Semester One Examination, 2021

Question/Answer Booklet

MATHEMATICS METHODS
ATAR Year 12
Section Two: Calculator-assumed

SOLUTIONS

Student Name: ____________________

Please circle your teacher's name

Teacher: Miss Hosking **Miss Rowden**

Time allowed for this paper
Reading time before commencing work: 10 minutes
Working time for paper: 100 minutes

Materials required/recommended for this paper
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

Number of additional answer booklets used (if applicable): ☐

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available	Percentage of examination
Section One: Calculator free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	96	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2021*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section Two: Calculator-assumed **65% (96 Marks)**

This section has thirteen (13) questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 100 minutes.

Question 9 **(7 marks)**

A hot potato was removed from an oven and placed on a cooling rack. Its temperature T, in degrees Celsius, t minutes after being removed from the oven was modelled by

$$T = 16 + 188e^{kt}.$$

The temperature of the potato halved between $t = 0$ and $t = 6.8$.

(a) Determine the value of the constant k. (3 marks)

Solution
$T_0 = 16 + 188 = 204$
$102 = 16 + 188e^{6.8k} \Rightarrow k = -0.115$
Specific behaviours
✓ indicates initial temperature
✓ equation for temperature halving
✓ solves for k

(b) The temperature of the potato eventually reached a steady state. Determine the time taken for its temperature to first fall to within $4\ °C$ of this steady state. (2 marks)

Solution
$T_\infty = 16$
$20 = 16 + 188e^{-0.115t} \Rightarrow t = 33.5 \text{ minutes}$
Specific behaviours
✓ indicates steady state temperature
✓ correct time, to at least 1 dp

(c) Determine the time at which the potato was cooling at a rate of $4\ °C$ per minute. (2 marks)

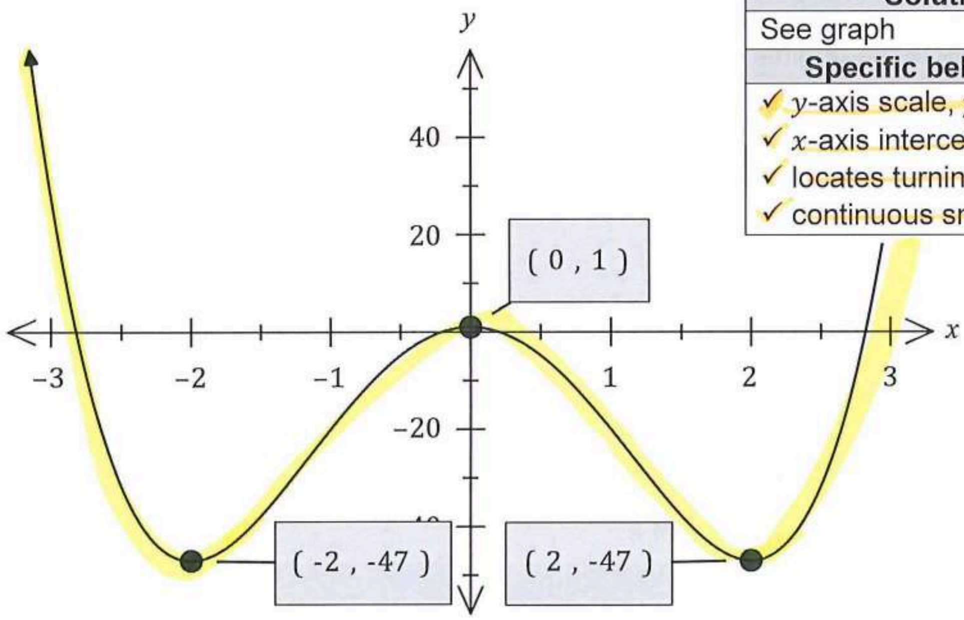
Solution
$\frac{dT}{dt} = -21.62e^{-0.115t}$
$-21.62e^{-0.115t} = -4 \Rightarrow t = 14.7 \text{ minutes}$
Specific behaviours
✓ indicates derivative
✓ correct time, to at least 1 dp

* correct soln.

Question 10 **(8 marks)**

Let $f(x) = 3x^4 + ax^2 + 1$.

(a) Sketch the graph of $y = f(x)$ when $a = -24$. (4 marks)



Solution
See graph
Specific behaviours
✓ y-axis scale, y-intercept
✓ x-axis intercepts
✓ locates turning points
✓ continuous smooth curve

(b) Show that the graph of $y = f(x)$ will always have a maximum turning point at $x = 0$ if $a < 0$. (4 marks)

Solution
$f'(x) = 12x^3 + 2ax$ $f'(0) = 0$ Hence curve always stationary when $x = 0$. $f''(x) = 36x^2 + 2a$ $f''(0) = 2a$ If $a < 0$ then $f''(0) < 0$ and so the curve will always be concave down. Hence a maximum at $x = 0$.
Specific behaviours
✓ shows $f'(0) = 0$ ✓ states always stationary when $x = 0$ ✓ shows $f''(0) = 2a$ ✓ justifies maximum using second derivative

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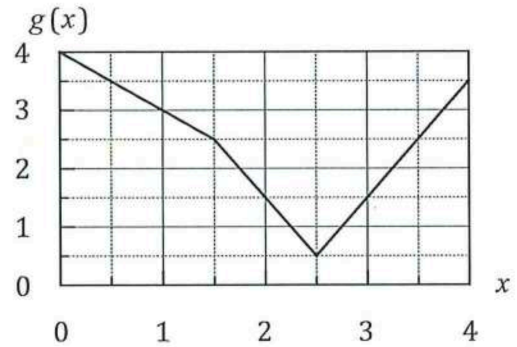
Question 11 **(8 marks)**

The graph of function g, and a table of values for function f and its derivatives are shown below.

x	1	2	3
$f(x)$	3	1	2
$f'(x)$	1	4	2
$f''(x)$	2	-1	-2



(a) Evaluate $h'(k)$ when

(i) $h(x) = f(g(x))$ and $k = 1$. (3 marks)

c
$h'(1) = f'(g(1)) \times g'(1)$ $= f'(3) \times (-1)$ $= (2)(-1) = -2$
Specific behaviours
✓ correct application of chain rule ✓ correct values for $g(x)$ and $g'(x)$ ✓ correct value

(ii) $h(x) = g(x) \div f(x)$ and $k = 2$. (3 marks)

Solution
$h'(2) = \dfrac{g'(2)f(2) - g(2)f'(2)}{f(2)^2}$ $= \dfrac{(-2)(1) - (1.5)(4)}{(1)^2} = -8$
Specific behaviours
✓ correct application of quotient rule ✓ correct values for $g(x)$ and $g'(x)$ ✓ correct value

(b) Evaluate $h''(3)$ when $h'(x) = f'(x) \times g'(x)$. (2 marks)

Solution
$h''(3) = f''(3)g'(3) + f'(3)g''(3)$ $= (-2)(2) + (3)(0)$ $= -4$
Specific behaviours
✓ uses product rule with at least two correct values ✓ correct result

Question 12 **(~~9~~ marks)** 10.

(a) If $x = \log_b 4$ and $y = \log_b 9$ then, in terms of x and y, determine:

(i) $\log_b 36$ (1 marks)

Solution
$\log_b 36 = \log_b 4 + \log_b 9$ $= x + y$
Specific behaviours
✓ correct expression

(ii) $\log_b \left(\dfrac{2}{3}\right)$ (2 marks)

Solution
$\log_b \dfrac{2}{3} = \dfrac{1}{2}\log_b \left(\dfrac{4}{9}\right)$ $= \dfrac{1}{2}(x - y)$
Specific behaviours
✓ applies the log laws of powers   ✓ correct expression

(iii) $\log_b 144b^3$ (2 marks)

* correct solns

Solution
$\log_b 144b^3 = \log_b 4^2 . 9 . b^3$ $= 2x + y + 3$
Specific behaviours
✓ correctly factorises 144   ✓ correct expression ✓correctly applies $3\log_b b = 3$

Question 12 continued

(b) The loudness L, in decibels, of sound is given by the equation

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

Where I is the intensity of sound and I_0 is the intensity of the sound just audible to the human ear.

(i) Find the loudness if the sound is 140 times as intense as I_0. (2 marks)

Solution
$L = 10\log_{10}\left(\frac{140I_0}{I_0}\right)$ $L = 21.46\,dB$
Specific behaviours
✓ substitutes for I ✓ correctly evaluates L

(ii) If the loudness was $28dB$ find in terms of I_0 intensity of sound. (2 marks)

Solution
$28 = 10\log_{10}\left(\frac{I}{I_0}\right)$ $2.8 = \log_{10}\left(\frac{I}{I_0}\right)$ $10^{2.8} = \frac{I}{I_0}$ $631I_0 = I$
Specific behaviours
✓ converts from a log to index statement   ✓ correctly expresses I in terms of I_0

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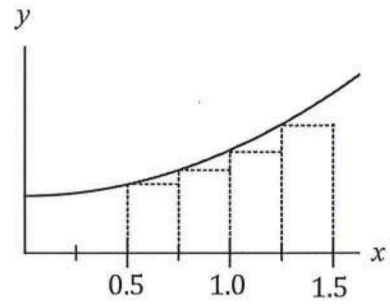
Question 13 **(8 marks)**

The graph of $y = f(x)$ is shown at right with 4 equal width inscribed rectangles. An estimate for the area under the curve between $x = 0.5$ and $x = 1.5$ is required.



The function f is defined as $f(x) = 2x^2 + 7$ and let the area sum of the 4 rectangles be S_4.

S_n, the area estimate using n inscribed rectangles can be calculated using

$$S_n = \sum_{i=1}^{i=n} f(x_i)\delta x$$

(a) State the values of x_1, x_2, x_3, x_4 and δx that should be used to determine S_4. (1 mark)

Solution
$x_1 = 0.5, x_2 = 0.75, x_3 = 1, x_4 = 1.25, \delta x = 0.25$
Specific behaviours
✓ correctly states all values

(b) Calculate the value of S_4. (3 marks)

Solution
$S_4 = 0.25\big((2(0.5)^2 + 7) + (2(0.75)^2 + 7) + (2(1)^2 + 7) + (2(1.25)^2 + 7)\big)$ $= 0.25(7.5 + 8.125 + 9 + 10.125)$ $= 0.25(34.75)$ $= \dfrac{139}{16} = 8.6875 \text{ u}^2$
Specific behaviours
✓ indicates correct calculation for one rectangle   ✓ correct heights of all rectangles   ✓ correct value

(c) Explain, with reasons, how the value of δx and the area estimate S_n will change as the number of inscribed rectangles increase. (2 marks)

Solution
δx is the width of each rectangle and so must decrease. S_n will increase, approaching true area under curve, as area 'lost' between curve and rectangles will decrease.
Specific behaviours
✓ indicates δx will decrease as it's the rectangle width   ✓ indicates S_n will increase

(d) Determine the limiting value of S_n as $n \to \infty$. (2 marks)

Solution
$S_\infty = \displaystyle\int_{0.5}^{1.5} f(x)\,dx = \frac{55}{6} = 9.1\overline{6} \text{ u}^2$
Specific behaviours
✓ correct integral   ✓ correct limiting value

See next page

Question 14 **(6 marks)**

The area A of a regular polygon with n sides of length x is given by

$$A = \frac{n\,x^2 \cos\left(\frac{\pi}{n}\right)}{4\sin\left(\frac{\pi}{n}\right)}$$

(a) Determine the exact area of a regular hexagon with side length 3 cm. (1 mark)

Solution
$A(x) = \dfrac{6\times 3^2\cos\left(\frac{\pi}{6}\right)}{4\sin\left(\frac{\pi}{6}\right)} = \dfrac{27\sqrt{3}cm^2}{2}$
Specific behaviours
✓ correct area (exact)

☆ correct sol'n's

(b) Simplify the above formula when $n = 12$ to obtain a function for the area of a regular dodecagon. (2 marks)

Solution
$A(x) = \dfrac{12\,x^2\cos\left(\frac{\pi}{12}\right)}{4\sin\left(\frac{\pi}{12}\right)} = 3x^2(\sqrt{3}+2)$
Specific behaviours
✓ correctly substitutes
✓ simplified function

$\dfrac{3x^2(\sqrt{3}+1)}{\sqrt{3}-1}$

up date soln

(c) Use the increments formula to estimate the change in area of a regular dodecagon when its side length increases from 10 cm to 10.3 cm. (3 marks)

$\dfrac{6x(\sqrt{3}+1)}{\sqrt{3}-1}$

Solution
$\dfrac{dA}{dx} = 12x(\sqrt{3}+2), \quad x = 10, \quad \delta x = 0.3$
$\delta A \approx \dfrac{dA}{dx}\delta x$
$\approx 12(10)(\sqrt{3}+2)(0.3)$
$\approx 18(\sqrt{3}+2) \approx 67.2\text{ cm}^2$
Specific behaviours
✓ derivative of A with respect to x
✓ correct use of increments formula
✓ calculates change

update solns.

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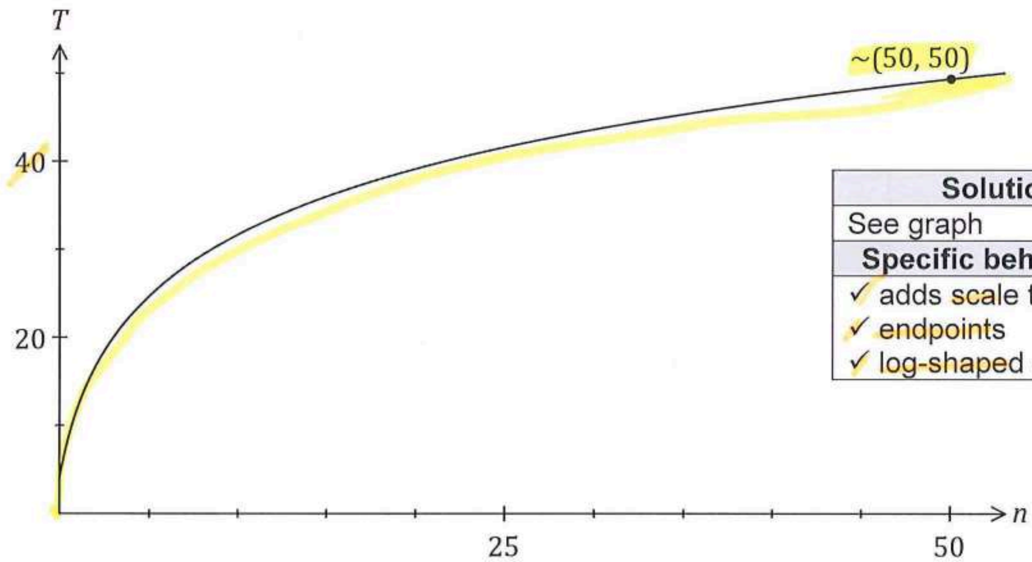
Question 15 **(8 marks)**

Hick's law, shown below, models the average time, T seconds, for a person to make a

$$T = a + b\log_2(n+1), \text{ where } a \text{ and } b \text{ are positive constants.}$$

(a) Draw the graph of T vs n on the axes below when $a = 4$ and $b = 8$. (3 marks)



Solution
See graph
Specific behaviours
✓ adds scale to y-axis
✓ endpoints
✓ log-shaped curve

(b) When a pizzeria had 10 choices of pizza, the average time for patrons to make a choice was 40 seconds. After doubling the number of choices, the average time to make their choice increased by 25%.

Modelling the relationship with Hick's law, predict the average time to make a choice if patrons were offered a choice of 35 pizzas. (5 marks)

Solution
$40 = a + b\log_2(10+1)$ $40 \times 1.25 = a + b\log_2(2 \times 10 + 1)$ $a = 2.917, b = 10.719$ $T = 2.917 + 10.719\log_2(35+1)$ $T = 58.34 \approx 58$ seconds
Specific behaviours
✓ writes first equation ✓ writes second equation ✓ solves for variables ✓ substitutes correctly ✓ states time, rounded to nearest second

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Question 16 **(8 marks)**

The volume, V litres, of fuel in a tank is reduced between $t = 0$ and $t = 48$ minutes so that

$$\frac{dV}{dt} = -175\pi \sin\left(\frac{\pi t}{48}\right)$$

(a) Determine, to the nearest litre, the amount of fuel emptied from the tank

(i) in the first minute. (3 marks)

Solution

$$\Delta V = \int_0^1 V'\,dt$$
$$= -17.985$$

Hence 18 litres were emptied.

Specific behaviours

✓ writes integral for change
✓ evaluates integral
✓ answers as positive number of litres

(ii) in the last 7 minutes. (1 mark)

Solution

$$\Delta V = \int_{41}^{48} V'\,dt = -866.3$$

Hence 866 litres were emptied.

Specific behaviours

✓ correct number of litres

The tank initially held $18\,600$ litres of fuel.

(b) Determine the volume of fuel in the tank 5 minutes after the volume in the tank reached $12\,000$ litres. (4 marks)

Solution

$$\int_0^T V'\,dt = -6\,600$$
$$T = 20.70$$

$$\Delta V = \int_{20.7}^{25.7} V'\,dt$$
$$= -2\,733$$

$$V(25.7) = 12\,000 - 2\,733$$
$$= 9\,267\text{ L}$$

Specific behaviours

✓ equation for $\Delta V = -6\,600$
✓ determines T
✓ determines ΔV
✓ correct volume

Alternative Solution

$$V(t) = \int V'\,dt = 8400\cos\left(\frac{\pi t}{48}\right) + c$$

$$V(0) = 18\,600 \Rightarrow c = 10\,200$$

$$V(T) = 12\,000 \Rightarrow T = 20.70$$

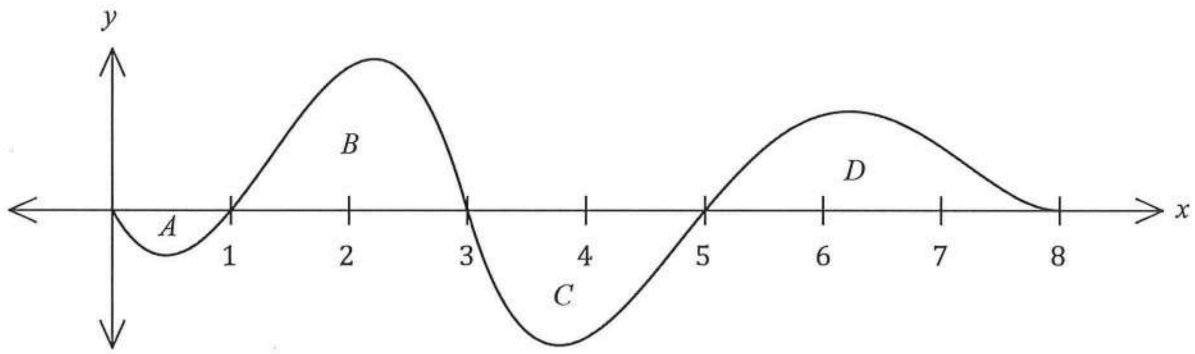
$$V(25.7) = 9\,267\text{ L}$$

Specific behaviours

✓ antiderivative for $V(t)$
✓ determines c
✓ determines T
✓ correct volume

Question 17 **(7 marks)**

Regions A, B, C and D bounded by the curve $y = f(x)$ and the x-axis are shown on this graph:



The areas of A, B, C and D are $5, 31, 27$ and 23 square units respectively.

(a) Determine the value of

(i) $\int_0^3 f(x)\,dx.$ (1 mark)

Solution
$I = -5 + 31 = 26$
Specific behaviours
✓ correct value

(ii) $\int_3^8 4f(x)\,dx.$ (2 marks)

Solution
$I = 4(-27 + 23) = 4(-4) = -16$
Specific behaviours
✓ shows sum of signed areas ✓ uses linearity to obtain correct value

(iii) $\int_1^8 (5 - f(x))\,dx.$ (2 marks)

Solution
$I = [5(7)] - [31 - 27 + 23]$ $I = 35 - 27 = 8$
Specific behaviours
✓ uses linearity to obtain two integrals ✓ correct value

(b) Explain why $\int_1^5 f'(x)\,dx = 0.$ (2 marks)

Solution
Using fundamental theorem, result is $f(5) - f(1)$. Since $f(1) = f(5) = 0$, then the difference is 0.
Specific behaviours
✓ uses fundamental theorem to obtain result ✓ explains value of 0 using the two roots

Question 18 **(7 marks)**

The table below shows the sign of the polynomial $f(x)$ and some of its derivatives at various values of x. There are no other zeroes of $f(x), f'(x)$ or $f''(x)$ apart from those shown in the table.

x	-2	-1	0	1	2	3	4
$f(x)$	$-$	0	$+$	$+$	$+$	0	$-$
$f'(x)$	$+$	$+$	0	$-$	$-$	0	$-$
$f''(x)$	$-$	$-$	$-$	0	$+$	0	$-$

(a) For what value(s) of x is the graph of the function concave down? (1 mark)

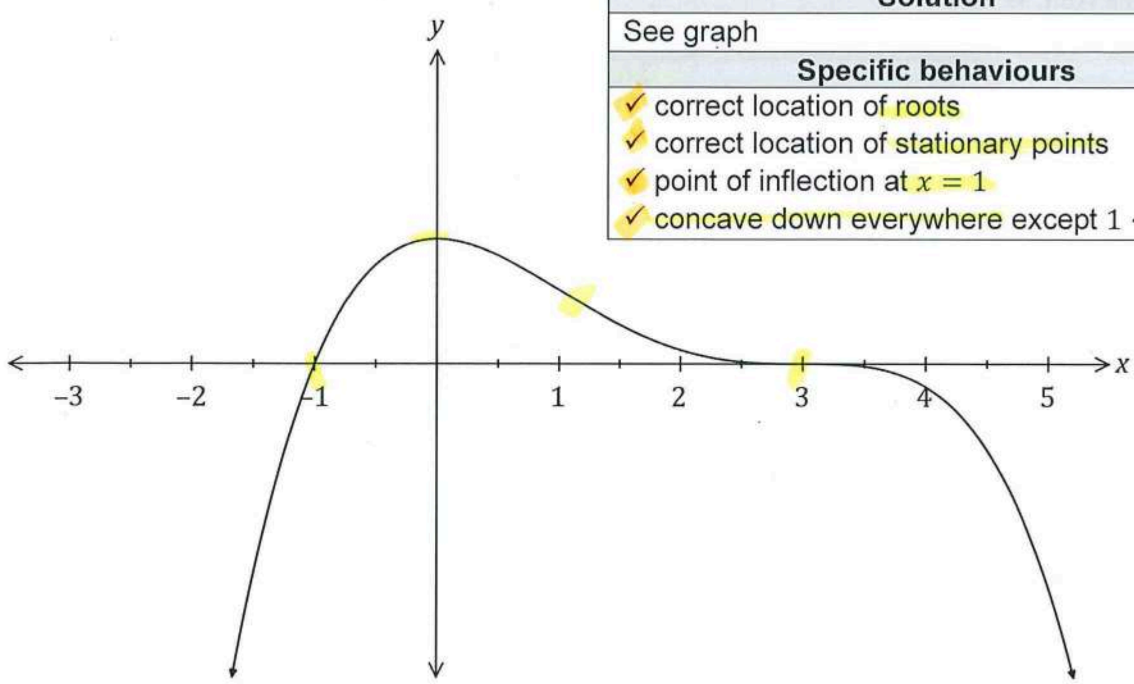
Solution
$x < 1$ and $x > 3$
Specific behaviours
✓ correct inequalities and domain

(b) At what location does the graph of f have a turning point? Explain your answer. (2 marks)

Solution
At $x = 0$. The gradient is zero and f is concave down on either side.
Specific behaviours
✓ location ✓ explanation

(c) Sketch a possible graph of $y = f(x)$ on the axes below. (4 marks)

Solution
See graph
Specific behaviours
✓ correct location of roots ✓ correct location of stationary points ✓ point of inflection at $x = 1$ ✓ concave down everywhere except $1 < x < 3$



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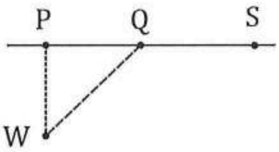
Question 19 **(8 marks)**

An offshore wind turbine W lies 12 km away from the nearest point P on a straight coast. It must be connected to a power storage facility S that lies on the coast 24 km away from P.



Engineers will lay the cable in two straight sections, from W to Q, where Q is a point on the coast x km from P, and then from Q to S.

The cost of installing cable along the coastline is $1000 per km and offshore is $2600 per km.

(a) Determine, to the nearest hundred dollars, the cost of installing the cable when Q lies midway from A to P. (2 marks)

Solution
$C = 1000 \times 12 + 2600 \times \sqrt{12^2 + 12^2}$ $= \$56\,100$
Specific behaviours
✓ correct expression ✓ calculates cost

(b) Show that C, the cost in hundreds of dollars, to run the cable from W to Q to S, is given by
$C = 26\sqrt{x^2 + 144} - 10x + 240.$ (2 marks)

Solution
$C_{WQ} = 26 \times QW = 26 \times \sqrt{x^2 + 12^2}$ $C_{QS} = 10(24 - x) = 240 - 10x$ Hence $C = C_{WQ} + C_{QS} = 26\sqrt{x^2 + 144} - 10x + 240$
Specific behaviours
✓ expression for cable from W to Q ✓ expression for cable from Q to S and shows sum

(c) Use calculus techniques to determine, with justification, the minimum cost of laying the cable from W to S. (4 marks)

Solution
$C'(x) = \dfrac{26x}{\sqrt{x^2 + 144}} - 10$ $C'(x) = 0 \Rightarrow x = 5$ $C(5) = 528$ $C''(5) \approx 1.7 \Rightarrow$ minimum, as +ve concavity Hence minimum cost is $52 800.
Specific behaviours
✓ correct derivative ✓ solves for optimum value of x ✓ justifies minimum ✓ states minimum cost

calc

$\dfrac{26x - 10\sqrt{x^2+144}}{\sqrt{x^2+144}}$

Question 20 **(8 marks)**

Small body A moves in a straight line with acceleration a cm/s^2 at time t s given by

$$a = pt + q$$

Initially, A has a displacement of 4 cm relative to a fixed point O and is moving with a velocity of 9 cm/s. Two seconds later, A has a displacement of 8.8 cm and a velocity of -3.6 cm/s.

(a) Determine the value of the constant p and the value of the constant q. (6 marks)

Solution

Velocity:

$$v = \int pt + q\, dt$$

$$v(t) = \frac{pt^2}{2} + qt + c$$

$$v(0) = 9 \Rightarrow c = 9$$

Displacement:

$$s(t) = \int \frac{pt^2}{2} + qt + 9\, dt$$

$$s(t) = \frac{pt^3}{6} + \frac{qt^2}{2} + 9t + k$$

$$s(0) = 4 \Rightarrow k = 4$$

$$v(2) = 2p + 2q + 9 = -3.6$$

$$s(2) = \frac{4p}{3} + 2q + 18 = 8.8 \quad ?$$

$+4$

Solve:

$$p = 0.9, \qquad q = -7.2$$

Specific behaviours

✓ antiderivative for velocity, constant evaluated
✓ integral for displacement
✓ displacement, constant evaluated
✓ expressions for $v(2)$ and $s(2)$
✓ value of p
✓ value of q

change solns.

(b) Determine the minimum velocity of A. (2 marks)

Solution

$$q \quad v' = 0 \Rightarrow 0.9t - 7.2 = 0 \Rightarrow t = 8$$

$$v(8) = -19.8 \text{ cm/s}$$

Specific behaviours

✓ indicates time for minimum
✓ correct minimum velocity

change soln.

Question 21 **(5 marks)**

(a) Determine the value of the constant a and the value of the constant b that make each of the following statements true, given that $f(x)$ is a polynomial:

(i) $\int_a^1 f(x)\,dx + \int_1^b f(x)\,dx = \int_{-3}^2 f(x)\,dx.$ (1 mark)

Solution
$a = -3, \quad b = 2$
Specific behaviours
✓ correct values

(ii) $\int_0^2 f(x)\,dx - \int_1^2 f(x)\,dx + \int_{-1}^0 f(x)\,dx = \int_a^b f(x)\,dx.$ (2 marks)

Solution
$a = -1, \quad b = 1$
Specific behaviours
✓ value of a
✓ value of b

(b) Determine $\frac{d}{dx}\left(\int_{h(x)}^3 f(t)\,dx.\right)$ (2 marks)

Solution
$\frac{d}{dx}\left[\int_{g(x)}^3 f(t)\,dt\right] = \frac{d}{dx}\left[-\int_3^{g(x)} f(t)\,dt\right]$ $= -f\big(g(x)\big).\,g'(x)$
Specific behaviours
✓ reverses the boundaries by introducing negative one
✓ correctly uses fundamental theorem

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