Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	52	35
Section Two: Calculator-assumed	12	12	100	96	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

Section One: Calculator-free **35% (52 Marks)**

This section has **six (6)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

It can be shown that for all $n \geq 0$,

$$^{n+1}P_r = \frac{n+1}{n-r+1} \times {}^nP_r$$

(a) Show that the identity is true when $n = 4$ and $r = 2$. (2 marks)

Solution
$LHS = {}^5P_2 = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 20$ $RHS = \frac{5}{3} \times {}^4P_2 = \frac{5}{3} \times \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 20$
Specific behaviours
✓ evaluates LHS ✓ evaluates RHS

Given that ${}^8P_4 = 1\,680$, ${}^{12}P_5 = 95\,040$ and ${}^{12}P_6 = 665\,280$, evaluate

(b) ${}^{11}P_6$. (2 marks)

Solution
${}^{12}P_6 = \frac{12}{6} \times {}^{11}P_6$ ${}^{11}P_6 = \frac{665280}{2} = 332\,640$
Specific behaviours
✓ relates ${}^{12}P_6$ and ${}^{11}P_6$ ✓ evaluates

(c) ${}^{10}P_4$. (3 marks)

Solution
${}^{10}P_4 = \frac{10}{6} \times {}^9P_4$ $= \frac{10}{6} \times \frac{9}{5} \times {}^8P_4$ $= 3 \times 1680 = 5040$
Specific behaviours
✓ expresses in terms of ${}^9P_4$ ✓ expresses in terms of ${}^8P_4$ ✓ evaluates

See next page

Question 2 **(11 marks)**

Three vectors are given by $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 7\mathbf{j}$ and $\mathbf{c} = 6\mathbf{i} + \mathbf{j}$.

(a) Determine

(i) $\mathbf{a} + \mathbf{b} + \mathbf{c}$. (1 mark)

Solution
$\mathbf{a} + \mathbf{b} + \mathbf{c} = 7\mathbf{i} + 3\mathbf{j}$
Specific behaviours
✓ states vector

(ii) $|\mathbf{c}|$. (1 mark)

Solution
$\|\mathbf{c}\| = \sqrt{36 + 1} = \sqrt{37}$
Specific behaviours
✓ states exact value

(iii) $2\mathbf{a} + 3\mathbf{b}$. (2 marks)

Solution
$2\mathbf{a} = 6\mathbf{i} - 10\mathbf{j}$ $3\mathbf{b} = -6\mathbf{i} + 21\mathbf{j}$ $2\mathbf{a} + 3\mathbf{b} = 11\mathbf{j}$
Specific behaviours
✓ determines scalar multiples ✓ determines sum

(b) Determine the unit vector $\hat{\mathbf{d}}$ that is parallel and in the same direction as $\mathbf{b} - \mathbf{a}$. (3 marks)

Solution
$\mathbf{d} = \mathbf{b} - \mathbf{a} = -5\mathbf{i} + 12\mathbf{j}$ $\|\mathbf{b} - \mathbf{a}\| = 13$ $\hat{\mathbf{d}} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$
Specific behaviours
✓ determines $\mathbf{b} - \mathbf{a}$ ✓ determines magnitude ✓ states unit vector

See next page

(c) Express **c** in terms of **a** and **b**. (4 marks)

Solution
Let $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$ **i**-coeffs: $3x - 2y = 6$ **j**-coeffs: $-5x + 7y = 1$ $15x - 10y = 30$ $-15x + 21y = 3$ $11y = 33 \Rightarrow y = 3$ $x = 4$ $\mathbf{c} = 4\mathbf{a} + 3\mathbf{b}$
Specific behaviours
✓ equates **i**-coeffs ✓ equates **j**-coeffs ✓ solves equations for first variable ✓ solves equations for second variable and states **c**

See next page

Question 3 **(8 marks)**

(a) Write the inverse of the following true statement and comment on the truth of the inverse statement. (2 marks)

"If the discriminant of the quadratic formula is zero, then the quadratic has just one real root."

Solution
If the discriminant of the quadratic formula is **not** zero, then the quadratic **does not have** just one real root. Statement is true.
Specific behaviours
✓ changes 'if P then Q' to 'if not P then not Q' ✓ indicates statement is true

(b) Write the converse of the following true statement and comment on the truth of the converse statement. (2 marks)

"If $x > 3$ then $x > 2$."

Solution
If $x > 2$ then $x > 3$. Statement is false.
Specific behaviours
✓ changes 'if P then Q' to 'if Q then P' ✓ indicates statement is false

(c) Determine the truth of the following statements, using an example or counter-example to support each answer.

(i) If $z \in \mathbb{R}$ and z^3 is an even number then z is an even number. (2 marks)

Solution
Statement is false. If $z^3 = 6$ (even) then $z = \sqrt[3]{6}$ (irrational, not even).
Specific behaviours
✓ states false ✓ supplies counter-example

(ii) If $x, y \in \mathbb{Z}$ and $x > y$ then $x^2 > y^2$. (2 marks)

Solution
Statement is false. If $x = 2, y = -3$ then $2 > -3$ but $2^2 \not> (-3)^2$.
Specific behaviours
✓ states false ✓ supplies counter-example using integers

See next page

Question 4 **(7 marks)**

(a) A body moves from $P(2, -3)$ to $Q(-2, 1)$.

(i) Determine the displacement vector $\overrightarrow{PQ}$ in component form. (1 mark)

Solution
$\overrightarrow{PQ} = \begin{pmatrix} -2-2 \\ 1--3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$
Specific behaviours
✓ expresses in component form

(ii) Determine the magnitude of the vector $\overrightarrow{PQ}$. (1 mark)

Solution
$\left\|\overrightarrow{PQ}\right\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$
Specific behaviours
✓ states magnitude

(b) A force of $6\mathbf{i} - 6\sqrt{3}\mathbf{j}$ N acts on a body. Determine the magnitude of the force and the angle its direction makes with the positive x-axis. (2 marks)

Solution
$\mathbf{F} = 6(\mathbf{i} - \sqrt{3})\mathbf{j}$ $\|\mathbf{F}\| = 12$ N $\theta = -60°$
Specific behaviours
✓ states magnitude ✓ states angle

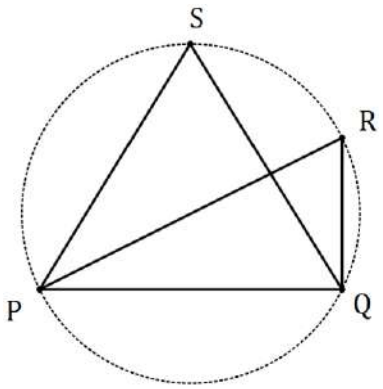
(c) A body moves with a velocity of 20 ms^{-1} at an angle of 135° with the positive x-axis. Express the velocity of the body in the form $a\mathbf{i} + b\mathbf{j}$, where a and b are constants. (3 marks)

Solution
$a = 20\cos 135 = -\dfrac{20\sqrt{2}}{2} = -10\sqrt{2}$ $b = 20\sin 135 = \dfrac{20\sqrt{2}}{2} = 10\sqrt{2}$ $v = -10\sqrt{2}\mathbf{i} + 10\sqrt{2}\mathbf{j}$ m/s
Specific behaviours
✓ determines expressions for a and b ✓ simplifies a and b ✓ states in required form

See next page

Question 5 **(10 marks)**

(a) In the diagram below, not drawn to scale, $PQRS$ is a cyclic quadrilateral such that $PS = QS$, $\angle RPQ = 34°$ and $\angle PQR$ is a right-angle.



Determine the sizes of

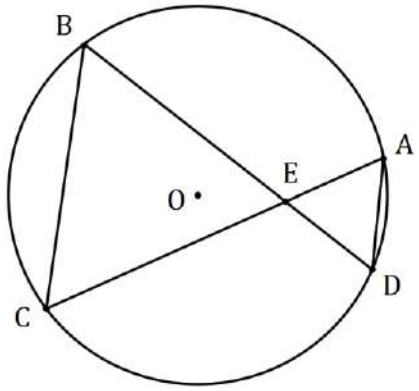
(i) $\angle PSQ$. (2 marks)

Solution
$\angle PRQ = 90 - 34 = 56°$ $\angle PSQ = \angle PRQ = 56°$
Specific behaviours
✓ determines $\angle PRQ$ ✓ states $\angle PSQ$

(ii) $\angle RPS$. (2 marks)

Solution
$\angle SPQ = \frac{180 - 56}{2} = 62°$ $\angle RPS = 62 - 34 = 28°$
Specific behaviours
✓ determines $\angle SPQ$ ✓ determines $\angle RPS$

(b) In the circle with centre O drawn below, chord AC intersects chord BD at E. Explain, with reasoning, why triangles AED and BEC are similar. (3 marks)



Solution
$\angle AED = \angle BEC$ (vertically opposite angles) $\angle CBD = \angle CAD$ (angles stand on same arc) Hence triangles are similar as three pairs of equal angles.
Specific behaviours
✓ one pair of angles, with reason ✓ second pair of angles, with reason ✓ summary, using AAA reasoning

(c) Prove that when two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord.

(3 marks)

Solution
Using diagram from (b), where $\Delta AED \sim \Delta BEC$. Then from ratio of corresponding sides $\frac{BE}{AE} = \frac{CE}{DE}$ Hence, $BE \times DE = AE \times CE$ and proof is complete.
Specific behaviours
✓ uses diagram/similar triangles ✓ uses ratio of sides ✓ shows products are equal

See next page

Question 6 **(9 marks)**

(a) Determine the number of different four-letter passwords that can be made by arranging a selection of four letters chosen from the list P, Q, R, R, R, R and S. (4 marks)

Solution
In each case, ways to choose R's×ways to choose others×arrangements: 1 R: $1 \times 1 \times 4! = 24$ 2 R's: $1 \times 3 \times \frac{4!}{2!} = 36$ 3 R's: $1 \times 3 \times \frac{4!}{3!} = 12$ 4 R's: $1 \times 1 \times 1 = 1$ Number of different passwords is 73.
Specific behaviours
✓ breaks in to cases ✓ calculates one case correctly ✓ calculates at least three cases correctly ✓ correct total

(b) Determine the number of positive integers between 1 and 240 inclusive that are **not** divisible by at least one of the integers 4, 5 or 6. (5 marks)

Solution
Multiples of 4, 5 or 6: $\frac{240}{4} + \frac{240}{5} + \frac{240}{6} = 60 + 48 + 40 = 148$ Multiples of 12, 20 or 30: $\frac{240}{12} + \frac{240}{20} + \frac{240}{30} = 20 + 12 + 10 + 8 = 40$ Multiples of 60: $\frac{240}{60} = 4$ Total divisible: $148 - 40 + 4 = 112$ Total **not** divisible: $240 - 112 = 128$
Specific behaviours
✓ uses multiples of one, two and three numbers ✓ uses LCM for two and three numbers ✓ calculates at least one set of multiples correctly ✓ uses inclusion-exclusion principle ✓ correct answer

End of questions

Additional working space

Question number: _____