Semester One Examination, 2021

Question/Answer Booklet

MATHEMATICS METHODS
ATAR Year 12
Section One: Calculator-free

SOLUTIONS

Student Name: ______________________________

Please circle your teacher's name

Teacher: Miss Hosking Miss Rowden

Time allowed for this paper
Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Materials required/recommended for this paper
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

Number of additional answer booklets used (if applicable):	

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Suggested working time (minutes)	Marks available	Percentage of examination
Section One: Calculator free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the ATAR course examinations are detailed in the *Year 12 Information Handbook 2021*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Supplementary pages for the use planning/continuing your answer to a question have been provided at the end of the Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has eight (8) questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $\frac{d}{dx}(\cos^4(x))$. (2 marks)

Solution
$-4\sin x\cos^3 x$
Specific behaviours
✓ indicates use of chain rule ✓ correct derivative

(b) Evaluate $f'\left(\frac{\pi}{2}\right)$ when $f(x) = \frac{x + \sin x}{\cos 2x}$. (4 marks)

Solution
$f'(x) = \frac{(1+\cos x)(\cos 2x) - (x+\sin x)(-2\sin 2x)}{\cos^2 2x}$ $f'\left(\frac{\pi}{2}\right) = \frac{(1+0)(-1)-0}{(-1)^2}$ $= -1$
Specific behaviours
✓ indicates use of quotient rule ✓ correct u' and v' ✓ correct derivative ✓ substitutes and simplifies

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Question 2 **(5 marks)**

A small body is initially at the origin. It is moving along the x-axis with velocity at time t seconds given by

$$v(t) = \left(\frac{t}{2} - 2\right)^3 \text{ cm/s}.$$

(a) Determine $x(t)$, a function for the displacement of the body at time t. (3 marks)

Solution
$x(t) = \int \left(\frac{t}{2} - 2\right)^3 dt$
$= \frac{2}{4}\left(\frac{t}{2} - 2\right)^4 + c$
$t = 0 \Rightarrow \frac{1}{2}(-2)^4 + c = 0 \Rightarrow c = -8$
$x(t) = \frac{1}{2}\left(\frac{t}{2} - 2\right)^4 - 8$
Specific behaviours
✓ reasonable attempt at using chain rule
✓ correct antiderivative
✓ correct displacement function

The small body is stationary when $t = T$.

(b) Determine the displacement of the body at $T + 8$ seconds. (2 marks)

Solution
$\frac{T}{2} - 2 = 0 \Rightarrow T = 4 \text{ s}$
$x(12) = \frac{1}{2}(4)^4 - 8$
$= \frac{1}{2}(256) - 8$
$= 120 \text{ cm}$
Specific behaviours
✓ correct value of T
✓ correct displacement

Question 3 **(6 marks)**

Determine the area of the finite region bounded by $y = \sqrt{2x}$ and $y = \frac{x}{2}$.

Solution
Points of intersection: $$\sqrt{2x} = \frac{x}{2}$$ $$x^2 - 8x = 0$$ $$x = 0, \qquad x = 8$$ Area: $$A = \int_0^8 \sqrt{2x} - \frac{x}{2}\,dx$$ $$= \left[\frac{(2x)^{\frac{3}{2}}}{3} - \frac{x^2}{4}\right]_0^8$$ $$= \left[\frac{(16)^{\frac{3}{2}}}{3} - \frac{8 \times 8}{4}\right] - 0$$ $$= \frac{64}{3} - 16$$ $$= \frac{16}{3}\ \mathrm{u}^2$$
Specific behaviours
✓ equates curves and squares ✓ points of intersection ✓ writes integral for area ✓ correct antiderivative ✓ substitutes ✓ simplifies to obtain area

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Question 4 **(8 marks)**

(a) Simplify $\log_2(32) \times \log_3(27^2)$. (3 marks)

Solution
$\log_2 2^5 \times \log_3 3^6 = 5 \times 6 = 30$
Specific behaviours
✓ expresses as powers of log bases ✓ uses log law of $\log_a a = 1$ ✓ simplifies

(b) Solve for x:

(i) $\log_2 \frac{x}{3} = 4$ (2 marks)

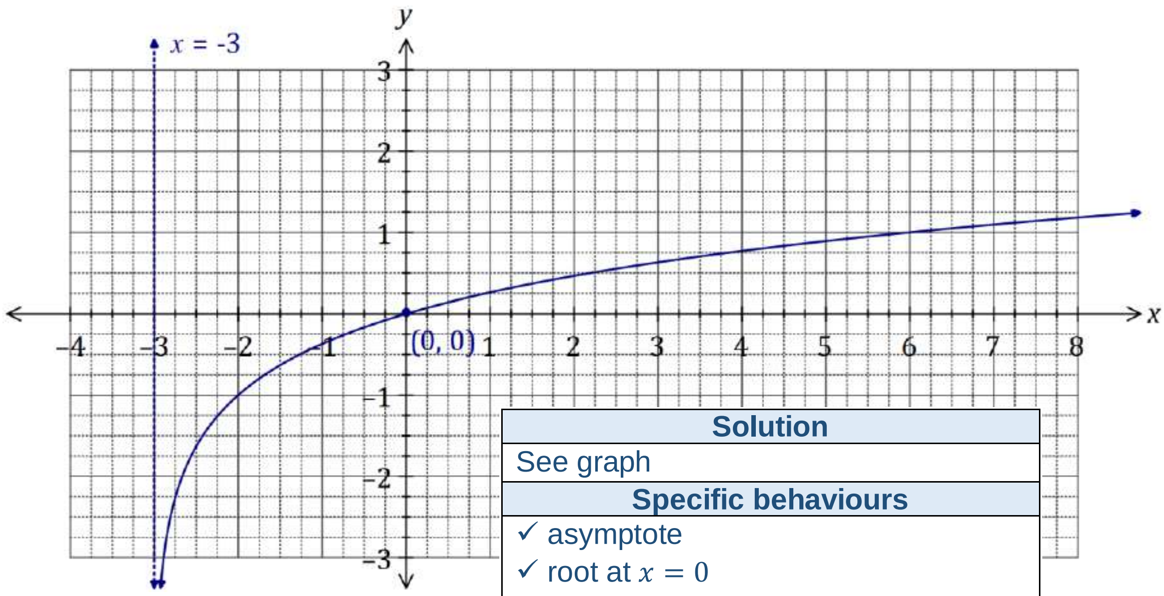
Solution
$2^4 = \frac{x}{3}$ ✓ $16 = \frac{x}{3}$ $x = 48$ ✓
Specific behaviours
✓ rewrites log into index form ✓ correct solution for x

(ii) $\log_m(x+2) - \log_m 4 = \log_m 3x$ (3 marks)

Solution
$\log_m \frac{(x+2)}{4} = \log_m 3x$ ✓ $\frac{x+2}{4} = 3x$ ✓ $x + 2 = 12x$ $2 = 11x$ $x = \frac{2}{11}$ ✓
Specific behaviours
✓ simplifies LHS using log laws ✓ equates ✓ correctly solves for x

Question 5 **(8 marks)**

(a) Sketch the graph of $y = \log_3(x + 3) - 1$ on the axes below, clearly showing the location of all asymptotes and axes intercepts. (3 marks)



Solution
See graph
Specific behaviours
✓ asymptote ✓ root at $x = 0$ ✓ curve, through $(-2, -1)$ and $(6, 1)$

(b) Determine the coordinates of the y-intercept of the graph of $y = 5\log_2(x + 0.5) + 1$. (2 marks)

Solution
$y = 5\log_2(0.5) + 1 = 5\log_2(2^{-1}) + 1 = -5 + 1 = -4$ At $(0, -4)$
Specific behaviours
✓ simplifies log term to -1 ✓ states coordinates of root

(c) The graph of $y = \log_a(x + a)$, where $a > 1$, passes through $(6, 2)$. Determine the coordinates of the root of the graph. (3 marks)

Solution
$2 = \log_a(6 + a) \Rightarrow a^2 - a - 6 = 0$ $(a - 3)(a + 2) = 0$ $a = 3 \text{ or } -2$ $a = 3, \quad (a > 1)$ Hence root at $(-2, 0)$
Specific behaviours
✓ forms quadratic equation ✓ solves for a with both solutions, then rejects -ve ✓ states coordinates of root

Question 6 **(5 marks)**

(a) Determine $\frac{d}{dx}(3x \cdot \sqrt[3]{e^x})$. (2 marks)

Solution
$\frac{d}{dx}\left(3x \cdot e^{\frac{x}{3}}\right) = 3e^{\frac{x}{3}} + xe^{\frac{x}{3}}$
Specific behaviours
✓ uses product rule ✓ obtains correct result

(b) Hence, or otherwise, determine $\int (3x \cdot \sqrt[3]{e^x})\, dx$. (3 marks)

Solution
$\int \frac{d}{dx}\left(3x \cdot e^{\frac{x}{3}}\right) dx = \int 3e^{\frac{x}{3}}\, dx + \int xe^{\frac{x}{3}}\, dx$ $3xe^{\frac{x}{3}} = 9e^{\frac{x}{3}} + \int xe^{\frac{x}{3}}\, dx$ $3\int xe^{\frac{x}{3}}\, dx = \int \left(3x \cdot \sqrt[3]{e^x}\right) dx = 9xe^{\frac{x}{3}} - 27e^{\frac{x}{3}} + c$
Specific behaviours
✓ integrates all terms of result from (a) ✓ uses fundamental theorem to simplify LHS ✓ obtains required result, with constant

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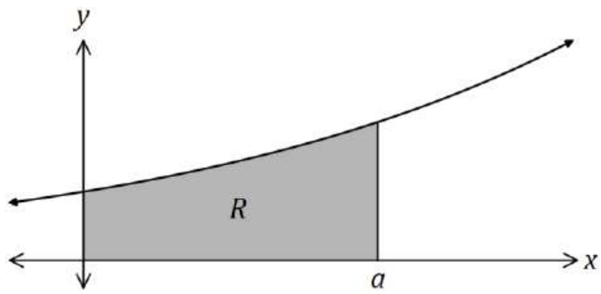
Question 7 **(6 marks)**

The shaded region R, shown on the graph below, is bounded by the curve $y = e^{3x}$ and the lines $y = 0,\ x = 0$ and $x = a$.



(a) Determine the area of R in terms of a. (3 marks)

Solution
$R = \int_0^a e^{3x}\,dx$ $= \left[\frac{e^{3x}}{3}\right]_0^a$ $= \frac{e^{3a}}{3} - \frac{e^0}{3} = \frac{e^{3a}}{3} - \frac{1}{3}$
Specific behaviours
✓ writes correct integral   ✓ antidifferentiates correctly   ✓ substitutes and simplifies

(b) Determine, in simplest form, the value of a for which the area of R is 21 square units. (3 marks)

Solution
$\frac{e^{3a}}{3} - \frac{1}{3} = 21 \Rightarrow e^{3a} = 64$ $\log e^{3a} = \log 64$ $3a \log e = \log 64$ $a = \frac{\log 64}{3 \log e}$
Specific behaviours
✓ isolates e^{3a} term   ✓ uses logs to obtain expression for a ✓ simplifies

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Question 8 **(8 marks)**

The function f is defined by $f(x) = \dfrac{4}{x^2 + 12}$, so that $f''(x) = \dfrac{24(x^2 - 4)}{(x^2 + 12)^3}$.

(a) Describe the concavity of the graph of $y = f(x)$. (4 marks)

Solution
$f''(x) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$ $x < -2, f''(x) > 0$ $-2 < x < 2, f''(x) < 0$ $x > 2, f''(x) > 0$ f is concave up when $x < -2$ and $x > 2$. f is concave down when $-2 < x < 2$.
Specific behaviours
✓ solves $f''(x) = 0$ ✓ indicates sign of $f''(x)$ in three intervals ✓ states domains for concave up, down ✓ uses correct inequalities in domains *(penalise ambiguous language such as between -2 and 2, etc.)*

(b) Determine, with justification, the range of $f'(x)$. (4 marks)

Solution
$f'(x) = \dfrac{-8x}{(x^2 + 12)^2}$ As $x \to \pm\infty, f'(x) \to 0$. Minimum and maximum of $f'(x)$ will be when its derivative $f''(x) = 0$, (i.e., at points of inflection) and from part (a) this is when $x = \pm 2$. $f'(\pm 2) = \pm \dfrac{-8 \times 2}{16 \times 16} = \mp \dfrac{1}{16}$ Hence the range is: $-\dfrac{1}{16} \le f'(x) \le \dfrac{1}{16}$.
Specific behaviours
✓ expression for $f'(x)$ ✓ states behaviour of $f'(x)$ for $x \to \pm\infty$ ✓ location of minimum and maximum values of $f'(x)$ ✓ correct range, as inequality

End of questions

Supplementary page

Question number: ____________________

Supplementary page

Question number: ____________________

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