# CS528 Midterm Exam

Andrew Struthers

October 2022

1. Analyze the following algorithms:

   (a) **for** $i \leftarrow 1$ **to** $n$ **do**
   **for** $j \leftarrow 1$ **to** 6 **do**
   **for** $k \leftarrow 1$ **to** $n$ **do**
   {constant time operation}

   The first for loop will run $n$ times. This will cause the second for loop to run $6n$ times, since the loop will run from $j = 1 \rightarrow 6$. Since the second loop runs $6n$ times, the third for loop will run $6n^2$ times, because $k = 1 \rightarrow n$ loops $n$ additional times. Therefore, the constant time operation will run $6n^2$ times. Thus, this algorithm will run in $\Theta(n^2)$ time.

   (b) **for** $i \leftarrow 1$ **to** $n$ **do**
   **for** $j \leftarrow 1$ **to** $i$ **do**
   **for** $k \leftarrow 1$ **to** $n$ **do**
   {constant time operation}

   The first loop will run $n$ times. Since the second loop is dependent on $i$ instead of n, the second loop will execute $\log(n)$ times, meaning the overall execution at this point will be $n\log n$ times. The third for loop executes a total of $n$ times, meaning the constant time operation will run $n^2\log(n)$ times. Thus, this algorithm will run in $\Theta(n^2\log(n))$ time.

   (c) **for** $i \leftarrow 1$ **to** $n$ **do**
   $j \leftarrow i$
   **while** $j > 0$ **do** $j \leftarrow j$ **div** 2
   {constant time operation}

   The for loop executes $n$ times. Since the while loop is dependent on $i$, and $j$ is getting halved each run of the while loop, the while loop will execute $\log(n)$ times. Thus, the runtime of the algorithm will be $\Theta(n\log(n))$ time.

2. Solve the following recurrences:

   (a) $t_n = 5t_{n-1} - 8t_{n-2} + 4t_{n-3}$, $n \geq 3$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$

   - We will use the characteristic method to solve this recurrence relation. First, we will turn the relation into it's homogeneous characteristic equation form. Observe:

$$\begin{aligned} t_n &= 5t_{n-1} - 8t_{n-2} + 4t_{n-3} \\ t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} &= 0 \\ (t-1)(t-2)(t-2) &= 0 \end{aligned} \tag{1}$$

     Therefore the homogeneous characteristic equation is $(t-1)(t-2)^2 = 0$. Since there is a repeated root, we will need to account for that in the general solution. The general solution will take the form:

$$t_n = c_1(r_1)^n + c_2(r_2)^n + c_3 n(r_2)^n$$

     Plugging in the known roots, we have:

$$\begin{aligned} t_n &= c_1(1)^n + c_2(2)^n + c_3 n(2)^n \\ t_n &= c_1 + c_2(2)^n + nc_3(2)^n \end{aligned} \tag{2}$$

     Now we must solve for $c_1$, $c_2$, and $c_3$. Observe:

$$\begin{aligned} t_0 = 0 &= c_1 + c_2(2)^0 + 0(c_3)(2)^0 \\ &= c_1 + c_2 \\ t_1 = 1 &= c_1 + c_2(2)^1 + 1c_3(2)^1 \\ &= c_1 + 2c_2 + 2c_3 \\ t_2 = 2 &= c_1 + c_2(2)^2 + 2c_3(2)^2 \\ &= c_1 + 4c_2 + 8c_3 \end{aligned} \tag{3}$$

     Now we can use an augmented matrix and rref to find values for $c_1$, $c_2$, and $c_3$. We can turn the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 4 & 8 & 2 \end{bmatrix} \tag{4}$$

     into the reduced row echelon form of

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \tag{5}$$

     Thus, $c_1 = -2$, $c_2 = 2$, and $c_3 = -\frac{1}{2}$. Plugging this into our general solution, we have the particular solution to our recurrence relation: $t_n = -2 + 2(2)^n - \frac{1}{2}n(2)^n$

(b) $t_n = 2t_{n-1} + 3^n$, $n \geq 1$, $t_0 = 0$

- We will use the characteristic method to solve this recurrence relation. First, we will turn the relation into it's homogeneous characteristic equation form. Observe:

$$\begin{aligned} t_n &= 2t_{n-1} + 3^n \\ t_n - 2t_{n-1} &= 3^n \\ (t-2) &= 3^n \end{aligned} \tag{6}$$

Now we must solve for the non-homogeneous characteristic solution. We must get the right side of the equation in the form $b^n p(n)$ where $b$ is a constant and $p(n)$ is a polynomial of degree $d$. We have $3^n \to 3^n(1)$ where $p(n) = 1$, thus having degree $d = 0$. Now that we have the right side in the proper form, we can multiply the homogeneous solution by $(x-b)^{d+1}$, or in our case $(x-3)^1$, leaving us with the characteristic equation:

$$(x-2)(x-3) = 0$$

Therefore, the general solution will take the form:

$$t_n = c_1(r_1)^n + c_2(r_2)^n$$

Plugging in the known roots, we have:

$$t_n = c_1(2)^n + c_2(3)^n \tag{7}$$

Now we must solve for $c_1$ and $c_2$. Observe:

$$\begin{aligned} t_0 = 0 &= c_1(2)^0 + c_3(3)^0 \\ &= c_1 + c_2 \\ t_1 = 3 &= c_1(2)^1 + c_2(3)^1 \\ &= 2c_1 + 3c_2 \end{aligned} \tag{8}$$

We can see that $c_1 = -c_2$ from the above relationship. Using substitution, we have:

$$\begin{aligned} 3 &= 2(-c_2) + 3c_2 \\ &= c_2 \end{aligned} \tag{9}$$

Therefore we know that $c_2 = 3$ and by the above relationship, $c_1 = -3$. Thus, we have the particular solution to our recurrence relation: $t_n = -3(2)^n + 3(3)^n$

(c) $T(n) = 3T\left(\frac{n}{2}\right) + n$, $n > 1$, $T(0) = 0$, $T(1) = 1$, for $n = 2^k$

- We will use the Master Theorem to solve this recurrence relation. From our relation, we know that $a = 3$ and $b = 2$. The Master Theorem states that $f(n) \in \Theta(n^d)$ where, in our case, $f(n) = n$, thus $d = 1$. Since $3 > 2^1$, we know that $T(n) \in \Theta\left(n^{\log_b a}\right)$. We also know, from the Master Theorem, that

$$T(n) = n^{\log_b(a)}\left[T(1) + \frac{b^d}{a}\frac{\frac{b^d}{a}^{\log_b(n)} - 1}{\frac{b^d}{a} - 1}\right]$$

solves the recurrence relation. Plugging in our a, b, and d values, we get:

$$\begin{aligned}
T(n) &= n^{\log_2(3)}\left[1 + \frac{2^1}{3}\frac{\frac{2^1}{3}^{\log_2(n)} - 1}{\frac{2^1}{3} - 1}\right] \\
&= n^{\log_2(3)}\left[1 + \frac{2}{3}\frac{\frac{2}{3}^{\log_2(n)} - 1}{-\frac{1}{3}}\right] \\
&= n^{\log_2(3)}\left[1 + \frac{2}{3}\left(-3\left(\frac{2}{3}^{\log_2(n)} - 1\right)\right)\right] \\
&= n^{\log_2(3)}\left[1 - 2\left(\frac{2}{3}^{\log_2(n)} - 1\right)\right] \\
&= n^{\log_2(3)}\left[1 - 2\left(\frac{2}{3}\right)^{\log_2(n)} + 2\right] \\
&= n^{\log_2(3)}\left[3 - 2\left(\frac{2}{3}\right)^{\log_2(n)}\right] \\
&= 3n^{\log_2(3)} - 2n^{\log_2(3)}\left(\frac{2}{3}\right)^{\log_2(n)} \\
&= 3n^{\log_2(3)} - 2n^{\log_2(3)}n^{\log_2\left(\frac{2}{3}\right)} \\
&= 3n^{\log_2(3)} - 2n
\end{aligned} \tag{10}$$

Since $T(n) = 3n^{\log_2(3)} - 2n$ is the most simplified form of the equation, we know that this is indeed the solution to the recurrence. Like the analysis above predicted, we can see that $T(n) \in \Theta\left(n^{log_b(a)}\right)$.

3.

Consider the following recursive algorithm for computing the sum of the first $n$ cubes: $S(n) = 1^3 + 2^3 + ... + n^3$.

**Algorithm** $S(n)$
//Input: A positive integer $n$
//Output: The sum of the first $n$ cubes
**if** $n = 1$ **return** 1
**else return** $S(n-1) + n * n * n$

Set up and solve a recurrence relation for the number of times the algorithm's basic operation is executed.

- Analyzing this algorithm, we can see that $S(n)$ will execute the basic operation $n$ times. Each recursive call adds one to the number of basic operations executed, so a recurrence relation that describes this algorithm is:

$$x(n) = x(n-1) + 1,\ x(1) = 1, \text{ for all } n \geq 1$$

To solve this recurrence relation, we will use backward substitution. We have the following relationship:

$$\begin{aligned} x(1) &= 1 \\ x(n) &= x(n-1) + 1 \end{aligned} \tag{11}$$

We can rewrite this relation to the following:

$$\begin{aligned} x(n-1) &= x(n-2) + 1 \\ x(n-2) &= x(n-3) + 1 \\ ... &= ... \\ x(n-k) &= x(n-(k-1)) + 1 \end{aligned} \tag{12}$$

where $k \geq 1$ is an integer. Using backward substitution, we get the following:

$$\begin{aligned} x(n) &= x(n-1) + 1 \\ &= x(n-2) + 1 + 1 \\ &= x(n-3) + 1 + 1 + 1 \\ &= ... \\ &= x(n-(n-1)) + 1 + ... + 1 \\ &= x(1) + (n-1) \\ &= 1 + (n-1) \\ &= n \end{aligned} \tag{13}$$

Therefore, we know the particular solution of this recursive algorithm is $x(n) = n$.