Lecture - Differential and Integral Calculus

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What does a function is?

First of all lets begin with one of the main problems when studying mathematics. The definition of a function:

$$X, Y \subseteq \mathbb{R} \tag{1}$$
$$f : X \to Y \tag{2}$$
$$x \mapsto f(x) = y \tag{3}$$

Let f be a function which takes a real value represented by x and transforms it in some other real value called y. We will conveniently denote f transforming x as $f(x)$ and such transformation will be equal y, hence the notation $f(x) = y$ will represent f converting an x in a y.

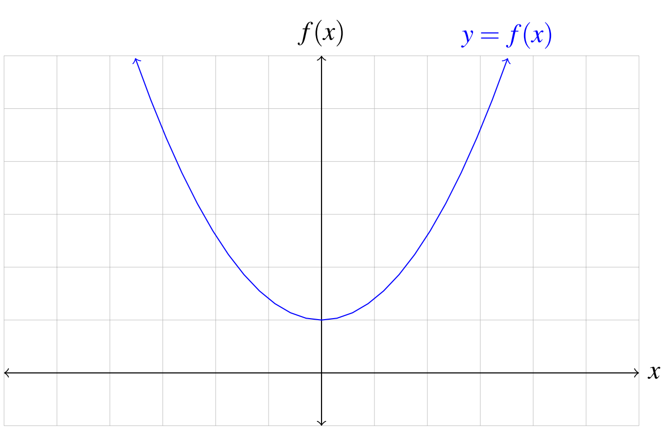
Now, the previous definition is good however its incomplete, we can also say that f is a function which transforms a set X into another set called Y, and f does it by converting each element $x \in X$ in some $y \in Y$.

Then we have defined two fundamental facts, first X is the domain of f and Y its range, and second, f is a function notated by $f(x) = y$. This will be useful for you and it will be when seeing further concepts in dept (not in this lecture of course).

Introduction to Differentiation

Differentiation is the process of finding a derivative, now what does a derivative is? It is the ratio respect the input of a variable and its result. In common words (monotone) the rate of change of the input respect its value.

The initial idea it that of measuring such ratio between two points in the function. Now let $f(x) = 0.4x^2 + 1 = y$ be our first example, it has the following graph

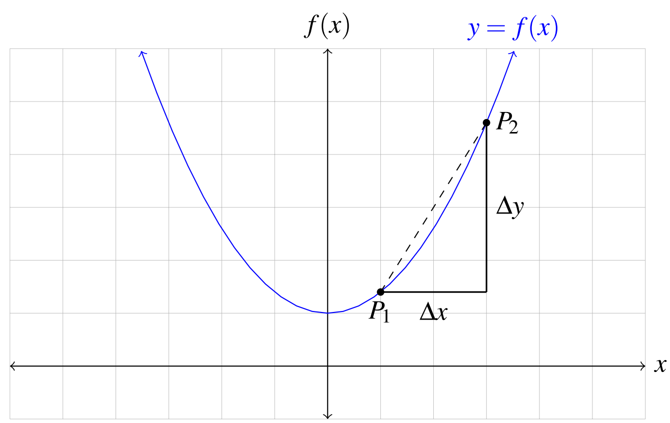


Our initial idea to find the rate of change of our function is: take two points $(x, f(x))$ and measure its difference, it is:

$$P_1 = (x_1, f(x_1)); \quad P_1(x_1, y_1) \tag{4}$$
$$P_2 = (x_2, f(x_2)); \quad P_2(x_2, y_2) \tag{5}$$

selecting arbitrarily $x_1 = 1, x_2 = 3$

$$P_1 = (1, f(1)) = (1, 0.4(1)^2 + 1) = (1, 1.4) \tag{6}$$
$$P_2 = (3, f(3)) = (3, 0.4(3)^2 + 1) = (3, 4.6) \tag{7}$$



we proceed to do the following difference:

$$m = \frac{\Delta y}{\Delta x} = \frac{P_{2y} - P_{1y}}{P_{2x} - P_{1x}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.6 - 1.4}{3 - 1} = \frac{8}{5} \tag{8}$$

and we find that the rate of change m between $f(1)$ and $f(3)$ is $8/5$. (Note: many texts mention m as the slope)

Now, that is the ratio between two inputs for f and its outcomes. What we want is to be able to formulate such rate change for a single point, it leads us to make smaller Δx and Δy values by choosing closer values of x. We can represent it as:

$$x_2 \to x_1 \text{ or } x_1 \to x_2 \tag{9}$$

x_2 tends to x_1 (or vice-versa), making those values infinitely close will reveal the instantaneous rate of change between x and y. How? Well, we first get a solution for particular cases and then extend it to more scenarios (over the same function by now).

Then, we represent getting each time closer from P_2 to P_1 as

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{10}$$

and so, we start to find the true origin of our derivatives:

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{11}$$

As $x_2 > x_1$ then we can rewrite our equation as

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1+h)-f(x_1)}{(x_1+h)-x_1}; \quad x_1+h = x_2 \tag{12}$$

and it turns out to be our famous limit

$$\frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{f(x_1+h)-f(x_1)}{h} \tag{13}$$

where naturally we would get an in-determination $\frac{0}{0}$ if we do not factorize. Additionally, given that our derivative is in terms of x_1 we can say for convenience that h is a constant which approximates zero and rewrite the derivative as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{14}$$

generalizing it for our function f.

Notations

Leibniz:

$\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f$	$\frac{d^n y}{dx^n}$, $\frac{d^n f}{dx^n}$, $\frac{d^n}{dx^n}f$	$\left.\frac{d^n y}{dx^n}\right\rvert_{x=k}$

to evaluate the result of a derivative being k a valid argument of y.

The advantage of this notation is that we can read the dependent function or variable in the numerator and the independent variable in the denominator.

Lagrange:

f',	f'',	f''',	$f^{(4)}$,	$\dots$,	$f^{(n)}$

Newton:

$\dot{y}$,	$\ddot{y}$,	$\dddot{y}$

(Note: Newton notation is sometimes used in physics)

Section Closing

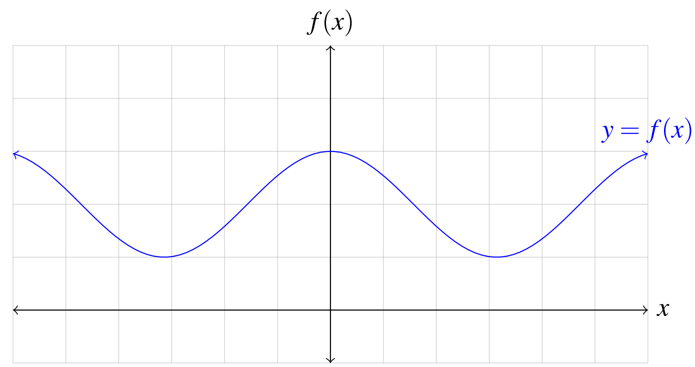
Well, here we reviewed how derivatives are found and which is the mathematical origin of the concept. Ages ago mathematicians have found many derivatives so, as a good engineer I will proceed to say, topic finished, we do not need to manually obtain the fundamental derivatives, they are all over the internet, go down in this lecture and you will find a list of main derivatives.

What is important when seeing a course of differential calculus is to take in consider the main derivatives and be able to understand those. Differentiation is simple in comparison with integration.

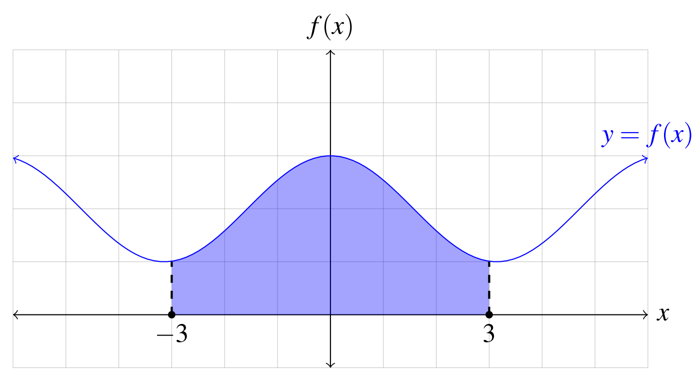
Introduction to Integration

The main idea behind integration is that of the area under the curve. What does it mean? It means that what we want from any function, the area enclosed between two points on its domain. Lets consider the function $\vartheta(\rho) = \psi + \cos(\rho) = \gamma$ where ψ is a real constant, we will rewrite this function as $y = f(x) = k + \cos x$ so that you understand that we can choose virtually any symbols we want for our functions and that would be valid.

Also, that you have not to fear the letters but the problem you are facing, the equation maybe simple but your concern about what does the symbols are will be a problem. Well, the graph of f if $k = 2$ is:

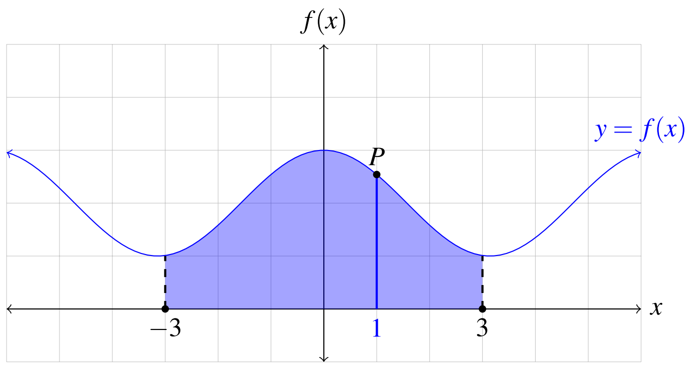


Now then, we want to calculate the area under the curve left by f in an interval $[a, b]$ of it. Suppose for this case that $a = -3, b = 3$, then



is the graphic representing which are we wish to find.

Now, we can we represent it in mathematical notation? Well, firstly we can approach the unconventional idea of seeing the result of a function as a distance, to illustrate it better, lets take $x = 1$ and consider their image on the graph



see how $f(1)$ has a value $y = 2+\cos(1)$ (x is taken by cos in radians) and it is a distance from the origin to the point $P(1, 2+\cos(1))$, here $f(1)$ is an infinitesimal part of the area of f in the interval $[-3,3]$.

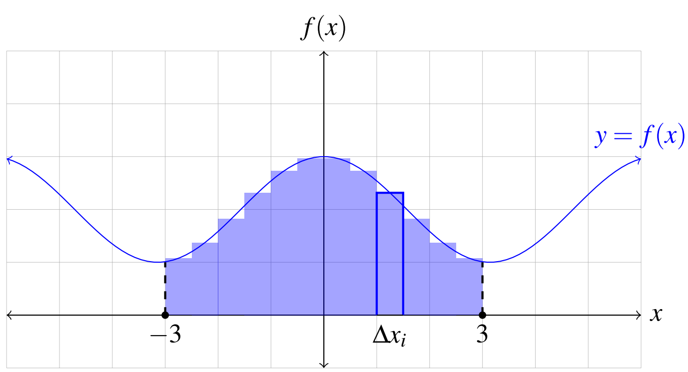
Now, we will say, what if we sum all these fragments of area, it will be a sum of all the results $f(x)$ in the interval $[-3,3]$ or:

$$A(f,I) = \sum_{x_i \in I} f(x_i); \quad I = [-3,3] \tag{15}$$

However it is impossible because there are infinitely uncountable numbers between two reals. Fortunately here is still another approach which reassembles our derivatives:

$$S(f) \approx \sum_{i=1}^{N} f(x_i^*)\Delta x_i \tag{16}$$

where $\Delta x_i = x_i - x_{i-1}$ is the width of a sub-interval, N is the number of sub-intervals from $[a,b]$ and $x_i^* \in [x_i, x_{i-1}]$ is a random value in such sub interval. This is called the *Riemann Sum*. If we were to apply it to our current function f, it will have a graphic representation



(Note: If we select $x_i^* = (x_i - x_{i-1})/2$ it is called the Midpoint Rule, a numerical method for integration)

This is a good approach but we can see that there will be an error when calculating the area in such way. Well, for now we can formulate an approximate calculus of the area:

$$S(f) = \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_i^*)\Delta x_i\right) \tag{17}$$

which are the implications of N tending to infinity? Well, it means that Δx_i tends to zero again but the gain is a more accurate result for the integral since the sub-rectangular areas are smaller.

Now, a problem arises because infinite sub-intervals leads to $\Delta x_i = 0$ and it turns out to be an area of zero! We know that we cannot calculate infinite sub-intervals, so Numerically/Computationally it is a good approach to approximate really hard integrals.

Then, how can we find the real area? Well, mathematicians has study this problem with convergence of series and a really interesting branch of its field called infinitesimal calculus. The explanation before was a very superficial introduction to such analysis, for now you need to know that there have been approximated the main results we need to know when finding the area under a curve which is also called "integration".

We use the following notation to indicate an integral:

$$F(x) = \int f(x)dx \tag{18}$$

where F is the integrated function and f the function being integrated. Now you may notice that we did not indicated the interval of integration (the interval we want to know the area under the curve of f), it is notated by:

$$A = \int_a^b f(x)dx \tag{19}$$

and its value is (for this context) a real number, Why? Because it is the estumated value of the area from zero to $f(x)$ in the interval $[a,b]$. (Note: F is a function because we are not specifying any interval while A is directly the value found)

We need to know a fundamental fact, derivatives and integrals are opposite operations. It implies that if you find a derivative of a function the integral of such must be the original function and vice-versa.

With that in mind, there integrals we can determine by means of differential analysis, in example: Given f and its derivative

$$f(x) = x^n + k$$
$$\frac{d}{dx}f = nx^{n-1}$$

we can write f as an *indefinite integral* of its derivative as follows

$$f(x) = \int f'(x)dx = \int nx^{n-1}dx$$
$$= x^n + C$$

Now, you may ask. Why did we wrote C instead of k? The answer is: Because we cannot directly estimate the value of some constant without at least one point of the original function.

Definite Integration

An definite integral is symbolically represented as

$$\int_a^b f(x)dx \tag{20}$$

where a, b are the lower and upper limits of the interval being calculated.

Properties:

$$\int_a^a f(x)dx = 0 \tag{21}$$

$$\int_a^b f(x)dx = -\int_a^b f(x)dx \tag{22}$$

$$\int_a^a f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx; \quad c \in (a,b) \tag{23}$$

(Remember: the interval (a,b) does not include a and b)

Indefinite Integration

An indefinite integral $F(x)$ is the anti-derivative of some given function f, the indefinite integral is the exact opposite of derivation/differentiation.

Properties: Indefinite integrals can be used to obtain definite integrals, it is to say: Areas for some interval

$$\int_a^b f(x)dx = F(b) - F(a) \tag{24}$$

$$= F(x)\Big|_a^b \tag{25}$$

(The last notation means F evaluated in the interval $[a,b]$)

Finally, these can represent many many things we wish to know about, forces, magnetic and electric fields, vector fields, etc.

Previous Definitions

Function shortcuts

$$k, a, c, C \in \mathbb{R} \tag{26}$$

$$\tau : \mathbb{R} \to \mathbb{R} \tag{27}$$

$$x \to \tau(x) \tag{28}$$

Considering τ as a class then let f, g be functions of same type that τ.

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \tag{29}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x} \tag{30}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x} \tag{31}$$

Differentiation

General

$$\frac{d}{dx} c = 0 \tag{32}$$

$$\frac{d}{dx} x = 1 \tag{33}$$

$$\frac{d}{dx} k f(x) = k f'(x) \tag{34}$$

$$\frac{d}{dx} f(x)^n = n f(x)^{n-1} \tag{35}$$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \tag{36}$$

$$\frac{d}{dx} f(x) g(x) = f(x) g'(x) + f'(x) g(x) \tag{37}$$

$$\frac{d}{dx} |f(x)| = \frac{f(x)}{|f(x)|} f'(x); \quad f(x) \neq 0 \tag{38}$$

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{39}$$

Transcendental Functions

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \tag{40}$$

$$\frac{d}{dx} \log_a f(x) = \frac{f'(x)}{f(x) \ln a} \tag{41}$$

$$\frac{d}{dx} e^{f(x)} = f'(x) e^f(x) \tag{42}$$

$$\frac{d}{dx} a^f(x) = a^f(x) f'(x) \ln a \tag{43}$$

Trigonometrical

$$\frac{d}{dx} \sin f(x) = f'(x) \cos f(x) \tag{44}$$

$$\frac{d}{dx} \cos f(x) = -f'(x) \sin f(x) \tag{45}$$

$$\frac{d}{dx} \tan f(x) = f'(x) \sec^2 f(x) \tag{46}$$

$$\frac{d}{dx} \cot f(x) = -f'(x) \csc^2 f(x) \tag{47}$$

$$\frac{d}{dx} \sec f(x) = f'(x) \sec f(x) \tan f(x) \tag{48}$$

$$\frac{d}{dx} \csc f(x) = -f'(x) \csc f(x) \cot f(x) \tag{49}$$

Inverse Trigonometrical

$$\frac{d}{dx} \arcsin\left(\frac{f(x)}{a}\right) = \frac{f'(x)}{\sqrt{a^2 - f(x)^2}} \tag{50}$$

$$\frac{d}{dx} \arccos\left(\frac{f(x)}{a}\right) = \frac{-f'(x)}{\sqrt{a^2 - f(x)^2}} \tag{51}$$

$$\frac{d}{dx} \arctan\left(\frac{f(x)}{a}\right) = \frac{a f'(x)}{a^2 + f(x)^2} \tag{52}$$

$$\frac{d}{dx} \operatorname{arccot}\left(\frac{f(x)}{a}\right) = \frac{-a f'(x)}{a^2 + f(x)^2} \tag{53}$$

$$\frac{d}{dx} \operatorname{arcsec}\left(\frac{f(x)}{a}\right) = \frac{a f'(x)}{f(x)\sqrt{f(x)^2 - a^2}} \tag{54}$$

$$\frac{d}{dx} \operatorname{arccsc}\left(\frac{f(x)}{a}\right) = \frac{-a f'(x)}{f(x)\sqrt{f(x)^2 - a^2}} \tag{55}$$

Hyperbolic

$$\frac{d}{dx} \sinh f(x) = f'(x) \cosh f(x) \tag{56}$$

$$\frac{d}{dx} \cosh f(x) = f'(x) \sinh f(x) \tag{57}$$

$$\frac{d}{dx} \tanh f(x) = f'(x) \operatorname{sech}^2 f(x) \tag{58}$$

$$\frac{d}{dx} \coth f(x) = -f'(x) \operatorname{csch}^2 f(x) \tag{59}$$

$$\frac{d}{dx} \operatorname{sech} f(x) = -f'(x) \operatorname{sech} f(x) \tanh f(x) \tag{60}$$

$$\frac{d}{dx} \operatorname{csch} f(x) = -f'(x) \operatorname{csch} f(x) \coth f(x) \tag{61}$$

Integration

General

$$\int dx = x + C \tag{62}$$

$$\int kf(x)dx = k\int f(x)dx \tag{63}$$

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx \tag{64}$$

$$\int x^n = \frac{x^{n+1}}{n+1} + C \tag{65}$$

Transcendental Functions

$$\int \frac{dx}{x} = \ln|x| + C \tag{66}$$

$$\int \frac{1}{x\ln x}dx = \ln\ln x + C \tag{67}$$

$$\int e^x = e^x + C \tag{68}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \tag{69}$$

Trigonometrical

$$\int \sin x dx = -\cos x + C \tag{70}$$

$$\int \cos x dx = \sin x + C \tag{71}$$

$$\int \tan x dx = -\ln|\cos x| + C \tag{72}$$

$$\int \cot x dx = \ln|\sin x| + C \tag{73}$$

$$\int \sec x \tan x dx = \sec x + C \tag{74}$$

$$\int \csc x \cot x dx = -\csc x + C \tag{75}$$

$$\int \sec^2 x dx = \tan x + C \tag{76}$$

$$\int \csc^2 x dx = -\cot x + C \tag{77}$$

$$\int \arcsin\left(\frac{x}{a}\right)dx = \sqrt{a^2-x^2} + x\arcsin\left(\frac{x}{a}\right) + C \tag{78}$$

$$\int \arccos\left(\frac{x}{a}\right)dx = x\arcsin\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + C \tag{79}$$

$$\int \arctan\left(\frac{x}{a}\right)dx = x\arctan\left(\frac{x}{a}\right) - \frac{a}{2}\ln\left(a^2+x^2\right) + C \tag{80}$$

$$\int \operatorname{arccot}\left(\frac{x}{a}\right)dx = x\operatorname{arccot}\left(\frac{x}{a}\right) + \frac{a}{2}\ln\left(a^2+x^2\right) + C \tag{81}$$

$$\begin{aligned}\int \operatorname{arcsec}\left(\frac{x}{a}\right)dx = x\operatorname{arcsec}\left(\frac{x}{a}\right)\\ -\frac{a}{2}\ln\left(\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}-x}\right) + C\end{aligned} \tag{82}$$

$$\begin{aligned}\int \operatorname{arccsc}\left(\frac{x}{a}\right)dx = x\operatorname{arccsc}\left(\frac{x}{a}\right)\\ +\frac{a}{2}\ln\left(\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}-x}\right) + C\end{aligned} \tag{83}$$

Hyperbolic

$$\int \sinh\left(\frac{x}{a}\right)dx = a\cosh\left(\frac{x}{a}\right) + C \tag{84}$$

$$\int \cosh\left(\frac{x}{a}\right)dx = a\sinh\left(\frac{x}{a}\right) + C \tag{85}$$

$$\int \tanh\left(\frac{x}{a}\right)dx = a\ln\left(\cosh\left(\frac{x}{a}\right)\right) + C \tag{86}$$

$$\int \coth\left(\frac{x}{a}\right)dx = a\ln\left(\tanh\left(\frac{x}{a}\right)\cosh\left(\frac{x}{a}\right)\right) + C \tag{87}$$

$$\int \operatorname{sech}\left(\frac{x}{a}\right)dx = a\arctan\left(\sinh\left(\frac{x}{a}\right)\right) + C \tag{88}$$

$$\int \operatorname{csch}\left(\frac{x}{a}\right)dx = a\ln\left(\tanh\left(\frac{x}{2a}\right)\right) + C \tag{89}$$

ADDITIONAL - TAYLOR SERIES

The Taylor series of a real (for this case) or complex function f that has infinite derivatives at an element a of its domain, the series is:

$$f:\mathbb{R}\to\mathbb{R};\quad \lim_{n\to\infty}\frac{d^n}{dx^n}f\neq 0 \tag{90}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n;\quad a\in\mathrm{dom}(f) \tag{91}$$

This formula is used to compute approximations for analytical integrals out of our actual reasoning capacity. (Note: $f^{(n)}$ represents the n-th derivative of f)

Now, the actual interest in this equation is that the convergence/similarity that some function f will have around a point a on its domain will be good as we add terms to the summation. Examples are:

$$e^x=\sum_{n=0}^{\infty}\frac{e^a}{n!}(x-a)^n \tag{92}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{\sin^{(n)}(a)}{n!}(x-a)^n \tag{93}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{\cos^{(n)}(a)}{n!}(x-a)^n \tag{94}$$

(Note: $\frac{d}{dx}e^x=e^x$ so there is no need to place the derivative in equation (92))

If we take $a=0$ then we get:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{95}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \tag{96}$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \tag{97}$$

Applications: When we face functions which analytically are virtually imposible to express in terms of elemental functions then we proceed as in the following example:

Find the definite integral of x^x, taking the Taylor Series and replacing:

$$\because x^x=\sum_{n=0}^{\infty}\frac{(x^x)^{(n)}(a)}{n!}(x-a)^n \tag{98}$$

$$=a^a+a^a(\ln a+1)(x-a)+\ldots\left.\frac{d^n}{dx^n}x^x\right|_{x=a}\frac{(x-a)^n}{n!} \tag{99}$$

we can integrate x^x as a series

$$\therefore\int_{x_a}^{x_b}x^x dx=\int_{x_a}^{x_b}\sum_{n=0}^{\infty}\frac{(x^x)^{(n)}(a)}{n!}(x-a)^n \tag{100}$$

$$=\sum_{n=0}^{\infty}\left(\left.\frac{d^n}{dx^n}x^x\right|_{x=a}\frac{1}{n!}\int_{x_a}^{x_b}(x-a)^n dx\right) \tag{101}$$

In addition with the help of Numerical Methods, we can compute algorithmically the integrals and derivatives in the summation and hence obtain answers; now those answers would get even more accurate if we evaluate a in the middle of our intervals:

$$\int_{x_a}^{x_b}x^x dx=\sum_{n=0}^{\infty}\left(\underbrace{\left.\frac{d^n}{dx^n}x^x\right|_{x=\frac{x_b-x_a}{2}}}_{\text{Constant term}}\frac{1}{n!}\int_{x_a}^{x_b}\left(x-\frac{x_b-x_a}{2}\right)^n dx\right) \tag{102}$$

(Note: The error ε could be extremely large for some special functions as $\psi_{n\ell m}$ in the scale of $a_0=\hbar^2/me^2\approx 0.0529mn$, the first Bohr Radius)

So, the general integral for some unknown function f in an interval $[a,b]$ using the Taylor Series and a point $\rho\in[a,b]$ is

$$\int_a^b f(x)dx=\sum_{n=0}^{\infty}\left(\left.\frac{d^n}{dx^n}f(x)\right|_{x=\rho}\frac{1}{n!}\int_a^b(x-\rho)\,dx\right) \tag{103}$$