CS528 Homework One

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Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures:

1. $f(n) = O(g(n))$ implies $g(n) = O(f(n))$
   Counterexample: Let $f(n) = n$ and $g(n) = n^2$. We know that $f(n) = O(g(n))$ is true because $n = O(n^2)$ since O-notation doesn't specify asymptotically tight upper bounds, just an asymptotic upper bound. We can clearly see, however, that $n^2 = O(n)$ is false because there is no constant multiplier c such that $0 \leq n^2 \leq cn$ for sufficiently large $n \geq n_0$, therefore $f(n) = O(g(n))$ implies $g(n) = O(f(n))$ is not a true statement.

2. $f(n) + g(n) = \Theta(\min(f(n), g(n)))$
   Counterexample: Let $f(n) = n$ and $g(n) = n^2$. By definition of Θ-notation, we know that

   $$\Theta(h(n)) = \{j(n) : \text{for some constants } c_1, c_2, \text{and } n_0,\ 0 \leq c_1 h(n) \leq j(n) \leq c_2 h(n)\}$$

   Using our $f(n)$ and $g(n)$, we would have $n^2 + n = \Theta(\min(n, n^2)) = \Theta(n)$, resulting in the following inequality:

   $$0 \leq c_1 n \leq n^2 + n \leq c_2 n$$

   for some constants c_1, c_2, and $n \geq n_0$ for some sufficiently large n_0. This inequality is false, since no real constant c_2 makes $n^2 + n \leq c_2 n$ a true statement. Therefore, $f(n) + g(n) = \Theta(\min(f(n), g(n)))$ is false.

3. $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n

   Proof. First, let $f(n) = O(g(n))$ such that, by definition of O-notation, there exist positive constants c_1 and n_1 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_1$. We will show that this statement implies $\lg(f(n)) = O(\lg(g(n)))$ where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n. Again, by O-notation, we know that $\lg(f(n)) = O(\lg(g(n)))$ means that there exists positive constants c_2 and n_2 such that $0 \leq \lg(f(n)) \leq c \cdot \lg(g(n))$ for all $n \geq n_2$. By the properties of logarithms, we can turn this inequality to $0 \leq \lg(f(n)) \leq \lg(g(n)^c)$. To satisfy our assumption condition of $f(n) = O(g(n))$, the asymptotically tightest function that $g(n)$ can equal is $f(n)$. Thus, in the case of the tightest upper bound, we would have

$0 \leq \lg(f(n)) \leq \lg(f(n)^c)$. Since c must be a positive constant, this statement is always true. In any case where the upper limit is not as tight as possible, $f(n) \leq g(n)$ implies $\lg(f(n)) \leq \lg(g(n))$ by transpose symmetry. Therefore, in all cases, $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$. $\square$

4. $f(n) = O(g(n))$ implies $2^{f(n)} = O\left(2^{g(n)}\right)$
   Counterexample: Let $f(n) = 2n$ and let $g(n) = n$. This satisfies the initial condition of $f(n) = O(g(n))$ because there exists many c_1 such that $0 \leq 2n \leq c_1 n$ for sufficiently large values of n. Take $c_1 = 5$ for example. However, $2^{f(n)} = O\left(2^{g(n)}\right)$ by definition means that $0 \leq 2^{2n} \leq c_2 2^n$ for some constant c_2 and sufficiently large n. This algebraically simplifies to $0 \leq 4^n \leq c_2 2^n$. No constant value for c_2 would make this inequality true, therefore we have found a counterexample to $f(n) = O(g(n))$ implies $2^{f(n)} = O\left(2^{g(n)}\right)$.

5. $f(n) = O\left((f(n))^2\right)$
   Counterexample: Let $f(n) = \frac{1}{n}$. O-notation states that there exists a constant c and an n_0 such that $0 \leq \frac{1}{n} \leq c\frac{1}{n}^2$, or $0 \leq \frac{1}{n} \leq c\frac{1}{n^2}$. There is no constant value of c that makes this inequality true, therefore we have found a counterexample to $f(n) = O\left((f(n))^2\right)$.

6. $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$

   Proof. One of the properties of Transpose Symmetry is $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$. Additionally, Since the properties of Transpose Symmetry hold true for asymptotic notations, we know that, by Transpose Symmetry, this statement must be true. $\square$