:

$$x(n) = 0 + \frac{1}{2}n + \frac{1}{2}n^2 = \frac{n(n+1)}{2}$$

(b) Backward Substitution Method:

We have the following relationship:

$$\begin{aligned} x(0) &= 0 \\ x(n) &= x(n-1) + n \end{aligned} \tag{14}$$

We can rewrite this relationship to the following:

$$\begin{aligned} x(n-1) &= x(n-2) + n \\ x(n-2) &= x(n-3) + n \\ ... &= ... \end{aligned} \tag{15}$$

Now, using backward substitution, we get the following:

$$\begin{aligned} x(n) &= x(n-1) + n \\ &= x(n-2) + (n-1) + n \\ &= x(n-3) + (n-2) + (n-1) + n \\ &= ... \\ &= x(n-n) + 1 + ... + n \\ &= x(0) + 1 + ... + n \\ &= 0 + 1 + 2 + 3 + ... + n \\ &= \frac{n(n+1)}{2} \end{aligned} \tag{16}$$

Thus we have the general solution that solves the recurrence relation is:

$$x(n) = \frac{n(n+1)}{2}$$

4. $x(n) = x(\frac{n}{2}) + n$ for $n > 1$, $x(1) = 1$, solve for $n = 2^k$

(a) Master Theorem Method:

The master theorem states that if we have a recurrence relationship in the form:

$$T(n) = aT(\frac{n}{b}) + f(n), T(1) = c$$

for some $n = b^k$, $k = 1, 2, ...$ where $a \geq 1$, $b \geq 2$, $c > 0$, and $f(n) \in \Theta(n^d)$ where $d \geq 0$, then:

$$T(n) = \left\{ \begin{array}{ll} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \text{log} n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{array} \right\}$$

We have the following:

$$x(n) = x(\frac{n}{2}) + n$$

So in our case, $a = 1$, $b = 2$, and $d = 1$. Since $1 < 2^1$, we know by the Master Theorem that the runtime of is $T(n) \in \Theta(n)$. The Master Theorem also states:

$$T(n) = \left\{ \begin{array}{ll} n^{\log_b(a)} \left[T(1) + \frac{b^d}{a} \frac{\frac{b^d}{a}^{\log_b(n)} - 1}{\frac{b^d}{a} - 1} \right] & \text{if } b^d \neq a \\ n^{\log_b(a)} \left[T(1) + \log_b(n) \right] & \text{if } b^d = a \end{array} \right\}$$

solves the recurrence relation. Plugging in our a, b, and d values, we get:

$$\begin{aligned} T(n) &= n^{\log_2(1)} \left[T(1) + \frac{2^1}{1} \frac{\frac{2^1}{1}^{\log_2(n)} - 1}{\frac{2^1}{1} - 1} \right] \\ &= n^0 \left[1 + 2 \frac{2^{\log_2(n)} - 1}{2 - 1} \right] \\ &= 1 + 2 \frac{2^{\log_2(n)} - 1}{1} \\ &= 1 + 2 \left(2^{\log_2(n)} - 1 \right) \\ &= 1 + 2 (n - 1) \\ &= 1 + 2n - 2 \\ &= 2n - 1 \end{aligned} \tag{17}$$

Therefore, the solution for the recurrence relation is:

$$x(n) = 2n - 1$$

(b) Backward Substitution Method:

We have the following relationship:

$$\begin{aligned} x(1) &= 1 \\ x(n) &= x(\frac{n}{2}) + n \end{aligned} \tag{18}$$

where $n = 2^k$. We can rewrite this relationship to the following:

$$\begin{aligned} x(\frac{n}{2}) &= x(\frac{n}{n^2}) + \frac{n}{2} \\ x(\frac{n}{4}) &= x(\frac{n}{n^4}) + \frac{n}{4} \\ ... &= ... \end{aligned} \tag{19}$$

Now, using backward substitution and $n = 2^k$, we get the following:

$$\begin{aligned} x(2^k) &= x(2^{k-1}) + 2^k \\ x(2^{x-1}) &= x(2^{k-2}) + 2^{k-1} + 2^k \\ ... &= ... \\ x(2^{k-k}) &= x(2^{k-k}) + 2^{k-(k-1)} + ... + 2^k \\ &= x(1) + 2^{k-(k-1)} + ... + 2^k \\ &= x(1) + 2^1 + 2^2 + 2^2 + ... + 2^k \\ &= 2^0 + 2^1 + 2^2 + ... + 2^k \\ &= 2^{k+1} - 1 \\ &= 2(2)^k - 1 \end{aligned} \tag{20}$$

We can then substitute the definition of $n = 2^k$ back into our solution, and thus we have the general solution that solves the recurrence relation:

$$x(n) = 2n - 1$$

5. $x(n) = x(\frac{n}{3}) + 1$ for $n > 1$, $x(1) = 1$, solve for $n = 3^k$

(a) Master Theorem Method:

Using the Master Theorem, we have $a = 1$, $b = 3$, and $d = 0$. Since $1 = 3^0$, we know by the Master Theorem that the runtime of is $T(n) \in \Theta(\log n)$. The Master Theorem also states:

$$T(n) = \left\{ \begin{array}{ll} n^{\log_b(a)} \left[T(1) + \frac{b^d}{a} \frac{\frac{b^d}{a}^{\log_b(n)} - 1}{\frac{b^d}{a} - 1} \right] & \text{if } b^d \neq a \\ n^{\log_b(a)} \left[T(1) + \log_b(n) \right] & \text{if } b^d = a \end{array} \right\}$$

solves the recurrence relation. Plugging in our a, b, and d values, we get:

$$\begin{aligned} T(n) &= n^{\log_3(1)} \left[T(1) + \log_3(n) \right] \\ &= 1 + \log_3 n \end{aligned} \tag{21}$$

Therefore, the solution for the recurrence relation is:

$$x(n) = 1 + \log_3(n)$$

(b) Backward Substitution Method:

We have the following relationship:

$$\begin{aligned} x(1) &= 1 \\ x(n) &= x(\frac{n}{3}) + 1 \end{aligned} \tag{22}$$

where $n = 3^k$, or $k = \log_3(n)$. We can rewrite this relationship to the following:

$$\begin{aligned} x(\frac{n}{3}) &= x(\frac{n}{3^2}) + 1 \\ x(\frac{n}{9}) &= x(\frac{n}{3^3}) + 1 \\ ... &= ... \end{aligned} \tag{23}$$

Now, using backward substitution and $n = 3^k$, we get the following:

$$\begin{aligned} x(3^k) &= x(3^{k-1}) + 1 \\ x(3^{x-1}) &= x(3^{k-2}) + 1 + 1 \\ ... &= ... \\ x(3^{k-(k-1)}) &= x(3^{k-k}) + 1 + ... + 1 \\ &= x(1) + k \\ &= 1 + k \end{aligned} \tag{24}$$

We can then substitute the definition of $k = \log_3 n$ back into our solution, and thus we have the general solution that solves the recurrence relation:

$$x(n) = 1 + \log_3(n)$$