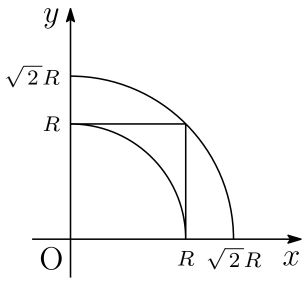
例 7.4

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

解答



$R>0$ として

$$D_1 : x^2+y^2 \leqq R^2,\ x \geqq 0,\ y \geqq 0$$

$$D_2 : 0 \leqq x \leqq R,\ 0 \leqq y \leqq R$$

$$D_3 : x^2+y^2 \leqq 2R^2,\ x \geqq 0,\ y \geqq 0$$

とおくと

$$D_1 \subset D_2 \subset D_3, \quad e^{-(x^2+y^2)} > 0$$

であるから

$$\iint_{D_1} e^{-(x^2+y^2)}dxdy \leqq \iint_{D_2} e^{-(x^2+y^2)}dxdy \leqq \iint_{D_3} e^{-(x^2+y^2)}dxdy$$

ここで

$$\iint_{D_2} e^{-(x^2+y^2)}dxdy = \int_0^R e^{-x^2}dx \times \int_0^R e^{-y^2}dy = \left(\int_0^R e^{-x^2}dx\right)^2$$

また，$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\text{ 周分} \end{pmatrix}$ とおくと

$$\begin{aligned}
\iint_{D_1} e^{-(x^2+y^2)}dxdy &= \iint_{D_1{}'} e^{-r^2}\cdot rdrd\theta \quad \left(D_1{}' : 0 \leqq \theta \leqq \frac{\pi}{2},\ 0 \leqq r \leqq R\right) \\
&= \int_0^{\frac{\pi}{2}} d\theta \times \int_0^R re^{-r^2}dr \\
&= \left[\theta\right]_0^{\frac{\pi}{2}} \times \left[-\frac{1}{2}e^{-r^2}\right]_0^R \\
&= \frac{\pi}{2} \times \left\{-\frac{1}{2}\left(e^{-R^2}-1\right)\right\} \\
&= \frac{\pi}{4}\left(1-e^{-R^2}\right) \to \frac{\pi}{4} \quad (R \to \infty)
\end{aligned}$$

同様に

$$\begin{aligned}
\iint_{D_3} e^{-(x^2+y^2)}dxdy &= \iint_{D_3{}'} e^{-r^2}\cdot rdrd\theta \quad \left(D_3{}' : 0 \leqq \theta \leqq \frac{\pi}{2},\ 0 \leqq r \leqq \sqrt{2}R\right) \\
&= \frac{\pi}{4}\left(1-e^{-2R^2}\right) \to \frac{\pi}{4} \quad (R \to \infty)
\end{aligned}$$

よって，はさみうちの定理より $\quad \left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}$

$\int_0^\infty e^{-x^2}dx > 0$ より $\quad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

公式

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{\pi}{2} & (n=0) \\ 1 & (n=1) \\ \dfrac{(n-1)\cdot(n-3)\cdot \cdots \cdot 1}{n\cdot(n-2)\cdot \cdots \cdot 2}\cdot\dfrac{\pi}{2} & (n=2,4,6,\dots) \\ \dfrac{(n-1)\cdot(n-3)\cdot \cdots \cdot 2}{n\cdot(n-2)\cdot \cdots \cdot 3}\cdot 1 & (n=3,5,7,\dots) \end{cases}$$

☆ 覚えておく

証明

$I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x dx$ とおくと，$I_0 = \dfrac{\pi}{2}$，$I_1 = 1$ はすぐわかる．また，$n \geqq 2$ のとき

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \cos^n x dx \\ &= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx \\ &= \Big[\cos^{n-1} x \sin x\Big]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx \\ &= (0\cdot 1 - 1\cdot 0) + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x(1-\cos^2 x)dx \\ &= (n-1)\left(\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - \int_0^{\frac{\pi}{2}} \cos^n x dx\right) \\ &= (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

$$\begin{array}{ccc} \cos^{n-1} x & & \sin x \\ (n-1)\cos^{n-2} x\cdot(-\sin x) & & \cos x \end{array}$$

であるから，漸化式 $I_n = \dfrac{n-1}{n} I_{n-2}$ $(n \geqq 2)$ が得られる．よって，$n = 2, 4, 6, \dots$ のとき

$$I_n = \frac{n-1}{n} I_{n-2} = \cdots = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{1}{2} I_0 = \frac{(n-1)\cdot(n-3)\cdot\cdots\cdot 1}{n\cdot(n-2)\cdot\cdots\cdot 2}\cdot\frac{\pi}{2}$$

また，$n = 3, 5, 7, \dots$ のとき

$$I_n = \frac{n-1}{n} I_{n-2} = \cdots = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{2}{3} I_1 = \frac{(n-1)\cdot(n-3)\cdot\cdots\cdot 2}{n\cdot(n-2)\cdot\cdots\cdot 3}\cdot 1$$

さらに，$x = \dfrac{\pi}{2} - t$ と置換すれば

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n t dt = I_n$$

もわかる． ∎

例 7.5

極座標変換を用いて次の重積分の値を求めよ.

(1) $\displaystyle\iint_D \frac{y^3}{x^6}dxdy \quad \big(D : 1 \leqq x^2+y^2 \leqq 4,\ 0 \leqq y \leqq x\big)$

(2) $\displaystyle\iint_D xydxdy \quad (D : x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$

(3) $\displaystyle\iint_D y(9x-4y)dxdy \quad \big(D : x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0\big)$

解答

(1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\ \text{周分} \end{pmatrix}$ とおくと

$$\iint_D \frac{y^3}{x^6}dxdy \quad \big(D : 1 \leqq x^2+y^2 \leqq 4,\ 0 \leqq y \leqq x\big)$$

忘れない

$$= \iint_{D'} \frac{r^3\sin^3\theta}{r^6\cos^6\theta}\cdot r\,drd\theta \quad \left(D' : 1 \leqq r \leqq 2,\ 0 \leqq \theta \leqq \frac{\pi}{4}\right)$$

$$= \left(\int_1^2 \frac{1}{r^2}dr\right) \times \left(\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos^6\theta}d\theta\right)$$

← 分離できる

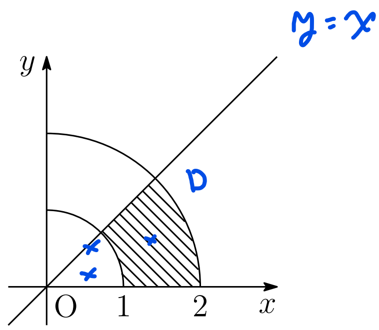
$$= \left[-\frac{1}{r}\right]_1^2 \times \int_0^{\frac{\pi}{4}} \frac{(1-\cos^2\theta)\sin\theta}{\cos^6\theta}d\theta$$

$$= -\left(\frac{1}{2}-1\right) \times \int_0^{\frac{\pi}{4}} \big\{-(\cos\theta)^{-6}\cdot(-\sin\theta)+(\cos\theta)^{-4}\cdot(-\sin\theta)\big\}d\theta$$

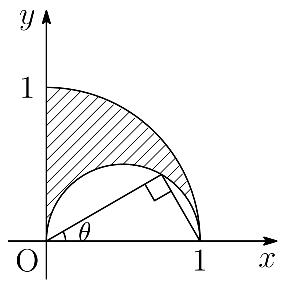
$$= \frac{1}{2} \times \left[\frac{1}{5}(\cos\theta)^{-5} - \frac{1}{3}(\cos\theta)^{-3}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\frac{1}{5}(4\sqrt{2}-1) - \frac{1}{3}(2\sqrt{2}-1)\right\}$$

$$= \frac{1+\sqrt{2}}{15}$$



(2) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix}$ とおくと



$$\iint_D xydxdy \quad (D : x \leqq x^2 + y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$$

$$= \iint_{D'} r\cos\theta \cdot r\sin\theta \cdot rdrd\theta \quad \left(D' : 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1\right)$$

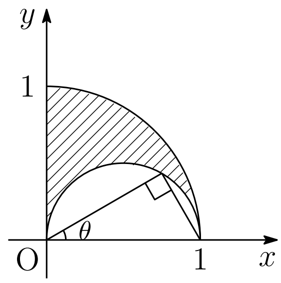
$$= \iint_{D'} r^3\cos\theta\sin\theta drd\theta = \int_0^{\frac{\pi}{2}} \left(\int_{\cos\theta}^1 r^3\cos\theta\sin\theta dr\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\cos\theta\sin\theta\right]_{r=\cos\theta}^{r=1} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}(1-\cos^4\theta)\cos\theta\sin\theta d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \{\sin\theta\cdot\cos\theta + \cos^5\theta\cdot(-\sin\theta)\}d\theta$$

$$= \frac{1}{4}\left[\frac{1}{2}\sin^2\theta + \frac{1}{6}\cos^6\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\left\{\frac{1}{2}(1-0) + \frac{1}{6}(0-1)\right\} = \frac{1}{12}$$

(3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix}$ とおくと



$$\iint_D y(9x-4y)dxdy \qquad \left(D : x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0\right)$$

$$= \iint_{D'} r\sin\theta(9r\cos\theta - 4r\sin\theta)\cdot rdrd\theta$$

$$\left(D' : 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1\right)$$

$$= \iint_{D'} r^3\sin\theta(9\cos\theta - 4\sin\theta)drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left\{\int_{\cos\theta}^1 r^3\sin\theta(9\cos\theta-4\sin\theta)dr\right\}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\sin\theta(9\cos\theta-4\sin\theta)\right]_{r=\cos\theta}^{r=1}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}(1-\cos^4\theta)\sin\theta(9\cos\theta-4\sin\theta)d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \{9(\cos\theta\sin\theta - \cos^5\theta\sin\theta) - 4(\sin^2\theta - \cos^4\theta\sin^2\theta)\}d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \Big[9\{-\cos\theta\cdot(-\sin\theta) + \cos^5\theta\cdot(-\sin\theta)\} - 4(\sin^2\theta - \cos^4\theta + \cos^6\theta)\Big]d\theta$$

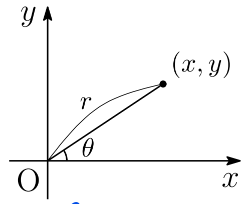
$$= \frac{1}{4}\left\{9\left[-\frac{1}{2}\cos^2\theta + \frac{1}{6}\cos^6\theta\right]_0^{\frac{\pi}{2}} - 4\left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2} + \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}\right)\right\}$$

$$= \frac{1}{4}\left[9\left\{-\frac{1}{2}(0-1) + \frac{1}{6}(0-1)\right\} - \frac{7}{8}\pi\right]$$

$$= \frac{3}{4} - \frac{7}{32}\pi$$

※（原点 O を極とする）極座標変換

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix} \quad \cdots\cdots①$$



において

r は原点 O から点 (x, y) までの距離

試験出る

である（これは勝手に変えられない！）.

例 7.5 (2),(3) の D は右図の斜線部であり

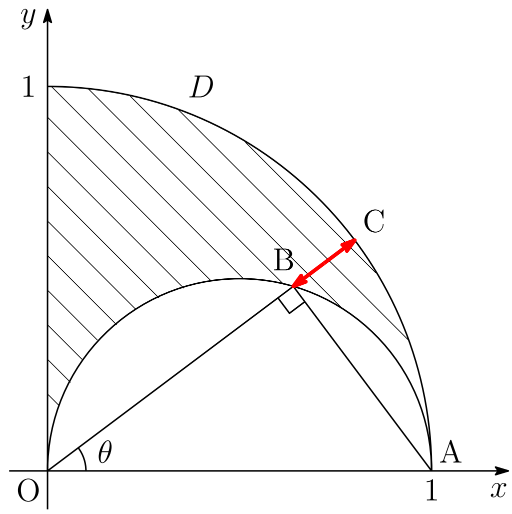
$$\mathrm{OB} = \mathrm{OA}\cos\theta = \cos\theta$$

$$\mathrm{OC} = 1$$

であるから，r の範囲は

$$\cos\theta \leqq r \leqq 1$$

となる．



赤線部の長さが $1 - \cos\theta$ であるから

$$0 \leqq r \leqq 1 - \cos\theta$$

と考えるのは間違いである．

もし，図形的に読み取ることが苦手であるならば，① を D の不等式に代入して r, θ の範囲を求めるとよい．

① を D の不等式に代入する．

$x \leqq x^2 + y^2$ より　　$r\cos\theta \leqq r^2$
$r \geqq 0$ より　　$\cos\theta \leqq r$　$\cdots\cdots②$

$x^2 + y^2 \leqq 1$ より　　$r^2 \leqq 1$
$r \geqq 0$ より　　$0 \leqq r \leqq 1$　$\cdots\cdots③$

$x \geqq 0$ より　　$r\cos\theta \geqq 0$
$r \geqq 0$ より　　$\cos\theta \geqq 0$　$\cdots\cdots④$

$y \geqq 0$ より　　$r\sin\theta \geqq 0$
$r \geqq 0$ より　　$\sin\theta \geqq 0$　$\cdots\cdots⑤$

④ であるから，② と ③ の共通部分は　　$\cos\theta \leqq r \leqq 1$

④ と ⑤ を同時に満たす θ（1 周分）は　　$0 \leqq \theta \leqq \dfrac{\pi}{2}$

例 7.6

極座標変換を用いて次の重積分の値を求めよ．

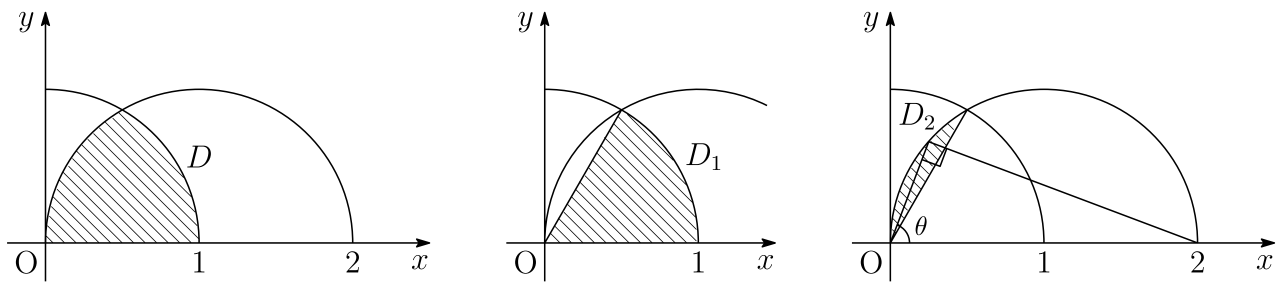
(1) $\displaystyle\iint_D xydxdy \quad \left(D : x^2+y^2 \leqq 1,\ x^2+y^2 \leqq 2x,\ y \geqq 0\right)$

(2) $\displaystyle\iint_D xydxdy \quad \left(D : 0 \leqq x \leqq 2,\ 0 \leqq y \leqq x,\ x^2+y^2 \geqq 1\right)$

(3) $\displaystyle\iint_D \frac{1}{(1+x^2+y^2)^2}dxdy \quad \left(D : (x^2+y^2)^2 \leqq x^2-y^2,\ x \geqq 0,\ y \geqq 0\right)$

解答

(1) 図のように D を D_1 と D_2 に分割する．



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix} \quad \text{とおくと}$$

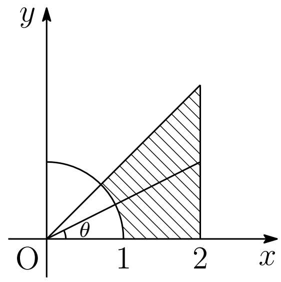
$$\begin{aligned}
\iint_{D_1} xydxdy &= \iint_{D_1'} r\cos\theta\cdot r\sin\theta\cdot rdrd\theta \quad \left(D_1' : 0 \leqq \theta \leqq \frac{\pi}{3},\ 0 \leqq r \leqq 1\right) \\
&= \left(\int_0^1 r^3dr\right)\times\left(\int_0^{\frac{\pi}{3}} \sin\theta\cdot\cos\theta d\theta\right) = \left[\frac{1}{4}r^4\right]_0^1 \times \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{3}} \\
&= \frac{1}{4}\times\frac{1}{2}\left(\frac{3}{4}-0\right) = \frac{3}{32}
\end{aligned}$$

$$\begin{aligned}
\iint_{D_2} xydxdy &= \iint_{D_2'} r\cos\theta\cdot r\sin\theta\cdot rdrd\theta \quad \left(D_2' : \frac{\pi}{3} \leqq \theta \leqq \frac{\pi}{2},\ 0 \leqq r \leqq 2\cos\theta\right) \\
&= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r^3\cos\theta\sin\theta dr\right)d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^{r=2\cos\theta} d\theta \\
&= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^5\theta\sin\theta d\theta = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^5\theta\cdot(-\sin\theta)d\theta = -\left[\frac{2}{3}\cos^6\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= -\frac{2}{3}\left(0-\frac{1}{64}\right) = \frac{1}{96}
\end{aligned}$$

よって

$$\iint_D xydxdy = \iint_{D_1} xydxdy + \iint_{D_2} xydxdy = \frac{3}{32}+\frac{1}{96} = \frac{5}{48}$$

(2) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\ \text{周分} \end{pmatrix}$ とおくと



$$\iint_D xydxdy \quad \big(D : 0 \leqq x \leqq 2,\ 0 \leqq y \leqq x,\ x^2+y^2 \geqq 1\big)$$

$$= \iint_{D'} r\cos\theta \cdot r\sin\theta \cdot rdrd\theta \quad \left(D' : 0 \leqq \theta \leqq \frac{\pi}{4},\ 1 \leqq r \leqq \frac{2}{\cos\theta}\right)$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_1^{\frac{2}{\cos\theta}} r^3 \sin\theta\cos\theta dr\right) d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4}\sin\theta\cos\theta\right]_{r=1}^{r=\frac{2}{\cos\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4}\sin\theta\cos\theta\left(\frac{16}{\cos^4\theta} - 1\right) d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\frac{16\sin\theta}{\cos^3\theta} - \sin\theta\cos\theta\right) d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \{-16(\cos\theta)^{-3} \cdot (-\sin\theta) - \sin\theta \cdot \cos\theta\} d\theta = \frac{1}{4}\left[8(\cos\theta)^{-2} - \frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left\{8(2-1) - \frac{1}{2}\left(\frac{1}{2} - 0\right)\right\} = \frac{31}{16}$$

(3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\ \text{周分} \end{pmatrix}$ とおく．

$x \geqq 0,\ y \geqq 0$ より　　$0 \leqq \theta \leqq \dfrac{\pi}{2}$

$(x^2+y^2)^2 \leqq x^2 - y^2$ より　　$(r^2)^2 \leqq r^2(\cos^2\theta - \sin^2\theta)$　　$\therefore\quad r^2 \leqq \cos 2\theta$

$\cos 2\theta \geqq 0,\ 0 \leqq \theta \leqq \dfrac{\pi}{2}$ より　　$0 \leqq \theta \leqq \dfrac{\pi}{4}$

このとき，$r^2 \leqq \cos 2\theta$ より　　$0 \leqq r \leqq \sqrt{\cos 2\theta}$

よって

$$\iint_D \frac{1}{(1+x^2+y^2)^2} dxdy \quad \big(D : (x^2+y^2)^2 \leqq x^2 - y^2,\ x \geqq 0,\ y \geqq 0\big)$$

$$= \iint_{D'} \frac{1}{(1+r^2)^2} \cdot rdrd\theta \quad \left(D' : 0 \leqq \theta \leqq \frac{\pi}{4},\ 0 \leqq r \leqq \sqrt{\cos 2\theta}\right)$$

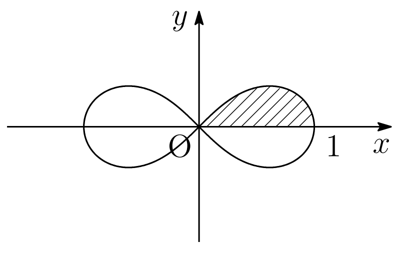
$$= \int_0^{\frac{\pi}{4}} \left\{\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2} dr\right\} d\theta = \int_0^{\frac{\pi}{4}} \left\{\int_0^{\sqrt{\cos 2\theta}} \frac{1}{2}(1+r^2)^{-2} \cdot 2rdr\right\} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[-\frac{1}{2}(1+r^2)^{-1}\right]_{r=0}^{r=\sqrt{\cos 2\theta}} d\theta = \int_0^{\frac{\pi}{4}} \left\{-\frac{1}{2}\left(\frac{1}{1+\cos 2\theta} - 1\right)\right\} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\tan\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\frac{\pi}{4} - \frac{1}{2}(1-0)\right\} = \frac{\pi}{8} - \frac{1}{4}$$

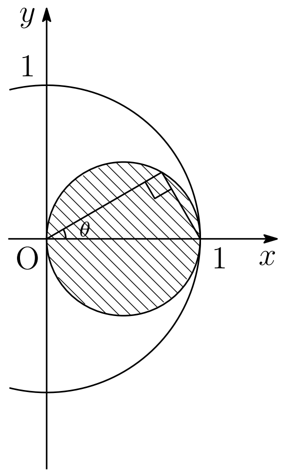
※ D は図の斜線部である．境界 $(x^2+y^2)^2 = x^2 - y^2$ は Lemniscate（レムニスケート）（連珠形）である．



例 7.7

xyz 空間において，球 $x^2+y^2+z^2 \leqq 1$ と円柱 $\left(x-\dfrac{1}{2}\right)^2+y^2 \leqq \dfrac{1}{4}$ の共通部分の体積 V を求めよ．

解答



球面の $z \geqq 0$ の部分を表す方程式は

$$z=\sqrt{1-x^2-y^2}$$

であり，円柱と xy 平面の共通部分は

$$D : \left(x-\frac{1}{2}\right)^2+y^2 \leqq \frac{1}{4}$$

である．対称性より

$$V = 2\iint_D \sqrt{1-x^2-y^2}\,dxdy$$

であるから，$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\text{ 周分} \end{pmatrix}$ とおくと

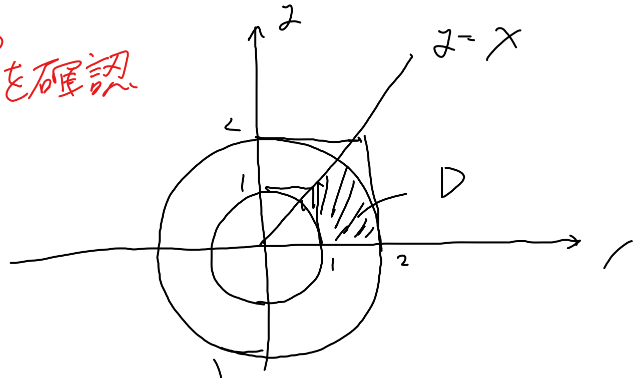
$$\begin{aligned}
V &= 2\iint_D \sqrt{1-x^2-y^2}\,dxdy \\
&= 2\iint_{D'} \sqrt{1-r^2}\cdot r\,dr\,d\theta \quad \left(D' : -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2},\ 0 \leqq r \leqq \cos\theta\right) \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{\cos\theta} r\sqrt{1-r^2}\,dr\right) d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{\cos\theta} (1-r^2)^{\frac{1}{2}}\cdot(-2r)\,dr\right\} d\theta \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_{r=0}^{r=\cos\theta} d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\left\{(\sin^2\theta)^{\frac{3}{2}}-1\right\} d\theta \\
&= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-|\sin^3\theta|)\,d\theta = \frac{4}{3}\int_0^{\frac{\pi}{2}} (1-\sin^3\theta)\,d\theta = \frac{4}{3}\left(\frac{\pi}{2}-\frac{2}{3}\cdot 1\right) = \frac{2}{3}\pi-\frac{8}{9}
\end{aligned}$$

【問題】

極座標変換を用いて次の重積分の値を求めよ.

tan θ の積分を確認

(1) $\displaystyle\iint_D \frac{y^2}{x^2}dxdy \quad (D: 1 \leqq x^2+y^2 \leqq 4,\ 0 \leqq y \leqq x)$



(1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ とおくと $\left(\begin{array}{l} r \geqq 0 \\ \theta \text{ は一周分} \end{array}\right)$

$$\iint_{D'} \tan^2\theta \, r\,dr\,d\theta \quad \left(D'\ 0 \leqq \theta \leqq \frac{\pi}{4},\ 1 \leqq r \leqq 2\right)$$

$$= \left(\int_1^2 r\,dr\right) \times \left(\int_0^{\frac{\pi}{4}} \tan^2\theta\,d\theta\right)$$

$$= \left[\frac{1}{2}r^2\right]_1^2 \times \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} - 1\right) d\theta$$

$2 - \frac{1}{2}$

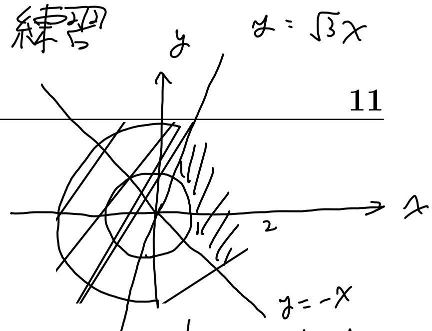
$$= \frac{3}{2} \times \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$$

$\frac{4}{2}$

$$= \frac{3}{2} \times \left(- \frac{\pi}{4}\right)$$

$$= \frac{3}{2} - \frac{3}{8}\pi$$

正確な計算ができるように練習



(2) $\iint_D \frac{y^5}{x^7}dxdy \quad \left(D : 1 \leqq x^2+y^2 \leqq 4,\ -x \leqq y \leqq \sqrt{3}\,x\right)$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ ($r \geqq 0$, θは一周分) とおくと

$$\iint_D \frac{y^5}{x^7}dxdy = \iint_{D'} \frac{1}{r^2}\cdot\frac{\sin^5\theta}{\cos^7\theta}\, r\, r\,d\theta = \iint_{D'} \frac{1}{r}\cdot\tan^5\theta\cdot\frac{1}{\cos^2\theta}\,drd\theta$$

$\left(D'\ 1 \leqq r \leqq 2,\ -\frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{3}\right)$

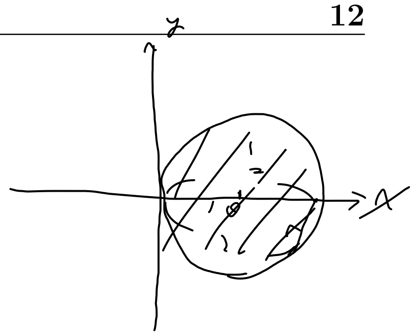
$$\iint_{D'} \frac{1}{r}\cdot\tan^5\theta\cdot\frac{1}{\cos^2\theta}\,drd\theta = \left(\int_1^2 \frac{1}{r}dr\right) \times \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^5\theta\cdot\frac{1}{\cos^2\theta}\,d\theta\right)$$

$$= \Big[\log|r|\Big]_1^2 \times \left[\frac{1}{6}\tan^6\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \log 2 \times \left(\frac{1}{6}\times 27 - \frac{1}{6}\times 1\right)$$

$$= \underline{\frac{13}{3}\log 2}$$

(3) $\iint_D (x^2+y^2)^{\frac{3}{2}}\,dxdy \quad (D: x^2+y^2 \leqq 2x)$



(3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ ($r \geqq 0$. θは一周分) とおくと

$\iint_{D'} (r^2)^{\frac{3}{2}}\, r\,dr\,d\theta \quad (D ; -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \quad 0 \leqq r \leqq 2\cos\theta)$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r^4\,dr \right) d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left[\frac{1}{5} r^5 \right]_0^{2\cos\theta} \right) d\theta$ $\qquad \frac{1}{5} \times 32\cos^5\theta$

$= \frac{32}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^5\theta)\, d\theta$

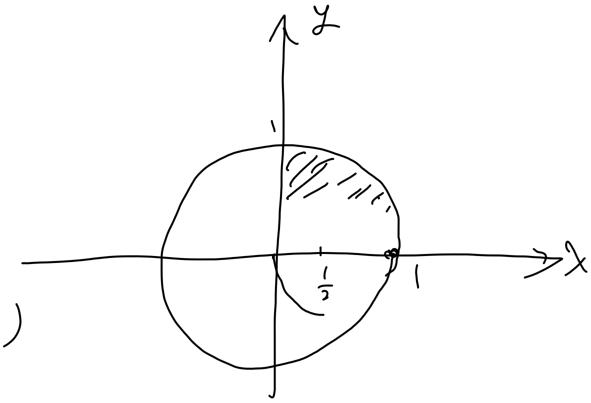
$= \frac{64}{5} \int_0^{\frac{\pi}{2}} \cos^5\, d\theta$

$= \frac{64}{5} \cdot \left(\frac{4 \cdot 2}{5 \cdot 3} \cdot 1 \right)$

$= \frac{64}{5} \cdot \frac{8}{15}$

$= \frac{512}{75}$

(4) $\iint_D x^2 dxdy \quad (D : x \leqq x^2 + y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ ($r \geqq 0$. θ は一周分) とおくと

$\iint_{D'} r^3 \cos^2\theta \, dr\, \theta \quad \left(0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1 \right)$

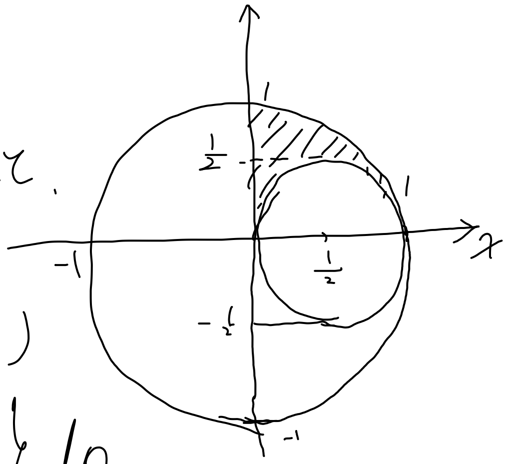
$= \int_0^{\frac{\pi}{2}} \left(\int_{\cos\theta}^{1} r^3 \cos^2\theta \, dr \right) d\theta$

$= \int_0^{\frac{\pi}{2}} \left(\left[\frac{1}{4} r^4 \cos^2\theta \right]_{\cos\theta}^{1} \right) d\theta$

2

$= \frac{3}{128}\pi$

(5) $\displaystyle\iint_D (7x-2y)\,dxdy \quad (D: x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $(r \geqq 0,$ θ は一周分) とおくと.

$\displaystyle\iint_{D'} (7r\cos\theta - 2r\sin\theta)\, r\,drd\theta \quad (D': 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1)$

$$\int_0^{\frac{\pi}{2}} \left\{ \int_{\cos\theta}^{1} (7r^2\cos\theta - 2r^2\sin\theta)\,dr \right\} d\theta$$

$$= \frac{11}{6} - \frac{7}{16}\pi$$

練習問題

極座標変換を用いて次の重積分の値を求めよ.

(1) $\displaystyle\iint_D \frac{1}{1+x^2+y^2}dxdy \quad (D: x^2+y^2 \leqq 1,\ y \geqq -x,\ x \geqq 0)$

(2) $\displaystyle\iint_D \frac{1}{(x^2+y^2)^3}dxdy \quad (D: 1 \leqq x^2+y^2 \leqq 9,\ y \geqq 0)$

(3) $\displaystyle\iint_D \frac{y}{x^3}dxdy \quad \left(D: 1 \leqq x^2+y^2 \leqq 4,\ -\sqrt{3}x \leqq y \leqq \frac{x}{\sqrt{3}}\right)$

(4) $\displaystyle\iint_D (5y-7)dxdy \quad (D: x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$

(5) $\displaystyle\iint_D (7x-2y)dxdy \quad (D: x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$

(6) $\displaystyle\iint_D x(3x-2y)dxdy \quad (D: x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$

(7) $\displaystyle\iint_D \frac{y}{x}dxdy \quad (D: 1 \leqq x^2+y^2 \leqq 2x,\ y \geqq 0)$

解答

(1) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1\ \text{周分} \end{pmatrix}$ とおくと

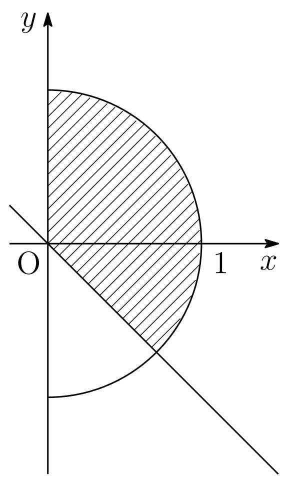
$$\iint_D \frac{1}{1+x^2+y^2}dxdy \quad (D: x^2+y^2 \leqq 1,\ y \geqq -x,\ x \geqq 0)$$

$$= \iint_{D'} \frac{1}{1+r^2}\cdot r dr d\theta \quad \left(D': 0 \leqq r \leqq 1,\ -\frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{2}\right)$$

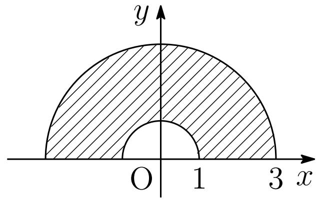
$$= \iint_{D'} \frac{r}{1+r^2}drd\theta = \left(\int_0^1 \frac{1}{2}\cdot\frac{2r}{1+r^2}dr\right)\times\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\right)$$

$$= \left[\frac{1}{2}\log(1+r^2)\right]_0^1 \times \left[\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2}(\log 2 - 0)\times\left\{\frac{\pi}{2}-\left(-\frac{\pi}{4}\right)\right\}$$

$$= \frac{3}{8}\pi\log 2$$



(2) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix}$ とおくと



$$\iint_D \frac{1}{(x^2+y^2)^3}dxdy \quad (D : 1 \leqq x^2+y^2 \leqq 9,\ y \geqq 0)$$

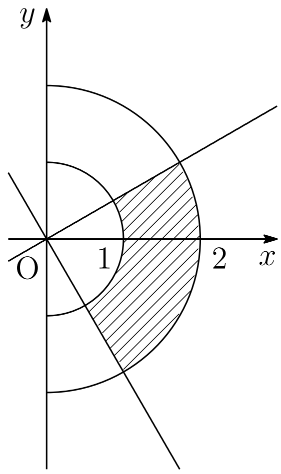
$$= \iint_{D'} \frac{1}{(r^2)^3} \cdot rdrd\theta \quad (D' : 1 \leqq r \leqq 3,\ 0 \leqq \theta \leqq \pi)$$

$$= \iint_{D'} \frac{1}{r^5}drd\theta = \left(\int_1^3 r^{-5}dr\right) \times \left(\int_0^\pi d\theta\right)$$

$$= \left[-\frac{1}{4}r^{-4}\right]_1^3 \times \left[\theta\right]_0^\pi = -\frac{1}{4}\left(\frac{1}{81}-1\right) \times \pi = \frac{20}{81}\pi$$

(3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta : 1 \text{ 周分} \end{pmatrix}$ とおくと

$$\iint_D \frac{y}{x^3}dxdy \quad \left(D : 1 \leqq x^2+y^2 \leqq 4,\ -\sqrt{3}x \leqq y \leqq \frac{x}{\sqrt{3}}\right)$$



$$= \iint_{D'} \frac{r\sin\theta}{r^3\cos^3\theta} \cdot rdrd\theta \quad \left(D' : 1 \leqq r \leqq 2,\ -\frac{\pi}{3} \leqq \theta \leqq \frac{\pi}{6}\right)$$

$$= \iint_{D'} \frac{1}{r} \cdot \frac{\sin\theta}{\cos^3\theta}drd\theta$$

$$= \left(\int_1^2 \frac{1}{r}dr\right) \times \left\{-\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} (\cos\theta)^{-3} \cdot (-\sin\theta)d\theta\right\}$$

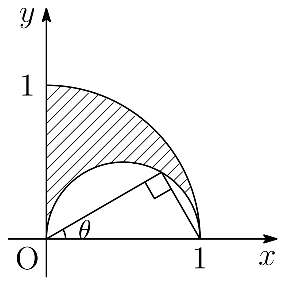
$$= \left[\log r\right]_1^2 \times \left\{-\left[-\frac{1}{2}(\cos\theta)^{-2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}\right\}$$

$$= (\log 2 - 0) \times \frac{1}{2}\left(\frac{4}{3}-4\right) = -\frac{4}{3}\log 2$$

※ $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sin\theta}{\cos^3\theta}d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \tan\theta \cdot \frac{1}{\cos^2\theta}d\theta = \left[\frac{1}{2}\tan^2\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{1}{3}-3\right) = -\frac{4}{3}$

でもよい．

(4) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{pmatrix} r \geqq 0 \\ \theta：1\text{ 周分} \end{pmatrix}$ とおくと



$$\iint_D (5y-7)dxdy \quad (D : x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$$

$$= \iint_{D'} (5r\sin\theta - 7)\cdot rdrd\theta \quad \left(D' : 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1\right)$$

$$= \iint_{D'} (5r^2\sin\theta - 7r)drd\theta = \int_0^{\frac{\pi}{2}} \left\{\int_{\cos\theta}^1 (5r^2\sin\theta - 7r)dr\right\} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{5}{3}r^3\sin\theta - \frac{7}{2}r^2\right]_{r=\cos\theta}^{r=1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left\{\frac{5}{3}(1-\cos^3\theta)\sin\theta - \frac{7}{2}(1-\cos^2\theta)\right\} d\theta$$

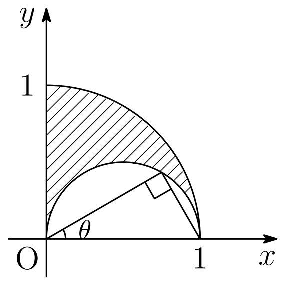
$$= \int_0^{\frac{\pi}{2}} \left[\frac{5}{3}\{\sin\theta + \cos^3\theta\cdot(-\sin\theta)\} - \frac{7}{2}\sin^2\theta\right] d\theta$$

$$= \frac{5}{3}\left[-\cos\theta + \frac{1}{4}\cos^4\theta\right]_0^{\frac{\pi}{2}} - \frac{7}{2}\cdot\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{5}{3}\left\{-(0-1) + \frac{1}{4}(0-1)\right\} - \frac{7}{8}\pi = \frac{5}{4} - \frac{7}{8}\pi$$

(5) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\begin{pmatrix} r \geqq 0 \\ \theta：1 \text{ 周分} \end{pmatrix}$ とおくと

$$\iint_D (7x - 2y)dxdy \quad (D : x \leqq x^2 + y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$$



$$= \iint_{D'} (7r\cos\theta - 2r\sin\theta) \cdot rdrd\theta$$

$$\left(D' : 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1\right)$$

$$= \iint_{D'} r^2(7\cos\theta - 2\sin\theta)drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left\{ \int_{\cos\theta}^1 r^2(7\cos\theta - 2\sin\theta)dr \right\} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 (7\cos\theta - 2\sin\theta) \right]_{r=\cos\theta}^{r=1} d\theta$$

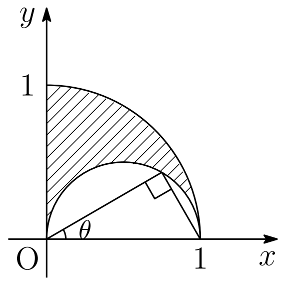
$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(1 - \cos^3\theta)(7\cos\theta - 2\sin\theta)d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \{7\cos\theta - 2\sin\theta - 7\cos^4\theta - 2\cos^3\theta \cdot (-\sin\theta)\}d\theta$$

$$= \frac{1}{3} \left\{ 7 \cdot 1 - 2 \cdot 1 - 7 \cdot \left(\frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \right) - \left[\frac{1}{2}\cos^4\theta \right]_0^{\frac{\pi}{2}} \right\}$$

$$= \frac{1}{3} \left\{ 5 - \frac{21}{16}\pi - \frac{1}{2}(0 - 1) \right\} = \frac{11}{6} - \frac{7}{16}\pi$$

(6) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\left(\begin{array}{l} r \geqq 0 \\ \theta：1 \text{ 周分} \end{array}\right)$ とおくと



$$\iint_D x(3x-2y)dxdy \quad (D : x \leqq x^2+y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0)$$

$$= \iint_{D'} r\cos\theta(3r\cos\theta - 2r\sin\theta)\cdot rdrd\theta$$

$$\left(D' : 0 \leqq \theta \leqq \frac{\pi}{2},\ \cos\theta \leqq r \leqq 1\right)$$

$$= \iint_{D'} r^3\cos\theta(3\cos\theta - 2\sin\theta)drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left\{\int_{\cos\theta}^1 r^3\cos\theta(3\cos\theta - 2\sin\theta)dr\right\}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}r^4\cos\theta(3\cos\theta-2\sin\theta)\right]_{r=\cos\theta}^{r=1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}(1-\cos^4\theta)\cos\theta(3\cos\theta-2\sin\theta)d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \{3(\cos^2\theta - \cos^6\theta) - 2\sin\theta\cdot\cos\theta - 2\cos^5\theta\cdot(-\sin\theta)\}d\theta$$

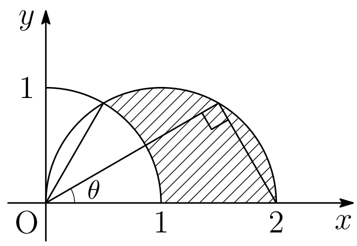
$$= \frac{1}{4}\left\{3\left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}\right) + \left[-\sin^2\theta - \frac{1}{3}\cos^6\theta\right]_0^{\frac{\pi}{2}}\right\}$$

$$= \frac{1}{4}\left\{\frac{9}{32}\pi - (1-0) - \frac{1}{3}(0-1)\right\} = \frac{9}{128}\pi - \frac{1}{6}$$

(7) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\begin{pmatrix} r \geqq 0 \\ \theta：1 \text{ 周分} \end{pmatrix}$ とおくと

$$\iint_D \frac{y}{x}dxdy \quad (D : 1 \leqq x^2 + y^2 \leqq 2x,\ y \geqq 0)$$

$$= \iint_{D'} \frac{r\sin\theta}{r\cos\theta} \cdot rdrd\theta \quad \left(D' : 0 \leqq \theta \leqq \frac{\pi}{3},\ 1 \leqq r \leqq 2\cos\theta\right)$$



$$= \iint_{D'} \frac{r\sin\theta}{\cos\theta}drd\theta = \int_0^{\frac{\pi}{3}} \left(\int_1^{2\cos\theta} r \cdot \frac{\sin\theta}{\cos\theta}dr\right)d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left[\frac{r^2}{2} \cdot \frac{\sin\theta}{\cos\theta}\right]_{r=1}^{r=2\cos\theta} d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{2}(4\cos^2\theta - 1) \cdot \frac{\sin\theta}{\cos\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \left(4\sin\theta \cdot \cos\theta + \frac{-\sin\theta}{\cos\theta}\right)d\theta$$

$$= \frac{1}{2}\Big[2\sin^2\theta + \log|\cos\theta|\Big]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left\{2\left(\frac{3}{4} - 0\right) + \left(\log\frac{1}{2} - 0\right)\right\} = \frac{3}{4} - \frac{1}{2}\log 2$$