Generalised Supersymmetric IWP Black Hole Solutions in (3+1) Dimensional Spacetime Using Spinorial Geometry

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Abstract

In this report we present a derivation of the Israel-Wilson-Perjés (IWP) class of black hole metrics using Supersymmetry (SUSY) and spinorial geometry. We derive the metric class by imposing that super-covariantly constant spinors must be admitted, and then naturally building the metric from spinor bilinear covariants. This avoids having to use Einstein's field equations altogether, relying essentially only on supersymmetric conditions for the underlying constraints on the geometry. This report begins with a review of differential geometry, gravity, and the tetrad formalism before introducing the Einstein-Maxwell action and the Killing Spinor Equation (KSE). We then solve the KSE for a specific canonical spinor orbit and build the metric from bilinear covariants, finalising by finding specific conditions that complete the metric. In this process we find that there is a layer of generality involved when picking the basis vectors to build the metric from, yielding new IWP-type solutions that are more general than those found in the literature.

Chapter 1

Introduction

Symmetries in nature have a profound effect on our understanding of Physics, and form the bases for many of our fundamental theories about the universe. They are defined as properties of mathematical or physical systems that are invariant under some transformation. They are extremely important, for example Noether's theorem states that every symmetry in our laws of Physics leads to a conserved quantity, like how time translation symmetry leads to the conservation of energy. Transformations that form these symmetries usually form a group, and continuous symmetries usually form continuous groups called Lie groups.

We will be using Supersymmetry (SUSY). At its core, SUSY is a proposed symmetry between 2 types of fundamental particles, fermions, which have half integer spin, and bosons, which have integer spin. It conjectures that every boson has a fermion superpartner that differs in spin by 1/2. When this idea is combined with General Relativity (GR), we get a theory of Supergravity (SUGRA), which predicts the graviton, and its fermion superpartner the gravitino. Just as the photon mediates electromagnetic interactions, the graviton mediates gravitational ones. The ideas of supersymmetry hold within them the important possibility of explaining as of yet unexplained Physics. In many supersymmetric theories, the lightest stable particles predicted are perfect candidates for dark matter, thought to make up most of the matter in the universe. In the realm of Particle Physics, SUSY was and still possibly could be a potential contender for unifying all fundamental forces that we know of in our universe except gravity at very high energies. A theory that unifies all the forces is called a Grand Unified Theory. An even more ambitious responsibility is given to SUGRA, an attempt to unify Quantum Mechanics and GR, creating a Theory of Everything.

And despite the disappointing lack of experimental evidence to date either categorically ruling out or supporting SUSY, one thing is for certain. It gives rise to some very interesting theoretical results.

We will be working within the realm of SUGRA to derive the Israel-Wilson-

Perjés (IWP) black hole metric, and a new generalisation in (3+1) dimensional spacetime. It is a metric class that provides stationary solutions to symmetrical distributions of charge and matter. This is not a new metric, it was first published back in 1971 by Perjés [1] as a stationary extension to the Majumdar-Papapetrou class of black holes, and independently in 1972 by Israel and Wilson [2] , who generated it from Laplace's equation. Both of these derivations did not use SUGRA, and required the use Einstein's field equations. In 1983, Tod [3] derived the metric using SUGRA and the condition that the metric must admit supercovariantly constant spinors using Twistor theory. We will be following a similar route, except where Tod used Twistor theory, which only works in 4D, we will be using a fairly recent collection of assembled methods that are together known as spinorial geometry [4] . The main advantage of this is that spinorial geometry is generaliseable to higher dimensions.

Normally when deriving metrics, one starts with an ansatz for the form of the metric, based off of some physical properties you want your metric to have, and then one would subject themselves to the task of solving Einstein's field equations for your metrics and some combination of stress-energy tensors. However, by far the most intriguing part of this entire endeavour of deriving the IWP metric in this way is the fact that using SUGRA as a base, you don't require the use of Einstein's field equations at any point in the calculation. The metric seemingly appears to build naturally only from SUSY conditions on the geometry of spacetime.

The main purpose of this project is to derive the IWP metric using the constraints imposed by SUGRA and the recent methods collectively known as being a part of spinorial geometry.

The text is organised as follows. Chapter 2 will begin by discussing and defining some theoretical tools for the project. A brief review of the tetrad formalism and spinors within the context of differential geometry will be included here, as well as the establishing of important notational conventions that we will continue to use further into the report. Chapter 3 is where we will introduce the Einstein-Maxwell action we are working within the realms of as well as the equations of motion for the theory. Chapter 4 will introduce the SUSY constraints and discuss their key relationship to the geometry of spacetime. Then in Chapter 5, the canonical spinor orbits are discussed and the SUSY constraints are reduced for the specific spinorial orbit that corresponds to the IWP metric. Chapter 6 follows naturally by building spinor bilinear covariants, investigating them, and using them to build the metric. In Chapter 7 we find further constraints on the metric, finalising the derivation for the black hole found by Tod, completing the main calculation. In Chapter 8 we go back into the derivation and introduce a layer of generality in picking basis vectors, in order to construct a more general IWP class of solutions. Chapter 9 concludes the report, with subsequent appendixes supplementing areas where calculations were left vague.

Chapter 2

Differential geometry

This chapter will focus on some tools we will use later in the report, assorted from differential and spinorial geometry. Differential forms provide a useful approach to multivariable calculus independent of coordinates. We have used them extensively throughout the project to do calculations as they are widely applicable to the language of GR and spinors. This chapter will introduce some actions on differential forms relevant to our subsequent calculations, and will then outline their use in GR by discussing the tetrad formalism, and their connection to spinors. Spinors importantly can be written as forms due to the isomorphism between Clifford algebras and exterior algebras [5].

2.1 Differential forms

In this subsection we will define the exterior derivative, a notion of the inner product useful for finding the partial derivatives from the basis forms, and the Hodge star operator[6][7].

Definition 2.1.1. The exterior derivative d that acts on r-forms on a manifold M is a map $\Omega^r(M) \to \Omega^{r+1}(M)$ whose action on

$$p = p_{\mu_1 \ldots \mu_r} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \tag{2.1.1}$$

is defined by

$$\mathrm{d}p = \left(\frac{\partial}{\partial x^\nu} p_{\mu_1 \ldots \mu_r} \right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}. \tag{2.1.2}$$

There is also a notion of an inner product between the tangent space and its dual, the cotangent space.

Definition 2.1.2. For the tangent space basis $e_\alpha \in T_qM$ and dual basis 1-forms $\omega^\alpha = \mathrm{d}x^\alpha \in T^*_qM$ on a manifold M at a point q we can equip an inner product $\langle \cdot, \cdot \rangle : T^*_qM \times T_qM \to \mathbb{C}$ such that

$$\langle \omega^\alpha, e_\beta \rangle = \delta^\alpha_\beta. \tag{2.1.3}$$

Additionally we can define a natural isomorphic operation between $\Omega^r(M)$ and $\Omega^{m-r}(M)$ on an m-dimensional manifold M called the Hodge star $*$ given a metric g. First we require the totally anti-symmetric tensor ε such that

$$\varepsilon_{\mu_1\mu_2...\mu_m} = \begin{cases} 1 & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an even permutation of } (12...m) \\ -1 & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an odd permutation of } (12...m) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.4}$$

Definition 2.1.3. The linear Hodge map $* : \Omega^r(M) \to \Omega^{m-r}(M)$ has an action

$$*\left(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r}\right) = \frac{\sqrt{|g|}}{(m-r)!}\varepsilon^{\mu_1\mu_2...\mu_r}{}_{\nu_r+1...\nu_m}\,\mathrm{d}x^{\nu_{r+1}} \wedge ... \wedge \mathrm{d}x^{\nu_m}. \tag{2.1.5}$$

2.2 The tetrad formalism

The tetrad formalism generalises the choice of basis in such a way that interfaces very well with spinors and makes the physics more transparent by embedding it in the tetrads themselves. In this section we will explore some key relations and then discuss the connection to spinors.

This is in essence an approach to GR that generalizes the choice of basis for the tangent bundle from a coordinate to a local basis[8][6]. This manifests in practise as writing the metric tensor as 2 vierbeins, in our case we will use orthonormal vierbein coordinates such that

$$g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}. \tag{2.2.1}$$

The orthonormality inherent in our conscious choice here is ingrained in the use of the flat Minkowski metric η_{ab} instead of - for example - a different metric g_{ab}. By imposing orthonormality we essentially push all the physics into the vierbeins, and this all comes with new relations, making this indeed a very different formalism.

Notational remark. To avoid confusion throughout this report, we will use the common notational convention of Latin letters for local Lorentz frame indices and Greek letters for the general coordinate basis.

In order to use this formalism effectively we will introduce some key relations. The vierbein indices can be lowered and raised in the usual way using the local and flat Minkowski metric η_{ab} and the coordinate metric $g_{\mu\nu}$. For example

$$e_{\nu a} = \eta_{ab} e_\nu^{\ b} \text{ and } e^{\mu a} = g^{\mu\nu} e_\nu^{\ a}. \tag{2.2.2}$$

Writing the vierbeins as differential forms

$$e^a = e_\mu^{\ a}\,\mathrm{d}x^\mu, \tag{2.2.3}$$

we can also introduce the spin connection 1-forms

$$\omega^{ab} = \omega_{\mu}^{\ ab}\,\mathrm{d}x^{\mu}. \tag{2.2.4}$$

This functions as a connection in a similar way to the Levi-Civita connection, except with forms. It is linked to the vierbeins themselves with the torsion 2-form

$$\Theta^{a} = \mathrm{d}e^{a} + \omega^{a}_{\ b} \wedge e^{b}. \tag{2.2.5}$$

If we take the torsion to be 0, which we will indeed do, this gives us a clear connection between the vierbeins and the spin connection coefficients. We also have the curvature 2-form

$$R^{a}_{\ b} = \mathrm{d}\omega^{a}_{\ b} + \omega^{a}_{\ c} \wedge \omega^{c}_{\ b} = \frac{1}{2}R^{a}_{\ bcd}e^{c} \wedge e^{d} \tag{2.2.6}$$

Equations 2.2.5 and 2.2.6 are called Cartan's structure equations, and we also necessarily have the 2 Bianchi identities that must be obeyed. If the torsion is 0, we have the first as

$$R^{a}_{\ b} \wedge e^{b} = 0, \tag{2.2.7}$$

and the second as

$$\mathrm{d}R^{a}_{\ b} + \omega^{a}_{\ c} \wedge R^{c}_{\ b} - R^{a}_{\ c} \wedge \omega^{c}_{\ b} = 0. \tag{2.2.8}$$

This in itself is enough to solve for many metrics, but shows more importantly the fundamental link between the spin connection and the geometry of spacetime. Spin connections, as their name suggests, also have a connection to the idea of spin, making the tetrad formalism extremely useful where supersymmetric conditions are concerned. The connection to the geometry is in practise due to the KSE. We will therefore now be defining how spinors, and the γ-matrices, operate in terms of forms, as this is the language the KSE is written in.

2.3 Spinors in 4 dimensions

The spinors that will appear in the supersymmetric condition, the Killing spinor equation, are Dirac spinors. From [5][9] we can write our spinors as complexified forms on $\mathbb{R}^2$. So in the space of Dirac spinors Δ we have for our spinors $\varepsilon \in \Delta = \bigwedge^{*}\left(\mathbb{R}^2\right) \otimes \mathbb{C}$, meaning they have the form

$$\varepsilon = \lambda \mathbb{1} + \mu^{1}e^{1} + \mu^{2}e^{2} + \sigma e^{12} \tag{2.3.1}$$

where e^1, e^2 are 1-forms on $\mathbb{R}^2$, and $e^{12} = e^1 \wedge e^2$. λ, μ^i, and σ are complex functions. Following in the same vein, we have the actions of γ-matrices on these forms given by

$$\begin{aligned} \gamma_0 &= -e^2 \wedge + i_{e^2}, \\ \gamma_1 &= e^1 \wedge + i_{e^1}, \\ \gamma_2 &= e^2 \wedge + i_{e^2}, \\ \gamma_3 &= i\left(e^1 \wedge - i_{e^1}\right), \end{aligned} \tag{2.3.2}$$

with γ^5 defined by $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and additionally $\gamma^{ab} = \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right)$, $\gamma^{ab} = -\gamma^{ba}$. All generated (in matrix form) from

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}. \tag{2.3.3}$$

They can also be raised and lowered in the usual way, for example $\gamma^a = \eta^{ab}\gamma_b$. All useful actions of combinations of these γ-matrices on ε can be found in the appendix for use in solving the KSE.

Chapter 3

Action and equations of motion

We are working with the Einstein-Maxwell theory in (3+1) dimensions, which has an action given by

$$\mathcal{S} = \int d^4x\sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu}), \tag{3.0.1}$$

where g is the metric, R is the Ricci scalar, and F is the U(1) gauge field strength. The signature of the metric is taken to be $(-,+,+,+)$. It represents both the theory of GR and electromagnetism. In this chapter we will derive the equations of motion, starting with Einstein's field equations and moving on to Maxwell's equations. Strictly, we do not require Einstein's field equations for deriving the IWP metric, but we will include its derivation for completeness and because it shows how powerful the tetrad formalism can be calculationally in certain circumstances[6].

3.1 Einstein's field equations

We will require the following results before beginning. They can be acquired by applying perturbation theory to the metric tensor and building up the perturbations through the Levi-Civita connection to the curvature.

$$\begin{aligned} \delta(\sqrt{-g}) &= \frac{-g_{\mu\nu}\sqrt{-g}\delta g_{\mu\nu}}{2} \\ \frac{\delta R}{\delta g^{\mu\nu}} &= R_{\mu\nu} \end{aligned} \tag{3.1.1}$$

For the time being, we will absorb the source-free lagrangian density into a single term so that $\mathcal{L}_M = -F_{\mu\nu}F^{\mu\nu}$, and rewriting the action as

$$\mathcal{S} = \int d^4x\sqrt{-g}\,(R + \mathcal{L}_M). \tag{3.1.2}$$

Einstein's field equations are derived by requiring that the variation in the overall action is null, i.e. $\delta\mathcal{S} = 0$. Proceeding in this way we vary the action $\mathcal{S}$,

$$\delta\mathcal{S} = \int \mathrm{d}^4x \left\{ \frac{\delta(\sqrt{-g})R}{\delta g^{\mu\nu}\sqrt{-g}} + \frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g})\mathcal{L}_M}{\delta g^{\mu\nu}\sqrt{-g}} + \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu}\sqrt{-g}. \tag{3.1.3}$$

Using equations 3.0.2 and reducing, this gives us

$$\delta\mathcal{S} = 0 \Leftrightarrow Rg_{\mu\nu} - 2R_{\mu\nu} + \mathcal{L}_M g_{\mu\nu} - 2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} = 0, \tag{3.1.4}$$

where

$$T_{\mu\nu} = \frac{1}{2}\mathcal{L}_M g_{\mu\nu} - \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} \tag{3.1.5}$$

is precisely the definition (up to a constant of proportionality) of the Hilbert Stress-Energy tensor for the nongravitational part of the lagrangian density $\mathcal{L}_M$. There is also the Einstein tensor defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{3.1.6}$$

Now for our case, where we have $\mathcal{L}_M = -F_{\mu\nu}F^{\mu\nu}$, we must find the form of $T_{\mu\nu}$, the electromagnetic stress-energy tensor. The first term is easy, it is impossible to simplify further,

$$\frac{1}{2}\mathcal{L}_M g_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{3.1.7}$$

but the second term

$$-\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}} = \frac{\delta}{\delta g^{\mu\nu}}\left(F_{\alpha\beta}F^{\alpha\beta}\right) \tag{3.1.8}$$

is still messy. We can reduce this, as it is usually done, by separating the flat and curved parts. For the field strength this means

$$F_{\alpha\beta}F^{\alpha\beta} = e^A_\alpha e^B_\beta e^\alpha_C e^\beta_D F_{AB}F^{CD}, \tag{3.1.9}$$

and for the metric

$$g^{\mu\nu} = e^\mu_M e^\nu_P \eta^{MP}. \tag{3.1.10}$$

We can use the chain rule for

$$\frac{\delta}{\delta g^{\mu\nu}} = \frac{\delta}{\delta e^\lambda_P}\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}}, \tag{3.1.11}$$

and then

$$\delta g^{\mu\nu} = 2\eta^{MP}e^\mu_M(\delta e^\lambda_P)\delta^\nu_\lambda \tag{3.1.12}$$

follows from the product rule. Using these ingredients, and substituting directly into 3.1.9 yields

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}}\left(F_{\alpha\beta}F^{\alpha\beta}\right) &= \frac{\delta}{\delta e^\lambda_P}\left(e^A_\alpha e^B_\beta e^\alpha_C e^\beta_D F_{AB}F^{CD}\right)\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}} \\ &= 4e^A_\alpha e^B_\beta e^\alpha_C \frac{\delta e^\beta_D}{\delta e^\lambda_P} F_{AB}F^{CD}\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}}, \end{aligned} \tag{3.1.13}$$

as $\frac{\delta}{\delta g^{\mu\nu}}$ acting on F_{AB} must be 0, due to it being in the flat Minkowski basis. We can proceed by consolidating some terms into δ-functions and using 3.1.12 such that:

$$\frac{\delta}{\delta g^{\mu\nu}}\left(F_{\alpha\beta}F^{\alpha\beta}\right) = 4e^A_\alpha e^B_\beta e^\alpha_C(\delta^\beta_\lambda)(\delta^P_D)F_{AB}F^{CD}\left(\frac{1}{2}\eta_{MP}e^M_\mu\delta^\lambda_\nu\right) \tag{3.1.14}$$

$$= 2e^A_\alpha e^B_\beta e^\alpha_C F_{AB}F^{CD}e^M_\mu\eta_{MD} \tag{3.1.15}$$

$$= 2e^A_\alpha e^B_\beta e^\alpha_C e_{D\mu}F_{AB}F^{CD} = 2F_{\alpha\nu}F^\alpha_{\ \mu} \tag{3.1.16}$$

Overall we are left with

$$G_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + 2F_{\alpha\nu}F^\alpha_{\ \mu} = T_{\mu\nu} \tag{3.1.17}$$

3.2 Sourceless Maxwell's equations

We will start with the matter part of the action

$$\mathcal{S}_M = \int \mathrm{d}^4x\mathcal{L}_M = -\int \mathrm{d}^4xF_{\alpha\beta}F^{\alpha\beta} \tag{3.2.1}$$

and the field strength defined as $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. As usual, we want the condition that requires $\delta\mathcal{S}_M = 0$, so varying the inside of the integral gives us

$$\begin{aligned} &\delta\left(F_{\alpha\beta}F^{\alpha\beta}\right) \\ &= 2F^{\alpha\beta}\delta F_{\alpha\beta} \\ &= 2F^{\alpha\beta}\left(\partial_\alpha\delta A_\beta - \partial_\beta\delta A_\alpha\right) \\ &= 4F^{\alpha\beta}\partial_\alpha\delta A_\beta \\ &= 4\left[\partial_\alpha\left(F^{\alpha\beta}\delta A_\beta\right) - \partial_\alpha F^{\alpha\beta}\delta A_\beta\right], \end{aligned} \tag{3.2.2}$$

using the fact that F is antisymmetric. Now we see that the first term vanishes at the integration boundary if we convert to a surface integral, and that the second term only vanishes for

$$\partial_\alpha F^{\alpha\beta} = 0, \tag{3.2.3}$$

or otherwise expressed as

$$\mathrm{d} * F = 0 \tag{3.2.4}$$

in differential form notation.

Chapter 4

Supersymmetric conditions and general KSE

Supersymmetric conditions are at the core of the report, and are what give us the fundamental geometric information required to build the metric. This chapter will outline how the conditions are acquired from SUSY considerations.

4.1 Supersymmetric background

In the context of SUSY, we are essentially imposing minimality on $N = 2$, $D = 4$ SUGRA and finding conditions for preserving half of the supersymmetry [10]. The Killing spinor equation then represents the vanishing of the gravitini SUSY transformation in a bosonic background. It is called $N = 2$ due to the fact that we are taking into account both gravitinos and photinos, both superpartners of gravitons and photons respectively. The minimality stems from lack of gauge coupling between particles. The underlying theory has 4 bosonic and 4 fermionic degrees of freedom, so if we combine the multiple gravitini Majorana spinors into a single Dirac spinor as

$$\psi_\mu = \psi_\mu^1 + i\psi_\mu^2, \tag{4.1.1}$$

we now have a resulting action

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho \\ &+\frac{i}{8}\left(F_{\mu\nu} + \hat{F}_{\mu\nu}\right)\bar{\psi}_\rho\gamma_{[\mu}\gamma^{\rho\sigma}\gamma_{\nu]}\psi_\sigma.\end{aligned} \tag{4.1.2}$$

Here we have

$$\mathcal{C}\bar{\psi} = \psi \tag{4.1.3}$$

under charge conjugation $\mathcal{C}$, and the additional relation

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}, \tag{4.1.4}$$

which is due to work by Kosmann being used to build the very essential spinor connection[7][11]. There is also

$$\gamma^{ab} = \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right) \tag{4.1.5}$$

from chapter 2, and

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \mathrm{Im}(\bar{\psi}_\mu\psi_\nu). \tag{4.1.6}$$

To now find the gravitino variation, you vary the action with respect to the various fields included, and impose the SUSY transformations, requiring that that the variation of the total action stays null throughout[12]. The action is invariant under the following local SUSY transformations

$$\begin{aligned} \delta_\varepsilon e^a_{\ \mu} &= \mathrm{Re}(\bar{\varepsilon}\gamma^a\psi_\mu), \\ \delta_\varepsilon A_\mu &= \mathrm{Im}(\bar{\varepsilon}\psi_\mu), \\ \delta_\varepsilon\psi_\mu &= \mathcal{D}_\mu\varepsilon. \end{aligned} \tag{4.1.7}$$

In 4.1.7 ε is an infinitesimal Dirac spinor, and the supercovariant derivative is given by

$$\mathcal{D}_\mu = \nabla_\mu + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu. \tag{4.1.8}$$

The vanishing of the gravitino SUSY transformation in a bosonic background of minimal $N = 2$, $D = 4$ SUGRA is then our KSE, given by

$$\mathcal{D}_\mu\varepsilon = 0 \Longleftrightarrow \left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab} + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu\right)\varepsilon = 0. \tag{4.1.9}$$

It's worth mentioning that this is the core of the entire spinorial geometry class of techniques, as outlined in [4]. Luckily the KSE is only a first order differential equation, and entirely solvable to the point of constraining the spin connections, which we will do in the next section. Following [4], it can also be shown that any geometry that satisfies the KSE is automatically a solution to the EM field equations.

4.2 Metric choice

As is common in the spinorial geometry technique, and following [9], we will choose what is commonly called the null basis to solve the KSE in, and then build our solutions. It is a simplifying choice that makes the γ-actions behave more nicely for the purposes of calculation. In that spirit we will define new

γ-actions so that

$$\begin{aligned}
\gamma_+ &= \frac{1}{\sqrt{2}}\left(\gamma_2 + \gamma_0\right) = \sqrt{2} i_{e^2}, \\
\gamma_- &= \frac{1}{\sqrt{2}}\left(\gamma_2 - \gamma_0\right) = \sqrt{2} e^2 \wedge, \\
\gamma_1 &= \frac{1}{\sqrt{2}}\left(\gamma_1 + i\gamma_3\right) = \sqrt{2} i_{e^1}, \\
\gamma_{\bar{1}} &= \frac{1}{\sqrt{2}}\left(\gamma_1 - i\gamma_3\right) = \sqrt{2} e^1 \wedge .
\end{aligned} \tag{4.2.1}$$

This new basis is defined by the non-zero components given by $g_{+-} = 1$, $g_{1\bar{1}} = 1$. The entire metric importantly therefore looks like

$$\mathrm{d}s^2 = 2\theta^+\theta^- + 2\theta^1\theta^{\bar{1}}, \tag{4.2.2}$$

where θ^a are 1-forms.

4.3 KSE for a general spinor

After a lengthy calculation, the 4*4 set of coupled algebraic and first order differential equations given by the KSE for the complexified spinor forms

$$\varepsilon = \lambda 1 + \mu^1 e^1 + \mu^2 e^2 + \sigma e^{12}, \tag{4.3.1}$$

are given by:

4.3.1 $\mu = +$

$$\partial_+(\lambda) + \frac{1}{2}\left(-\lambda\omega_{++-} + 2\sigma\omega_{+-\bar{1}} - \lambda\omega_{+1\bar{1}}\right) - \frac{\mu^2 i}{\sqrt{2}}\left(F_{+-} + F_{1\bar{1}}\right) = 0 \tag{4.3.2}$$

$$\partial_+(\mu^1) + \frac{1}{2}\left(-\mu^1\omega_{++-} - 2\mu^2\omega_{+-1} + \mu^1\omega_{+1\bar{1}}\right) + \frac{\sigma i}{\sqrt{2}}\left(F_{+-} - F_{1\bar{1}}\right) = 0 \tag{4.3.3}$$

$$\partial_+(\mu^2) + \frac{1}{2}\left(\mu^2\omega_{++-} + 2\mu^1\omega_{++\bar{1}} - \mu^2\omega_{+1\bar{1}}\right) - \sigma i\sqrt{2}F_{+\bar{1}} = 0 \tag{4.3.4}$$

$$\partial_+(\sigma) + \frac{1}{2}\left(\sigma\omega_{++-} - 2\lambda\omega_{++1} + \sigma\omega_{+1\bar{1}}\right) - \mu^2 i\sqrt{2}F_{+1} = 0 \tag{4.3.5}$$

4.3.2 $\mu = -$

$$\partial_-(\lambda) + \frac{1}{2}\left(-\lambda\omega_{-+-} + 2\sigma\omega_{--\bar{1}} - \lambda\omega_{-1\bar{1}}\right) - \mu^1 i\sqrt{2}F_{-\bar{1}} = 0 \tag{4.3.6}$$

$$\partial_-(\mu^1) + \frac{1}{2}\left(-\mu^1\omega_{-+-} - 2\mu^2\omega_{--1} + \mu^1\omega_{-1\bar{1}}\right) - \lambda i\sqrt{2}F_{-1} = 0 \tag{4.3.7}$$

$$\partial_-(\mu^2) + \frac{1}{2}\left(\mu^2\omega_{-+-} + 2\mu^1\omega_{-+\bar{1}} - \mu^2\omega_{-1\bar{1}}\right) + \frac{\lambda i}{\sqrt{2}}\left(F_{+-} - F_{1\bar{1}}\right) = 0 \quad (4.3.8)$$

$$\partial_-(\sigma) + \frac{1}{2}\left(\sigma\omega_{-+-} - 2\lambda\omega_{-+1} + \sigma\omega_{-1\bar{1}}\right) - \frac{\mu^1 i}{\sqrt{2}}\left(F_{+-} + F_{1\bar{1}}\right) = 0 \quad (4.3.9)$$

4.3.3 $\mu = 1$

$$\partial_1(\lambda) + \frac{1}{2}\left(-\lambda\omega_{1+-} + 2\sigma\omega_{1-\bar{1}} - \lambda\omega_{11\bar{1}}\right) - \frac{\mu^1 i}{\sqrt{2}}\left(F_{+-} + F_{1\bar{1}}\right) = 0 \quad (4.3.10)$$

$$\partial_1(\mu^1) + \frac{1}{2}\left(-\mu^1\omega_{1+-} - 2\mu^2\omega_{1-1} + \mu^1\omega_{11\bar{1}}\right) - \sigma i\sqrt{2}F_{-1} = 0 \quad (4.3.11)$$

$$\partial_1(\mu^2) + \frac{1}{2}\left(\mu^2\omega_{1+-} + 2\mu^1\omega_{1+\bar{1}} - \mu^2\omega_{11\bar{1}}\right) + \frac{\sigma i}{\sqrt{2}}\left(F_{+-} - F_{1\bar{1}}\right) = 0 \quad (4.3.12)$$

$$\partial_1(\sigma) + \frac{1}{2}\left(\sigma\omega_{1+-} - 2\lambda\omega_{1+1} + \sigma\omega_{11\bar{1}}\right) - \mu^1 i\sqrt{2}F_{+1} = 0 \quad (4.3.13)$$

4.3.4 $\mu = \bar{1}$

$$\partial_{\bar{1}}(\lambda) + \frac{1}{2}\left(-\lambda\omega_{\bar{1}+-} + 2\sigma\omega_{\bar{1}-\bar{1}} - \lambda\omega_{\bar{1}1\bar{1}}\right) + \mu^2 i\sqrt{2}F_{-\bar{1}} = 0 \quad (4.3.14)$$

$$\partial_{\bar{1}}(\mu^1) + \frac{1}{2}\left(-\mu^1\omega_{\bar{1}+-} - 2\mu^2\omega_{\bar{1}-1} + \mu^1\omega_{\bar{1}1\bar{1}}\right) + \frac{\lambda i}{\sqrt{2}}\left(F_{1\bar{1}} - F_{+-}\right) = 0 \quad (4.3.15)$$

$$\partial_{\bar{1}}(\mu^2) + \frac{1}{2}\left(\mu^2\omega_{\bar{1}+-} + 2\mu^1\omega_{\bar{1}+\bar{1}} - \mu^2\omega_{\bar{1}1\bar{1}}\right) + \lambda i\sqrt{2}F_{+\bar{1}} = 0 \quad (4.3.16)$$

$$\partial_{\bar{1}}(\sigma) + \frac{1}{2}\left(\sigma\omega_{\bar{1}+-} - 2\lambda\omega_{\bar{1}+1} + \sigma\omega_{\bar{1}1\bar{1}}\right) + \frac{\mu^2 i}{\sqrt{2}}\left(F_{1\bar{1}} + F_{+-}\right) = 0 \quad (4.3.17)$$

Here we used the γ-actions given in Appendix A. These are effectively the SUSY conditions that we use to build the metric. However first we need to specify the spinorial orbit that gives us the IWP metric.

Chapter 5

Spinor orbits and reduced KSE

In this chapter we will find the 3 canonical spinor orbits as outlined in [9] as a means of reducing the SUSY conditions obtained. This step is essential in specifying the SUSY conditions for the IWP metric, as the 3 canonical spin orbits relate to 3 different metrics. All 3 orbits which we will derive are given by:

- $\varepsilon = e^2$ - Flat Minkowski metric
- $\varepsilon = 1 + \alpha e^1$ - PP Wave metric
- $\varepsilon = 1 + \beta e^2$ - IWP metric

The overall method in the derivation is to use specific gauge transformations on the original spinor form $\varepsilon = \lambda \mathbb{1} + \mu^1 e^1 + \mu^2 e^2 + \sigma e^{12}$ in order to reduce it into one of the 3 specific forms, showing in the process that they cannot be changed further in any meaningful way.

5.1 Gauge transformations

There are 2 classes of gauge transformations on ε that leave the KSE invariant, in this section we will list them and their actions, for use in the reduction. First, there are local $U(1)$ gauge transformations of the form

$$\varepsilon \longrightarrow e^{i\theta}\varepsilon \tag{5.1.1}$$

for real functions θ, and also the local $Spin(3,1)$ gauge transformations of the form

$$\varepsilon \longrightarrow e^{\frac{1}{2} f^{\mu\nu}\gamma_{\mu\nu}}\varepsilon \tag{5.1.2}$$

for real functions $f^{\mu\nu}$. The individual $Spin(3,1)$ gauge transformation actions are then essentially summed up by the following:

γ_{01}

$$\begin{aligned}
e^{x\gamma_{01}}\mathbb{1} &= \cosh(x)\mathbb{1} + \sinh(x)e^1 \wedge e^2 \\
e^{x\gamma_{01}}e^1 &= \cosh(x)e^1 - \sinh(x)e^2 \\
e^{x\gamma_{01}}e^2 &= \cosh(x)e^2 - \sinh(x)e^1 \\
e^{x\gamma_{01}}e^1 \wedge e^2 &= \cosh(x)e^1 \wedge e^2 + \sinh(x)\mathbb{1}
\end{aligned} \tag{5.1.3}$$

γ_{02}

$$\begin{aligned}
e^{x\gamma_{02}}\mathbb{1} &= e^x\mathbb{1} \\
e^{x\gamma_{02}}e^1 &= e^x e^1 \\
e^{x\gamma_{02}}e^2 &= e^{-x}e^2 \\
e^{x\gamma_{02}}e^1 \wedge e^2 &= e^{-x}e^1 \wedge e^2
\end{aligned} \tag{5.1.4}$$

γ_{03}

$$\begin{aligned}
e^{x\gamma_{03}}\mathbb{1} &= \cosh(x)\mathbb{1} + i\sinh(x)e^1 \wedge e^2 \\
e^{x\gamma_{03}}e^1 &= \cosh(x)e^1 + i\sinh(x)e^2 \\
e^{x\gamma_{03}}e^2 &= \cosh(x)e^2 - i\sinh(x)e^1 \\
e^{x\gamma_{03}}e^1 \wedge e^2 &= \cosh(x)e^1 \wedge e^2 - i\sinh(x)\mathbb{1}
\end{aligned} \tag{5.1.5}$$

γ_{12}

$$\begin{aligned}
e^{x\gamma_{12}}\mathbb{1} &= \cos(x)\mathbb{1} + \sin(x)e^1 \wedge e^2 \\
e^{x\gamma_{12}}e^1 &= \cos(x)e^1 - \sin(x)e^2 \\
e^{x\gamma_{12}}e^2 &= \cos(x)e^2 + \sin(x)e^1 \\
e^{x\gamma_{12}}e^1 \wedge e^2 &= \cos(x)e^1 \wedge e^2 - \sin(x)\mathbb{1}
\end{aligned} \tag{5.1.6}$$

γ_{13}

$$\begin{aligned}
e^{x\gamma_{13}}\mathbb{1} &= e^{ix}\mathbb{1} \\
e^{x\gamma_{13}}e^1 &= e^{-ix}e^1 \\
e^{x\gamma_{13}}e^2 &= e^{ix}e^2 \\
e^{x\gamma_{13}}e^1 \wedge e^2 &= e^{-ix}e^1 \wedge e^2
\end{aligned} \tag{5.1.7}$$

γ_{23}

$$\begin{aligned}
e^{x\gamma_{23}}\mathbb{1} &= \cos(x)\mathbb{1} - i\sin(x)e^1 \wedge e^2 \\
e^{x\gamma_{23}}e^1 &= \cos(x)e^1 - i\sin(x)e^2 \\
e^{x\gamma_{23}}e^2 &= \cos(x)e^2 - i\sin(x)e^1 \\
e^{x\gamma_{23}}e^1 \wedge e^2 &= \cos(x)e^1 \wedge e^2 - i\sin(x)\mathbb{1}
\end{aligned} \tag{5.1.8}$$

Note in particular how these actions sometimes act simultaneously on $1, e^1$ and e^2, e^{12}, and how $\gamma_{12}, \gamma_{13}, \gamma_{23}$ generate $SU(2)$ transformations. Noticing this aspect makes it easier to find useful actions that reduce our general spinor form.

5.2 The 3 canonical orbits

We begin with our general spinor form

$$\varepsilon = \lambda \mathbb{1} + \mu^1 e^1 + \mu^2 e^2 + \sigma e^{12}, \tag{5.2.1}$$

and act on it with a generalised local gauge transformations defined by $\gamma_{01}, \gamma_{03}, \gamma_{12}, \gamma_{23}$. We will do this to leading orders in the series expansions of the trigonometric and hyperbolic functions. The transformations defined by γ_{02} and γ_{13} we will use separately as they give a simple scaling and rotation in the complex plane, and are more useful in their current forms. Our general $\gamma_{01}, \gamma_{03}, \gamma_{12}, \gamma_{23}$ transformation up to an overall unimportant constant of proportionality and to leading order is given as follows:

$$\begin{aligned} e^{a\gamma_{01}+b\gamma_{03}+c\gamma_{12}+d\gamma_{23}}\mathbb{1} &= \mathbb{1} + \left[(a+c) + i(b-d)\right] e^{12} \\ e^{a\gamma_{01}+b\gamma_{03}+c\gamma_{12}+d\gamma_{23}}e^1 &= e^1 + \left[-(a+c) + i(b-d)\right] e^2 \\ e^{a\gamma_{01}+b\gamma_{03}+c\gamma_{12}+d\gamma_{23}}e^2 &= e^2 + \left[-(a-c) - i(b+d)\right] e^1 \\ e^{a\gamma_{01}+b\gamma_{03}+c\gamma_{12}+d\gamma_{23}}e^{12} &= e^{12} + \left[(a-c) - i(b+d)\right] \mathbb{1} \end{aligned} \tag{5.2.2}$$

Using a transformation defined by $a = b = c = d = x \in \mathbb{R}$ on our general spinor, we get

$$\begin{aligned} \mathbb{1} &\longrightarrow \mathbb{1} + 2xe^{12}, \\ e^1 &\longrightarrow e^1 - 2xe^2, \\ e^2 &\longrightarrow e^2 - 2ixe^1, \\ e^{12} &\longrightarrow e^{12} - 2ix\mathbb{1}. \end{aligned} \tag{5.2.3}$$

This allows us to set any one of $\lambda, \mu^1, \mu^2, \sigma = 0$, due to the ability to always regenerate the lost parameter by going backwards a step. We therefore choose $\sigma = 0$, and $\lambda \in \mathbb{R}$, the latter by using the $SU(2)$ rotation defined by the action of γ_{13}. We are left with

$$\varepsilon = \lambda \mathbb{1} + \mu^1 e^1 + \mu^2 e^2. \tag{5.2.4}$$

We can then say there are 2 distinct cases to try to reduce further, one where $\mu^2 = 0$, and one where $\mu^2 \neq 0$. We will consider the latter first. Acting on (5.2.4) with the transformation defined by $a = b = d = x$, and $c = -x$, we see

$$\begin{aligned} \mathbb{1} &\longrightarrow \mathbb{1}, \\ e^1 &\longrightarrow e^1, \\ e^2 &\longrightarrow e^2 - 2x(1+i)e^1, \end{aligned} \tag{5.2.5}$$

allowing us to set $\mu^1 = 0$, by the same logic as before, so we have

$$\varepsilon = \lambda \mathbb{1} + \mu^2 e^2. \tag{5.2.6}$$

Then if $\lambda \neq 0$, we can use the general re-scaling generated by γ_{02}, to get $\lambda = 1$, so that

$$\varepsilon = 1 + \mu^2 e^2. \tag{5.2.7}$$

The re-scaling by γ_{02} works in this case due to its property of scaling $\mathbb{1}, e^1$, and e^2, e^{12} in opposite directions, which can be clearly seen in eq (5.1.4).

However, if instead $\lambda = 0$, we can combine the general $SU(2)$ rotation and re-scaling defined by γ_{13}, and γ_{02} respectively, to set $\mu^2 = 1$:

$$\varepsilon = e^2 \tag{5.2.8}$$

This is explained essentially by first rotating μ^2 so that it is fully real, and then re-scaling it to unity.

If instead we have $\mu^2 = 0$ in (5.2.4), there are again the 2 cases of $\lambda \neq 0$, and $\lambda = 0$. Taking first the former, we can use the rotation and scaling again so that

$$\varepsilon = 1 + \mu^1 e^1. \tag{5.2.9}$$

If $\lambda = 0$ however, we can use any transformation that generates an e^2 form together with a scaling and rotation so that we are left with

$$\varepsilon = e^2 \tag{5.2.10}$$

as before.

Overall, we have found 3 canonical spinor orbits that cannot be reduced further, we can always use the $Spin(3,1)$ gauge transformation to write a general spinor in differential forms as

$$\begin{aligned} \varepsilon &= e^2, \\ \varepsilon &= 1 + \alpha e^1, \\ \varepsilon &= 1 + \beta e^2, \end{aligned} \tag{5.2.11}$$

for functions $\alpha, \beta \in \mathbb{C}$.

5.3 Reduced IWP KSE

We know from [4] that the spinor orbit that relates to the IWP metric is

$$\varepsilon = 1 + \beta e^2. \tag{5.3.1}$$

So setting $\lambda = 0$, $\mu^1 = 0$, $\mu^2 = \beta$, and $\sigma = 0$ in our general KSE from section (4.3), we discover the KSE for our specific metric, given by:

5.3.1 $\mu = +$

$$(\omega_{++-} + \omega_{+1\bar{1}}) + \beta i\sqrt{2}\,(F_{+-} + F_{1\bar{1}}) = 0 \tag{5.3.2}$$

$$\omega_{+-1} = 0 \tag{5.3.3}$$

$$2\partial_{+}(\beta) + \beta\,(\omega_{++-} - \omega_{+1\bar{1}}) = 0 \tag{5.3.4}$$

$$\omega_{++1} + \beta i\sqrt{2}F_{+1} = 0 \tag{5.3.5}$$

5.3.2 $\mu = -$

$$\omega_{-+-} + \omega_{-1\bar{1}} = 0 \tag{5.3.6}$$

$$\beta\omega_{--1} + i\sqrt{2}F_{-1} = 0 \tag{5.3.7}$$

$$2\partial_{-}(\beta) + \beta\,(\omega_{-+-} - \omega_{-1\bar{1}}) + i\sqrt{2}\,(F_{+-} - F_{1\bar{1}}) = 0 \tag{5.3.8}$$

$$\omega_{-+1} = 0 \tag{5.3.9}$$

5.3.3 $\mu = 1$

$$\omega_{1+-} - \omega_{11\bar{1}} = 0 \tag{5.3.10}$$

$$\omega_{1-1} = 0 \tag{5.3.11}$$

$$2\partial_{1}(\beta) + \beta\,(\omega_{1+-} - \omega_{11\bar{1}}) = 0 \tag{5.3.12}$$

$$\omega_{1+1} = 0 \tag{5.3.13}$$

5.3.4 $\mu = \bar{1}$

$$(\omega_{\bar{1}+-} + \omega_{\bar{1}1\bar{1}}) - \beta i2\sqrt{2}F_{-\bar{1}} = 0 \tag{5.3.14}$$

$$\beta\sqrt{2}\omega_{\bar{1}-1} + i\,(F_{+-} - F_{1\bar{1}}) = 0 \tag{5.3.15}$$

$$2\partial_{\bar{1}}(\beta) + \beta\,(\omega_{\bar{1}+-} - \omega_{\bar{1}1\bar{1}}) + i2\sqrt{2}F_{+\bar{1}} = 0 \tag{5.3.16}$$

$$-\sqrt{2}\omega_{\bar{1}+1} + \beta i\,(F_{+-} + F_{1\bar{1}}) = 0 \tag{5.3.17}$$

5.4 IWP metric constraints

To solve the above equations, we require 2 key pieces of information about how the gauge field and spin connections behave. First, they are antisymmetric:

$$\begin{gathered}\omega_{abc} = -\omega_{acb} \text{ (in the interior indices only)} \\ F_{ab} = -F_{ba}\end{gathered} \tag{5.4.1}$$

And due to our choice of basis, where the 1 and $\bar{1}$ components are complex conjugates of each other, we can use complex conjugation in the indices. For example

$$\begin{gathered}(F_{+1})^* = F_{+\bar{1}}, \\ \text{or} \\ (\omega_{1\bar{1}1})^* = \omega_{\bar{1}1\bar{1}}.\end{gathered} \tag{5.4.2}$$

This is summarised easily as

$$\begin{gathered}(+)^* \longrightarrow +, \\ (-)^* \longrightarrow -, \\ (1)^* \longrightarrow \bar{1}, \\ (\bar{1})^* \longrightarrow 1.\end{gathered} \tag{5.4.3}$$

where the complex conjugation acts on the object the indices are a part of, and $+, -, 1, \bar{1}$ are the index.

Solving the set of 16 simultaneous equations from the previous section using the above information, we have the constraints on the field and geometry we require for building the metric, given in sets of 4 by:

Gauge field strength components

$$\begin{gathered}F_{+-} = -\frac{i\sqrt{2}}{2}\partial_-(\bar{\beta} - \beta) \\ F_{+1} = -\frac{i\sqrt{2}}{2}\partial_1(\bar{\beta}) \\ F_{-1} = -\frac{i\sqrt{2}}{2\beta\bar{\beta}}\partial_1(\beta) \\ F_{1\bar{1}} = -\frac{i\sqrt{2}}{2}\partial_-(\bar{\beta} + \beta)\end{gathered} \tag{5.4.4}$$

Spin connection components

$$\begin{gathered}\omega_{++-} = -(\bar{\beta}\partial_-(\beta) + \beta\partial_-(\bar{\beta})) \\ \omega_{++1} = -\beta\partial_1(\bar{\beta}) \\ \omega_{+-1} = 0 \\ \omega_{+1\bar{1}} = \bar{\beta}\partial_-(\beta) - \beta\partial_-(\bar{\beta})\end{gathered} \tag{5.4.5}$$

$$
\begin{aligned}
\omega_{-+-} &= 0\\
\omega_{-+1} &= 0\\
\omega_{--1} &= -\frac{1}{\beta^2\bar{\beta}}\partial_1(\beta)\\
\omega_{-1\bar{1}} &= 0
\end{aligned}
\tag{5.4.6}
$$

$$
\begin{aligned}
\omega_{1+-} &= -\frac{1}{\beta}\partial_1(\beta)\\
\omega_{1+1} &= 0\\
\omega_{1-1} &= 0\\
\omega_{11\bar{1}} &= \frac{1}{\beta}\partial_1(\beta)
\end{aligned}
\tag{5.4.7}
$$

$$
\begin{aligned}
\omega_{\bar{1}+-} &= -\frac{1}{\bar{\beta}}\partial_{\bar{1}}\bar{\beta}\\
\omega_{\bar{1}+1} &= \beta\partial_-(\bar{\beta})\\
\omega_{\bar{1}-1} &= \frac{1}{\beta}\partial_-(\beta)\\
\omega_{\bar{1}1\bar{1}} &= -\frac{1}{\bar{\beta}}\partial_{\bar{1}}\bar{\beta}
\end{aligned}
\tag{5.4.8}
$$

There is also an extra relation for the partial differentials:

$$
\partial_+(\beta) - \beta\bar{\beta}\partial_-(\beta) = 0 \tag{5.4.9}
$$

In conclusion of this chapter, any metric that satisfies the above constraints automatically also satisfies the EM-field equations, meaning it is an allowed geometry in GR. Our next step is to build the metric using these constraints, and we will do that using spinor bilinear covariants.

Chapter 6

Bilinear covariants and metric

Spinor bilinear covariants are essential in building the form of the IWP metric. They are at their simplest, combinations of γ-actions together with our general IWP spinor form $1 + \beta e^2$ that transform in certain ways under certain group transformations. They act as building blocks from which we can naturally extract geometrical notions about the metric, for example possible killing and closed vector fields. Using these killing and closed vector fields allows us to choose a new curvilinear basis that makes sense for our metric. There are many more bilinear covariants than the specific ones we will use but they are not relevant to this calculation. This chapter is dedicated to finding the useful bilinears and their properties, and building the metric in alternative curvilinear coordinates using those bilinears. The 2 bilinear covariants that are useful to us are

$$V_a = \langle \gamma_0 \varepsilon, \gamma_a \varepsilon \rangle, \tag{6.0.1}$$

and

$$W_a = \langle \gamma_0 \varepsilon, \gamma_5 \gamma_a \varepsilon \rangle. \tag{6.0.2}$$

Here we have $a = +, -, 1, \bar{1}$, and we use $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ for simplicity. The inner product with angled brackets is defined for forms simply by $\langle \alpha_i \theta_i, \beta_j \theta_j \rangle = \alpha_i^* \beta_i$. From now on we will also use the inner product as defined in definition 2.1.2 liberally.

6.1 Form and properties of V

With V being of the form

$$V_a = \langle \gamma_0 \varepsilon, \gamma_a \varepsilon \rangle, \tag{6.1.1}$$

we easily find that

$$V = \sqrt{2}(\beta\bar{\beta}\partial_- - \partial_+) = \sqrt{2}(\beta\bar{\beta}\theta^+ - \theta^-). \tag{6.1.2}$$

This vector is manifestly time-like, which can be seen simply by

$$|V|^2 = V_a V^a = -4|\beta|^2 < 0. \tag{6.1.3}$$

An important property of V is that it is also a Killing vector. Killing vectors for our metric are key pieces of information that we did not know previously. We will use V and the fact that it is a time-like Killing vector to introduce a curvilinear coordinate that is a good choice for time. To show that V is a killing vector it must satisfy the Killing equation, which we will give conveniently as

$$\nabla_a V_b + \nabla_b V_a = 0, \tag{6.1.4}$$

with

$$\nabla_a V_b = \partial_a V_b - \omega^c{}_{ab} V_c. \tag{6.1.5}$$

This gives us 7 independent equations. When using the constraints on the spin connections from the KSE found in the last chapter, they are all found to be 0, which confirms that V is indeed a time-like Killing vector.

6.2 Form and properties of W

Similarly to the previous section, we have a vector W of the form

$$W_a = \langle \gamma_0 \varepsilon, \gamma_5 \gamma_a \varepsilon \rangle, \tag{6.2.1}$$

which after a short calculation shows that

$$W = \sqrt{2}(\beta\bar{\beta}\partial_- + \partial_+) = \sqrt{2}(\beta\bar{\beta}\theta^+ + \theta^-). \tag{6.2.2}$$

W is instead space-like, which can be seen simply from

$$|W|^2 = 4|\beta|^2 > 0 \tag{6.2.3}$$

This again allows us to later use it as a tool to build a certain curvilinear space coordinate. It also happens to be a closed vector. For a vector to be closed it must satisfy

$$\nabla_a W_b - \nabla_b W_a = 0. \tag{6.2.4}$$

After another 7 independent equations, all shown to be 0 using the previous chapter's spin connection constraints, we have a space-like closed vector.

6.3 Building the metric - θ^+ and θ^-

Our final metric will be made of 4 curvilinear coordinates, (t,x,y,z). This section is about picking suitable choices for them all, based on the information we currently have, and relating them to the so called null basis defined by the basis forms θ^a. Once we have connections between the (t,x,y,z) coordinates and the θ^a coordinates, we can simply substitute in our θ^a's in terms of (t,x,y,z) into the null metric

$$\mathrm{d}s^2 = 2\theta^+\theta^- + 2\theta^1\theta^{\bar{1}}, \tag{6.3.1}$$

finding the form of our IWP black hole.

We begin with the real forms θ^+, θ^-. Since V is a time-like Killing vector, we introduce the coordinate t such that

$$V = q\partial_t, \tag{6.3.2}$$

for a constant q. Similarly, since W is a space-like closed vector, we can introduce the coordinate z such that

$$W = kdz, \tag{6.3.3}$$

for a constant k. Now given

$$|V|^2 = -4|\beta|^2, \tag{6.3.4}$$

and

$$W = \frac{4|\beta|^2}{k}\partial_z, \tag{6.3.5}$$

a short calculation allows us to have, at their most general,

$$V = \frac{-4|\beta|^2}{q}(dt + \phi), \tag{6.3.6}$$

where ϕ is a 1-form independent of time, and

$$W = \frac{4|\beta|^2}{k}\partial_z. \tag{6.3.7}$$

Now going back to the original forms of V, and W from the last sections, notice we have a direct connection between t and z, and the null basis forms θ^a, where

$$\begin{aligned} V &= \sqrt{2}(\beta\bar{\beta}\theta^+ - \theta^-), \\ W &= \sqrt{2}(\beta\bar{\beta}\theta^+ + \theta^-). \end{aligned} \tag{6.3.8}$$

Solving V and W simultaneously for θ^+, and θ^- yields

$$\begin{aligned} \theta^+ &= \frac{V+W}{\sqrt{2}\beta\bar{\beta}}, \\ \theta^- &= \frac{W-V}{\sqrt{2}}. \end{aligned} \tag{6.3.9}$$

In the end, this simply follows with some algebra to leave us with

$$\theta^+ = \frac{1}{\sqrt{2}\beta\bar{\beta}}(-\beta\bar{\beta}(dt+\phi)+dz), \tag{6.3.10}$$

and

$$\theta^- = \frac{1}{\sqrt{2}}(\beta\bar{\beta}(dt+\phi)+dz). \tag{6.3.11}$$

These are our final forms of θ^+, θ^-, and we picked $k=q=2$ for simplicity, and without any loss of generality, for no other reason than the easing of algebra later on.

6.4 Building the metric - θ^1 and $\theta^{\bar{1}}$

The 2 coordinates we have not explicitly introduced are x and y. θ^1 and $\theta^{\bar{1}}$ also have no tangible relation to V and W, and as such will require a slightly different approach.

6.4.1 Preamble - vanishing of the torsion

We will use the vanishing of the torsion to get a condition that we can use to constrain our possible choices of θ^1.

Using Cartan's first structure equation we have

$$\Theta^a = \mathrm{d}e^a + \omega^a_{\ b} \wedge e^b. \tag{6.4.1}$$

For $\Theta^a = 0$, and picking $a=1$ components specifically, we are left with:

$$\mathrm{d}\theta^1 = -\omega^1_{\ a} \wedge \theta^a \tag{6.4.2}$$

This implies, after a bit of algebra and the use of the relation relating actions of ∂_- and ∂_+ on β, i.e.

$$\partial_+(\beta) = \beta\bar{\beta}\partial_-(\beta) \tag{6.4.3}$$

that

$$\mathrm{d}\theta^1 = -\left(\frac{\partial_+\bar{\beta}}{\bar{\beta}}\theta^+ + \frac{\partial_-\bar{\beta}}{\bar{\beta}}\theta^- + \frac{\partial_{\bar{1}}\bar{\beta}}{\bar{\beta}}\theta^{\bar{1}}\right) \wedge \theta^1 \tag{6.4.4}$$

or more simply and in a coordinate-less way that

$$\mathrm{d}\theta^1 = -\,\mathrm{d}\left(\ln\bar{\beta}\right)\wedge\theta^1. \tag{6.4.5}$$

This shows that θ^1 is hypersurface orthogonal, meaning that we can introduce it as

$$\theta^1 = f\,\mathrm{d}\zeta, \tag{6.4.6}$$

for some function f, and ζ.

6.4.2 Constraints on θ^1

We start with the ansatz

$$\theta^1 = f(a\,\mathrm{d}x + b\,\mathrm{d}y) \tag{6.4.7}$$

for a function f and $a, b \in \mathbb{C}$, using the 2 coordinates we have not introduced explicitly yet, x and y.

Using condition 6.4.5, we get that for the left side,

$$\mathrm{d}\theta^1 = (b\partial_x f - a\partial_y f)\,\mathrm{d}x\wedge\mathrm{d}y - a\partial_z f\,\mathrm{d}x\wedge\mathrm{d}z - b\partial_z f\,\mathrm{d}y\wedge\mathrm{d}z \tag{6.4.8}$$

and for the right side that

$$-\,\mathrm{d}\left(\ln\bar{\beta}\right)\wedge\theta^1 = f\left\{\left(\frac{a\partial_y\bar{\beta}}{\bar{\beta}} - \frac{b\partial_x\bar{\beta}}{\bar{\beta}}\right)\mathrm{d}x\wedge\mathrm{d}y + \frac{a\partial_z\bar{\beta}}{\bar{\beta}}\,\mathrm{d}x\wedge\mathrm{d}z + \frac{b\partial_z\bar{\beta}}{\bar{\beta}}\,\mathrm{d}y\wedge\mathrm{d}z\right\}. \tag{6.4.9}$$

This gives us 3 (2 unique) equations we must adhere to when picking a θ^1 for our ansatz:

$$\begin{aligned} xy\ :&\quad b\partial_x f - a\partial_y f = f\left(\frac{a\partial_y\bar{\beta}}{\bar{\beta}} - \frac{b\partial_x\bar{\beta}}{\bar{\beta}}\right)\\ xz\ :&\quad -a\partial_z f = af\frac{\partial_z\bar{\beta}}{\bar{\beta}}\\ yz\ :&\quad -b\partial_z f = bf\frac{\partial_z\bar{\beta}}{\bar{\beta}} \end{aligned} \tag{6.4.10}$$

We can use xz and yz to get rid of the f easily, by showing easily that

$$f = \frac{c}{\bar{\beta}} \tag{6.4.11}$$

is consistent for a $c \in \mathbb{C}$. This also happens to completely satisfy the xy component, so we can take any a and b we want, requiring that θ^1 has nonzero both real and imaginary parts as it is a complex basis by construction.

To be consistent with Tod, we will pick

$$\theta^1 = \frac{1}{\bar{\beta}\sqrt{2}}(\mathrm{d}x - i\,\mathrm{d}y). \tag{6.4.12}$$

$\theta^{\bar{1}}$ is then simply the complex conjugate of θ^1, given by

$$\theta^{\bar{1}} = \frac{1}{\beta\sqrt{2}}(dx + idy). \tag{6.4.13}$$

6.5 The metric in t, x, y, z coordinates

With our previous results of

$$\begin{aligned}
\theta^+ &= \frac{1}{\sqrt{2}\beta\bar{\beta}}(-\beta\bar{\beta}(dt + \phi) + dz),\\
\theta^- &= \frac{1}{\sqrt{2}}(\beta\bar{\beta}(dt + \phi) + dz),\\
\theta^1 &= \frac{1}{\bar{\beta}\sqrt{2}}(\mathrm{d}x - i\,\mathrm{d}y),\\
\theta^{\bar{1}} &= \frac{1}{\beta\sqrt{2}}(dx + idy),
\end{aligned} \tag{6.5.1}$$

we have our final form of the metric as

$$\mathrm{d}s^2 = -\beta\bar{\beta}(dt + \phi_i dx^i)^2 + \frac{1}{\beta\bar{\beta}}(dx^i)^2, \tag{6.5.2}$$

where the i index runs between 1 and 3, and denotes partial derivatives for ϕ. Our work is not done yet though, as the functions ϕ and β still have little to no constraints on what they could be.

Chapter 7

Extra conditions on the metric

This chapter completes the derivation of the IWP metric by finding the conditions on ϕ and β. Intuitively, conditions on functions we have not yet fully constrained are generally found in 2 ways. The first, which we will use to find the so called ϕ-condition, is due to essentially noticing that we now have 2 different ways to express the same thing. Namely the $\mathrm{d}\theta^a$'s. For the β-condition will use the second way, which requires using information we have not yet, but will now, impose. In this case meaning one of the equations of motion we derived, the Maxwell equation, $\mathrm{d} * F = 0$, as well as it's partner the Bianchi identity, $\mathrm{d}F = 0$ [6]. However to complete this task, calculationally we first require relations between our partial derivatives in the null basis, and curvilinear basis. This chapter will begin there and then finalise the derivation for the IWP metric with the requisite conditions.

7.1 ∂_a relations

7.1.1 ∂_+, ∂_-

Similar to the way we found θ^+, θ^-, the simplest way uses our spinor bilinears V and W. Clearly

$$\begin{aligned} V &= \sqrt{2}(\beta\bar{\beta}\partial_- - \partial_+) = 2\partial_t, \\ W &= \sqrt{2}(\beta\bar{\beta}\partial_- + \partial_+) = 2|\beta|^2\partial_z, \end{aligned} \tag{7.1.1}$$

allows us to simply solve for ∂_+ and ∂_-, so that

$$\begin{aligned} \partial_- &= \frac{1}{\sqrt{2}\beta\bar{\beta}}\left(\partial_t + \beta\bar{\beta}\partial_z\right), \\ \partial_+ &= \frac{1}{\sqrt{2}}\left(\beta\bar{\beta}\partial_z - \partial_t\right). \end{aligned} \tag{7.1.2}$$

It is also worth noticing that ϕ is time independent by construction, and β is also time independent due to eq 5.4.9, making the partial derivatives effectively

$$\partial_- = \frac{1}{\sqrt{2}}\partial_z, \tag{7.1.3}$$

$$\partial_+ = \frac{\beta\bar{\beta}}{\sqrt{2}}\partial_z, \tag{7.1.4}$$

due to the lack of other functions in the rest of the calculations.

7.1.2 ∂_1, $\partial_{\bar{1}}$

As before, the bases for ∂_1 and $\partial_{\bar{1}}$ are more tricky. We start with

$$\theta^1 = \frac{1}{\bar{\beta}\sqrt{2}}(dx - idy), \tag{7.1.5}$$

and again use the notion of the inner product between the tangent and dual space introduced in chapter 2 so that we require

$$\langle \theta^1, \partial_1 \rangle = 1. \tag{7.1.6}$$

So starting with the ansatz

$$\partial_1 = a\partial_x + b\partial_y, \tag{7.1.7}$$

eq 7.1.6 requires that

$$\frac{a}{\bar{\beta}\sqrt{2}} - \frac{bi}{\bar{\beta}\sqrt{2}} = 1. \tag{7.1.8}$$

Taking

$$a = \frac{\bar{\beta}}{\sqrt{2}},\ b = \frac{i\bar{\beta}}{\sqrt{2}} \tag{7.1.9}$$

solves (1.4) and leaves us with

$$\begin{gathered} \partial_1 = \frac{\bar{\beta}}{\sqrt{2}}\left(\partial_x + i\partial_y\right), \\ \text{and it's complex conjugate} \\ \partial_{\bar{1}} = \frac{\beta}{\sqrt{2}}\left(\partial_x - i\partial_y\right). \end{gathered} \tag{7.1.10}$$

7.2 ϕ-condition

We mentioned that this relation stems from the ability to write the $\mathrm{d}\theta^a$'s in 2 manifestly different ways. Equating those 2 different ways leaves us with the ϕ-condition. The first way to write them comes from the vanishing of the torsion,

$$\mathrm{d}\theta^a + \omega^a_{\ b} \wedge \theta^b = 0, \tag{7.2.1}$$

and the KSE spin connection constraints, so that:

$$d\theta^+ = \omega_{+-} \wedge \theta^+ - \omega_{-1} \wedge \theta^1 - \omega_{-\bar{1}} \wedge \theta^{\bar{1}} \tag{7.2.2}$$

$$\begin{aligned} \implies d\theta^+ = &-(\frac{1}{\bar{\beta}}\partial_{\bar{1}}(\bar{\beta})\theta^{\bar{1}} + \frac{1}{\beta}\partial_1(\beta)\theta^1) \wedge \theta^+ + (\frac{1}{\beta^2\bar{\beta}}\partial_1(\beta)\theta^- \\ &- \frac{1}{\beta}\partial_-(\beta)\theta^{\bar{1}}) \wedge \theta^1 + (\frac{1}{\bar{\beta}^2\beta}\partial_{\bar{1}}(\bar{\beta})\theta^- - \frac{1}{\bar{\beta}}\partial_-(\bar{\beta})\theta^1) \wedge \theta^{\bar{1}} \end{aligned} \tag{7.2.3}$$

$$d\theta^- = -\omega_{+-} \wedge \theta^- - \omega_{+1} \wedge \theta^1 - \omega_{+\bar{1}} \wedge \theta^{\bar{1}} \tag{7.2.4}$$

$$\begin{aligned} \implies d\theta^- = &((\bar{\beta}\partial_-(\beta) + \beta\partial_-(\bar{\beta}))\theta^+ + \frac{1}{\beta}\partial_1(\beta)\theta^1 + \frac{1}{\bar{\beta}}\partial_{\bar{1}}(\bar{\beta})\theta^{\bar{1}}) \wedge \theta^- \\ &+ (\beta\partial_1(\bar{\beta})\theta^+ - \beta\partial_-(\bar{\beta})\theta^{\bar{1}}) \wedge \theta^1 + (\bar{\beta}\partial_{\bar{1}}(\beta)\theta^+ - \bar{\beta}\partial_-(\beta)\theta^1) \wedge \theta^{\bar{1}} \end{aligned} \tag{7.2.5}$$

Note that $\mathrm{d}\theta^+$ and $\mathrm{d}\theta^-$ turn out to be sufficient for the ϕ-condition, so we will not take up space with $\mathrm{d}\theta^1$, $\mathrm{d}\theta^{\bar{1}}$, which do not give any new information.

The second way of expressing them comes from our new basis, hence the requirement for being able to relate our partial derivatives. We take the exterior derivative of each of our basis forms θ^+, θ^- in the new basis, as given by (6.5.1):

$$\begin{aligned} d\theta^+ = &\frac{1}{\sqrt{2}}(dxdy(-\partial_x\phi_y + \partial_y\phi_x) + dxdz(-\partial_x\phi_z + \partial_z\phi_x + \partial_x(\frac{1}{\beta\bar{\beta}})) \\ &+ dydz(-\partial_y\phi_z + \partial_z\phi_y + \partial_y(\frac{1}{\beta\bar{\beta}}))) \end{aligned} \tag{7.2.6}$$

$$\begin{aligned} d\theta^- = &\frac{1}{\sqrt{2}}(dxdy(\phi_y\partial_x(|\beta|^2) - \phi_x\partial_y(|\beta|^2) - |\beta|^2\partial_x(\phi_y) + |\beta|^2\partial_y(\phi_x)) \\ &+ dxdz(\phi_z\partial_x(|\beta|^2) - \phi_x\partial_z(|\beta|^2) - |\beta|^2\partial_x(\phi_z) + |\beta|^2\partial_z(\phi_x)) \\ &+ dydz(\phi_z\partial_y(|\beta|^2) - \phi_y\partial_z(|\beta|^2) - |\beta|^2\partial_y(\phi_z) + |\beta|^2\partial_z(\phi_y)) \\ &+ dxdt(\partial_x(|\beta|^2)) + dydt(\partial_y(|\beta|^2)) + dzdt(\partial_z(|\beta|^2))) \end{aligned} \tag{7.2.7}$$

Comparing the 2 different forms, after a lengthy calculation requiring also

$$\theta^1 \wedge \theta^{\bar{1}} = \frac{i}{\beta\bar{\beta}} dxdy \tag{7.2.8}$$

$$\theta^+ \wedge \theta^- = dzdt + \phi_x dzdx + \phi_y dzdy \tag{7.2.9}$$

$$\begin{aligned} \theta^+ \wedge \theta^1 = \frac{1}{2\bar{\beta}|\beta|^2}(&dxdy(|\beta|^2(\phi_y + i\phi_x)) + dxdz(|\beta|^2\phi_z - 1) + dydz(i(1-|\beta|^2\phi_z)) \\ &+ dxdt(|\beta|^2) + dydt(-i|\beta|^2)) \end{aligned} \tag{7.2.10}$$

$$\begin{aligned} \theta^- \wedge \theta^1 = \frac{1}{2\bar{\beta}}(&dxdy(-|\beta|^2(i\phi_x + \phi_y)) + dxdz(-(|\beta|^2\phi_z + 1)) + dydz(i(|\beta|^2\phi_z + 1)) \\ &+ dxdt(-|\beta|^2) + dydt(i|\beta|^2)) \end{aligned} \tag{7.2.11}$$

to change from θ^a coordinates to t, x, y, z, we now extract the relations

$$\partial_z\phi_x - \partial_x\phi_z = -\frac{i}{(\beta\bar{\beta})^2}(\bar{\beta}\partial_y(\beta) - \beta\partial_y(\bar{\beta})) \tag{7.2.12}$$

from the xz part,

$$\partial_y\phi_z - \partial_z\phi_y = -\frac{i}{(\beta\bar{\beta})^2}(\bar{\beta}\partial_x(\beta) - \beta\partial_x(\bar{\beta})) \tag{7.2.13}$$

from the yz part, and

$$\partial_x\phi_y - \partial_y\phi_x = -\frac{i}{(\beta\bar{\beta})^2}(\bar{\beta}\partial_z(\beta) - \beta\partial_z(\bar{\beta})) \tag{7.2.14}$$

from the xy part. This is more compactly written as

$$\vec{\nabla} \times \vec{\phi} = -\frac{i}{\beta\bar{\beta}}\vec{\nabla} ln(\frac{\beta}{\bar{\beta}}), \tag{7.2.15}$$

and we thus conclude the derivation for the ϕ-relation.

7.3 β-condition

The condition on β is gathered from extra conditions on the metric. The first is the Bianchi identity,

$$\mathrm{d}F = 0, \tag{7.3.1}$$

and the second is

$$\mathrm{d} * F = 0. \tag{7.3.2}$$

Finding the β-condition is as simple as imposing these constraints. We know that

$$F = F_{+-}\theta^+\wedge\theta^- + F_{+1}\theta^+\wedge\theta^1 + F_{+\bar{1}}\theta^+\wedge\theta^{\bar{1}} + F_{-1}\theta^-\wedge\theta^1 + F_{-\bar{1}}\theta^-\wedge\theta^{\bar{1}} + F_{1\bar{1}}\theta^1\wedge\theta^{\bar{1}}, \tag{7.3.3}$$

and with the definition of the Hodge star operator introduced in chapter 2 we also know that

$$*F = i(-F_{+-}\theta^1\wedge\theta^{\bar{1}} - F_{+1}\theta^+\wedge\theta^1 + F_{+\bar{1}}\theta^+\wedge\theta^{\bar{1}} + F_{-1}\theta^-\wedge\theta^1 - F_{-\bar{1}}\theta^-\wedge\theta^{\bar{1}} - F_{1\bar{1}}\theta^+\wedge\theta^-). \tag{7.3.4}$$

From the KSE and new partial derivative, we have

$$F_{+-} = -\frac{i}{2}\partial_z(\bar{\beta} - \beta), \tag{7.3.5}$$

$$F_{+1} = -\frac{i\bar{\beta}}{2}(\partial_x(\bar{\beta}) + i\partial_y(\bar{\beta})), \tag{7.3.6}$$

$$F_{-1} = -\frac{i}{2\beta}(\partial_x(\beta) + i\partial_y(\beta)), \tag{7.3.7}$$

$$F_{1\bar{1}} = -\frac{i}{2}\partial_z(\bar{\beta} + \beta). \tag{7.3.8}$$

Together with the θ^a wedge products from the previous section, we have our F and $*F$ reduced in a form we can use as

$$\begin{aligned}
F = {} & \frac{1}{2}dxdy(i(\phi_x\partial_y(\bar{\beta} - \beta) - \phi_y\partial_x(\bar{\beta} - \beta)) + \frac{1}{|\beta|^2}\partial_z(\bar{\beta} + \beta)) \\
& + \frac{1}{2}dxdz(i(\phi_x\partial_z(\bar{\beta} - \beta) - \phi_z\partial_x(\bar{\beta} - \beta)) - \frac{1}{|\beta|^2}\partial_y(\bar{\beta} + \beta)) \\
& + \frac{1}{2}dydz(i(\phi_y\partial_z(\bar{\beta} - \beta) - \phi_z\partial_y(\bar{\beta} - \beta)) + \frac{1}{|\beta|^2}\partial_x(\bar{\beta} + \beta)) \\
& + dxdt(-\frac{i}{2}\partial_x(\bar{\beta} - \beta)) + dydt(-\frac{i}{2}\partial_y(\bar{\beta} - \beta)) + dzdt(-\frac{i}{2}\partial_z(\bar{\beta} - \beta))
\end{aligned} \tag{7.3.9}$$

$$\begin{aligned}
* F = {} & \frac{1}{2}dxdy(\phi_x\partial_y(\bar{\beta} + \beta) - \phi_y\partial_x(\bar{\beta} + \beta) - \frac{i}{|\beta|^2}\partial_z(\bar{\beta} - \beta)) \\
& + \frac{1}{2}dxdz(\phi_x\partial_z(\bar{\beta} + \beta) - \phi_z\partial_x(\bar{\beta} + \beta) + \frac{i}{|\beta|^2}\partial_y(\bar{\beta} - \beta)) \\
& + \frac{1}{2}dydz(\phi_y\partial_z(\bar{\beta} + \beta) - \phi_z\partial_y(\bar{\beta} + \beta) - \frac{i}{|\beta|^2}\partial_x(\bar{\beta} - \beta)) \\
& + dxdt(-\frac{1}{2}\partial_x(\bar{\beta} + \beta)) + dydt(-\frac{1}{2}\partial_y(\bar{\beta} + \beta)) + dzdt(-\frac{1}{2}\partial_z(\bar{\beta} + \beta))
\end{aligned} \tag{7.3.10}$$

Therefore using the Maxwell equation and the Bianchi identity we find

$$
\begin{gathered}
dF = \frac{1}{2}dxdydz(i((-\partial_x(\phi_z)+\partial_z(\phi_x))\partial_y(\bar{\beta}-\beta)+(\partial_y(\phi_z)-\partial_z(\phi_y))\partial_x(\bar{\beta}-\beta)+ \\
(\partial_x(\phi_y)-\partial_y(\phi_x))\partial_z(\bar{\beta}-\beta)) \\
+\partial_z(\frac{1}{|\beta|^2})\partial_z(\bar{\beta}+\beta)+\partial_y(\frac{1}{|\beta|^2})\partial_y(\bar{\beta}+\beta)+\partial_x(\frac{1}{|\beta|^2})\partial_x(\bar{\beta}+\beta) \\
+\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\bar{\beta}+\beta)) = 0
\end{gathered}
\tag{7.3.11}
$$

and

$$
\begin{gathered}
d*F = \frac{1}{2}dxdydz((\partial_z(\phi_x)-\partial_x(\phi_z))\partial_y(\bar{\beta}+\beta)+(\partial_y(\phi_z)-\partial_z(\phi_y))\partial_x(\bar{\beta}+\beta)+ \\
(\partial_x(\phi_y)-\partial_y(\phi_x))\partial_z(\bar{\beta}+\beta) \\
-i(\partial_z(\frac{1}{|\beta|^2})\partial_z(\bar{\beta}-\beta)+\partial_y(\frac{1}{|\beta|^2})\partial_y(\bar{\beta}-\beta)+\partial_x(\frac{1}{|\beta|^2})\partial_x(\bar{\beta}-\beta) \\
+\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\bar{\beta}-\beta))) = 0.
\end{gathered}
\tag{7.3.12}
$$

From now on we only have to simplify, first by using the ϕ-relation, and then noticing

$$
-\frac{i}{\beta\bar{\beta}}\partial_j ln(\frac{\beta}{\bar{\beta}}) = -\frac{i}{(\beta\bar{\beta})^2}(\bar{\beta}\partial_j(\beta)-\beta\partial_j(\bar{\beta})), \tag{7.3.13}
$$

$$
\partial_i(\frac{1}{|\beta|^2}) = -\frac{1}{(\beta\bar{\beta})^2}(\bar{\beta}\partial_i(\beta)+\beta\partial_i(\bar{\beta})), \tag{7.3.14}
$$

we can get 2 equations

$$
\begin{aligned}
\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\bar{\beta}+\beta) - \frac{2}{(\bar{\beta}\beta)^2}(\beta(\partial_x(\bar{\beta}))^2+\bar{\beta}(\partial_x(\beta))^2+\beta(\partial_y(\bar{\beta}))^2 \\
+\bar{\beta}(\partial_y(\beta))^2+\beta(\partial_z(\bar{\beta}))^2+\bar{\beta}(\partial_z(\beta))^2) = 0,
\end{aligned}
\tag{7.3.15}
$$

and

$$
\begin{aligned}
\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\bar{\beta}-\beta) + \frac{2}{(\bar{\beta}\beta)^2}(-\beta(\partial_x(\bar{\beta}))^2+\bar{\beta}(\partial_x(\beta))^2-\beta(\partial_y(\bar{\beta}))^2 \\
+\bar{\beta}(\partial_y(\beta))^2-\beta(\partial_z(\bar{\beta}))^2+\bar{\beta}(\partial_z(\beta))^2) = 0.
\end{aligned}
\tag{7.3.16}
$$

The first of the above 2 comes from d$F = 0$, and the second comes from d$*F = 0$. Adding them together yields

$$\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\bar{\beta})-\frac{2}{\bar{\beta}^2\beta}((\partial_x(\bar{\beta}))^2+(\partial_y(\bar{\beta}))^2+(\partial_z(\bar{\beta}))^2)=0, \quad (7.3.17)$$

and subtracting yields

$$\frac{1}{|\beta|^2}(\partial_x^2+\partial_y^2+\partial_z^2)(\beta)-\frac{2}{\bar{\beta}\beta^2}((\partial_x(\beta))^2+(\partial_y(\beta))^2+(\partial_z(\beta))^2)=0. \quad (7.3.18)$$

These are essentially what make up the β-relation, but if we notice that

$$\nabla^2(\frac{1}{\beta})=\partial_k^2(\frac{1}{\beta})=\partial_k(\partial_k(\frac{1}{\beta})=\partial_k(-\frac{1}{\beta^2}\partial_k(\beta))=\frac{2}{\beta^3}(\partial_k(\beta))^2-\frac{1}{\beta^2}\partial_k^2(\beta), \quad (7.3.19)$$

we manage to find the more compact form

$$\nabla^2(\frac{1}{\beta})=\nabla^2(\frac{1}{\bar{\beta}})=0. \quad (7.3.20)$$

So the inverse of β is harmonic.

7.4 The IWP metric

The IWP metric class as derived by Tod, which we have now found, is then completely given by

$$ds^2=-\beta\bar{\beta}(dt+\phi_i dx^i)^2+\frac{1}{\beta\bar{\beta}}(dx^i)^2, \quad (7.4.1)$$

together with the ϕ and β relations

$$\vec{\nabla}\times\vec{\phi}=-\frac{i}{\beta\bar{\beta}}\vec{\nabla}ln(\frac{\beta}{\bar{\beta}}), \quad (7.4.2)$$

and

$$\nabla^2(\frac{1}{\beta})=\nabla^2(\frac{1}{\bar{\beta}})=0. \quad (7.4.3)$$

It reduces to the more well known Majumdar-Papapetrou solution [13] for $\phi^i=0$,

$$\mathrm{d}s^2=-\beta\bar{\beta}\,\mathrm{d}t^2+\frac{1}{\beta\bar{\beta}}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right). \quad (7.4.4)$$

Chapter 8

Generalised IWP solutions

This chapter will generalise the IWP solution commonly found in the literature.

Back in section 6.4.2, we make the concious choice to pick a,b,f for our θ^1 basis form

$$\theta^1 = f(a\,\mathrm{d}x + b\,\mathrm{d}y), \tag{8.0.1}$$

where we used

$$f = \frac{1}{\sqrt{2}\bar{\beta}},\ a = 1\ , b = -i. \tag{8.0.2}$$

This recovered the IWP metric, as derived by Tod and others in the literature. However in this process we showed that we can indeed pick any a,b, as long as there are nonzero real and imaginary parts to θ^1. For the purposes of this project, and due to time constraints we did not manage to generalise the IWP solution as far as we would have liked, however we did do a simple test case, not straying too far from the original IWP solution.

Examining the $2\theta^1\theta^{\bar{1}}$ we see that

$$2\theta^1\theta^{\bar{1}} = 2|f|^2(|a|^2dx^2 + |b|^2dy^2 + (a\bar{b} + \bar{a}b)dxdy) \tag{8.0.3}$$

Comparing this to the original IWP metric along with the terms for $\theta^+\theta^-$, we will use the following totally arbitrary constraining conditions, as a means to test our case before generalising the metric further:

$$a\bar{b} + \bar{a}b = 0 \tag{8.0.4}$$

$$|a|^2 = |b|^2 = 1 \tag{8.0.5}$$

$$|f|^2 = \frac{1}{2|\beta|^2|a|^2} \tag{8.0.6}$$

Following a similar and lengthy process to the derivation of Tod's IWP metric, we eventually get the slightly generalised IWP class of metrics (sgIWP)

$$ds^2 = -\beta\bar{\beta}(dt + \phi_i dx^i)^2 + \frac{1}{\beta\bar{\beta}}(dx^i)^2, \quad (8.0.7)$$

with the conditions

$$\vec{\nabla} \times \vec{\phi} = \frac{-a}{b|\beta|^2}\vec{\nabla} ln(\frac{\beta}{\bar{\beta}}), \quad (8.0.8)$$

and

$$\nabla^2(\frac{1}{\beta}) = \nabla^2(\frac{1}{\bar{\beta}}) = 0, \quad (8.0.9)$$

as well as the arbitrary extra conditions imposed for the purpose of the test

$$a\bar{b} + \bar{a}b = 0, \quad (8.0.10)$$

and

$$|a|^2 = |b|^2 = 1. \quad (8.0.11)$$

The derivation of the sgIWP ϕ-condition is in Appendix B.

The sgIWP metric has the same form and β-condition as Tod's, but the ϕ-condition is different, allowing for greater variety in picking a, b. This reduces to Tod's metric with the choices $a = 1$, $b = -i$. We found a single paper [14] that seemingly also diverts from Tod, however the author does not mention the generality in picking the free parameters a, b, and instead just uses the also quite simple but logical choice $a = 1$,$b = i$. We believe this gives some credibility to the validity of our extended sgIWP metric class, however more work is needed to generalise it further.

Chapter 9

Conclusion

This report had the main aim of deriving the IWP black hole metric as found by Tod from SUSY conditions on the geometry of spacetime. In order to achieve this goal, we introduced all the concepts required in order of use, and successfully built the metric, using the spinorial geometry method outlined in [4]. Alongside this, we also discovered a possible new point of generalisability in the choice of basis forms for the metric, allowing us to derive a slightly generalised IWP black hole class of solutions. More work is needed in this area, as we only had the time for a small testing case. Our test gave us positive results, reducing to the known IWP solution for certain parameters, and also reducing to a seperate, specific IWP solution found in another published paper [14]. The focus of the project was not the IWP metric itself, but its ability to be derived from SUSY conditions without the use of the EM field equations, as well as the spinorial geometry methods that were employed. It is very possible that our work will lead to a new black hole metric class, which reduces to IWP for certain conditions.

Appendix A

Various γ actions

These actions are used in the reduction of the KSE.

A.1 The actions of γ_a

The actions of γ_a on our forms is given by

$$
\begin{aligned}
\gamma_0 &= -e^2 \wedge + i_{e^2}, \\
\gamma_1 &= e^1 \wedge + i_{e^1}, \\
\gamma_2 &= e^2 \wedge + i_{e^2}, \\
\gamma_3 &= i \left(e^1 \wedge - i_{e^1} \right).
\end{aligned}
\tag{A.1.1}
$$

It is easiest to proceed with a change in basis so that we have

$$
\begin{aligned}
\gamma_+ &= \frac{1}{\sqrt{2}} \left(\gamma_2 + \gamma_0 \right) = \sqrt{2} i_{e^2}, \\
\gamma_- &= \frac{1}{\sqrt{2}} \left(\gamma_2 - \gamma_0 \right) = \sqrt{2} e^2 \wedge, \\
\gamma_1 &= \frac{1}{\sqrt{2}} \left(\gamma_1 + i\gamma_3 \right) = \sqrt{2} i_{e^1}, \\
\gamma_{\bar{1}} &= \frac{1}{\sqrt{2}} \left(\gamma_1 - i\gamma_3 \right) = \sqrt{2} e^1 \wedge .
\end{aligned}
\tag{A.1.2}
$$

Now raising the gamma matrices as you would typically, using $\gamma^a = g^{ab}\gamma_b$,

$$
\begin{aligned}
\gamma^+ &= \gamma_- = \sqrt{2} e^2 \wedge, \\
\gamma^- &= \gamma_+ = \sqrt{2} i_{e^2}, \\
\gamma^1 &= \gamma_{\bar{1}} = \sqrt{2} e^1 \wedge, \\
\gamma^{\bar{1}} &= \gamma_1 = \sqrt{2} i_{e^1}.
\end{aligned}
\tag{A.1.3}
$$

A.2 The actions of γ_a on ϵ

In Minkowski basis:

$$\begin{aligned}\gamma_0\epsilon &= \mu^2 - \sigma e^1 - \lambda e^2 + \mu^1 e^1 \wedge e^2 \\ \gamma_1\epsilon &= \mu^1 + \lambda e^1 + \sigma e^2 + \mu^2 e^1 \wedge e^2 \\ \gamma_2\epsilon &= \mu^2 - \sigma e^1 + \lambda e^2 - \mu^1 e^1 \wedge e^2 \\ \gamma_3\epsilon &= i\left(-\mu^1 + \lambda e^1 - \sigma e^2 + \mu^2 e^1 \wedge e^2\right)\end{aligned} \tag{A.2.1}$$

and for convenience:

$$\begin{aligned}\gamma_0\mathbb{1} &= -e^2 \\ \gamma_0 e^1 &= e^1 \wedge e^2 \\ \gamma_0 e^2 &= \mathbb{1} \\ \gamma_0 e^1 \wedge e^2 &= -e^1\end{aligned} \tag{A.2.2}$$

$$\begin{aligned}\gamma_1\mathbb{1} &= e^1 \\ \gamma_1 e^1 &= \mathbb{1} \\ \gamma_1 e^2 &= e^1 \wedge e^2 \\ \gamma_1 e^1 \wedge e^2 &= e^2\end{aligned} \tag{A.2.3}$$

$$\begin{aligned}\gamma_2\mathbb{1} &= e^2 \\ \gamma_2 e^1 &= -e^1 \wedge e^2 \\ \gamma_2 e^2 &= \mathbb{1} \\ \gamma_2 e^1 \wedge e^2 &= -e^1\end{aligned} \tag{A.2.4}$$

$$\begin{aligned}\gamma_3\mathbb{1} &= ie^1 \\ \gamma_3 e^1 &= -i \\ \gamma_3 e^2 &= ie^1 \wedge e^2 \\ \gamma_3 e^1 \wedge e^2 &= -ie^2\end{aligned} \tag{A.2.5}$$

and also for $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

$$\begin{aligned}\gamma_5\mathbb{1} &= 1 \\ \gamma_5 e^1 &= -e^1 \\ \gamma_5 e^2 &= -e^2 \\ \gamma_5 e^1 \wedge e^2 &= e^1 \wedge e^2\end{aligned} \tag{A.2.6}$$

In null basis:

$$
\begin{aligned}
\gamma_+\epsilon &= \sqrt{2}\left(\mu^2 - \sigma e^1\right) \\
\gamma_-\epsilon &= \sqrt{2}\left(\lambda e^2 - \mu^1 e^1 \wedge e^2\right) \\
\gamma_1\epsilon &= \sqrt{2}\left(\mu^1 + \sigma e^2\right) \\
\gamma_{\bar{1}}\epsilon &= \sqrt{2}\left(\lambda e^1 + \mu^2 e^1 \wedge e^2\right)
\end{aligned}
\tag{A.2.7}
$$

A.3 The actions of γ_{ab} on ϵ

01

$$
\begin{aligned}
\gamma_{01}\mathbb{1} &= e^1 \wedge e^2 \\
\gamma_{01}e^1 &= -e^2 \\
\gamma_{01}e^2 &= -e^1 \\
\gamma_{01}e^1 \wedge e^2 &= \mathbb{1}
\end{aligned}
\tag{A.3.1}
$$

02

$$
\begin{aligned}
\gamma_{02}\mathbb{1} &= \mathbb{1} \\
\gamma_{02}e^1 &= e^1 \\
\gamma_{02}e^2 &= -e^2 \\
\gamma_{02}e^1 \wedge e^2 &= -e^1 \wedge e^2
\end{aligned}
\tag{A.3.2}
$$

03

$$
\begin{aligned}
\gamma_{03}\mathbb{1} &= ie^1 \wedge e^2 \\
\gamma_{03}e^1 &= ie^2 \\
\gamma_{03}e^2 &= -ie^1 \\
\gamma_{03}e^1 \wedge e^2 &= -i
\end{aligned}
\tag{A.3.3}
$$

12

$$
\begin{aligned}
\gamma_{12} &= e^1 \wedge e^2 \\
\gamma_{12}e^1 &= -e^2 \\
\gamma_{12}e^2 &= e^1 \\
\gamma_{12}e^1 \wedge e^2 &= -\mathbb{1}
\end{aligned}
\tag{A.3.4}
$$

13

$$
\begin{aligned}
\gamma_{13}\mathbb{1} &= i \\
\gamma_{13}e^1 &= -ie^1 \\
\gamma_{13}e^2 &= ie^2 \\
\gamma_{13}e^1 \wedge e^2 &= -ie^1 \wedge e^2
\end{aligned}
\tag{A.3.5}
$$

23

$$
\begin{aligned}
\gamma_{23}\mathbb{1} &= -ie^1 \wedge e^2 \\
\gamma_{23}e^1 &= -ie^2 \\
\gamma_{23}e^2 &= -ie^1 \\
\gamma_{23}e^1 \wedge e^2 &= -i
\end{aligned}
\tag{A.3.6}
$$

A.4 The actions of γ^{ab} on ϵ

Reminder that $\gamma^{ab} = \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right)$, and $\gamma^{ab} = -\gamma^{ba}$, so there are exactly 6 fully unique nonzero actions on ϵ.

$$
\begin{aligned}
\gamma^{+-}\epsilon &= -\lambda - \mu^1 e^1 + \mu^2 e^2 + \sigma e^1 \wedge e^2 \\
\gamma^{+1}\epsilon &= -2\lambda e^1 \wedge e^2 \\
\gamma^{+\bar{1}}\epsilon &= 2\mu^1 e^2 \\
\gamma^{-1}\epsilon &= -2\mu^2 e^1 \\
\gamma^{-\bar{1}}\epsilon &= 2\sigma \\
\gamma^{1\bar{1}}\epsilon &= -\lambda + \mu^1 e^1 - \mu^2 e^2 + \sigma e^1 \wedge e^2
\end{aligned}
\tag{A.4.1}
$$

A.5 The actions of $\gamma^{ab}\gamma_\mu$ on ϵ

$\mu = +$

$$
\begin{aligned}
\gamma^{+-}\gamma_+\epsilon &= \sqrt{2}\left(-\mu^2 + \sigma e^1\right) \\
\gamma^{+1}\gamma_+\epsilon &= -2\sqrt{2}\mu^2 e^1 \wedge e^2 \\
\gamma^{+\bar{1}}\gamma_+\epsilon &= -2\sqrt{2}\sigma e^2 \\
\gamma^{-1}\gamma_+\epsilon &= 0 \\
\gamma^{-\bar{1}}\gamma_+\epsilon &= 0 \\
\gamma^{1\bar{1}}\gamma_+\epsilon &= \sqrt{2}\left(-\mu^2 - \sigma e^1\right)
\end{aligned}
\tag{A.5.1}
$$

$\mu = -$

$$
\begin{aligned}
\gamma^{+-}\gamma_-\epsilon &= \sqrt{2}\left(\lambda e^2 - \mu^1 e^1 \wedge e^2\right) \\
\gamma^{+1}\gamma_-\epsilon &= 0 \\
\gamma^{+\bar{1}}\gamma_-\epsilon &= 0 \\
\gamma^{-1}\gamma_-\epsilon &= -2\sqrt{2}\lambda e^1 \\
\gamma^{-\bar{1}}\gamma_-\epsilon &= -2\sqrt{2}\mu^1 \\
\gamma^{1\bar{1}}\gamma_-\epsilon &= \sqrt{2}\left(-\lambda e^2 - \mu^1 e^1 \wedge e^2\right)
\end{aligned}
\tag{A.5.2}
$$

$\mu = -$

$$
\begin{aligned}
\gamma^{+-}\gamma_1\epsilon &= -2\sqrt{2}\mu^1 e^1 \wedge e^2 \\
\gamma^{+1}\gamma_1\epsilon &= \sqrt{2}\left(-\mu^1 + \sigma e^2\right) \\
\gamma^{+\bar{1}}\gamma_1\epsilon &= 0 \\
\gamma^{-1}\gamma_1\epsilon &= -2\sqrt{2}\sigma e^1 \\
\gamma^{-\bar{1}}\gamma_1\epsilon &= 0 \\
\gamma^{1\bar{1}}\gamma_1\epsilon &= \sqrt{2}\left(-\mu^1 - \sigma e^2\right)
\end{aligned}
\tag{A.5.3}
$$

$\mu = \bar{1}$

$$
\begin{aligned}
\gamma^{+-}\gamma_{\bar{1}}\epsilon &= \sqrt{2}\left(-\lambda e^1 + \mu^2 e^1 \wedge e^2\right) \\
\gamma^{+1}\gamma_{\bar{1}}\epsilon &= 0 \\
\gamma^{+\bar{1}}\gamma_{\bar{1}}\epsilon &= 2\sqrt{2}\lambda e^2 \\
\gamma^{-1}\gamma_{\bar{1}}\epsilon &= 0 \\
\gamma^{-\bar{1}}\gamma_{\bar{1}}\epsilon &= 2\sqrt{2}\mu^2 \\
\gamma^{1\bar{1}}\gamma_{\bar{1}}\epsilon &= \sqrt{2}\left(\lambda e^1 + \mu^2 e^1 \wedge e^2\right)
\end{aligned}
\tag{A.5.4}
$$

Appendix B

Generalised IWP solution: ϕ-condition

We proceed similarly as with the Tod IWP solution.
First, we calculate the exterior derivatives of the θ's.

$$\begin{aligned} d\theta^+ = \frac{1}{\sqrt{2}}(dxdy(-\partial_x\phi_y + \partial_y\phi_x) + dxdz(-\partial_x\phi_z + \partial_z\phi_x + \partial_x(\frac{1}{\beta\bar{\beta}})) \\ + dydz(-\partial_y\phi_z + \partial_z\phi_y + \partial_y(\frac{1}{\beta\bar{\beta}}))) \end{aligned} \tag{B.0.1}$$

$$\begin{aligned} d\theta^- = -\frac{1}{\sqrt{2}}(dxdy(\phi_y\partial_x(|\beta|^2) - \phi_x\partial_y(|\beta|^2) - |\beta|^2\partial_x(\phi_y) + |\beta|^2\partial_y(\phi_x)) \\ + dxdz(\phi_z\partial_x(|\beta|^2) - \phi_x\partial_z(|\beta|^2) - |\beta|^2\partial_x(\phi_z) + |\beta|^2\partial_z(\phi_x)) \\ + dydz(\phi_z\partial_y(|\beta|^2) - \phi_y\partial_z(|\beta|^2) - |\beta|^2\partial_y(\phi_z) + |\beta|^2\partial_z(\phi_y)) \\ + dxdt(\partial_x(|\beta|^2)) + dydt(\partial_y(|\beta|^2)) + dzdt(\partial_z(|\beta|^2))) \end{aligned} \tag{B.0.2}$$

Now we need to compare this with our other $d\theta$'s, however requiring needing to calculate all of the wedge products of θs in t,x,y,z:

$$\theta^1 \wedge \theta^{\bar{1}} = |f|^2(a\bar{b} - \bar{a}b)dxdy \tag{B.0.3}$$

$$\theta^+ \wedge \theta^- = dzdt + \phi_x dzdx + \phi_y dzdy \tag{B.0.4}$$

$$\begin{aligned} \theta^+ \wedge \theta^1 = \frac{f}{\sqrt{2}|\beta|^2}(dxdy(|\beta|^2(a\phi_y - b\phi_x)) + dxdz(a(|\beta|^2\phi_z - 1)) + dydz(-b(1 - |\beta|^2\phi_z)) \\ + dxdt(a|\beta|^2) + dydt(b|\beta|^2)) \end{aligned} \tag{B.0.5}$$

$$\theta^{-}\wedge\theta^{1} = \frac{f}{\sqrt{2}}(dxdy(|\beta|^2(-b\phi_x+a\phi_y))+dxdz(-a(|\beta|^2\phi_z+1))+dydz(-b(|\beta|^2\phi_z+1)) + dxdt(-a|\beta|^2) + dydt(-b|\beta|^2)) \tag{B.0.6}$$

Before we compare the two $d\theta$s we need to get the partial derivatives in t,x,y,z. We know that

$$V = 2\partial_t \tag{B.0.7}$$

and

$$V = |\beta|^2(dt+\phi) \tag{B.0.8}$$

as well as

$$W = 2|\beta|^2\partial_z \tag{B.0.9}$$

and

$$W = 2dz \tag{B.0.10}$$

Therefore

$$|\beta|^2(dt+\phi) + dz = 2\partial_t + |\beta|^2\partial_z \tag{B.0.11}$$

Using this with the θs we have we find

$$\partial_{+} = \frac{1}{\sqrt{2}}(2\partial_t + |\beta|^2\partial_z) \tag{B.0.12}$$

$$\partial_{-} = \frac{1}{\sqrt{2}}(-\frac{2}{|\beta|^2}\partial_t + \partial_z) \tag{B.0.13}$$

$$\partial_1 = \frac{1}{2f}\left(\frac{1}{a}\partial_x + \frac{1}{b}\partial_y\right). \tag{B.0.14}$$

$$\begin{aligned} d\theta^{+} &= dxdy(-\frac{|f|^2(a\bar{b}-\bar{a}b)}{\sqrt{2}}(\frac{1}{\bar{\beta}}\partial_{-}(\bar{\beta}) - \frac{1}{\beta}\partial_{-}(\beta))) \\ &\quad + dxdz(-\frac{1}{\sqrt{2}|\beta|^2}(\frac{1}{\bar{\beta}}\partial_x(\bar{\beta}) + \frac{1}{\beta}\partial_x(\beta) + \frac{\bar{a}}{\bar{\beta}\bar{b}}\partial_y(\bar{\beta}) + \frac{a}{\beta b}\partial_y(\beta))) \\ &\quad + dydz(-\frac{1}{\sqrt{2}|\beta|^2}(\frac{\bar{b}}{\bar{\beta}\bar{a}}\partial_x(\bar{\beta}) + \frac{b}{a\beta}\partial_x(\beta) + \frac{1}{\bar{\beta}}\partial_y(\bar{\beta}) + \frac{1}{\beta}\partial_y(\beta))) \end{aligned} \tag{B.0.15}$$

Comparing our two different dθ^{+}s we find

$$\partial_z\phi_x - \partial_x\phi_z = -\frac{1}{(\beta\bar{\beta})^2}(\frac{a}{b}\bar{\beta}\partial_y(\beta) + \frac{\bar{a}}{\bar{b}}\beta\partial_y(\bar{\beta})) \tag{B.0.16}$$

from the dxdz coefficients.

$$\partial_y\phi_z - \partial_z\phi_y = \frac{1}{(\beta\bar{\beta})^2}(\frac{b}{a}\bar{\beta}\partial_x(\beta) + \frac{\bar{b}}{\bar{a}}\beta\partial_x(\bar{\beta})) \tag{B.0.17}$$

from the dydz coefficients and

$$\partial_x\phi_y - \partial_y\phi_x = -\frac{|f|^2(a\bar{b} - \bar{a}b)}{|\beta|^2}(\bar{\beta}\partial_z(\beta) - \beta\partial_z(\bar{\beta})) \tag{B.0.18}$$

from the dxdy coordinates. Using the relation

$$\bar{a}b + \bar{b}a = 0 \tag{B.0.19}$$

implies

$$\frac{\bar{a}}{\bar{b}} = -\frac{a}{b}. \tag{B.0.20}$$

Using this and

$$|f|^2 = \frac{1}{2|\beta|^2|b|^2} \tag{B.0.21}$$

implies the condition.

$$\vec{\nabla} \times \vec{\phi} = -\frac{a}{b|\beta|^2}\vec{\nabla}ln(\frac{\beta}{\bar{\beta}}) \tag{B.0.22}$$

This also holds for $d\theta^-$.

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