Lecture - Geometry in the Space

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Previous Definitions

Advice

The following is a compilation of formulas, its derivations and explanations. From those, what you need to know is which terms are you going to substitute and you will get the answer, some derivations seems complex, do not worry just move on and as a good engineer (I'm also one) use the formulas.

Canonical Vectors

There are some concepts you should see before

$$\mathbf{i} = \langle 1,0,0\rangle\,; \quad \mathbf{j} = \langle 0,1,0\rangle\,; \quad \mathbf{k} = \langle 0,0,1\rangle \tag{1}$$

These vectors are called canonical vectors, the standard base for $\mathbb{R}^3$ or the generators of $\mathbb{R}^3$.

Orthogonal

A vector or object (vector, line, plane, etc.) is said to be orthogonal if and only if its dot product respect to another object is equal zero.

Norm

The norm of a vector can be notated using $|\mathbf{v}|$, $\|\mathbf{v}\|$ or if we know that $\mathbf{v}$ is a vector then by writing v we may infer given the context that v is the norm.

The Straight Line

We start from the following equation

$$(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k} = t(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}) \tag{2}$$

Rearanging terms we find the following equation

$$\Longrightarrow x\mathbf{i}+y\mathbf{j}+z\mathbf{k} = x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}+t(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}) \tag{3}$$

$$\Longrightarrow \langle x,y,z\rangle = \langle x_0+tv_1, y_0+tv_2, z_0+tv_3\rangle \tag{4}$$

$$\Longrightarrow \mathbf{r}(t) = \mathbf{r_0}+t\mathbf{v} \tag{5}$$

and then we can indicate the vector $\langle x,y,z\rangle$ (notated in canonical vectors terms) as a *function* of an initial point/vector $\mathbf{r_0}$ with a direction vector $\mathbf{v}$ which freely constructs the line a variable called t.

Now, often we find the following equations system:

$$\begin{cases} x = x_0+tv_1 \\ y = y_0+tv_2 \\ z = z_0+tv_3 \end{cases} \Longleftrightarrow \begin{cases} t = \frac{x-x_0}{v_1} \\ t = \frac{y-y_0}{v_2} \\ t = \frac{z-z_0}{v_3} \end{cases} \tag{6}$$

$$\Longrightarrow t = \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3} \tag{7}$$

this is a way to indicate the correspondences given in eq (3). If you see well, it is obtained by analyzing the terms multiplying each canonical vector separately:

$$x\mathbf{i} = x_0\mathbf{i}+tv_1\mathbf{i} \tag{8}$$

$$y\mathbf{j} = y_0\mathbf{j}+tv_2\mathbf{j} \tag{9}$$

$$z\mathbf{k} = z_0\mathbf{k}+tv_3\mathbf{k} \tag{10}$$

and so neglecting $\mathbf{i},\mathbf{j},\mathbf{k}$ we obtain the system.

Constructing a Line with Two Points

Let P,Q be two points in a line $\ell \subset \mathbb{R}^3$

$$P = \langle p_1,p_2,p_3\rangle \tag{11}$$

$$Q = \langle q_1,q_2,q_3\rangle \tag{12}$$

Then we can find the equation of ℓ next way:

$$\Longrightarrow \mathbf{v} = \vec{PQ} \tag{13}$$

$$\mathbf{v} = \langle q_1-p_1, q_2-p_2, q_3-p_3\rangle \tag{14}$$

$$\Longrightarrow \ell : \begin{cases} x = p_1+t(q_1-p_1) \\ y = p_2+t(q_2-p_2) \\ z = p_3+t(q_3-p_3) \end{cases} \tag{15}$$

A Point and an Orthogonal Vector

Let $P,\mathbf{v}_\perp$

$$P = \langle p_1,p_2,p_3\rangle \tag{16}$$

$$\mathbf{v}_\perp = \langle v_{1\perp}, v_{2\perp}, v_{3\perp}\rangle \tag{17}$$

be two vectors, $P \in \ell$ and $\mathbf{v}_\perp$ an orthogonal vector to ℓ. Because P dot $\mathbf{v}_\perp$ is equal 0, then we can say

$$\because P\cdot\mathbf{v}_\perp = 0 \tag{18}$$

$$\therefore Q\cdot\mathbf{v}_\perp = \langle x,y,z\rangle\cdot\langle v_{1\perp}, v_{2\perp}, v_{3\perp}\rangle = 0 \tag{19}$$

$$= v_{1\perp}x+v_{2\perp}y+v_{3\perp}z = 0 \tag{20}$$

and taking in consider that geometrically we have infinite orthogonal lines to $\mathbf{v}_\perp$, then we can understand that a plane is formed.

Returning to our problem, we can form a line if we take two random values for two of our three x,y,z variables, I will select x and y to be substituted for those values

$$x = x_\omega \neq p_1; \quad y = y_\omega \neq p_2 \tag{21}$$

(x_ω, y_ω are those random values)

$$\langle x_\omega, y_\omega, z\rangle\cdot\langle v_{1\perp}, v_{2\perp}, v_{3\perp}\rangle = 0 \tag{22}$$

$$\Longrightarrow x_\omega v_{1\perp}+y_\omega v_{2\perp}+zv_{3\perp} = 0 \tag{23}$$

$$\Longrightarrow z = -\frac{x_\omega v_{1\perp}+y_\omega v_{2\perp}}{v_{3\perp}} \tag{24}$$

we obtain a fixed value for z and we can form a vector

$$Q = \left\langle x_\omega, y_\omega, -\frac{x_\omega v_{1\perp} + y_\omega v_{2\perp}}{v_{3\perp}} \right\rangle \tag{25}$$

and then form a line using the previous topic

$$\ell : \begin{cases} x = p_1 + t(x_\omega - p_1) \\ y = p_2 + t(y_\omega - p_2) \\ z = p_3 + t\left(-\frac{x_\omega v_{1\perp} + y_\omega v_{2\perp}}{v_{3\perp}} - p_3\right) \end{cases} \tag{26}$$

Distance from point to a line

Let $\ell \subset \mathbb{R}^3$ be a line and $R \notin \ell$ a point, then the distance $d(R,\ell)$ can be calculated next way:

$$\ell \equiv \mathbf{r}(t) \tag{27}$$

$$\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v} \tag{28}$$

being $\mathbf{v}$ the direction vector;

$$\mathbf{v}' = \frac{\mathbf{v} \cdot R}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{\mathbf{v} \cdot R}{\|\mathbf{v}\|^2}\mathbf{v} \tag{29}$$

the projection of R onto $\mathbf{v}$, and

$$d(R,\ell) = \left\|R - \mathbf{v}'\right\| = \left\|R - \frac{\mathbf{v} \cdot R}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\right\| = \left\|R - \frac{\mathbf{v} \cdot R}{\|\mathbf{v}\|^2}\mathbf{v}\right\| \tag{30}$$

the distance from R to ℓ.
(Note: this method works for objects in $\mathbb{R}^n$)

The Plane

A plane in $\mathbb{R}^3$ is described by the equations:

$$\mathscr{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{31}$$

$$\mathscr{P} : ax + by + cz = d \tag{32}$$

where we can say that it has a normal vector (orthogonal)

$$\mathbf{n} = \langle a, b, c \rangle \tag{33}$$

or for the first equation

$$P = \langle x_0, y_0, z_0 \rangle \in \mathscr{P} \tag{34}$$

$$\mathbf{n} = \langle A, B, C \rangle \tag{35}$$

where P is a point in the plane.

Distance From Point to a Plane

Let $\mathscr{P}$ be a plane and $P \notin \mathscr{P}$ be a point, then the distance from P to $\mathscr{P}$ is obtained next way:

$$\because \mathscr{P} : ax + by + cz = d \quad \therefore \mathbf{n} = \langle a, b, c \rangle \tag{36}$$

Consider the vector $\mathbf{n}$, normal (orthogonal) to the plane $\mathscr{P}$, then we can project P in $\mathbf{n}$ and measure the norm of such vector:

$$d(\mathscr{P}, P) = \left\|\frac{P \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n}\right\| = \left\|\frac{P \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}\right\| \tag{37}$$

Or we simply take the absolute value of the scalar projection of P onto $\mathbf{n}$

$$d(\mathscr{P}, P) = \left|\frac{P \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\right| = \frac{\left|\sum_{i=1}^n p_i n_i\right|}{\left|\sum_{i=1}^n n_i^2\right|} \tag{38}$$

(n in the summation indicates the dimension, n_i is an entry of $\mathbf{n}$)

Building a Plane from Three Points

Let $P, Q, R \in \mathscr{P}$ be points in a given plane

$$\because \overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{n} \tag{39}$$

$$\therefore \mathscr{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{40}$$

then we can directly obtain the equation of the plane from P, Q, R.

Angle Between Planes

Let $\mathscr{P}_1, \mathscr{P}_2$ be two planes, the angle between them is obtained from its normal vectors $\mathbf{n}_1, \mathbf{n}_2$ as follows:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{41}$$

$$\implies \theta = \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right) \tag{42}$$

alternatively there exists the notation:

$$\measuredangle(\mathscr{P}_1, \mathscr{P}_2) = \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}\right) \tag{43}$$

Intersection of Two Planes

Let $\mathscr{P}_1, \mathscr{P}_2$ be two planes in $\mathbb{R}^3$:

$$\mathscr{P}_1 : a_1x + b_1y + c_1z = d_1; \quad \mathbf{n}_1 = \langle a_1, b_1, c_1 \rangle \tag{44}$$

$$\mathscr{P}_2 : a_2x + b_2y + c_2z = d_2; \quad \mathbf{n}_2 = \langle a_2, b_2, c_2 \rangle \tag{45}$$

Now, remember that the cross product of two vectors generates an orthogonal one to those. Hence:

$$\mathbf{n}_1 \times \mathbf{n}_2 = \langle a_1, b_1, c_1 \rangle \times \langle a_2, b_2, c_2 \rangle = \mathbf{v} \tag{46}$$

is the direction vector. And what remains is to find a point in the intersection line. This is solved by doing:

$$z' = 0 \tag{47}$$

$$\therefore a_1x' + b_1y' = d_1 \tag{48}$$

$$\therefore a_2x' + b_2y' = d_2 \tag{49}$$

obtaining a system of linear equations. If we take the first equation and solve by substitution:

$$\implies x' = \frac{d_1 - b_1y'}{a_1} \tag{50}$$

$$\implies a_2\frac{d_1 - b_1y'}{a_1} + b_2y = d_2 \tag{51}$$

$$\implies \frac{-a_2b_1}{a_1}y' + b_2y' = d_2 - \frac{a_2d_1}{a_1} \tag{52}$$

$$\implies \frac{a_1b_2 - a_2b_1}{a_1}y' = d_2 - \frac{a_2d_1}{a_1} \tag{53}$$

$$\implies y' = \frac{a_1}{a_1b_2 - a_2b_1}\left(d_2 - \frac{a_2d_1}{a_1}\right) \tag{54}$$

$$\implies x' = \frac{d_1 - b_1}{a_1b_2 - a_2b_1}\left(d_2 - \frac{a_2d_1}{a_1}\right) \tag{55}$$

we get the values x', y', z' to substitute in the position vector for constructing the line of intersection between the planes:

$$\mathscr{P}_1 \cap \mathscr{P}_2 = \left\langle x', y', z' \right\rangle + t\mathbf{v} \tag{56}$$

Now, the intersection of two planes has the general form of:

$$\begin{aligned}\mathscr{P}_1 \cap \mathscr{P}_2 = \Bigg\langle & \frac{d_1 - b_1}{a_1 b_2 - a_2 b_1}\left(d_2 - \frac{a_2 d_1}{a_1}\right), \\ & \frac{a_1}{a_1 b_2 - a_2 b_1}\left(d_2 - \frac{a_2 d_1}{a_1}\right), 0 \Bigg\rangle + t(\mathbf{n}_1 \times \mathbf{n}_2)\end{aligned} \tag{57}$$

when we fix $z' = 0$ to have a concrete answer. Note that we could had selected x' or z' and assign any random value.

You may ask, why did I changed x, y, z for x', y', z'? The reason is: Because x, y, z commonly refer to variables, it means they have a domain and we consider them in a different way, here x', y', z' are constants, its value is determined, that's the reason.

Intersection of Three Planes

Let $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ be planes in $\mathbb{R}^3$

$$\mathscr{P}_1 : a_1 x + b_1 y + c_1 z = d_1 \tag{58}$$

$$\mathscr{P}_2 : a_2 x + b_2 y + c_2 z = d_2 \tag{59}$$

$$\mathscr{P}_3 : a_3 x + b_3 y + c_3 z = d_3 \tag{60}$$

The intersection of those exists if and only if

$$\mathbf{n}_1 \cdot \mathbf{n}_2 \neq 1; \quad \mathbf{n}_1 \cdot \mathbf{n}_3 \neq 1; \quad \mathbf{n}_2 \cdot \mathbf{n}_3 \neq 1 \tag{61}$$

and is found by solving the following augmented matrix

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right] \tag{62}$$

Intersection of a Line with a Plane

Letting ℓ be a line, $\mathscr{P}$ be a plane, then we have

$$\ell : \begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases} ; \quad \mathscr{P} : ax + by + cz = d \tag{63}$$

and we can do the clever substitution:

$$a(x_0 + t v_1) + b(y_0 + t v_2) + c(z_0 + t v_3) = d \tag{64}$$

and find the value of t

$$\implies t(a v_1 + b v_2 + c v_3) = d - (a x_0 + b y_0 + c z_0) \tag{65}$$

$$\implies t = \frac{d - (a x_0 + b y_0 + c z_0)}{a v_1 + b v_2 + c v_3} \tag{66}$$

and so, we replace t in the original line ℓ

$$\begin{cases} x' = x_0 + \left(\frac{d - (a x_0 + b y_0 + c z_0)}{a v_1 + b v_2 + c v_3}\right) v_1 \\ y' = y_0 + \left(\frac{d - (a x_0 + b y_0 + c z_0)}{a v_1 + b v_2 + c v_3}\right) v_2 \\ z' = z_0 + \left(\frac{d - (a x_0 + b y_0 + c z_0)}{a v_1 + b v_2 + c v_3}\right) v_3 \end{cases} \tag{67}$$

obtaining so the vector

$$\mathscr{P} \cap \ell = \langle x', y', z' \rangle \tag{68}$$

(Note that there exists an intersection while $\mathbf{v} \cdot \mathbf{n} \neq 1$)