応用数学1

第13回目

演習レポート課題　解答

(1) $y'' - 6y' - 16y = 8$

$\boxed{1}$ $y'' - 6y' - 16y = 0$：

$y = e^{px}$とすると、

固有方程式：$p^2 - 6p - 16 = 0$

$$(p-8)(p+2) = 0$$

$$\therefore p = 8, -2$$

基本解：e^{8x}, e^{-2x}

一般解：$y = Ae^{8x} + Be^{-2x}$ （A, Bは任意定数）

$\boxed{2}$ $A, B \to A(x), B(x)$：

$y = A(x)e^{8x} + B(x)e^{-2x}$

$\begin{cases} A'(x)e^{8x} + B'(x)e^{-2x} = 0 \quad \text{―――} ① \end{cases}$

$\boxed{2}$ $A, B \to A(x), B(x)$:

$$y = A(x)e^{8x} + B(x)e^{-2x}$$

$$\begin{cases} A'(x)e^{8x} + B'(x)e^{-2x} = 0 & \text{——①} \\ A'(x)\cdot 8e^{8x} + B'(x)\cdot(-2e^{-2x}) = 8 & \text{——②} \end{cases}$$

①より、$A'(x) = -B'(x)e^{-10x}$ ——③

②に代入、$-B'(x)\cdot 8e^{-2x} - 2B'(x)e^{-2x} = 8$

$$B'(x) = -\frac{4}{5}e^{2x}$$

③に代入、$A'(x) = \frac{4}{5}e^{2x}\cdot e^{-10x} = \frac{4}{5}e^{-8x}$

$$A(x) = \frac{4}{5}\int e^{-8x}dx = \frac{4}{5}\cdot\left(-\frac{1}{8}\right)e^{-8x} + C_1$$

$$= -\frac{1}{10}e^{-8x} + C_1$$

$$B(x) = -\frac{4}{5}\int e^{2x}dx = -\frac{4}{5}\cdot\frac{1}{2}e^{2x} + C_2$$

$$= -\frac{2}{5}e^{2x} + C_2 \quad (C_1, C_2: 積分定数)$$

ゆえに、与式の一般解は、

$$y = \left(-\frac{1}{10}e^{-8x} + C_1\right)e^{8x} + \left(-\frac{2}{5}e^{2x} + C_2\right)e^{-2x}$$

②に代入、$-B'(x)\cdot 8e^{-2x}-2B'(x)e^{-2x}=8$

$$B'(x)=-\frac{4}{5}e^{2x}$$

③に代入、$A'(x)=\frac{4}{5}e^{2x}\cdot e^{-10x}=\frac{4}{5}e^{-8x}$

$$A(x)=\frac{4}{5}\int e^{-8x}dx=\frac{4}{5}\cdot\left(-\frac{1}{8}\right)e^{-8x}+C_1$$

$$=-\frac{1}{10}e^{-8x}+C_1$$

$$B(x)=-\frac{4}{5}\int e^{2x}dx=-\frac{4}{5}\cdot\frac{1}{2}e^{2x}+C_2$$

$$=-\frac{2}{5}e^{2x}+C_2 \quad (C_1, C_2:\text{積分定数})$$

ゆえに、与式の一般解は、

$$y=\left(-\frac{1}{10}e^{-8x}+C_1\right)e^{8x}+\left(-\frac{2}{5}e^{2x}+C_2\right)e^{-2x}$$

$$=-\frac{1}{10}-\frac{2}{5}+C_1e^{8x}+C_2e^{-2x}$$

$$=\underline{-\frac{1}{2}+C_1e^{8x}+C_2e^{-2x}}\ /\!/$$

(2) $y''-3y'+2y=e^{3x}$

$\boxed{1}$ $y''-3y'+2y=0$:

$y=e^{px}$ とすると、

固有方程式 : $p^3-3p+2=0$

$(p-2)(p-1)=0$

$\therefore p=1, 2$

基本解 : e^{x}, e^{2x}

一般解 : $y=Ae^{x}+Be^{2x}$ (A, B : 任意定数)

$\boxed{2}$ $A, B \longrightarrow A(x), B(x)$:

$y=A(x)e^{x}+B(x)e^{2x}$

$$\begin{cases} A'(x)e^{x}+B'(x)e^{2x}=0 & \text{—— ①} \\ A'(x)e^{x}+B'(x)\cdot 2e^{2x}=e^{3x} & \text{—— ②} \end{cases}$$

①より、$A'(x)=-B'(x)e^{x}$ —— ③

②に代入、$-B'(x)e^{2x}+2B'(x)e^{2x}=e^{3x}$

$\boxed{2}$ $A, B \to A(x), B(x)$:

$$y = A(x)e^{x} + B(x)e^{2x}$$

$$\begin{cases} A'(x)e^{x} + B'(x)e^{2x} = 0 & \text{①} \\ A'(x)e^{x} + B'(x)\cdot 2e^{2x} = e^{3x} & \text{②} \end{cases}$$

①より、$A'(x) = -B'(x)e^{x}$ ③

②に代入、$-B'(x)e^{2x} + 2B'(x)e^{2x} = e^{3x}$

$$B'(x) = e^{x}$$

③に代入、$A'(x) = -e^{x}\cdot e^{x} = -e^{2x}$

$$A(x) = -\int e^{2x}dx = -\frac{1}{2}e^{2x} + C_1$$

$$B(x) = \int e^{x}dx = e^{x} + C_2 \quad (C_1, C_2: \text{積分定数})$$

ゆえに、与式の一般解は、

$$y = \left(-\frac{1}{2}e^{2x} + C_1\right)e^{x} + \left(e^{x} + C_2\right)e^{2x}$$

$$= -\frac{1}{2}e^{3x} + e^{3x} + C_1e^{x} + C_2e^{2x}$$

$$= \frac{1}{2}e^{3x} + C_1e^{x} + C_2e^{2x}$$

[2] $A, B \to A(x), B(x)$:

$$y = A(x)e^{x} + B(x)e^{2x}$$

$$\begin{cases} A'(x)e^{x} + B'(x)e^{2x} = 0 & \text{———①} \\ A'(x)e^{x} + B'(x)\cdot 2e^{2x} = e^{3x} & \text{———②} \end{cases}$$

①より、$A'(x) = -B'(x)e^{x}$ ———③

②に代入、$-B'(x)e^{2x} + 2B'(x)e^{2x} = e^{3x}$

$$B'(x) = e^{x}$$

③に代入、$A'(x) = -e^{x}\cdot e^{x} = -e^{2x}$

$$A(x) = -\int e^{2x}dx = -\frac{1}{2}e^{2x} + C_1$$

$$B(x) = \int e^{x}dx = e^{x} + C_2 \quad (C_1, C_2 : 積分定数)$$

ゆえに、与式の一般解は、

$$y = \left(-\frac{1}{2}e^{2x} + C_1\right)e^{x} + (e^{x} + C_2)e^{2x}$$

$$= -\frac{1}{2}e^{3x} + e^{3x} + C_1e^{x} + C_2e^{2x}$$

$$= \frac{1}{2}e^{3x} + C_1e^{x} + C_2e^{2x}$$

(3) $y'' - 2y' + y = e^x$

$\boxed{1}$ $y'' - 2y' + y = 0$：

$y = e^{px}$とすると、

固有方程式：$p^2 - 2p + 1 = 0$

$(p-1)^2 = 0$

$\therefore p = 1$：重解

基本解：e^x, xe^x

一般解：$y = Ae^x + Bxe^x$ (A, B：任意定数)

$\boxed{2}$ $A, B \longrightarrow A(x), B(x)$：

$y = A(x)e^x + B(x)xe^x$

$$\begin{cases} A'(x)e^x + B'(x)xe^x = 0 & \text{——① } \\ A'(x)e^x + B'(x)(e^x + xe^x) = e^x & \text{——② } \end{cases}$$

①より、$A'(x) = -B'(x)x$ ——③

②に代入、$\cancel{-B'(x)xe^x} + B'(x)(e^x \cancel{+ xe^x}) = e^x$

$\boxed{2}$ $A, B \longrightarrow A(x), B(x)$:

$$y = A(x)e^x + B(x)xe^x$$

$$\begin{cases} A'(x)e^x + B'(x)xe^x = 0 & \text{——— ①} \\ A'(x)e^x + B'(x)(e^x + xe^x) = e^x & \text{——— ②} \end{cases}$$

①より、$A'(x) = -B'(x)x$ ——— ③

②に代入、$\cancel{-B'(x)xe^x} + B'(x)(e^x \cancel{+xe^x}) = e^x$

$$B'(x) = 1$$

③に代入、$A'(x) = -x$

$$\begin{cases} A(x) = -\int x\,dx = -\frac{1}{2}x^2 + c_1 \\ B(x) = \int 1\,dx = x + c_2 \end{cases} \quad (c_1, c_2 : 積分定数)$$

ゆえに、与式の一般解は、

$$y = \left(-\frac{1}{2}x^2 + c_1\right)e^x + (x + c_2)xe^x$$

$$= \frac{1}{2}x^2e^x + c_1e^x + c_2xe^x$$

(4) $y'' + 8y' + 17y = 2e^{-3x}$

$\boxed{1}$ $y'' + 8y' + 17y = 0$:

$y = e^{\rho x}$ とすると、

固有方程式 : $\rho^2 + 8\rho + 17 = 0$

$$\rho = -4 \pm \sqrt{-1} = -4 \pm i$$

基本解 : $e^{(-4+i)x}$, $e^{(-4-i)x}$

一般解 : $y = Ae^{(-4+i)x} + Be^{(-4-i)x}$ (A, B : 任意定数)

$$= Ae^{-4x}e^{ix} + Be^{-4x}e^{-ix}$$

$$= e^{-4x}\{A(\cos x + i\sin x) + B(\cos x - i\sin x)\}$$

$\downarrow$ $C \equiv A + B$, $D \equiv (A - B)i$

$$= e^{-4x}(C\cos x + D\sin x)$$

$\boxed{2}$ $C, D \longrightarrow C(x), D(x)$:

$y = e^{-4x}(C(x)\cos x + D(x)\sin x)$

$\boxed{2}$ $C, D \to C(x), D(x)$：

$$y = e^{-4x}\left(C(x)\cos x + D(x)\sin x\right)$$

$$\begin{cases} C'(x)e^{-4x}\cos x + D'(x)e^{-4x}\sin x = 0 & \text{①} \\ C'(x)\left(-4e^{-4x}\cos x - e^{-4x}\sin x\right) \\ \quad + D'(x)\left(-4e^{-4x}\sin x + e^{-4x}\cos x\right) = 2e^{-3x} & \text{②} \end{cases}$$

①より、$C'(x) = -D'(x)\dfrac{\sin x}{\cos x}$ ③

(② $\times\ e^{4x}$) に代入、

$$-D'(x)\left(\cancel{-4\sin x} - \frac{\sin^2 x}{\cos x}\right) + D'(x)\left(\cancel{-4\sin x} + \cos x\right) = 2e^{x}$$

$$D'(x)(\sin^2 x + \cos^2 x) = 2e^{x}\cos x$$

$$D'(x) = 2e^{x}\cos x$$

③に代入、$C'(x) = -2e^{x}\sin x$

$$\begin{cases} C(x) = -2\displaystyle\int e^{x}\sin x\,dx \\ D(x) = 2\displaystyle\int e^{x}\cos x\,dx \end{cases}$$

$\boxed{2}$ $C, D \to C(x), D(x)$:

$$y = e^{-4x}\left(C(x)\cos x + D(x)\sin x\right)$$

$$\begin{cases} C'(x)e^{-4x}\cos x + D'(x)e^{-4x}\sin x = 0 & \text{— ①} \\ C'(x)\left(-4e^{-4x}\cos x - e^{-4x}\sin x\right) \\ \quad + D'(x)\left(-4e^{-4x}\sin x + e^{-4x}\cos x\right) = 2e^{-3x} & \text{— ②} \end{cases}$$

①より、$C'(x) = -D'(x)\dfrac{\sin x}{\cos x}$ — ③

(② $\times e^{4x}$) に代入、

$$-D'(x)\left(\cancel{-4\sin x} - \frac{\sin^2 x}{\cos x}\right) + D'(x)\left(\cancel{-4\sin x} + \cos x\right) = 2e^{x}$$

$$D'(x)\left(\sin^2 x + \cos^2 x\right) = 2e^{x}\cos x$$

$$D'(x) = 2e^{x}\cos x$$

③に代入、$C'(x) = -2e^{x}\sin x$

$$\begin{cases} C(x) = -2\displaystyle\int e^{x}\sin x\,dx \\ D(x) = 2\displaystyle\int e^{x}\cos x\,dx \end{cases}$$

$$
\begin{array}{rl}
 & (e^x \sin x)' = e^x \sin x + e^x \cos x \\
-) & (e^x \cos x)' = e^x \cos x - e^x \sin x \\
\hline
 & (e^x \sin x)' - (e^x \cos x)' = 2e^x \sin x
\end{array}
$$

$$\Downarrow$$

$$e^x \sin x - e^x \cos x = 2\int e^x \sin x \, dx$$

$$(e^x \sin x)' = e^x \sin x + e^x \cos x$$

$$-\,\big)\; (e^x \cos x)' = e^x \cos x - e^x \sin x$$

$$(e^x \sin x)' - (e^x \cos x)' = 2e^x \sin x$$

$$\Downarrow$$

$$e^x \sin x - e^x \cos x = 2\int e^x \sin x \, dx$$

$$
\begin{array}{r}
+) \quad \\
\hline
\end{array}
$$

$$(e^x \sin x)' + (e^x \cos x)' = 2e^x \cos x$$

$$\Downarrow$$

$$e^x \sin x + e^x \cos x = 2\int e^x \cos x \, dx$$

$$
\begin{cases}
c(x) = e^x(\cos x - \sin x) + c_1 \\
b(x) = e^x(\cos x + \sin x) + c_2 \quad (c_1, c_2 \text{：積分定数})
\end{cases}
$$

ゆえに、与式の一般解は、

$$
\begin{aligned}
y &= e^{-4x}\{(e^x(\cos x - \sin x) + c_1)\cos x \\
&\quad + (e^x(\cos x + \sin x) + c_2)\sin x\} \\
&= e^{-4x}\{e^x(\cos^2 x - \cancel{\sin x \cos x} + c_1 \cos x \\
&\quad + \cancel{\sin x \cos x} + \sin^2 x + c_2 \sin x\}
\end{aligned}
$$

$$\Downarrow$$

$$e^x\sin x + e^x\cos x = 2\int e^x \cos x\,dx$$

$$\begin{cases} a(x) = e^x(\cos x - \sin x) + c_1 \\ b(x) = e^x(\cos x + \sin x) + c_2 \end{cases} \quad (c_1, c_2：積分定数)$$

ゆえに、与式の一般解は、

$$\begin{aligned} y &= e^{-4x}\left\{\left(e^x(\cos x - \sin x) + c_1\right)\cos x \right.\\ &\qquad \left. + \left(e^x(\cos x + \sin x) + c_2\right)\sin x\right\} \\ &= e^{-4x}\left\{e^x\left(\cos^2 x - \cancel{\sin x\cos x}\right) + c_1\cos x \right.\\ &\qquad \left. + \cancel{\sin x\cos x} + \sin^2 x + c_2\sin x\right\} \\ &= \underline{e^{-3x} + e^{-4x}(c_1\cos x + c_2\sin x)} \end{aligned}$$

(5) $y'' + y = \sin x$

$\boxed{1}$ $y'' + y = 0$:

$y = e^{px}$とすると、

固有方程式 : $p^2 + 1 = 0$

$\therefore p = \pm i \longrightarrow$ 基本解 : e^{ix}, e^{-ix}

一般解 : $y = Ae^{ix} + Be^{-ix}$ (A, B : 任意定数)

$= A(\cos x + i\sin x) + B(\cos x - i\sin x)$

$C \equiv A + B,\ D \equiv (A - B)i$

$= C\cos x + D\sin x$

$\boxed{2}$ $C, D \rightarrow C(x), D(x)$:

$y = C(x)\cos x + D(x)\sin x$

$$\begin{cases} C'(x)\cos x + D'(x)\sin x = 0 & \text{①} \\ -C'(x)\sin x + D'(x)\cos x = \sin x & \text{②} \end{cases}$$

$\boxed{2}$ $C, D \to C(x), D(x)$:

$$y = C(x)\cos x + D(x)\sin x$$

$$\begin{cases} C'(x)\cos x + D'(x)\sin x = 0 & \text{——— ①} \\ -C'(x)\sin x + D'(x)\cos x = \sin x & \text{——— ②} \end{cases}$$

①より、$C'(x) = -D'(x)\dfrac{\sin x}{\cos x}$ ——— ③

②に代入、$D'(x)\dfrac{\sin^2 x}{\cos x} + D'(x)\cos x = \sin x$

$$D'(x)\sin^2 x + D'(x)\cos^2 x = \sin x\cos x$$

$$D'(x) = \sin x\cos x$$

③に代入、$C'(x) = -\sin^2 x$

$$C(x) = -\int \sin^2 x\,dx$$

$$\begin{aligned} \cos 2x &= \cos(x+x) \\ &= \cos x\cos x - \sin x\sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

③に代入、$C'(x) = -\sin^2 x$

$$C(x) = -\int \sin^2 x \, dx$$

$$\begin{aligned}\cos 2x &= \cos(x+x) \\ &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x\end{aligned}$$

$$= \frac{1}{2}\int (\cos 2x - 1)\, dx$$

$$= \frac{1}{4}\sin 2x - \frac{1}{2}x + C_1$$

$$D(x) = \int \sin x \cos x \, dx$$

$$= \frac{1}{2}\int \sin 2x \, dx$$

$$D(x) = \int \sin x \cos x \, dx$$

$$= \frac{1}{2}\int \sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + C_2 \quad (C_1, C_2 : \text{積分定数})$$

ゆえに、与式の一般解は、

$$y = \left(\frac{1}{4}\sin 2x - \frac{1}{2}x + C_1\right)\cos x$$

$$+\left(-\frac{1}{4}\cos 2x + C_2\right)\sin x$$

$$= \frac{1}{4}(\sin 2x \cos x - \cos 2x \sin x) - \frac{1}{2}x\cos x$$

$$+ C_1 \cos x + C_2 \sin x$$

$$= \frac{1}{4}\sin(2x - x) - \frac{1}{2}x\cos x + C_1\cos x + C_2 \sin x$$

$$= \frac{1}{4}\sin x - \frac{1}{2}x\cos x + C_1 \cos x + C_2 \sin x$$

$$= -\frac{1}{2}x\cos x + C_1\cos x + C_3 \sin x \quad \left(C_3 = \frac{1}{4} + C_2\right)$$