Semester Two Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student number: In figures

In words ____________________

Your name ____________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet

Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen.
Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Markers use only		
Question	Maximum	Mark
9	4	
10	4	
11	7	
12	8	
13	6	
14	7	
15	8	
16	8	
17	6	
18	8	
19	12	
20	11	
21	9	
S2 Total	98	
S2 Wt (×0.6633)	65%	

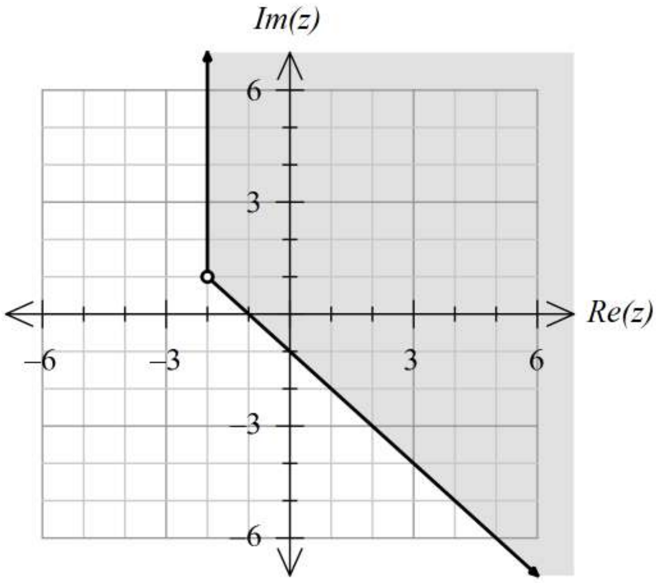
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(4 marks)**

The locus of a complex number z is shown below.



(a) Without using $\mathrm{Re}(z)$ or $\mathrm{Im}(z)$, write an inequality in terms of z for the locus. (3 marks)

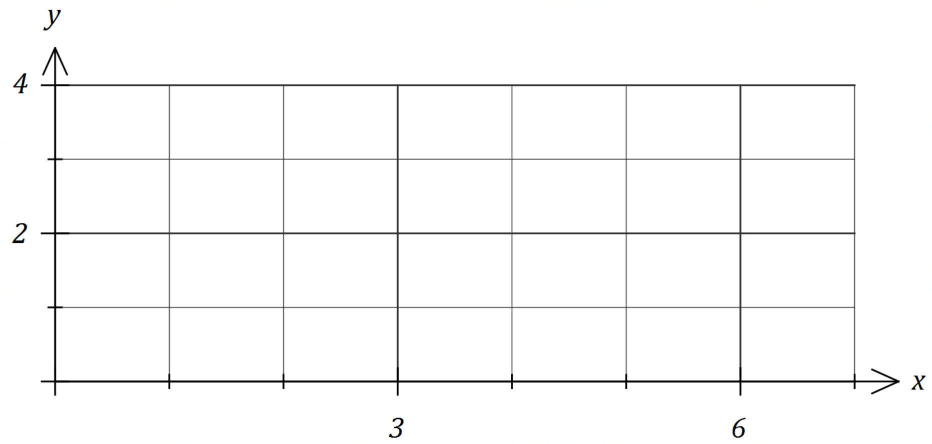
(b) Add the locus for $|z| = |z + 4i|$ to the diagram above. (1 mark)

Question 10 **(4 marks)**

Solve the equation $z^5 = -16 + 16\sqrt{3}i$, giving solutions in polar form $r\,\text{cis}\,\theta$ where $-\pi < \theta \le \pi$ and $r > 0$.

Question 11 **(7 marks)**

(a) Sketch the graph of $y = \dfrac{e^{0.5x}}{1+x}$ on the axes below. (2 marks)



The Trapezoidal Rule can be used to determine the numerical approximation of a definite integral when an antiderivative cannot be found. When a continuous interval $[a_0,\ a_n]$ is divided into n smaller intervals of equal width w, the bounds of these smaller intervals can be denoted by $a_0,\ a_1,\ a_2, \dots,\ a_{n-1},\ a_n$. The Trapezoidal Rule is then expressed as follows:

$$\int_{a_0}^{a_n} f(x)\,dx = \frac{w}{2}[f(a_0) + 2f(a_1) + 2f(a_2) + \cdots + 2f(a_{n-1}) + f(a_n)]$$

(b) Use the above rule to determine an estimate to 4 decimal places for $\displaystyle\int_0^6 \frac{e^{0.5x}}{1+x}\,dx$ using

(i) 2 intervals. (2 marks)

(ii) 24 intervals. (3 marks)

Question 12 **(8 marks)**

The diameter of copper wire produced by a machine is normally distributed with a mean of $485\ \mu\mathrm{m}$ and a variance of $396\ \mu\mathrm{m}^2$.

A production supervisor routinely takes a random sample of 45 diameters and calculates their mean, $\bar{X}$.

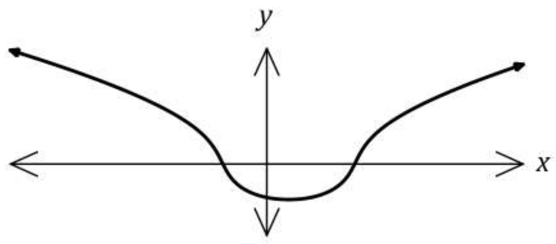
(a) Describe the distribution of $\bar{X}$. (3 marks)

(b) Determine the probability that the mean of a random sample of 45 diameters is greater than $490\ \mu\mathrm{m}$. (1 mark)

(c) Repeated random sampling of n diameters from the machine shows that there is a 17% chance that the sample mean is less than $482\ \mu\mathrm{m}$. Determine n. (4 marks)

Question 13 **(6 marks)**

A particle is moving along the curve shown below with equation $x^2 - 2x = y^3 + 3y + 8$.



The x-coordinate of the particle is changing at a constant rate given by $\frac{dx}{dt} = -4$.

Determine the rates at which the y-coordinate of the particle is changing when $y = 0$.

Question 14 **(7 marks)**

Water, containing 8 grams of dissolved sugar per litre, flows into a tank at a constant rate of 15 litres per hour.

Water is drawn from the tank, initially containing 300 litres of water with no dissolved sugar, at the same constant rate of 15 litres per hour.

Let the weight of sugar in the tank after t hours be w grams and assume that the sugar is always evenly dissolved throughout the water in the tank.

(a) By considering the rate at which dissolved sugar flows in and out of the tank, show that (2 marks)

$$\frac{dw}{dt} = \frac{2400 - w}{20}.$$

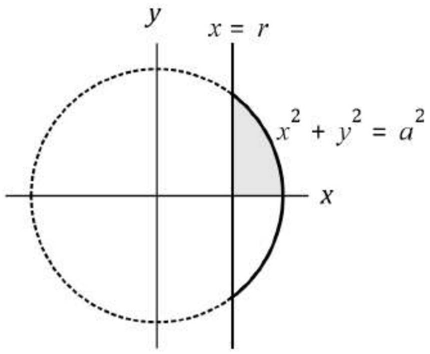
The water drawn from the tank can be used in a manufacturing process once the level of dissolved sugar exceeds 1.5 grams per litre.

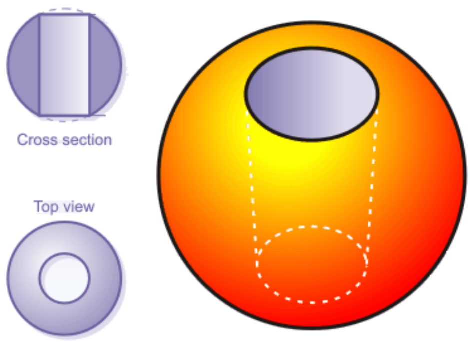
(b) Derive an equation for w in terms of t and hence determine how long this will take. (5 marks)

Question 15 **(8 marks)**

A manufacturer drills a hole of radius r through the centre of a solid sphere of radius a.

To determine the volume V of the remaining solid, consider the circle with equation $x^2 + y^2 = a^2$ and the line with equation $x = r$, as shown in the cross section diagram below.





(a) Determine the coordinates of the point of intersection of the line and the circle in the first quadrant in terms of r and a. (2 marks)

(b) Construct a single integral to determine the volume of revolution obtained when the shaded region in the diagram is rotated about the y-axis. (2 marks)

(c) Evaluate your integral from (b) and hence determine a formula for V, the volume of the remaining solid. (3 marks)

(d) Hence, or otherwise, determine the exact volume of the remaining solid when a hole of radius 8 cm is drilled through the centre of a solid sphere of radius 17 cm. (1 mark)

Question 16 **(8 marks)**

A researcher used data from a sample of 162 newborn babies in order to estimate the mean weight and length of newborns in a large city.

(a) The weights of the babies in the sample had a mean of 3.35 kg and a standard deviation of 0.58 kg.

(i) Use this data to obtain a 90% confidence interval for the mean weight of a newborn baby in the city. (2 marks)

(ii) State two assumptions made when constructing your confidence interval. (2 marks)

(b) The 95% confidence interval for the mean length L cm of newborn babies derived from the sample was $(50.82, 51.58)$. Determine the sample mean and standard deviation used to construct this interval. (4 marks)

Question 17 **(6 marks)**

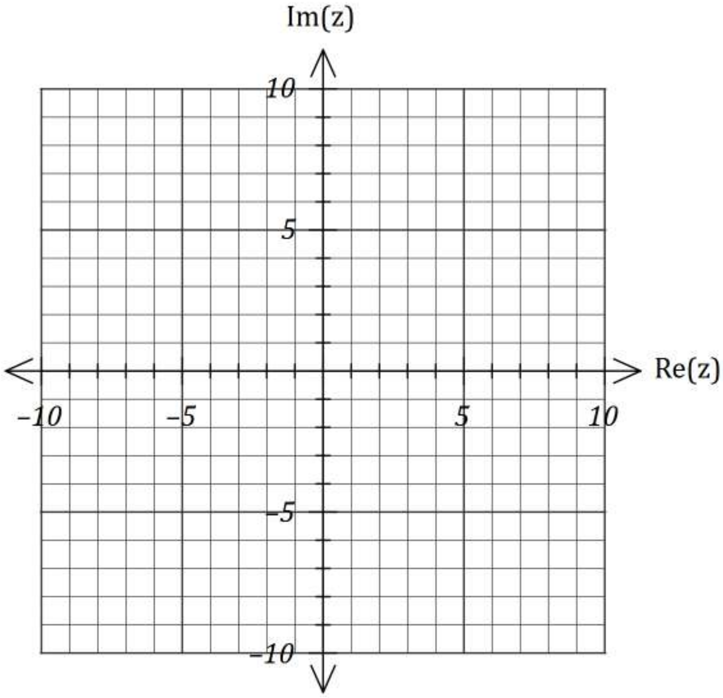
(a) State the equations of all asymptotes of the graph of $y = \dfrac{8x^2 - 75}{x^2 - 25}$. (2 marks)

(b) Let $f(x) = \dfrac{ax^2 + bx + c}{x + d}$.

The graph of $y = f(x)$ has no roots, a y-intercept of 4 and two asymptotes (with equations $x = -2$ and $y = 3x - 1$). Determine the value of each of the constants a, b, c and d. (4 marks)

Question 18 **(8 marks)**

(a) Indicate the subset of points in the complex plane that satisfy $|z + 3 - 2i| \leq 4$ on the axes below. (3 marks)



(b) Given that $|z + 3 - 2i| \leq 4$, determine

(i) the maximum value of $\text{Im}(z)$. (1 mark)

(ii) the minimum value of $|z - 2|$. (2 marks)

(iii) the minimum value of $\text{Re}(iz)$. (2 marks)

Question 19 **(12 marks)**

Points A, B and C lie in plane Π and have position vectors $\begin{pmatrix} -2 \\ 0 \\ 9 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ respectively.

Point B also lies on the sphere S that has centre A.

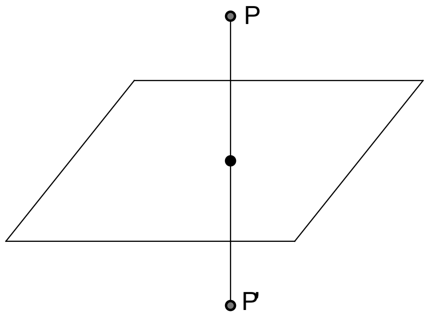
(a) Determine the vector equation of S. (3 marks)

(b) Determine the Cartesian equation for plane Π. (4 marks)

A point and its reflection in a plane are equidistant from the plane and lie on a line that is perpendicular to the plane.

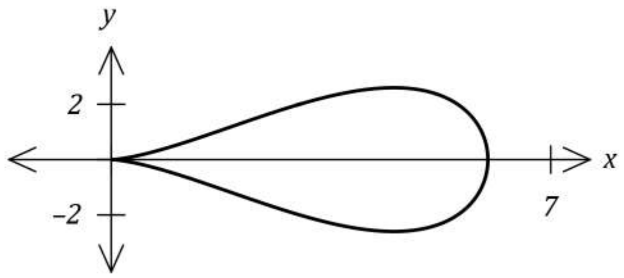
Point P has position vector $\begin{pmatrix} 9 \\ -9 \\ 16 \end{pmatrix}$.

(c) Determine the position vector of P', the reflection of P in plane Π. (5 marks)



Question 20 **(11 marks)**

The path of a particle is shown in the diagram below.



The position vector of the particle after t seconds is given by $\mathbf{r}(t) = \begin{pmatrix} 3 - 3\sin t \\ 2\cos t - \sin 2t \end{pmatrix}$ centimetres, for $t \geq 0$.

(a) Determine the initial position of the particle. (1 mark)

(b) Determine the acceleration vector of the particle at the instant it first reaches the origin. (3 marks)

(c) Determine the distance travelled by the particle from the time it leaves its initial position until the time it first reaches the origin. (3 marks)

(d) The Cartesian equation of the path of the particle is $ay^2 + bx^3 + 4x^4 = 0$. Determine the value of each of the constants a and b. (4 marks)

Question 21 **(9 marks)**

Researchers used a simulation to model the population of foxes $F(t)$ and the population of rabbits $R(t)$ on an island. The rates of change of each population after t years are given by

$$\frac{dF}{dt} = 0.32R \text{ and } \frac{dR}{dt} = -0.08F.$$

(a) Briefly explain how the rabbit population is changing. (1 mark)

(b) Show that $\frac{d^2R}{dt^2} = -0.0256R$. (2 marks)

The equation in part (b) suggests that a model of the form $R(t) = k\cos(at + b)$ would be appropriate, where a, b and k are positive constants.

(c) Explain this choice of model. (1 mark)

The research model used $b = 0.27$ and the initial size of the rabbit population was 800.

(d) Determine the value of a and the value of k. (2 marks)

(e) Determine an equation for $F(t)$ in terms of t and hence calculate the number of years it takes for the initial fox population to double in size. (3 marks)

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

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Question number: _________