



Input (x)	Hidden (z)	Output (y)
1	2	1

Network Architecture

A 2-2-1 neural network.

- 2 input neurons
- 2 hidden neurons (with sigmoid activation)
- 1 output neuron (with linear activation)

Initial Weights and Biases (Randomly Chosen)

- Input - Hidden Layer**
 - $w_{11} = 0.1, w_{12} = 0.2$ (hidden neuron 1)
 - $w_{21} = 0.3, w_{22} = 0.4$ (hidden neuron 2)
 - $b_{11} = 0.1, b_{12} = 0.1$
- Hidden - Output Layer**
 - $w_{31} = 0.5, w_{32} = 0.5$
 - $b_{31} = 0.1$

Step 1: Forward Propagation

Input: $x = [1, 0] = x$

Hidden Layer Activation:

$$z_1 = w_{11}x_1 + w_{12}x_2 + b_{11} = 0.1 + 0.2 + 0.1 = 0.3$$

$$z_2 = w_{21}x_1 + w_{22}x_2 + b_{12} = 0.3 + 0.4 + 0.1 = 0.8$$

Apply Sigmoid Activation:

$$a_1 = \sigma(z_1) = \frac{1}{1 + e^{-z_1}} \approx 0.57$$

$$a_2 = \sigma(z_2) = \frac{1}{1 + e^{-z_2}} \approx 0.72$$

Output Layer Activation:

$$z_3 = w_{31}a_1 + w_{32}a_2 + b_{31} = 0.5 \cdot 0.57 + 0.5 \cdot 0.72 + 0.1 \approx 0.77$$

Apply Linear Activation:

$$\hat{y} = z_3 \approx 0.77$$

Prediction: 0.77

Step 2: Loss Calculation (Binary Cross-Entropy)

For $x_i = 1, y = 1$

$$Loss = -y \log(\hat{y}) - (1 - y) \log(1 - \hat{y}) = -1 \log(0.77) - 0 \log(0.23) \approx 0.25$$

Step 3: Backward Propagation

Calculate error at output and propagate back to hidden layer.

Output Layer Gradient:

$$\frac{\partial Loss}{\partial z_3} = \frac{\partial Loss}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial z_3} = (1 - \hat{y}) \cdot 1 = 0.23$$

Hidden Layer Gradients:

$$\delta_1 = \frac{\partial Loss}{\partial z_1} = \frac{\partial Loss}{\partial z_3} \cdot w_{31} = 0.23 \cdot 0.5 = 0.115$$

$$\delta_2 = \frac{\partial Loss}{\partial z_2} = \frac{\partial Loss}{\partial z_3} \cdot w_{32} = 0.23 \cdot 0.5 = 0.115$$

Step 4: Weight Update (Gradient Descent)

Learning rate $\eta = 0.1$

$$w_{11} = w_{11} - \eta \cdot \delta_1 \cdot x_1 = 0.1 - 0.1 \cdot 0.115 = 0.0885$$

Final Weights and Biases:

- $w_{11} = 0.0885, w_{12} = 0.2, b_{11} = 0.1$
- $w_{21} = 0.3, w_{22} = 0.4, b_{12} = 0.1$
- $w_{31} = 0.5, w_{32} = 0.5, b_{31} = 0.1$