Physics with the Banach Fixed Point Theorem: A Complementary Framework for Deriving Formulas

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Abstract

This note explores how the Banach Fixed Point Theorem (BFPT), first established by Banach [1], provides a complementary framework for analyzing physical systems. While differential equations often serve as the foundation for modeling dynamics, the BFPT offers an alternative perspective by framing problems in terms of metric spaces and contraction mappings. By revisiting the general particle half-life formula [2], this approach demonstrates the iterative convergence of closed-form solutions and their stability, providing insights into the underlying mathematical structures of these systems.

1 Introduction

Differential equations are the cornerstone of mathematical modeling in physics, describing phenomena such as particle decay. However, the process of solving these equations can sometimes obscure the iterative and structural nature of the underlying solutions. The Banach Fixed Point Theorem (BFPT), introduced by Banach [1], provides an alternative lens for understanding physical phenomena by emphasizing iterative convergence and fixed-point stability.

This note aims to clarify the connection between BFPT and differential equations while framing its pedagogical value as a complementary tool for understanding physical systems. By revisiting the general particle half-life formula, historically modeled by exponential decay [2], the study demonstrates how contraction mappings within metric spaces naturally yield solutions that align with classical results.

2 Preliminaries

To establish the foundation for this approach, key definitions and theorems related to BFPT are provided.

2.1 Metric Spaces

A pair (X, d) is a metric space if X is a set and $d : X \times X \to \mathbb{R}$ is a distance metric. The metric d must satisfy:

1. Non-negativity: For all $x, y \in X$,

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \iff x = y. \tag{1}$$

2. Symmetry: For all $x, y \in X$,

$$d(x, y) = d(y, x). \tag{2}$$

3. Triangle inequality: For all $x, y, z \in X$,

$$d(x, z) \leq d(x, y) + d(y, z). \tag{3}$$

2.2 Complete Metric Spaces

A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X. Completeness ensures that iterative mappings remain well-defined within the space.

2.3 Banach Fixed Point Theorem

[Banach Fixed Point Theorem [1]] Let (X, d) be a complete metric space, and let $T : X \to X$ be a contraction mapping. That is, there exists $c \in [0, 1)$ such that

$$d(T(x), T(y)) \leq c \cdot d(x, y) \quad \forall x, y \in X. \tag{4}$$

Then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the sequence $(T^n(x_0))$ converges to x^*.

3 Application to Particle Decay

This section applies BFPT to particle decay. While this problem is traditionally modeled using differential equations, fixed-point methods provide an alternative perspective.

3.1 General Particle Half-Life Formula

The particle decay formula is given by:

$$P(t) = P_0 \alpha^t, \quad 0 < \alpha < 1, \tag{5}$$

where P_0 is the initial quantity of the sample, and α is the decay constant.

Metric Space and Completeness. Define $E = \{(t, P) \mid t \geq 0, P \geq 0\}$ and

$$d((t_1, P_1), (t_2, P_2)) = |t_1 - t_2| + |P_1 - P_2|. \tag{6}$$

This is a standard metric on $\mathbb{R}^2$, and E is complete because $\mathbb{R}$ is complete.

Contraction Mapping. Define the iterative mapping:

$$T(t, P) = (t + \beta, \alpha P), \tag{7}$$

where $\beta > 0$ and $0 < \alpha < 1$. The contraction property holds because:

$$d(T(t_1, P_1), T(t_2, P_2)) = |t_1 - t_2| + \alpha|P_1 - P_2| \leq d((t_1, P_1), (t_2, P_2)). \tag{8}$$

Iterative Convergence. Starting from $(0, P_0)$, repeated application of T yields:

$$t_n = n\beta, \quad P_n = P_0\alpha^n. \tag{9}$$

Substituting $t = n\beta$, the decay formula is recovered:

$$P(t) = P_0\alpha^{t/\beta}. \tag{10}$$

Derivation of the Half-Life Formula. The half-life $t_{1/2}$ is the time at which $P(t_{1/2}) = \frac{P_0}{2}$. Substituting into the decay formula:

$$\frac{P_0}{2} = P_0\alpha^{t_{1/2}/\beta}. \tag{11}$$

Dividing through by P_0 and taking the natural logarithm:

$$\ln\left(\frac{1}{2}\right) = \frac{t_{1/2}}{\beta}\ln(\alpha). \tag{12}$$

Rearranging gives:

$$t_{1/2} = \frac{\ln(2)}{\ln(1/\alpha)}\beta. \tag{13}$$

This is the general formula for the half-life.

4 Conclusion

The Banach Fixed Point Theorem provides an intuitive framework for analyzing iterative convergence in physical systems. While this example aligns with traditional differential equation-based results, this perspective highlights stability and equilibrium in a mathematically rigorous manner. Future work may explore how this approach extends to more complex systems or offers computational advantages and new closed-form formulas for physical phenomena.

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References

[1] Sutherland, W. A. (2009). *Introduction to Metric and Topological Spaces* (2nd ed.). Oxford University Press, pp. 39, 192.

[2] E. Rutherford and F. Soddy, "The Cause and Nature of Radioactivity," *Philosophical Magazine*, 1902.