第3講

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(例題1)

$f(z) = \cos z \quad (z = x + yi)$

$= \cos(x + iy) \quad \to \quad u(x,y) + i\,v(x,y)$ の形に

↓ 加法定理

$= \cos x \underbrace{\cos iy}_{\cosh y} - \sin x \underbrace{\sin iy}_{i\sinh y}$

$= \underbrace{\cos x \cosh y}_{u(x,y)} - i \underbrace{\sin x \sinh y}_{v(x,y)}$ //

$\left(\begin{aligned} \cosh x &= \frac{e^{x}+e^{-x}}{2} \\ \sinh x &= \frac{e^{x}-e^{-x}}{2} \end{aligned} \right)$

$(e^{ix} = \cos x + i\sin x)$

(例題2)

$f = \sinh(x + yi)$

$= \sin i(y - ix) = i\sin(y - ix)$

↓ 減法定理

$= i(\sin y \underbrace{\cos ix}_{\cosh x} - \cos y \underbrace{\sin ix}_{i\sinh x})$

$= i\sin y \cosh x - i^2 \cos y \sinh x$

$= \sinh x \cos y + i\cosh x \sin y$ //

$\times \dfrac{1 \times i}{i \times i}$

$-i$

(演習)

問1 次の加法定理を証明せよ.

(1) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(2) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

問2 例題1にならって. $\sin(x+iy)$ を $\sin x$. $\cos x$ $\sinh y$, $\cosh y$ で表式を導け.

問3 例題2にならって. $\cosh(x+iy)$ を $\cosh x$、$\sinh x$, $\cos y$, $\sin y$ で表式を導け.

問4 例題1.2にならって、次の複素数を $u+iv$ の形で表せ.

(1) $\cos(2+i)$

(2) $\sinh(1+2i)$

(3) $\sin\left(\frac{\pi}{4} + 2i\right)$

(4) $\cosh\left(2 + \frac{\pi}{4}i\right)$

$\left(\begin{aligned} \cos x &= \frac{e^{ix}+e^{-ix}}{2} = \cosh ix \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} = -i\sinh ix \end{aligned} \right)$

問2

$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$ (isinh y)

$= \sin x \cosh x + i\cos x \sinh y$

$\left(\begin{aligned} \cos iy &= \frac{e^{-y}+e^{y}}{2} = \cosh x \\ \sin iy &= \frac{e^{-y}-e^{y}}{2i} = +i\sinh x \end{aligned} \right)$

問3

$\cosh(x+iy) = \cos(hx + hiy)$

$= \cosh x \cos hiy + \sinh x \sin ihy$

$= \cosh x \cos y + i\sinh x \sin y$

問1

(1)

$\cosh(z_1 + z_2) = \cos(hz_1 + hz_2) = \dfrac{e^{i(hz_1+hz_2)} + e^{-i(hz_1+hz_2)}}{2}$

$= \dfrac{e^{ihz_1}\cdot e^{ihz_2} + e^{-ihz_1} + e^{-ihz_2}}{2}$

$\cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2 = \dfrac{e^{ihz_1}+e^{-ihz_1}}{2} \cdot \dfrac{e^{ihz_2}+e^{-ihz_2}}{2} - \dfrac{e^{ihz_1}-e^{-ihz_1}}{2i} \cdot \dfrac{e^{ihz_2}-e^{-ihz_2}}{2i}$

$= \dfrac{e^{ihz_1}\cdot e^{ihz_2} + \cancel{e^{ihz_1}\cdot e^{-ihz_2}} + \cancel{e^{-ihz_1}\cdot e^{ihz_2}} + e^{-ihz_1}\cdot e^{-ihz_2}}{4} + \dfrac{e^{ihz_1}\cdot e^{ihz_2} - \cancel{e^{ihz_1}\cdot e^{-ihz_2}} - \cancel{e^{-ihz_1}\cdot e^{ihz_2}} + e^{-ihz_1}\cdot e^{-ihz_2}}{4}$

$= \dfrac{e^{ihz_1}\cdot e^{ihz_2} + e^{-ihz_1}\cdot e^{-ihz_2}}{2}$

$= \cosh(z_1 + z_2)$

よって 証明された

(2) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

(左辺) $= \sinh(z_1 + z_2) \cdot \sin(hz_1 + hz_2)$

$= \dfrac{e^{i(hz_1+hz_2)} - e^{-i(hz_1+hz_2)}}{2i}$

$= \dfrac{e^{ihz_1}\cdot e^{ihz_2} - e^{-ihz_1}\cdot e^{-ihz_2}}{2i}$

$$\overline{\qquad\qquad}$$
$$2i$$

$$= \frac{e^{ihz_1}\cdot e^{ihz_2} - e^{-ihz_1}\cdot e^{-ihz_2}}{2i}$$

(右辺). $\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$$= \frac{e^{ihz_1} - e^{-ihz_1}}{2i}\cdot\frac{e^{ihz_2}+e^{-ihz_2}}{2} + \frac{e^{ihz_1}+e^{-ihz_1}}{2}\cdot\frac{e^{ihz_2}-e^{-ihz_2}}{2i}$$

$$= \frac{e^{ihz_1}\cdot e^{ihz_2} + \cancel{e^{ihz_1}\cdot e^{-ihz_2}} - \cancel{e^{-ihz_1}\cdot e^{ihz_2}} - e^{-ihz_1}\cdot e^{-ihz_2}}{4i} + \frac{e^{ihz_1}\cdot e^{ihz_2} - \cancel{e^{ihz_1}\cdot e^{-ihz_2}} + \cancel{e^{-ihz_1}\cdot e^{ihz_2}} - e^{-ihz_1}\cdot e^{-ihz_2}}{4i}$$

$$= \frac{e^{ihz_1}e^{ihz_2} - e^{-ihz_1}e^{-ihz_2}}{2i}$$

$=$ (左辺)

よって 証明された.

$$\left(\begin{array}{l} \cos x = \cosh ix \\ \sin x = -i\sinh ix \\ \cos ix = \cosh x \\ \sin ix = i\sinh x \end{array}\right)$$

問4

(1) $\cos(2+i) = \cos 2\cos i - \sin 2\sin i$

$= \cos 2\cdot\cosh 1 - \sin 2\cdot i\sinh 1$

$= \dfrac{e^2+1}{2e}\cos 2 - i\dfrac{e^2-1}{2e}\sin 2$

$= \dfrac{e^2+1}{2e}\cos 2 + i\left(\dfrac{1-e^2}{2e}\right)\sin 2$

(2)

$\sinh(1+2i) = \sinh(i(2-i))$

$= i\sin(2-i)$

$= i(\sin 2\cos i - \cos 2\sin i)$

$= i(\sin 2\cdot\cosh 1 - \cos 2\cdot i\sinh 1)$

$= i\left(\dfrac{e^2+1}{2e}\cdot\sin 2 - i\left(\dfrac{e^2-1}{2e}\right)\cos 2\right)$

$= \left(\dfrac{e^2-1}{2e}\right)\cos 2 + i\left(\dfrac{e^2+1}{2e}\right)\sin 2$

(3) $\sin\left(\frac{\pi}{4}+2i\right) = \sin\frac{\pi}{4}\cos 2i + \cos\frac{\pi}{4}\sin 2i$

$= \dfrac{1}{\sqrt{2}}\cos 2i + \dfrac{1}{\sqrt{2}}\sin 2i$

$= \dfrac{1}{\sqrt{2}}\cdot\cosh 2 + \dfrac{1}{\sqrt{2}}\cdot i\sinh 2$

$= \dfrac{\cosh 2}{\sqrt{2}} + i\cdot\dfrac{\sinh 2}{\sqrt{2}}$

(4) $\cosh\left(2+\frac{\pi}{4}i\right) = \cosh i\left(-2i+\frac{\pi}{4}\right)$

$= \cosh i\left(\frac{\pi}{4}-2i\right)$

$= \cos\left(\frac{\pi}{4}-2i\right)$

$= \cos\frac{\pi}{4}\cos 2i + \sin\frac{\pi}{4}\sin 2i$

$= \dfrac{1}{\sqrt{2}}\cos 2i + \dfrac{1}{\sqrt{2}}\sin 2i$

$= \dfrac{1}{\sqrt{2}}\cosh 2 + i\cdot\dfrac{1}{\sqrt{2}}\sinh 2$