CS528 Homework Two

Andrew Struthers

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Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \geq 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T\left(\frac{n}{2}\right) + n^4$

- The master theorem states that if we have a recurrence relationship in the form:

$$T(n) = aT(\frac{n}{b}) + f(n), T(1) = c$$

for some $n = b^k$, $k = 1, 2, ...$ where $a \geq 1$, $b \geq 2$, $c > 0$, and $f(n) \in \Theta(n^d)$ where $d \geq 0$, then:

$$T(n) = \left\{ \begin{array}{ll} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{array} \right\}$$

In this example, we have $a = 2$, $b = 2$, and $f(n) = n^4$ so we know that $d = 4$. Since $2 < 2^4$ in this example, we know that $T(n) = \Theta\left(n^4\right)$

b. $T(n) = T\left(\frac{7n}{10}\right) + n$

- In this example, we have $a = 1$, $b = \frac{10}{7}$, and $f(n) = n$ so we know that $d = 1$. Since $1 < \frac{10}{7}$ in this example, we know that $T(n) = \Theta\left(n\right)$

c. $T(n) = 16T\left(\frac{n}{4}\right) + n^2$

- In this example, we have $a = 16$, $b = 4$, and $f(n) = n^2$ so we know that $d = 2$. Since $16 = 4^2$ in this example, we know that $T(n) = \Theta\left(n^2 \log n\right)$

d. $T(n) = 7T\left(\frac{n}{3}\right) + n^2$

- In this example, we have $a = 7$, $b = 3$, and $f(n) = n^2$ so we know that $d = 2$. Since $7 < 3^2$ in this example, we know that $T(n) = \Theta\left(n^2\right)$

e. $T(n) = 7T\left(\frac{n}{2}\right) + n^2$

- In this example, we have $a = 7$, $b = 2$, and $f(n) = n^2$ so we know that $d = 2$. Since $7 > 2^2$ in this example, we know that $T(n) = \Theta\left(n^{\log_2 7}\right)$

f. $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$

- In this example, we have $a = 2$, $b = 4$, and $f(n) = \sqrt{n}$ so we know that $d = \frac{1}{2}$. Since $2 = 4^{\frac{1}{2}}$ in this example, we know that $T(n) = \Theta\left(n^4 \text{log} n\right)$

g. $T(n) = T(n-2) + n^2$

- In this example, we must use the characteristic method instead of the master theorem. To solve this relation, first we separate the equation into the homogeneous part and the non-homogeneous part. First we will get the characteristic equation of the homogeneous part. Observe:

$$\begin{aligned} T(n) &= T(n-2) + n^2 \\ T(n) - T(n-2) &= n^2 \\ r^2 - 1 = n^2(r-1)(r+1) &\quad = n^2 \text{Homogeneous Characteristic Equation} \end{aligned} \tag{1}$$

Therefore the homogeneous characteristic equation is $(t-1) = 0$. Now we will determine the characteristic equation for the hon-homogeneous part. We must get the right side of the equation in the form of $b^n p(n)$ where b is a constant and $p(n)$ is a polynomial in n of degree d. Since the right side of the equation is n^2, we have $1^n n^2$, where $p(n) = n^2$ is a polynomial of degree 2. We can turn the right hand side to $(r-b)^{d+1}$, so in our case we have $(r-1)^3$, which is the non-homogeneous characteristic equation. Multiplying the homogeneous characteristic equation with the non-homogeneous characteristic equation, and get:

$$(r-1)(r+1)(r-1)^3 = (r+1)(r-1)^4 = 0$$

The roots of this equation are $r_1 = -1, r_2 = 1$, so our solution is in the form of:

$$T(n) = c_1(r_1)^n + c_2(r_2)^n + c_3 n^2 (r_2)^n + c_4 n^2 (r_2)^n + c_5 n^3 (r_2)^n$$

Simplifying this, we have:

$$T(n) = c_1(-1)^n + c_2 + c_3 n + c_4 n^2 + c_5 n^3$$

Since the highest term in the polynomial is n^3, we know that the asymptotic behaviour of $T(n)$ is bound by $T(n) = \Theta(n^3)$