# Numerical Integration Methods

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## 1 Introduction

In this project, we were expected to use numerical integration methods to estimate the integral of

$$g(x) = \frac{1}{1+e^{-3x}}$$

on the interval $[-1, 3]$. We will use Compozite Trapezoidal and Composite Simpson's methods of numerical integration.

## 2 Analysis

First, we will calculate the integral of $g(x)$ exactly. Observe:

$$\begin{aligned} g(x) &= \int_{-1}^{3} \frac{1}{1+e^{-3x}}\, dx \\ &= \int_{-1}^{3} \frac{e^{3x}}{1+e^{3x}}\, dx \end{aligned} \tag{1}$$

Now we do u-substitution, so let $u = e^{3x} + 1$. Then $\frac{du}{dx} = 3e^{3x}$ and $dx = \frac{3^{-3x}}{3} du$. Substituting back into the equation above we have:

$$\begin{aligned} &= \int_{-1}^{3} \frac{e^{3x}}{1+e^{3x}} * \frac{e^{-3x}}{3}\, du \\ &= \frac{1}{3}\int_{-1}^{3} \frac{1}{1+e^{3x}}\, du \\ &= \frac{1}{3}\int_{-1}^{3} \frac{1}{u}\, du \end{aligned} \tag{2}$$

$$
\begin{aligned}
&= \frac{1}{3} ln(u)\Big|_{-1}^{3} \\
&= \frac{ln(u)}{3}\Big|_{-1}^{3} \\
&= \frac{ln(e^{3x}+1}{3}\Big|_{-1}^{3} \\
&= \frac{ln(e^{3(3)})}{3} - \frac{ln(e^{3(-1)})}{3} \\
&\approx 2.98384329566880
\end{aligned}
\tag{3}
$$

Looking at the error term for the Composite Trapezoid Method, we have the formula:

$$
\frac{g''(\xi)h^2(b-a)}{12}
$$

where $a \leq \xi \leq b$ and $h$ is the step size between our bounds. We will be looking at 3 cases, $n = 11, n = 41$, and then calculating a value of $n$ such that our error is less than $10^{-4}$. The second derivative of $g(x)$ has a maximum value of $\approx 0.866$. The corresponding $h$-value for 11 points is $h = 0.4$. Therefore the error is bounded by $\frac{0.866*0.4^2*(3-(-1)}{12} \approx 0.0462$. Likewise, the $h$-value for $n = 41$ is $h = 0.1$, so the error is bounded by $\frac{0.866*0.1^2*4}{12} \approx 0.00289$. Using the above error formula, we can calculate the appropriate $h$-value such that our error is less than $10^{-4}$. We get $h = \sqrt{\frac{10^{-4}*12}{(3-(-1))0.866}} \approx 0.0186$. We can get our number of points by taking $n = \frac{3-(-1)}{0.0186} + 1$ and rounding it up to the nearest whole number to get $n = 216$. This is the number of points needed to get an error less than $10^{-4}$ using the Composite Trapezoidal method.

Similarly, looking at the error term for the Composite Simpson's Method, we have the formula:

$$
\frac{g''''(\xi)h^4(b-a)}{180}
$$

where $a \leq \xi \leq b$ and $h$ is the step size between our bounds. We will again be looking at 3 cases, $n = 11, n = 41$, and then calculating a value of $n$ such that our error is less than $10^{-4}$. The fourth derivative of $g(x)$ has a maximum value of $\approx 10.342$. The corresponding $h$-value for 11 points is $h = 0.4$. Therefore the error is bounded by $\frac{10.342*0.4^4*(3-(-1)}{180} \approx 0.00588$. Likewise, the $h$-value for $n = 41$ is $h = 0.1$, so the error is bounded by $\frac{10.342*0.1^4*4}{180} \approx 0.000023$. Using the above error formula, we can calculate the appropriate $h$-value such that our error is less than $10^{-4}$. We can see from $n = 41$ that our error bound is already less than $10^{-4}$, so we should expect an $n$-value less than 41. We get $h = \sqrt[4]{\frac{10^{-4} * 180}{(3-(-1))10.342}} \approx 0.1444$. We can get our number of points by taking $n = \frac{3-(-1)}{0.1444} + 1$ and rounding it up to the nearest whole number to get $n = 29$. This is the number of points needed to get an error less than $10^{-4}$ using the Composite Simpson's method.

We know that the Composite Simpson's method will be more accurate than the Composite Trapezoid method due to the error term. We can see in the error term for the Simpson's method that it behaves like $O(h^4)$ whereas the Trapezoid error behaves like $O(h^2)$. As $h \to 0$, the error for Simpson's will go to zero faster than for Trapezoid. This means that we can get a smaller error faster using Simpson's method, so we know that the Simpson's method is more accurate. Because of this and the analysis of our step sizes and their resulting error bounds found above, I am going to predict that the Simpson's will be much better of an approximation than the Trapezoid, and it will use less points to get an answer within the tolerance.

## 3 Computer Program

In Figure 1, we can see the output of the code when computing the integral with the Composite Trapezoidal Method with $n = 11, n = 41, n = 216$ points. Notice the only value that was within tolerance was $n = 216$.

```
=-=-=-=-=-=
Composite Trapezoidal:
=-=-=-=-=-=
Required h = 0.0186121401926444
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
  n          h    Real Value    Calculated Value    Absolute Error    Relative Error   Within Tolerance
---  ---------  ------------  ------------------  ----------------  ----------------  ------------------
 11  0.4             2.98385             2.98207        0.00177056        0.0593382   False
 41  0.1             2.98385             2.98373        0.00011251        0.00377064  False
216  0.0186047       2.98385             2.98384        3.89846e-06       0.000130652 True

=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
```

Figure 1: Composite Trapezoidal Method at $n = 11, n = 41, n = 216$

In Figure 2, we can see the output of the code when computing the integral with the Composite Simpson's Method with $n = 11, n = 41, n = 29$ points.

```
=-=-=-=-=-=
Composite Simpson's:
=-=-=-=-=-=
Required h = 0.144427164687629
=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
  n         h     Real Value    Calculated Value    Absolute Error    Relative Error  Within Tolerance
---  --------  ------------  ------------------  ----------------  ----------------  ------------------
 11  0.4            2.98385              2.9841       0.000251352        0.00842375  False
 41  0.1            2.98385             2.98384       4.92819e-07       1.65162e-05  True
 29  0.142857       2.98385             2.98384       2.05454e-06       6.88553e-05  True

=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=
```

Figure 2: Composite Simpson's Method at $n = 11, n = 41, n = 29$

## 4 Results

Hello

## References

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