代表的な関数の **Maclaurin** 展開（再掲載）

覚える必要なし

(1) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R})$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \quad (x \in \mathbb{R})$$

(2) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad (x \in \mathbb{R})$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \cdots \quad (x \in \mathbb{R})$$

(3) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (x \in \mathbb{R})$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots \quad (x \in \mathbb{R})$$

(4) $\log(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad (-1 < x \leqq 1)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad (-1 < x \leqq 1)$$

(5) $(1+x)^\alpha = 1 + \displaystyle\sum_{n=1}^{\infty} \binom{\alpha}{n} x^n \quad (-1 < x < 1)$

ただし，$\alpha \neq 0, 1, 2, 3, \ldots$ とし，

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdot\cdots\cdot(\alpha-n+1)}{n!}$$

は一般二項係数である．

(6) $\sqrt{1+x} = 1 + \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n} x^n \quad (-1 \leqq x \leqq 1)$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots \quad (-1 \leqq x \leqq 1)$$

(7) $\dfrac{1}{\sqrt{1+x}} = 1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n} x^n \quad (-1 < x \leqq 1)$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \cdots \quad (-1 < x \leqq 1)$$

(8) $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \quad (-1 < x < 1)$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots \quad (-1 < x < 1)$$

(9) $$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad (-1 \leqq x \leqq 1)$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad (-1 \leqq x \leqq 1)$$

(10) $$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n} \cdot \frac{x^{2n+1}}{2n+1} \quad (-1 \leqq x \leqq 1)$$
$$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots \quad (-1 \leqq x \leqq 1)$$

例 4.4

代表的な関数の Maclaurin 展開を用いて，次の関数の Maclaurin 展開をかっこ内の項まで求めよ．ただし，係数は既約分数にすること．

(1) $\dfrac{e^x}{\sqrt{1+x}}$ （4 次以下）　(2) $\sqrt{1+\dfrac{2x}{3}-x^2}$ （4 次以下）

(3) $e^{x\cos x}$ （5 次以下）　(4) $\dfrac{1}{\cos x}$ （6 次以下）

解答

(1) e^x と $\dfrac{1}{\sqrt{1+x}}$ の Maclaurin 展開の式をかける．そのとき，5 次以上の項は省略すれば

e^x　　$\frac{1}{\sqrt{1+x}}$

$$\frac{e^x}{\sqrt{1+x}} = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)\left(1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4-\cdots\right)$$

$$\begin{array}{rcccccccccc}
= 1 & - & \frac{1}{2}x & + & \frac{3}{8}x^2 & - & \frac{5}{16}x^3 & + & \frac{35}{128}x^4 & + & \cdots \\
 & + & x & - & \frac{1}{2}x^2 & + & \frac{3}{8}x^3 & - & \frac{5}{16}x^4 & + & \cdots \\
 & & & + & \frac{1}{2}x^2 & - & \frac{1}{4}x^3 & + & \frac{3}{16}x^4 & + & \cdots \\
 & & & & & + & \frac{1}{6}x^3 & - & \frac{1}{12}x^4 & + & \cdots \\
 & & & & & & & + & \frac{1}{24}x^4 & + & \cdots
\end{array}$$

同類項を縦に並べて書く.

$$= 1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{48}x^3+\frac{41}{384}x^4+\cdots$$

(2) $\sqrt{1+x}$ の Maclaurin 展開の式

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots$$

において x を $\dfrac{2x}{3} - x^2$ におきかえる．そのとき，5 次以上の項は省略すれば

$$\begin{aligned}
\sqrt{1+\frac{2x}{3}-x^2} &= 1 + \frac{1}{2}\left(\frac{2x}{3}-x^2\right) - \frac{1}{8}\left(\frac{2x}{3}-x^2\right)^2 + \frac{1}{16}\left(\frac{2x}{3}-x^2\right)^3 \\
&\qquad - \frac{5}{128}\left(\frac{2x}{3}-x^2\right)^4 + \cdots \\
&= 1 + \frac{1}{2}\left(\frac{2}{3}x - x^2\right) - \frac{1}{8}\left(\frac{4}{9}x^2 - \frac{4}{3}x^3 + x^4\right) \\
&\qquad + \frac{1}{16}\left(\frac{8}{27}x^3 - \frac{4}{3}x^4 + \cdots\right) - \frac{5}{128}\left(\frac{16}{81}x^4 + \cdots\right) + \cdots \\
&= 1 + \frac{1}{3}x - \frac{1}{2}x^2 \\
&\qquad - \frac{1}{18}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 \\
&\qquad + \frac{1}{54}x^3 - \frac{1}{12}x^4 + \cdots \\
&\qquad - \frac{5}{648}x^4 + \cdots \\
&= 1 + \frac{1}{3}x - \frac{5}{9}x^2 + \frac{5}{27}x^3 - \frac{35}{162}x^4 + \cdots
\end{aligned}$$

(3) $e^{x} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \cdots$

代入

$\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \dfrac{x^8}{40320} - \cdots$

より

$$
\begin{aligned}
e^{x\cos x} &= 1 + x\cos x + \frac{1}{2}(x\cos x)^2 + \frac{1}{6}(x\cos x)^3 + \frac{1}{24}(x\cos x)^4 + \frac{1}{120}(x\cos x)^5 + \cdots \\
&= 1 + \left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots\right) + \frac{1}{2}\left(x - \frac{1}{2}x^3 + \cdots\right)^2 \\
&\qquad + \frac{1}{6}\left(x - \frac{1}{2}x^3 + \cdots\right)^3 + \frac{1}{24}(x - \cdots)^4 + \frac{1}{120}(x - \cdots)^5 + \cdots \\
&= 1 + \left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + \cdots\right) + \frac{1}{2}(x^2 - x^4 + \cdots) \\
&\qquad + \frac{1}{6}\left(x^3 - \frac{3}{2}x^5 + \cdots\right) + \frac{1}{24}(x^4 + \cdots) + \frac{1}{120}(x^5 + \cdots) + \cdots
\end{aligned}
$$

$$
\begin{array}{rcccccccccccc}
= & 1 & + & x & & & - & \frac{1}{2}x^3 & & & + & \frac{1}{24}x^5 & + \cdots \\
 & & & & + & \frac{1}{2}x^2 & & & - & \frac{1}{2}x^4 & & & + \cdots \\
 & & & & & & + & \frac{1}{6}x^3 & & & - & \frac{1}{4}x^5 & + \cdots \\
 & & & & & & & & + & \frac{1}{24}x^4 & & & + \cdots \\
 & & & & & & & & & & + & \frac{1}{120}x^5 & + \cdots
\end{array}
$$

$$
= 1 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{11}{24}x^4 - \frac{1}{5}x^5 + \cdots
$$

(4) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots$$

より

$$\begin{aligned}
\frac{1}{\cos x} &= \frac{1}{1 + (\cos x - 1)} \\
&= 1 - (\cos x - 1) + (\cos x - 1)^2 - (\cos x - 1)^3 + \cdots \\
&= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\
&\qquad\qquad + \left(-\frac{x^2}{2} + \frac{x^4}{24} + \cdots \right)^2 - \left(-\frac{x^2}{2} + \cdots \right)^3 + \cdots \\
&= 1 - \left(-\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \right) \\
&\qquad\qquad + \left(\frac{1}{4}x^4 - \frac{1}{24}x^6 + \cdots \right) - \left(-\frac{1}{8}x^6 + \cdots \right) + \cdots \\
&= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots \\
&\qquad\qquad + \frac{1}{4}x^4 - \frac{1}{24}x^6 + \cdots \\
&\qquad\qquad\qquad + \frac{1}{8}x^6 + \cdots \\
&= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots
\end{aligned}$$

$$
\begin{aligned}
※\ \tan x &= \sin x \times \frac{1}{\cos x} \\
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots\right) \\
&= x + \frac{1}{2}x^3 + \frac{5}{24}x^5 + \frac{61}{720}x^7 + \cdots \\
&\qquad - \frac{1}{6}x^3 - \frac{1}{12}x^5 - \frac{5}{144}x^7 + \cdots \\
&\qquad\qquad + \frac{1}{120}x^5 + \frac{1}{240}x^7 + \cdots \\
&\qquad\qquad\qquad - \frac{1}{5040}x^7 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots
\end{aligned}
$$

※ $\tan x$ は奇関数である．そこで，$\tan x$ の Maclaurin 展開を

$\tan x = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots$

とすると

$$
\begin{aligned}
\tan x \cos x &= (a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) \\
&= a_1x + \left(-\frac{a_1}{2} + a_3\right)x^3 + \left(\frac{a_1}{24} - \frac{a_3}{2} + a_5\right)x^5 \\
&\qquad + \left(-\frac{a_1}{720} + \frac{a_3}{24} - \frac{a_5}{2} + a_7\right)x^7 + \cdots
\end{aligned}
$$

これと

$\sin x = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + \cdots$

の係数を比較すると

$$
\begin{cases}
a_1 = 1 \\
-\dfrac{a_1}{2} + a_3 = -\dfrac{1}{6} \\
\dfrac{a_1}{24} - \dfrac{a_3}{2} + a_5 = \dfrac{1}{120} \\
-\dfrac{a_1}{720} + \dfrac{a_3}{24} - \dfrac{a_5}{2} + a_7 = -\dfrac{1}{5040} \\
\cdots
\end{cases}
$$

順に求めると　$a_1 = 1,\ a_3 = \dfrac{1}{3},\ a_5 = \dfrac{2}{15},\ a_7 = \dfrac{17}{315}, \ldots$

よって　$\tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 + \cdots$

【問題】

代表的な関数の Maclaurin 展開を用いて，次の関数の Maclaurin 展開をかっこ内の項まで求めよ．ただし，係数は既約分数にすること．

(1) $\dfrac{\log(1+x)}{\sqrt{1+x}}$　（5 次以下）

$= x - x^2 + \frac{23}{24}x^3 - \frac{11}{12}x^4 + \frac{563}{640}x^5 + \cdots$

(2) $\sqrt{1+x-\dfrac{7x^2}{3}}$ （4 次以下）

$\div\ 1+\frac{1}{2}x-\frac{31}{24}x^2+\frac{31}{48}x^3-\frac{1333}{1152}x^4+\cdots$

(3) $e^{\arctan x}$ （5 次以下）

$= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{7}{24}x^4 + \frac{1}{24}x^5 + \cdots$