PERTH MODERN SCHOOL

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INDEPENDENT PUBLIC SCHOOL

WAEP Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ____

Your name ____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

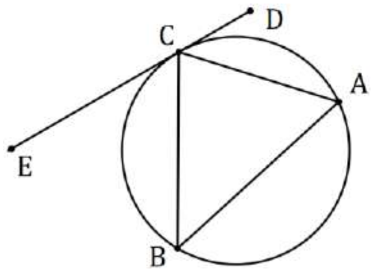
Section Two: Calculator-assumed **65% (97 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

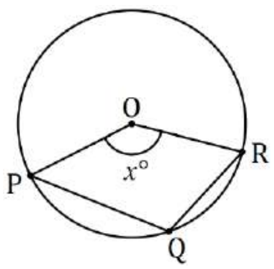
Question 9 **(7 marks)**

(a) In the diagram below, points A and B lie on a circle, DE is a tangent to the circle at C, $\angle ACD = 64°$ and $\angle BAC = 49°$. Determine the sizes of $\angle ABC, \angle BCE$ and $\angle BCA$. (3 marks)



Solution
$\angle ABC = \angle ACD = 64°$ $\angle BCE = \angle BAC = 49°$ $\angle BCA = \angle 180 - 64 - 49 = 67°$
Specific behaviours
✓ $\angle ABC$ ✓ $\angle BCE$ ✓ $\angle BCA$

(b) In the next diagram, P, Q and R lie on a circle with centre O and $\angle POR = x°$. Determine, with reasons, the size of $\angle PQR$ in terms of x. (4 marks)



Solution
Reflex $\angle POR = \alpha = 360 - x$ (Angle sum at point) $\angle PQR = \frac{1}{2}\alpha$ (Angle on arc at centre twice that on circumference) $\angle PQR = \frac{1}{2}(360 - x) = 180 - \frac{1}{2}x$
Specific behaviours
✓ expression for reflex angle ✓ reason ✓ $\angle PQR$ ✓ reason *(Or other methods)*

Question 10 **(7 marks)**

(a) A body travels with a velocity $12\mathbf{i} - 5\mathbf{j}$ ms^{-1}. Determine its speed and the bearing on which it is moving, assuming the positive y-axis to be due north. (3 marks)

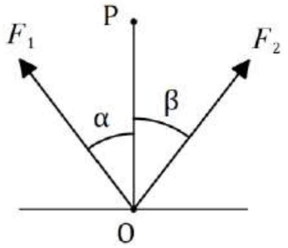
Solution
$\text{Speed} = \sqrt{12^2 + (-5)^2} = 13 \text{ m/s}$ $\text{Angle} = \tan^{-1}\left(\frac{-5}{12}\right) = -22.6°$ $\text{Bearing} = 360n - (-22.6 - 90) = 112.6°$
Specific behaviours
✓ speed ✓ angle ✓ bearing

(b) Given that $\lambda(5\mathbf{i} - 2\mathbf{j}) + \mu(-7\mathbf{i} + 4\mathbf{j}) = 25\mathbf{i} - 13\mathbf{j}$, determine the values of λ and μ. (4 marks)

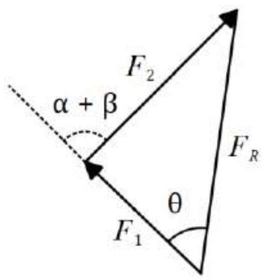
Solution
$5\lambda - 7\mu = 25$ $-2\lambda + 4\mu = -13$ $\lambda = 1.5$ $\mu = -2.5$
Specific behaviours
✓ equates **i**-coefficients ✓ equates **j**-coefficients ✓ value of λ ✓ value of μ

Question 11 **(8 marks)**

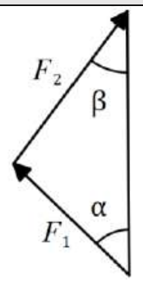
Two forces, $F_1 = 230\text{ N}$ and $F_2 = 272\text{ N}$, act on a body at O, and make angles of $\alpha = 15°$, and $\beta = 20°$ respectively with the vertical OP, as shown in the diagram below.



(a) Determine the magnitude of the resultant force and the angle it makes with the vertical. (5 marks)

Solution	
	$F_R = \sqrt{230^2 + 272^2 - 2(230)(272)\cos 145}$ $F_R = 479\ N$ $\frac{\sin\theta}{272} = \frac{\sin 145}{478.9}$ $\theta = 19°$ $\text{Angle} = 19° - 15° = 4°$
Specific behaviours	
✓ sketch with forces nose to tail ✓ indicates use of cosine rule for magnitude ✓ magnitude ✓ indicates use of sine rule for angle ✓ angle with vertical	

(b) The magnitude of F_2 is to be adjusted so that the direction of the resultant is vertical. Determine the required magnitude of F_2. (3 marks)

Solution	
	$\frac{\sin 15}{F_2} = \frac{\sin 20}{230}$ $F_2 = 174\text{ N}$
Specific behaviours	
✓ sketch ✓ indicates use of sine rule ✓ magnitude	

Question 12 **(6 marks)**

The largest Australian family recently met with the largest English family. Between them, these two families had 37 children.

(a) Three of the children were chosen at random to feature in a TV documentary about the two families. Determine the number of different selections of three children that were possible. (1 mark)

Solution
$\binom{37}{3} = 7\ 770$
Specific behaviours
✓ correct number

(b) Prove that at least four of the children were born in the same month of the year. (2 marks)

Solution
Let the 12 months of the year be the pigeon-holes and the 37 children the pigeons. If 3 pigeons are placed in each of the 12 pigeon-holes, then there is still one left over, and so at least one of the pigeon-holes must have at least 4 pigeons (children).
Specific behaviours
✓ defines pigeons and pigeon-holes ✓ uses pigeon-hole principle

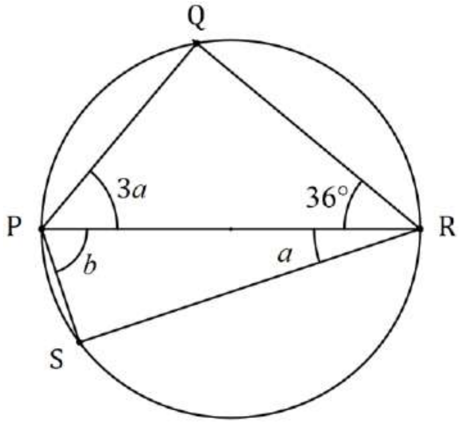
There were more children in the English family than the Australian family and the English children all had blue, brown, hazel or grey coloured eyes.

(c) Show that at least five English children had the same eye colour. (3 marks)

Solution
$37 \div 2 = 18.5 \Rightarrow \text{minimum of } 19 \text{ from England}$ 4 eye colours are the pigeon-holes and 19 children are pigeons. If 4 pigeons placed in each pigeon hole there are 3 left over and so at least one of the pigeon-holes must have at least 5 pigeons (children with same coloured eyes).
Specific behaviours
✓ minimum number of English children ✓ defines pigeons and pigeon-holes ✓ uses pigeon-hole principle

Question 13 **(8 marks)**

(a) Determine the size of angles a and b in the diagram below, where Q and S lie on the circumference of the circle with diameter PR. (3 marks)



Solution
$3a + 36 = 90$ $a = 18°$ $b = 90 - 18 = 72°$
Specific behaviours
✓ uses angle in semi-circle ✓ value of a ✓ value of b

(b) Triangle ABC has sides of length $AB = 5$ cm, $BC = 7$ cm and $AC = 9$ cm. Prove, using the method of contradiction, that if AC is a diameter of a circle then B does not lie on the circumference of the circle. (5 marks)

Solution
Assume that B does lie on the circumference of the circle. Then $\angle ABC = 90°$ (angle in semi-circle) Hence $AB^2 + BC^2 = AC^2$ (Pythagoras' theorem) But $5^2 + 7^2 = 74 \neq 81 = 9^2$. Hence assumption must be incorrect and so B does not lie on the circumference.
Specific behaviours
✓ assumes contradictory statement ✓ states angle in semi-circle ✓ deduces relationship between sides ✓ shows contradiction ✓ summarises

Question 14 **(8 marks)**

(a) Simplify $(4\mathbf{a} - 2\mathbf{b}) \cdot (\mathbf{a} - 3\mathbf{b})$ given that $|\mathbf{a}| = 5, |\mathbf{b}| = 3$ and vector $\mathbf{a}$ is parallel and in the opposite direction to vector $\mathbf{b}$. (4 marks)

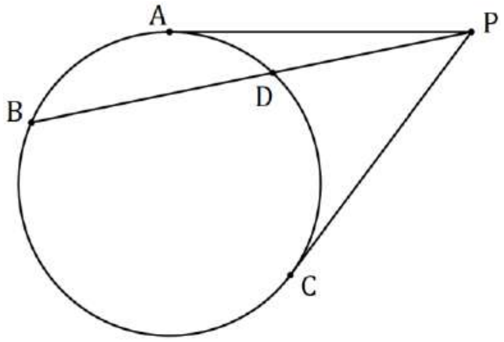
Solution
$(4\mathbf{a} - 2\mathbf{b}) \cdot (\mathbf{a} - 3\mathbf{b}) = 4\mathbf{a} \cdot \mathbf{a} - 12\mathbf{a} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{a} + 6\mathbf{b} \cdot \mathbf{b}$ $= 4a^2 - 12(-ab) - 2(-ab) + 6b^2$ $= 4(5^2) + 14(3 \times 5) + 6(3^2)$ $= 364$
Specific behaviours
✓ expands scalar product   ✓ indicates $\mathbf{a} \cdot \mathbf{b} = -ab$ ✓ substitutes magnitudes   ✓ simplifies

(b) Using $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, demonstrate a vector method to show that if the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ of parallelogram $OABC$ are perpendicular, then the parallelogram is a rhombus. (4 marks)

Solution
$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ Given $\overrightarrow{OB}$ and $\overrightarrow{AC}$ are perpendicular then $(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = 0$ $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a} = 0$ $|\mathbf{c}|^2 = |\mathbf{a}|^2$ Hence lengths of sides of $OABC$ are congruent and so $OABC$ is a rhombus.
Specific behaviours
✓ determines vectors for diagonals   ✓ uses scalar product   ✓ expands scalar product   ✓ explains that sides must be congruent

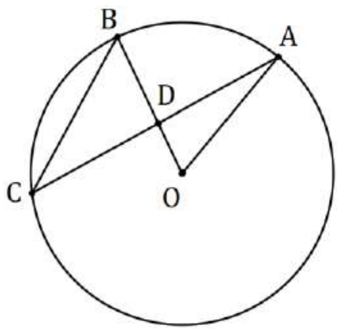
Question 15 **(9 marks)**

(a) In the diagram below, PA and PC are tangents to the circle, with $PA = 58$ cm. Secant PB cuts the circle at D, so that $PD = 40$ cm. Determine the lengths of PC and BD. (4 marks)



Solution
$PC = PA = 58\text{ cm}$ $PD \times PB = AP^2$ $40(40 + BD) = 58^2$ $BD = 44.1\text{ cm}$
Specific behaviours
✓ value of PC ✓ indicates use of tangent-secant theorem ✓ equation for BD ✓ value of BD

(b) In the diagram below, A, B and C lie on the circumference of the circle with centre O, with AC intersecting OB at D. Prove that $\angle DAO = \angle DBC - \angle DCB$. (5 marks)



Solution
$\angle DAO + \angle DOA = \angle BDA = \angle DBC + \angle DCB$ (sum of exterior angles equal) But $\angle DOA = \angle BOA = 2\,\angle ACB = 2\angle DCB$ (angle at centre-circumference) Hence $\angle DAO + 2\angle DCB = \angle DBC + \angle DCB$ And so $\angle DAO = \angle DBC - \angle DCB$
Specific behaviours
✓ derives first equation ✓ reasoning for first equation ✓ uses angle at centre-circumference ✓ substitutes ✓ simplifies

Question 16 **(9 marks)**

(a) Determine the number of integers between 1 and 370 that are divisible by 4 or 7. (4 marks)

Solution
$370 \div 4 = 92.5 \Rightarrow 92$ divisible by 4 $370 \div 7 = 52.8 \ldots \Rightarrow 52$ divisible by 7 $370 \div 28 = 13.2 \ldots \Rightarrow 13$ divisible by both $n = 92 + 52 - 13 = 131$
Specific behaviours
✓ divisible by 4 & 7 ✓ divisible by 28 ✓ use of inclusion-exclusion principle ✓ correct number

(b) A pigeon fancier has 5 Florentine, 6 King and 8 Maltese pigeons and must choose three of them to enter in a local show. Determine the number of different ways the three pigeons can be chosen if

(i) there are no restrictions. (1 mark)

Solution
$\binom{19}{3} = 969$
Specific behaviours
✓ correct number

(ii) the fancier decides to take one of each breed. (2 marks)

Solution
$\binom{5}{1} \times \binom{6}{1} \times \binom{8}{1} = 240$
Specific behaviours
✓ uses multiplication principle ✓ correct number

(iii) the fancier decides to take at least two Maltese pigeons. (2 marks)

Solution
$\binom{8}{2}\binom{11}{1} + \binom{8}{3}\binom{11}{0} = 308 + 56 = 364$
Specific behaviours
✓ indicates two cases ✓ correct number

Question 17 **(6 marks)**

Three vectors are $\mathbf{u} = a\mathbf{i} + b\mathbf{j}, \mathbf{v} = -3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j}$.

(a) Determine the vector projection of $\mathbf{v}$ on $\mathbf{w}$ in exact form. (2 marks)

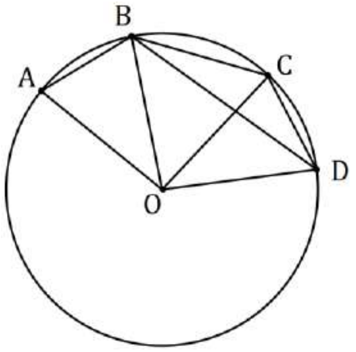
Solution
$[\mathbf{v} \cdot \mathbf{w}] \times \dfrac{\mathbf{w}}{\lvert\mathbf{w}\rvert^2} = \left[\begin{pmatrix}-3\\5\end{pmatrix} \cdot \begin{pmatrix}-1\\4\end{pmatrix}\right] \times \dfrac{1}{17}\begin{pmatrix}-1\\4\end{pmatrix}$ $= -\dfrac{23}{17}\mathbf{i} + \dfrac{92}{17}\mathbf{j}$
Specific behaviours
✓ indicates suitable form of projection   ✓ solution in exact form

(b) If $\mathbf{u}$ has the same magnitude as $\mathbf{v}$ and is perpendicular to $\mathbf{w}$, determine the exact values of the coefficients a and b. (4 marks)

Solution
$a^2 + b^2 = (-3)^2 + 5^2 = 34$ $-a + 4b = 0$ Using CAS, $a = 4\sqrt{2}$ and $b = \sqrt{2}$ or $a = -4\sqrt{2}$ and $b = -\sqrt{2}$
Specific behaviours
✓ equation from magnitudes   ✓ equation from perpendicular   ✓ one solution   ✓ both solutions

Question 18 **(7 marks)**

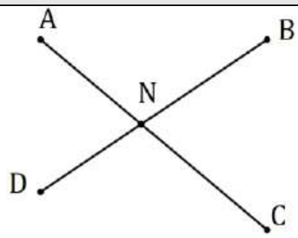
(a) In the diagram below, points B and C lie on the minor arc AD of the circle with centre O. The lengths of chords AB and CD are congruent, $\angle BOC = 37°$ and $\angle AOD = 163°$. Determine the size of $\angle CBD$. (3 marks)



Solution
$\angle AOB = \angle COD = \dfrac{163 - 37}{2} = 63°$ $\angle CBD = \dfrac{1}{2}\angle COD = 31.5°$
Specific behaviours
✓ indicates equal angles on equal chords   ✓ size of $\angle COD$ ✓ size of $\angle CBD$

(b) Line segment AC intersects line segment BD at N. Given that AC and BD are non-parallel and the lengths AN, AC, BN and BD are 6, 41, 21 and 31 cm respectively, explain whether the points A, B, C and D are concyclic. (4 marks)

Solution



$$CN = 41 - 6 = 35, DN = 31 - 21 = 10$$

$$AN.CN = BN.DN$$

$$6 \times 35 = 210 = 21 \times 10$$

Concyclic, as interval lengths satisfy the intersecting chord theorem.

Specific behaviours

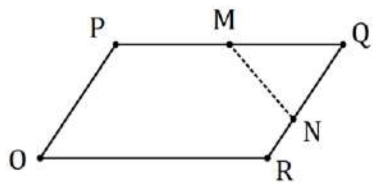
✓ sketch
✓ uses correct chord lengths
✓ uses property of intersecting chords
✓ explanation

Question 19 **(8 marks)**

(a) Triangle ABC has vertices with position vectors $A(2,-6), B(-3,14)$ and $C(6,8)$. Point P lies on side BC so that $2\overrightarrow{BP} = \overrightarrow{PC}$. Determine the vector $\overrightarrow{AP}$. (4 marks)

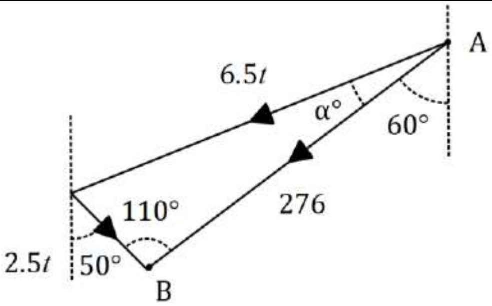
Solution
$\overrightarrow{BC} = \begin{pmatrix}6\\8\end{pmatrix} - \begin{pmatrix}-3\\14\end{pmatrix} = \begin{pmatrix}9\\-6\end{pmatrix}$ $\overrightarrow{BP} = \frac{1}{3}\overrightarrow{BC}$ $= \begin{pmatrix}3\\-2\end{pmatrix}$ $\overrightarrow{AB} = \begin{pmatrix}-3\\14\end{pmatrix} - \begin{pmatrix}2\\-6\end{pmatrix} = \begin{pmatrix}-5\\20\end{pmatrix}$ $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$ $= \begin{pmatrix}-5\\20\end{pmatrix} + \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-2\\18\end{pmatrix}$
Specific behaviours
✓ $\overrightarrow{BC}$ ✓ $\overrightarrow{BP}$ ✓ $\overrightarrow{AB}$ ✓ $\overrightarrow{AP}$

(b) $OPQR$ is a parallelogram. Point M is the midpoint of side PQ and point N is on side QR so that $QN = \frac{3}{5}QR$. If $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OR} = \mathbf{r}$, determine $\overrightarrow{MN}$ in terms of $\mathbf{p}$ and $\mathbf{r}$. (4 marks)

Solution	
	$\overrightarrow{OM} = \mathbf{p} + \frac{1}{2}\mathbf{r}$ $\overrightarrow{ON} = \mathbf{r} + \frac{2}{5}\mathbf{p}$ $\overrightarrow{MN} = \left(\mathbf{r} + \frac{2}{5}\mathbf{p}\right) - \left(\mathbf{p} + \frac{1}{2}\mathbf{r}\right)$ $= \frac{1}{2}\mathbf{r} - \frac{3}{5}\mathbf{p}$
Specific behaviours	
✓ sketch ✓ $\overrightarrow{OM}$ ✓ $\overrightarrow{ON}$ ✓ $\overrightarrow{MN}$	

Question 20 **(7 marks)**

A small boat leaves jetty A to travel to jetty B, 276 m away on a bearing of $240°$. A steady current of 2.5 ms^{-1} runs in the river between the jetties on a bearing $130°$. If the small boat travels at a constant speed of 6.5 ms^{-1}, determine the bearing it should steer to reach jetty B and how long the journey will take.

Solution
 $\dfrac{\sin 110}{6.5t} = \dfrac{\sin \alpha}{2.5t}$ $\alpha = 21.2°$ $180 - 110 - 21.2 = 48.8$ $\dfrac{\sin 110}{6.5t} = \dfrac{\sin 48.8}{276}$ $t = 53 \text{ seconds}$ Bearing: $240 + 21.2 = 261.2°$
Specific behaviours
✓ diagram ✓ angle in triangle between current and AB ✓ equation using sin rule for α ✓ solves for angle offset α ✓ equation using sin rule for t ✓ correct time ✓ correct bearing

Question 21 **(7 marks)**

A child is playing with thirteen coloured cubes, all the same size. There are six pink cubes, three navy and one each of red, blue, orange and green.

(a) If the child stacks cubes one on top of another to make a column, determine the number of different coloured columns that can be made using

(i) all the red, blue and green cubes. (1 mark)

Solution
$3! = 6$
Specific behaviours
✓ number

(ii) all the pink, red and orange cubes. (2 marks)

Solution
$\dfrac{(6+1+1)!}{6!} = \dfrac{8!}{6!} = 56$
Specific behaviours
✓ numerator ✓ correct number

(iii) all the cubes. (2 marks)

Solution
$\dfrac{13!}{6!\,3!} = 1\ 441\ 440$
Specific behaviours
✓ expression ✓ correct number

(b) If all but one of the cubes are used to make a column, determine the number of different coloured columns that can now be made. Justify your answer. (2 marks)

Solution
$1\ 441\ 400$ columns All the columns 13 tall with a pink on top must have a difference in the 12 cubes beneath and so if the top pink is removed, the remaining columns will still be different. The same is true for columns with other coloured top cubes, and the remaining 12 tall columns will have one less cube of the top colour and so must be different to all other columns. So, no change.
Specific behaviours
✓ correct number ✓ justification

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________