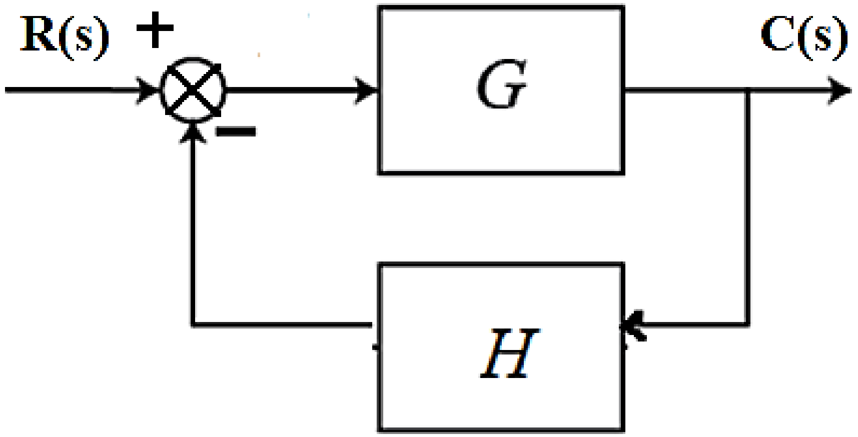
MODULE 2

Time Response Analysis

Time Response

- It is defined as the output of a closed loop system as a function of time.
- Closed loop transfer function ,



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = T(s)$$

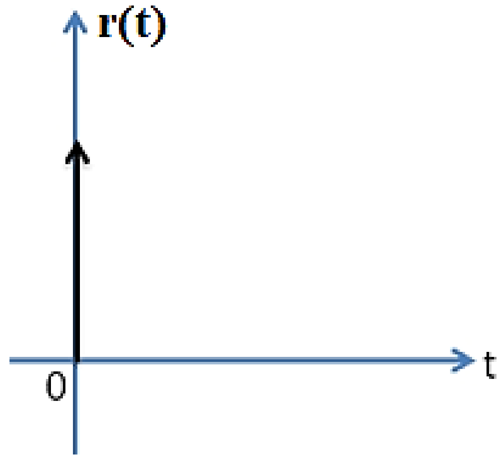
- Response in S domain : C(s) = R(s) × T(s)
- Response in time domain : c(t) = L^{-1} {R(s) × T(s) }
- Time response has 2 parts :

1. Transient response : response of the system when i/p changes from 1 state to the other
2. Steady state response : response of system as t approaches infinity.

Test Signals

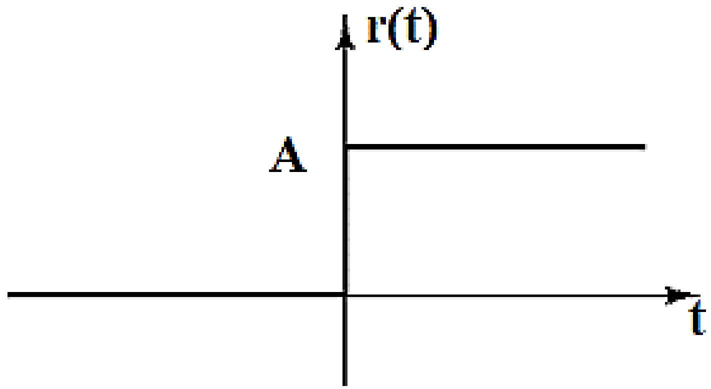
1. Impulse signal

$$r(t) = \infty \,;\, t = 0$$
$$\quad = 0 \,;\, t \neq 0$$
$$R(s) = 1$$



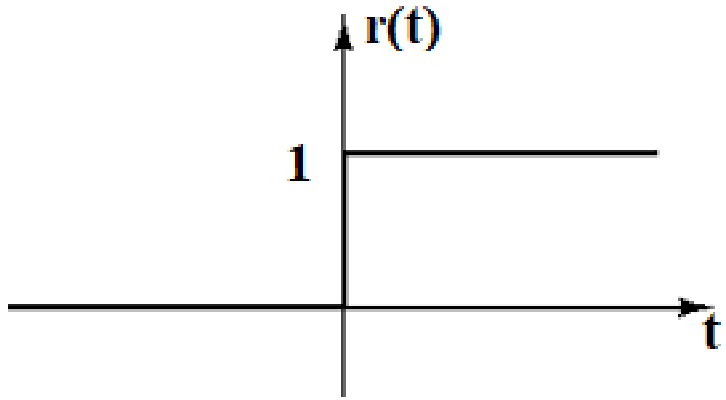
2. Step signal

$$r(t) = A \,;\, t \geq 0$$
$$\quad = 0 \,;\, t < 0$$
$$R(s) = \frac{A}{s}$$



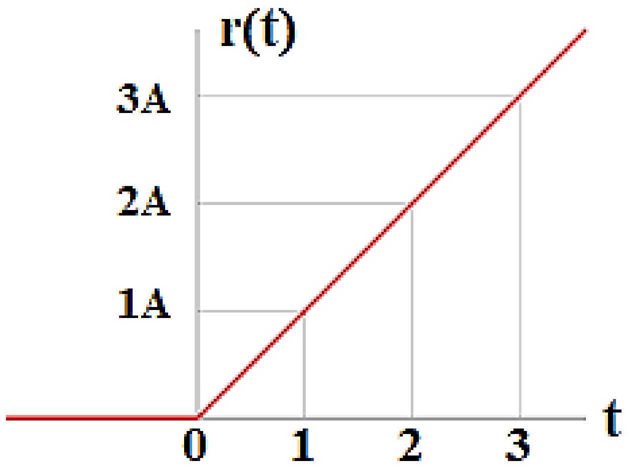
Unit Step signal

$$r(t) = 1 \,;\, t \geq 0$$
$$\quad = 0 \,;\, t < 0$$
$$R(s) = \frac{1}{s}$$



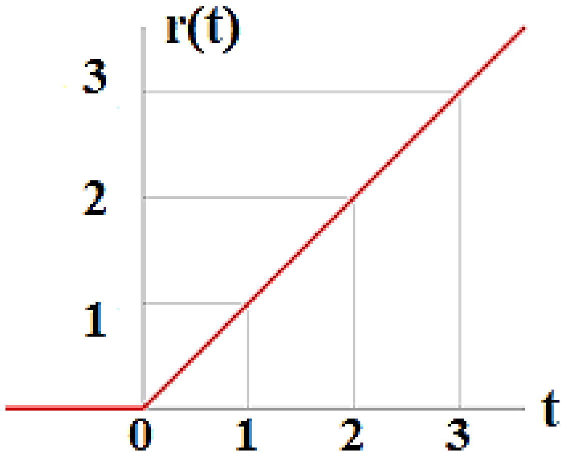
3. Ramp signal

$$r(t) = At\,;t \geq 0$$
$$= 0\ ;\ t < 0$$
$$R(s) = \frac{A}{s^2}$$



Unit ramp signal

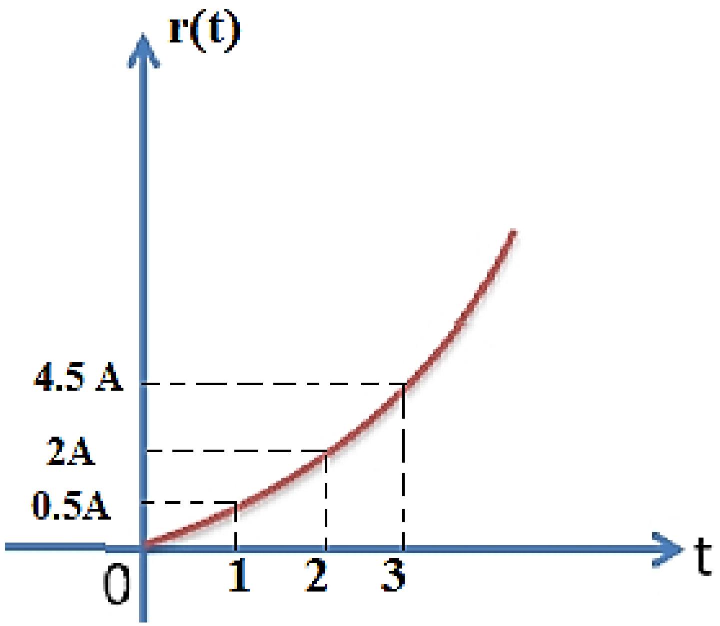
$$r(t) = t\,;t \geq 0$$
$$= 0;\ t < 0$$
$$R(s) = \frac{1}{s^2}$$



4. Parabolic signal

$$r(t) = \frac{At^2}{2} \; ; t \geq 0$$
$$= 0 \quad ; \; t < 0$$

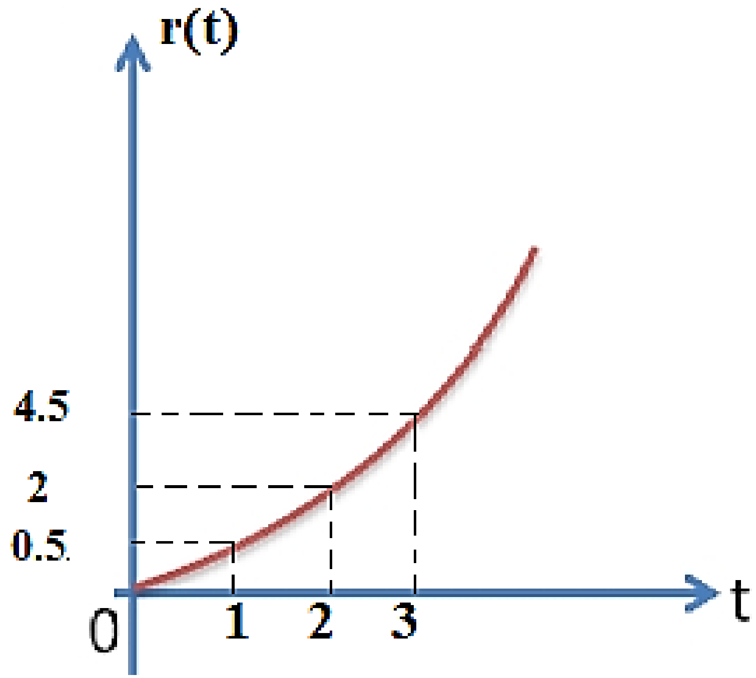
$$R(s) = \frac{A}{s^3}$$



Unit Parabolic signal

$$r(t) = \frac{t^2}{2} \; ; t \geq 0$$
$$= 0 \; ; \; t < 0$$

$$R(s) = \frac{1}{s^3}$$



Impulse response

- Response of a system when impulse signal is given as input.

c(t) = L^{-1} {R(s) × T(s) }   R(s) =1

∴ c(t) = $\boldsymbol{L^{-1}}$ {T(s) }

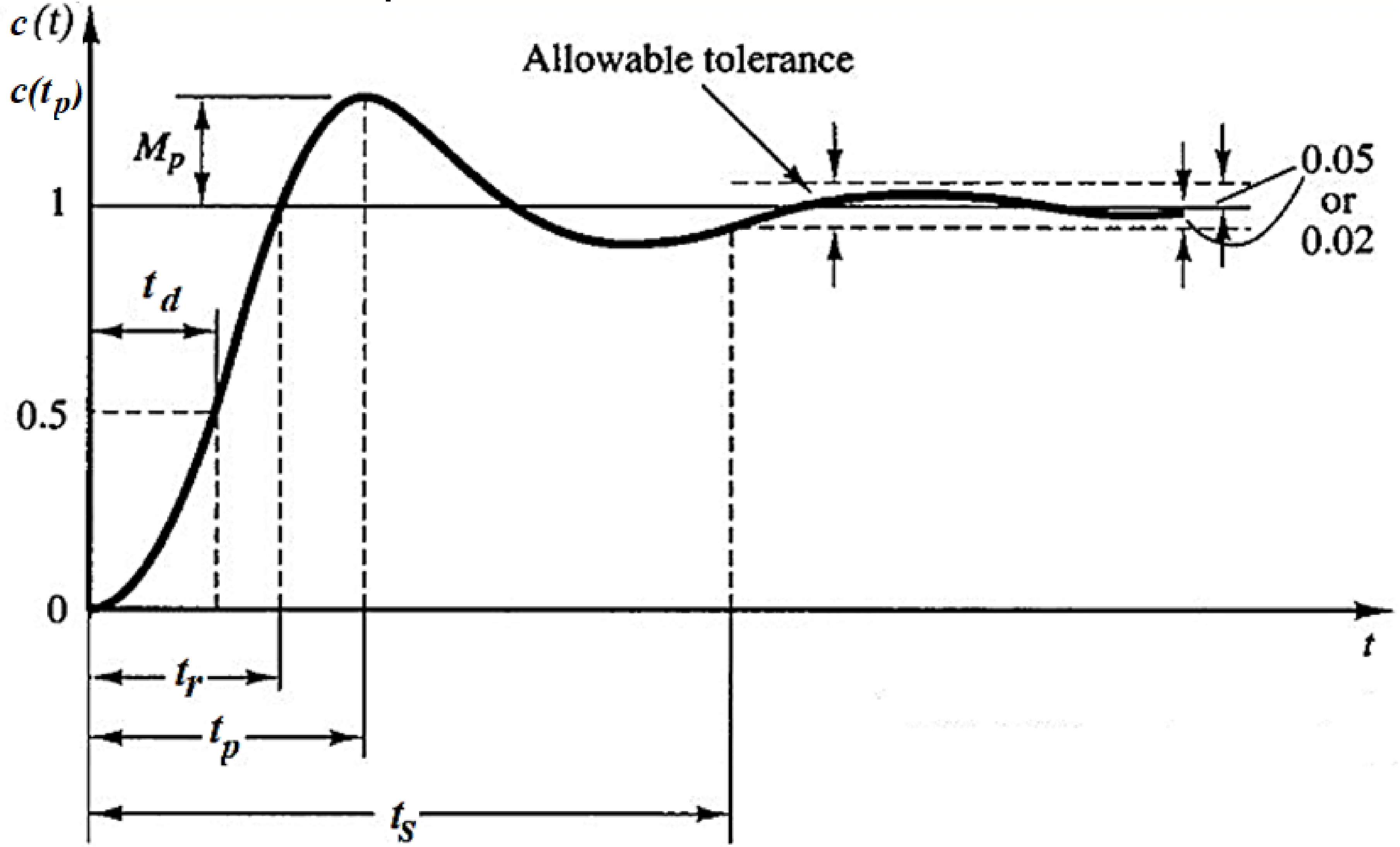
Order of a system

- Transfer Function , $\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots \ldots\ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots \ldots\ldots + a_{n-1} s + a_n}$
- Order of the system= maximum power of s in the denominator = n
- n = 0 (zeroth order system)
- n = 1 (first order s/m)
- n also gives the number of poles

- Poles: values of s in the denominator at which transfer function becomes ∞
- Zeros: values of s in the numerator at which transfer function becomes 0

Time domain specifications

- Performance characteristics of a linear control system are specified in terms of time domain specifications.



1. **Delay Time (t_d)** : Time taken for the response to reach 50% of the final value, for the very first time.
2. **Rise Time (t_r)** : Time taken for the response to rise from 0 to 100% for the very first time. For underdamped system it is the time taken for the response to rise from 0 to 100%. For overdamped systems, rise time is calculated from 10% to 90%. For critically damped systems, it is the time taken by the response to rise from 5% to 95%.
3. **Peak Time (t_p)** : Time taken for the response to reach the peak value for the very first time.

4. Peak Overshoot (M_p) : It is defined as the ratio of the maximum peak value to the final value, where peak value is measured from final value.

Let c(∞) be the final value of c(t) and $c(t_p)$ be the maximum value of c(t)

Peak Overshoot (M_p) = $\frac{c(t_p)-c(\infty)}{c(\infty)}$

% Peak Overshoot (M_p) = $\frac{c(t_p)-c(\infty)}{c(\infty)} \times 100$

5. Settling time (t_s) : Time taken by the response to reach and stay within a specified error. Usually expressed as % of final value. The usual tolerable error is 2% or 5% of the final value

Response of a First Order System for Impulse Input (impulse response)

- Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

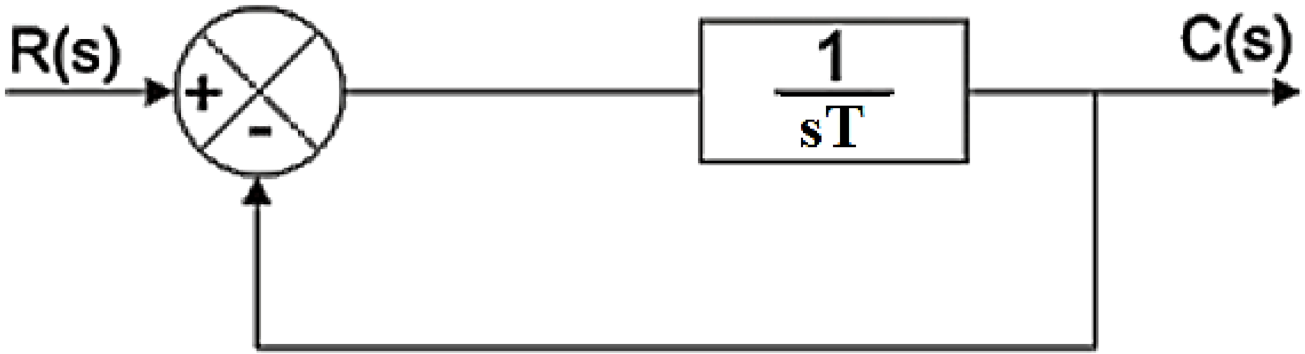
$$= \frac{1}{1 + Ts}$$

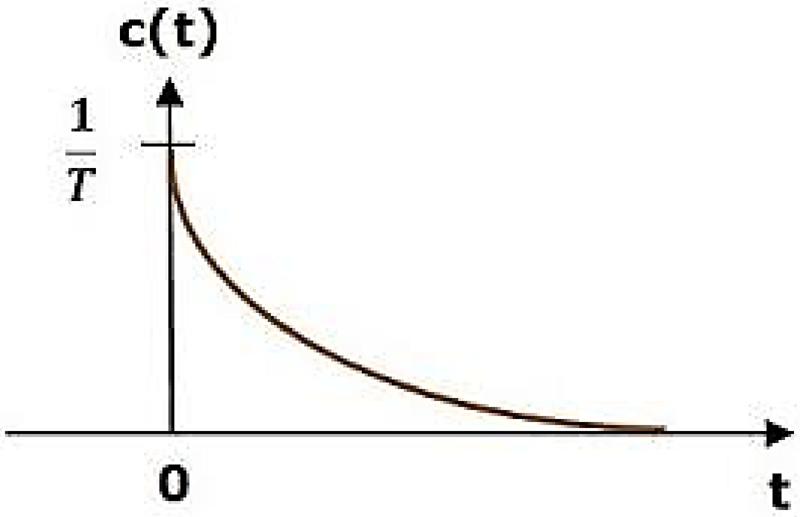
$$C(s) = R(s).\frac{1}{1 + Ts}$$

$$R(s) = 1$$

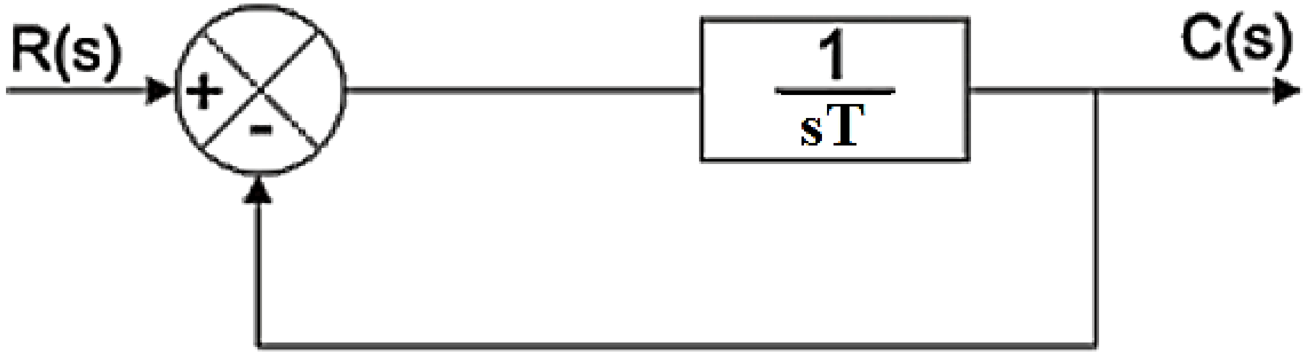
$$C(s) = \frac{1}{1 + Ts} \quad = \frac{1}{T(s + {}^{1}/_{T})}$$

Impulse response , $\quad c(t) = \frac{1}{T}\, e^{-\frac{1}{T}t}$





Response of a First Order System for Step Input



- Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + Ts}$$

$$C(s) = R(s).\frac{1}{1 + Ts}$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s}.\frac{1}{1 + Ts} \quad = \frac{1}{T}\frac{1}{s(s + \frac{1}{T})} = \frac{1/T}{s(s + \frac{1}{T})}$$

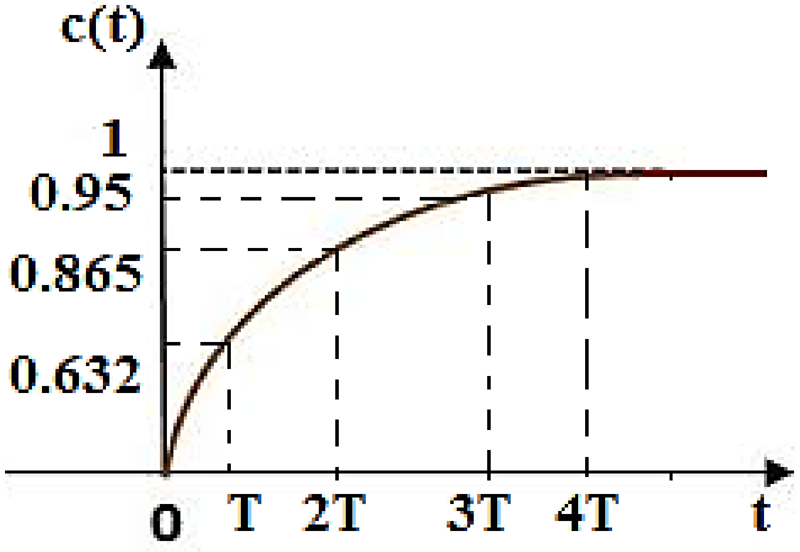
$$C(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{T}}$$

$$A = \quad 1, \qquad B = -1$$

$$C(s) = \frac{1}{s} + \frac{-1}{s + \frac{1}{T}}$$

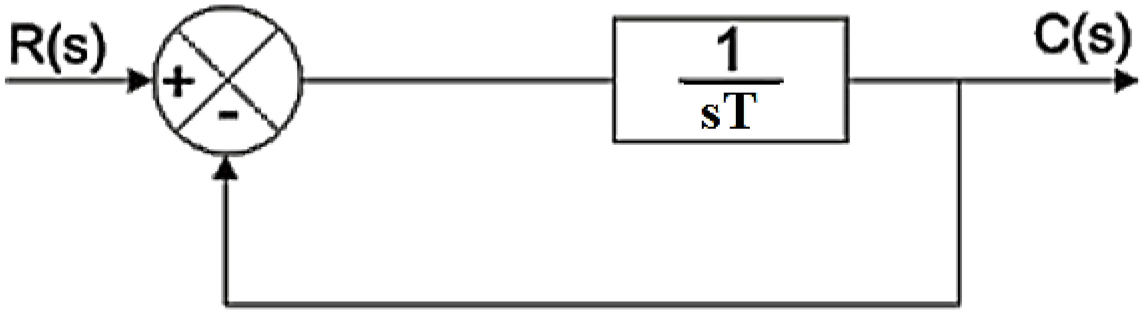
Step response

$$c(t) = 1 - e^{-\frac{1}{T}t}$$


c(t)
1
0.95
0.865
0.632
0
T
2T
3T
4T
t

Response of a First Order System for Ramp Input

- Closed loop transfer function



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{1}{1+Ts}$$

$$C(s) = R(s).\frac{1}{1+Ts}$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2}.\frac{1}{1+Ts} = \frac{1}{T}\frac{1}{s^2(s+\frac{1}{T})} = \frac{1/T}{s^2(s+\frac{1}{T})}$$

$$C(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+\frac{1}{T}}$$

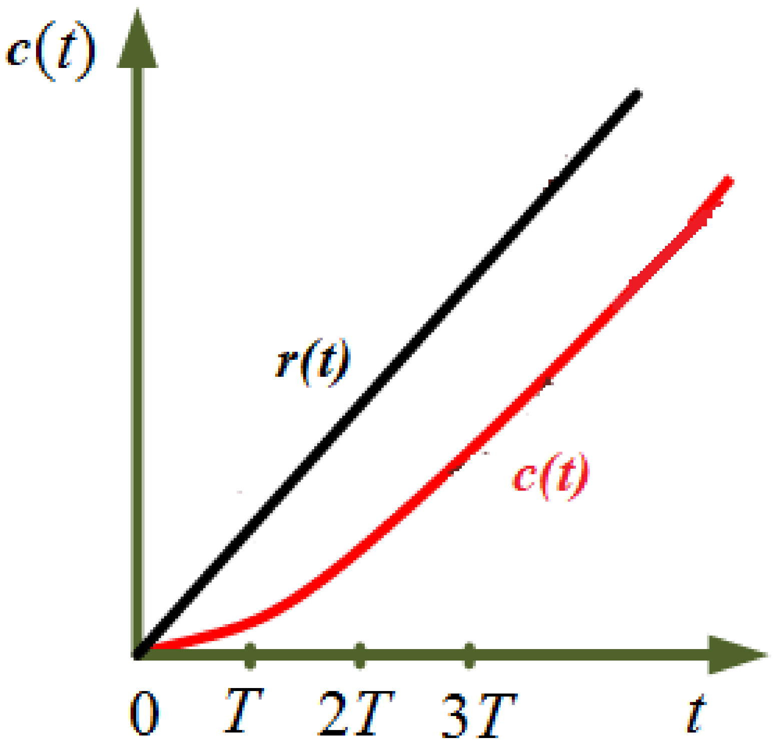
$$A = -T$$

$$B = 1$$

$$\mathrm{C} = \mathrm{T}$$

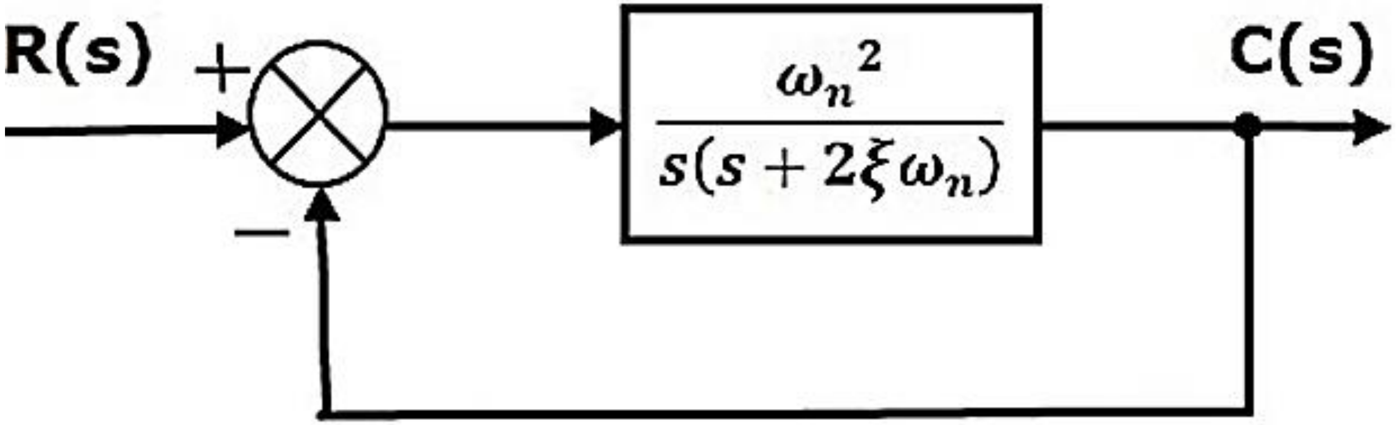
$$C(s) = \frac{-T}{s} + \frac{1}{s^2} + \frac{T}{s+\frac{1}{T}}$$

$$c(t) = t - T + Te^{-\frac{1}{T}t}$$



$c(t)$
$r(t)$
$c(t)$
0
T
$2T$
$3T$
t

Second Order Systems



- Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

- ω_n = undamped natural frequency (rad/sec)
- ξ = damping ratio
- The response of a second order system depends on the value of damping ratio.

  Case 1 : Undamped system, $\xi = 0$

  Case 2 : Under damped system, $0 < \xi < 1$

  Case 3 : Critically damped system, $\xi = 1$

  Case 4 : Over damped system, $\xi > 1$

Response of Undamped Second Order System for Unit Step Input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- For undamped systems ξ =0

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$C(s) = R(s).\frac{\omega_n^2}{s^2 + \omega_n^2}$$

For unit step input, $R(s) = \frac{\mathbf{1}}{\boldsymbol{s}}$

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2}$$

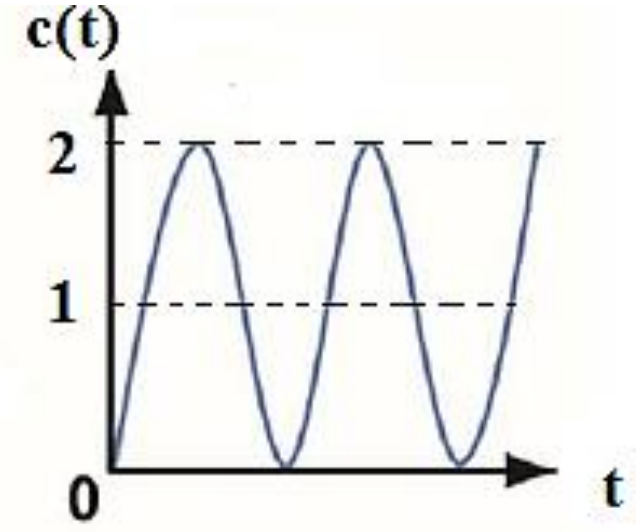
$$A = 1$$

$$B = -1$$

$$\mathrm{C} = 0$$

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$c(t) = 1 - \cos\omega_n \mathrm{t}$$



Response of Under damped Second Order System for Unit Step Input

- Closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$
- For under damped systems $0 < \xi < 1$ and the roots of denominator are complex conjugates.
- The roots of the denominator are $s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$
- Since $\xi < 1$, ξ^2 is also less than 1, so $1 - \xi^2$ is always positive
- Hence roots are $s = -\xi\omega_n \pm \omega_n\sqrt{(-1)(1 - \xi^2)} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$
- Damped frequency of oscillation $\omega_d = \omega_n\sqrt{1 - \xi^2}$

$$\therefore\ s = -\xi\omega_n \pm j\omega_d$$

$$C(s) = R(s).\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For unit step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} . \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$A = 1$$

$$B = -1$$

$$C = -2\xi\omega_n$$

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n + j\omega_d)(s + \xi\omega_n + j\omega_d)}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + {\omega_d}^2} = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + {\omega_d}^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + {\omega_d}^2}$$

Multiply and divide by ω_d in the 3rd term

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + {\omega_d}^2} - \frac{\xi\omega_n}{\omega_d} \frac{\omega_d}{(s + \xi\omega_n)^2 + {\omega_d}^2}$$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right] = 1 - e^{-\xi\omega_n t}[\cos\omega_d t + \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}\sin\omega_d t]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t]$$

$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}[\sin\theta\ \cos\omega_d t + \cos\theta\sin\omega_d t]$$

$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

Where $\theta = tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

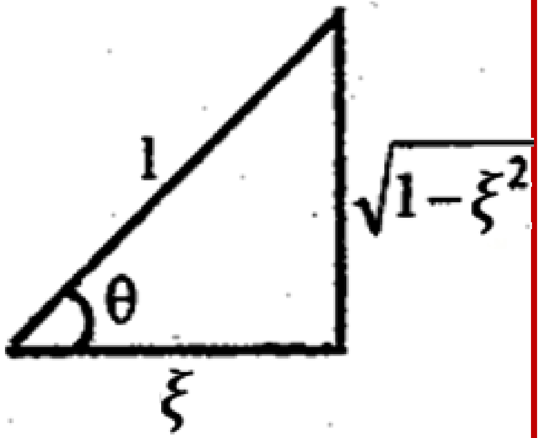
Note :

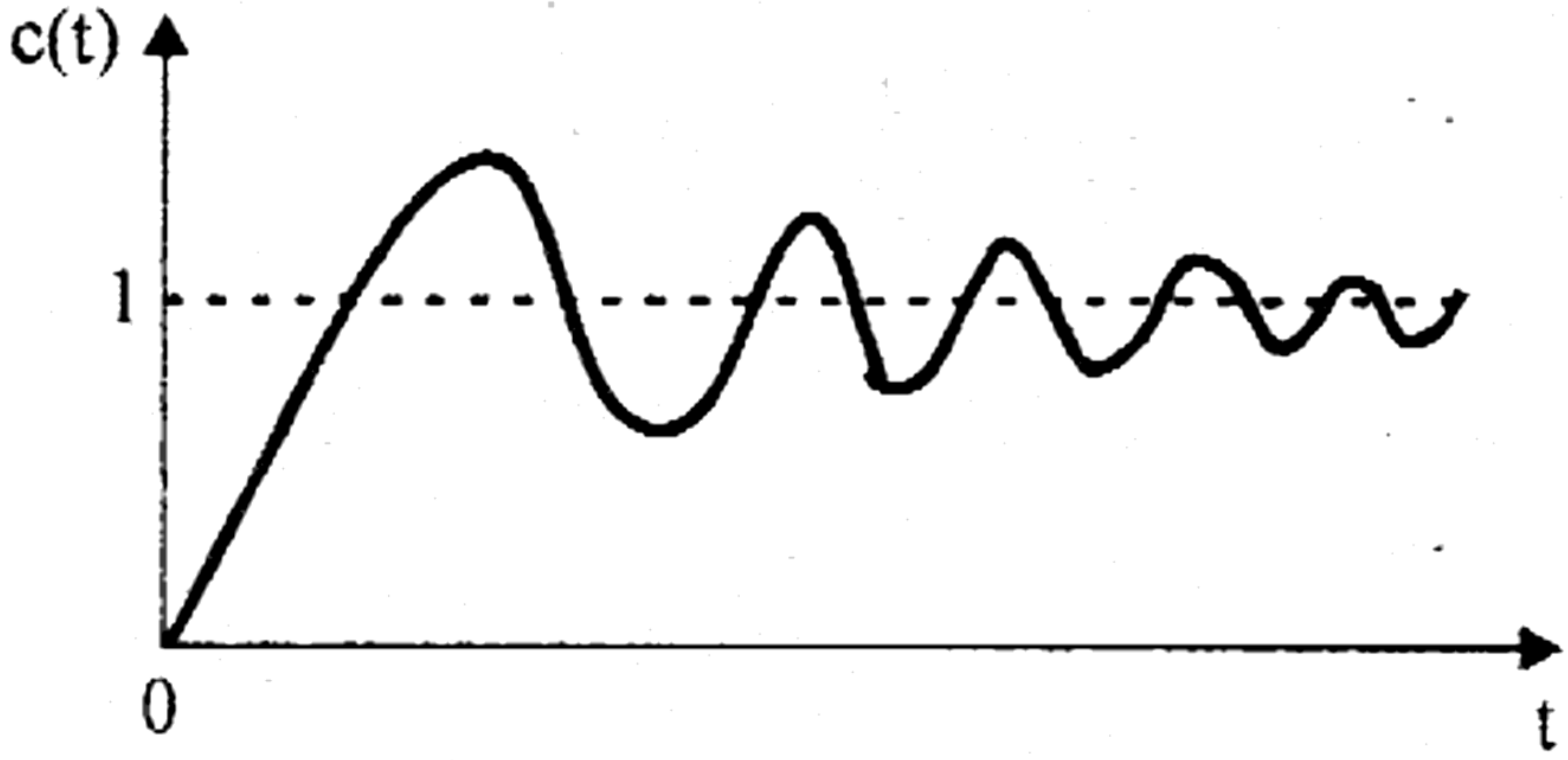
On constructing a right angle triangle with ξ & $\sqrt{1-\xi^2}$, we get

$\cos\theta = \xi$

$\sin\theta = \sqrt{1-\xi^2}$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$




c(t)
1
0
t

Response of Critically Damped Second Order System for Unit Step Input

- Closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$
- For critically damped systems $\xi = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$C(s) = R(s).\frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

For unit step input, $R(s) = \dfrac{\mathbf{1}}{\mathbf{s}}$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

$$C(s) = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$A = 1$$

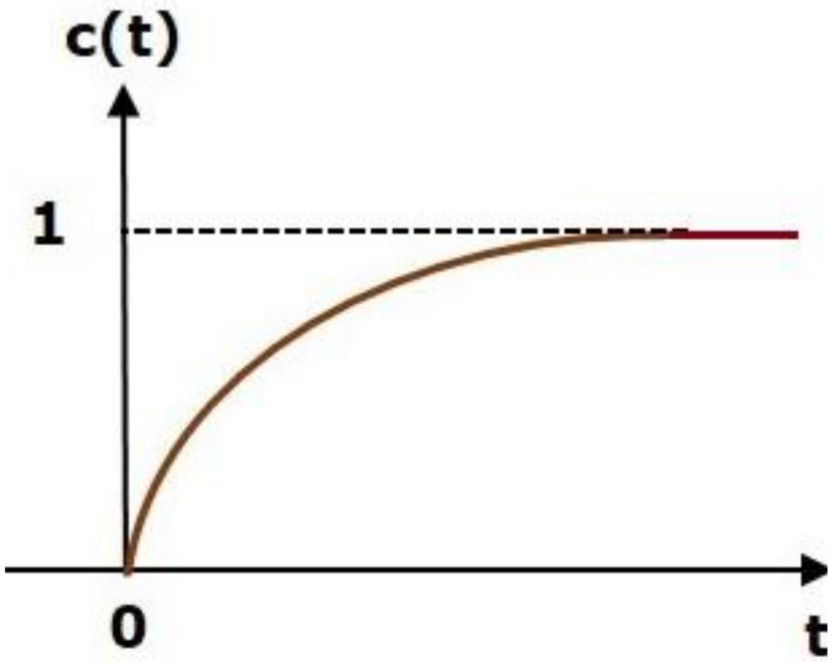
$$B = -1$$

$$C = -\omega_n$$

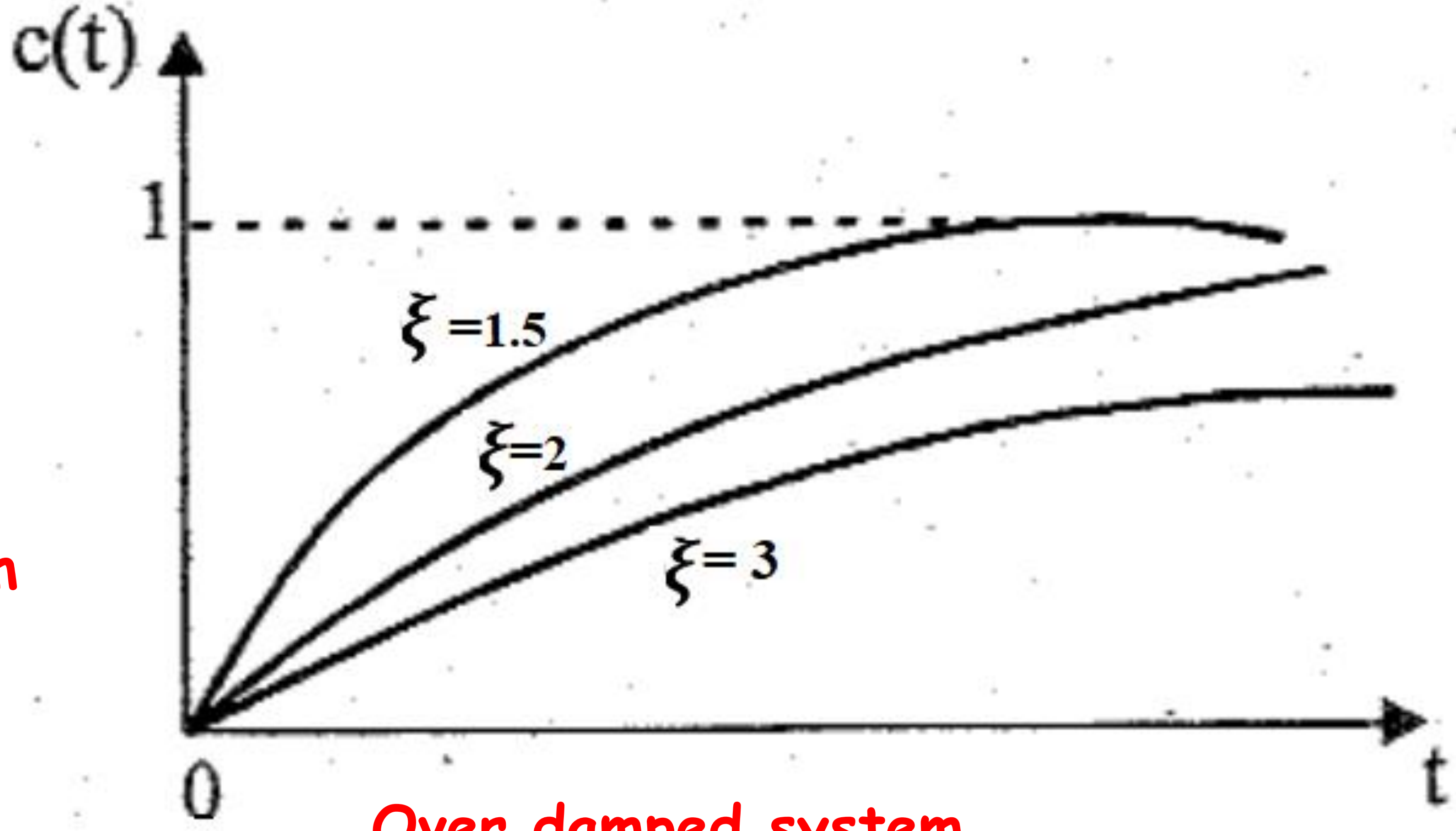
$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}(1 + \omega_n t)$$



Critically damped system



Over damped system

Q. The forward path transfer function of a unity feedback control system is given by G(s) = $\frac{\mathbf{4}}{\boldsymbol{s(s+5)}}$. Obtain the response of the system to unit step input. Also find damping ratio and $\boldsymbol{\omega_n}$

Soln :

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 5s + 4}$$

$$C(s) = R(s).\frac{4}{s^2 + 5s + 4} = \frac{4}{s(s^2 + 5s + 4)} = \frac{4}{s\,(s + 1)(s + 4)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 4}$$

$$A = 1$$

$$B = -4/3$$

$$C = 1/3$$

$$C(s) = \frac{1}{s} + \frac{-4/3}{s + 1} + \frac{1/3}{s + 4}$$

$$c(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

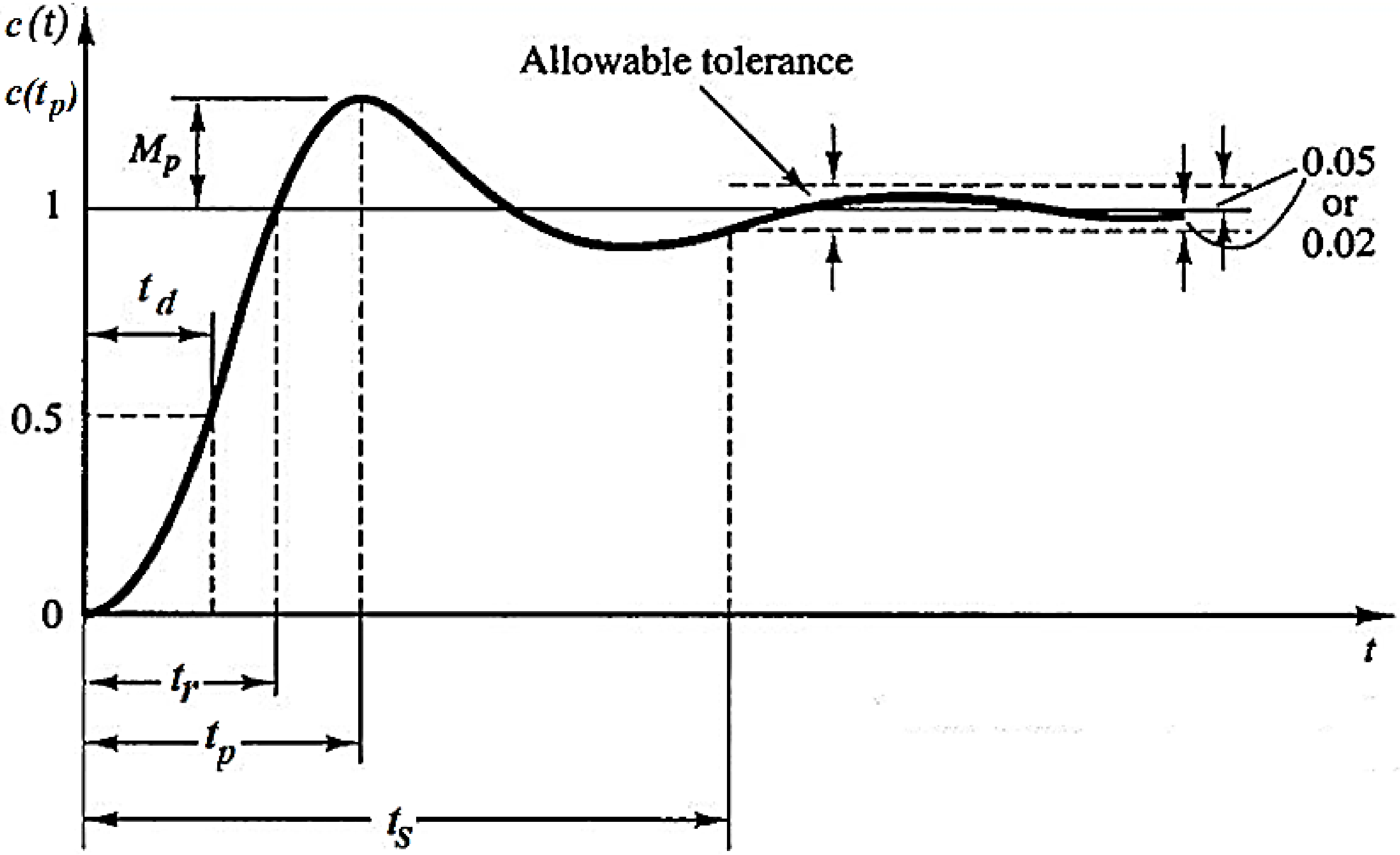
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \therefore \boldsymbol{\omega_n} = 2 \text{ rad/sec}$$

$$2\xi\omega_n = 5$$

$$\therefore\ \xi = 1.25$$

Time domain specifications for a second order system



1. Rise Time (t_r):

The unit step response c(t) for a second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

At t = t_r, c(t) = c(t_r) = 1

$$c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}}\sin(\omega_d t_r + \theta) = 1$$

$$e^{-\xi\omega_n t_r} \neq 0 \quad \therefore \sin(\omega_d t_r + \theta) = 0$$

sin ϕ = 0, when ϕ = π, 2π, 3π...

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

Rise Time, $t_r = \dfrac{\pi - \theta}{\omega_d}$

Note:

$\theta = tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$ (rad)

$\omega_d = \omega_n\sqrt{1-\xi^2}$

2. Peak Time (t_p):

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{c(t)}\Big|_{\boldsymbol{t=t_p}} = \mathbf{0}$$

$$\frac{d}{dt}c(t) = \frac{d}{dt}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

$$-\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}.\cos(\omega_d t + \theta).(\omega_d) + \sin(\omega_d t + \theta).\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}.(-\xi\omega_n)\right]$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$-\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}.\cos(\omega_d t + \theta).(\omega_n\sqrt{1-\xi^2}) + \sin(\omega_d t + \theta).\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}.(-\xi\omega_n)\right]$$

$$= -e^{-\xi\omega_n t}.\cos(\omega_d t+\theta).(\omega_n) + \sin(\omega_d t+\theta).\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}.(\xi\omega_n)$$

$$= \frac{e^{-\xi\omega_n t}.\omega_n}{\sqrt{1-\xi^2}}\left[\xi.\ \sin(\omega_d t+\theta) - \sqrt{1-\xi^2}.\cos(\omega_d t+\theta)\right]$$

$$= \frac{e^{-\xi\omega_n t}.\omega_n}{\sqrt{1-\xi^2}}[cos\ \theta\ .\ \sin(\omega_d t+\theta) - \sin\theta\,.\cos(\omega_d t+\theta)]$$

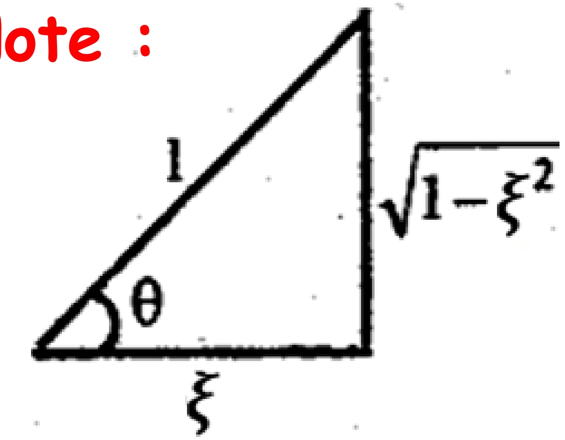
$$= \frac{e^{-\xi\omega_n t}.\omega_n}{\sqrt{1-\xi^2}}[\sin(\omega_d t+\theta) - \theta] = \frac{e^{-\xi\omega_n t}.\omega_n}{\sqrt{1-\xi^2}}[\sin\omega_d t]$$

At t = t_p, $\frac{d}{dt}c(t) = 0$ $\quad \therefore \frac{e^{-\xi\omega_n t_p}.\omega_n}{\sqrt{1-\xi^2}}[\sin\omega_d t_p] = 0$

$e^{-\xi\omega_n t_p} \neq 0,\quad \therefore \sin\omega_d t_p = 0 \rightarrow \omega_d t_p = \pi, 2\pi, 3\pi\ldots$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Note :



cos $\theta = \xi$

sin $\theta = \sqrt{1-\xi^2}$

tan $\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

3. Peak Overshoot (M_p):

Peak Overshoot (M_p) = $\frac{c(t_p) - c(\infty)}{c(\infty)}$

$c(t_p)$=peak response at t = t_p

$c(\infty)$=final steady state response

The unit step response c(t) for a second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

At t =∞, $c(t) = c(\infty) = 1 - \frac{e^{-\xi\omega_n \infty}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) = 1 - 0 = 1$

At t =t_p, $c(t) = c(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin(\omega_d t_p + \theta)$

$$c(t_p) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$= 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

$$= 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\theta)$$

$$= 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} . \sqrt{1-\xi^2} \quad = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

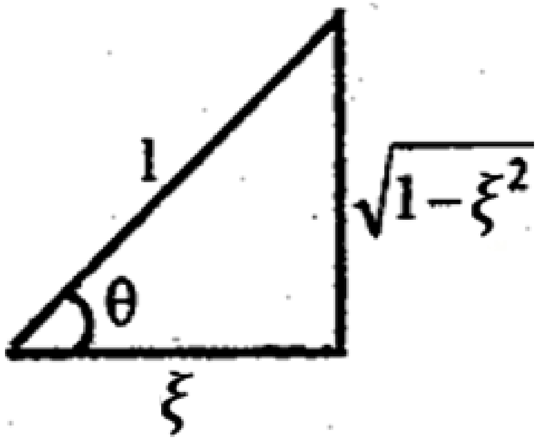
Note:

$$t_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\sin(\pi + \theta) = -\sin\theta$$

$$\sin\theta = \sqrt{1-\xi^2}$$



Peak Overshoot (M_p) = $\dfrac{c(t_p) - c(\infty)}{c(\infty)} = \dfrac{\left[1 + e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}}\right] - 1}{1}$

$$M_p = e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}}$$

$$\% \, M_p = e^{-\frac{\xi \pi}{\sqrt{1-\xi^2}}} \times 100$$

4. Settling Time (t_s):

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

The response of a second order system has 2 components:

i. Decaying exponential component : $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

ii. Sinusoidal component : $\sin(\omega_d t + \theta)$

Settling time is decided by the decaying exponential component.

For 2% tolerance error band

At t = $\boldsymbol{t_s}$, $\boldsymbol{e^{-\xi\omega_n t_s}} = 0.02$

$$-\xi\omega_n t_s = \ln(0.02) = -4$$

$$t_s = \frac{4}{\xi\omega_n}$$

For 5% tolerance error band

At t = $\boldsymbol{t_s}$, $\boldsymbol{e^{-\xi\omega_n t_s}} = 0.05$

$$-\xi\omega_n t_s = \ln(0.05) = -3$$

$$t_s = \frac{3}{\xi\omega_n}$$

Q. The response of a servomechanism is c(t) = $\mathbf{1 + 0.2e^{-60t} - 1.2e^{-10t}}$ when subject to a unit step input. Obtain the expression for closed loop transfer function.

Soln:

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

- Closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{\dfrac{600}{s(s+60)(s+10)}}{R(s)} = \dfrac{\dfrac{600}{s(s+60)(s+10)}}{1/s}$

$$\frac{C(s)}{R(s)} = \frac{600}{(s+60)(s+10)} = \frac{600}{s^2 + 70s + 600}$$

Q. A unity feedback control system has an open loop transfer function $G(s) = \frac{10}{s(s+2)}$. Find the rise time, peak time, peak overshoot & settling time for an error of 2% if the input is a step signal of 12 units. Also sketch the response

Soln :

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{10}{s^2 + 2s + 10}$$

$$\omega_n^2 = 10 \therefore \boldsymbol{\omega_n} = 3.162 \text{ rad/sec}$$

$$2\xi\omega_n = 2$$

$$\therefore \xi = 0.3162$$

$$\theta = tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \text{ (rad)}= 1.249 \text{ rad}$$

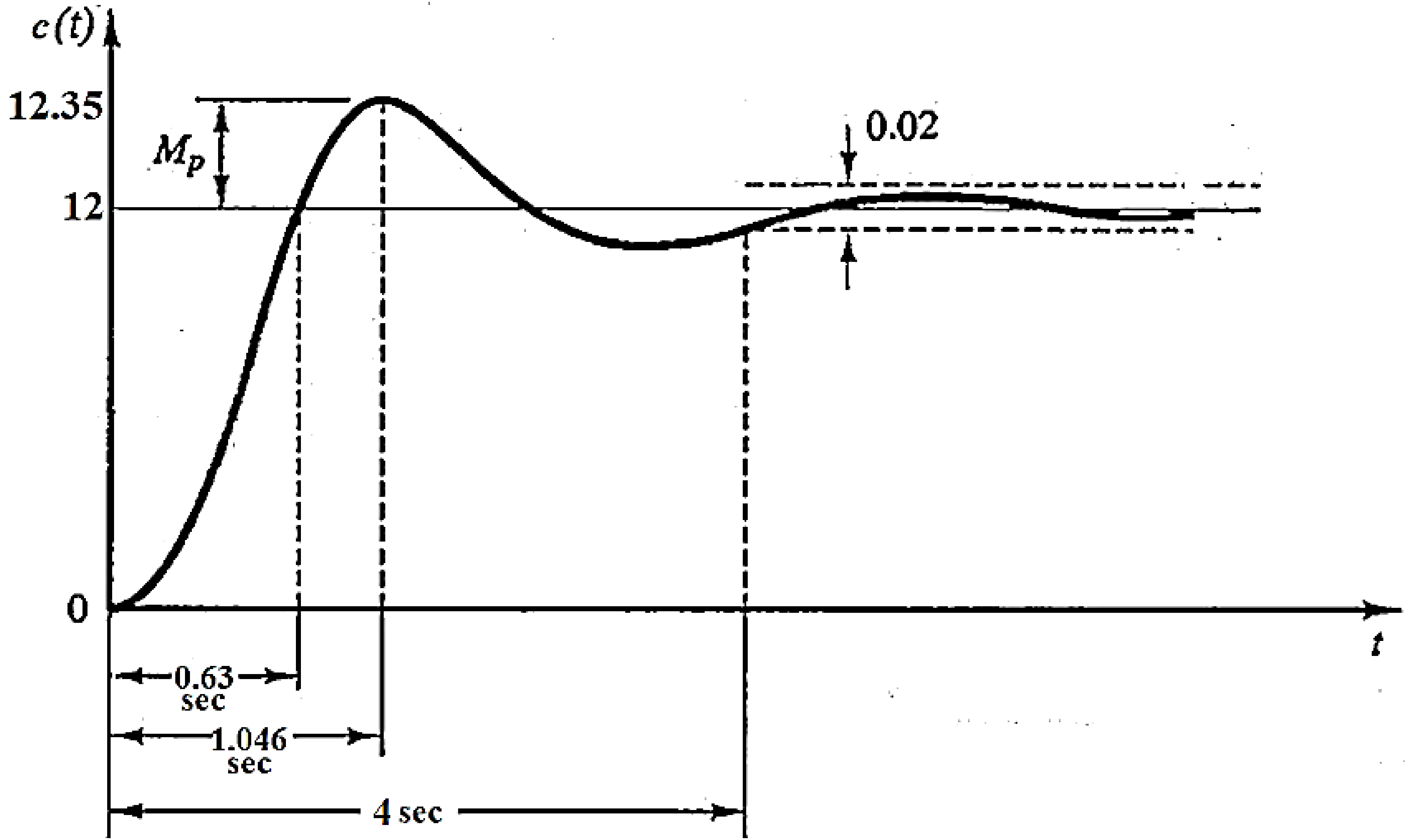
$$\omega_d = \omega_n\sqrt{1 - \xi^2} = 3 \text{ rad/sec}$$

$$\boldsymbol{t_r} = \frac{\pi - \theta}{\omega_d} = .63 \text{ sec}$$

$$\boldsymbol{t_p} = \frac{\pi}{\omega_d} = 1.046 \text{ sec}$$

$$M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 0.35$$

$$\boldsymbol{t_s} = \frac{4}{\xi\omega_n} = 4 \text{ sec}$$



c(t)
12.35
M_p
12
0
0.02
t
0.63 sec
1.046 sec
4 sec

Type Number of a System

- Type is specified for the loop function G(s).H(s)
- The number of poles of the loop transfer function G(s).H(s) at the origin decides the type of system.
- If G(s).H(s) has N poles at the origin , then the type of the system is N.
- The general form of a loop transfer function is

$$G(s).H(s) = K\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s^N(s+p_1)(s+p_2)(s+p_3)\ldots\ldots}$$

$z_1, z_2, z_3 \ldots\ldots \textit{are the zeros}$

$p_1, p_2, p_3 \ldots\ldots \textit{are the poles}$

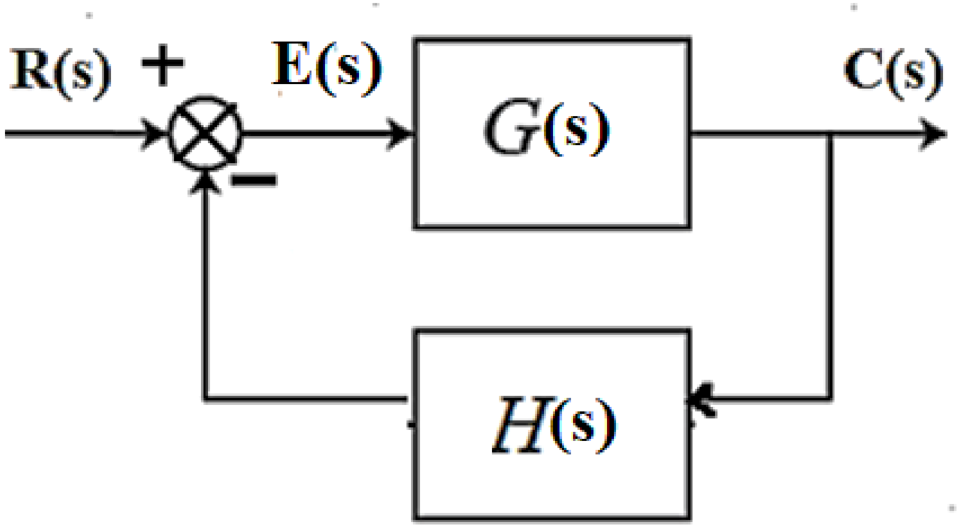
N=Number of poles on the origin=type of system

N=0 ; type 0 system

N=1 ; type 1 system

N=2 ; type 2 system

Steady State Error



- It is the value of error signal as t tends to infinity.
- Errors arise from nature of input, system type and from non linearity of system components
- Steady state performance of a stable control system is judged by its steady state error to step, ramp & parabolic inputs.
- Error signal $\mathrm{E(s)} = R(s) - C(s).H(s) ---(1)$

$$C(s) = \mathrm{E(s).G(s)}$$

Substituting for C(s),

$$\mathrm{E(s)} = R(s) - \mathrm{E(s).G(s)}.H(s)$$

$$\mathrm{E(s)[1 + G(s)}.H(s)] = R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Error signal in time domain

$$\mathrm{e(t)} = L^{-1}\left\{\frac{R(s)}{1 + G(s)H(s)}\right\}$$

- Let e_{ss} be the steady state error (error when t tends to infinity)

$$e_{ss} = \lim_{t\to\infty} e(t)$$

- According to final value theorem

$$L\left\{\lim_{t\to\infty} f(t)\right\} = \lim_{s\to 0} s\,F(s)$$

$$\therefore e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s.E(s) = \lim_{s\to 0} \left\{\frac{s.R(s)}{1 + G(s)H(s)}\right\}$$

Static Error Constants

Positional error constant : $K_p = \lim_{s\to 0} G(s).H(s)$

Velocity error constant : $K_v = \lim_{s\to 0} s.G(s).H(s)$

Acceleration error constant : $K_a = \lim_{s\to 0} s^2.G(s).H(s)$

Steady State Error When Input is Unit Step Signal

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s.R(s)}{1+G(s)H(s)}\right\}$$

For unit step input, $R(s) = \frac{1}{s}$

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s.\frac{1}{s}}{1+G(s)H(s)}\right\} = \lim_{s\to 0}\frac{1}{1+G(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0}[G(s)H(s)]} = \frac{1}{1+K_p}$$

Type 0 System

$$K_p = \lim_{s\to 0} G(s).H(s)$$

$$= \lim_{s\to 0} K.\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{(s+p_1)(s+p_2)(s+p_3)\ldots\ldots}$$

$$K_p = K.\frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = constatnt$$

$$e_{ss} = \frac{1}{1+K_p} = constant$$

Type 1 System

$$K_p = \lim_{s\to 0} G(s).H(s)$$

$$= \lim_{s\to 0} K.\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s.(s+p_1)(s+p_2)(s+p_3)\ldots\ldots}$$

$$= \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0$$

Steady State Error When Input is Unit Ramp Signal

$$e_{ss} = \lim_{s \to 0} \left\{ \frac{s.R(s)}{1 + G(s)H(s)} \right\}$$

For unit ramp input, $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} \left\{ \frac{s.\frac{1}{s^2}}{1 + G(s)H(s)} \right\} = \lim_{s \to 0} \frac{1}{s[1 + G(s)H(s)]}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s + s.G(s)H(s)} = \frac{1}{\lim_{s \to 0} [s.G(s)H(s)]}$$

$$= \frac{1}{K_v}$$

Type 0 System

$$K_v = \lim_{s \to 0} s.G(s).H(s)$$

$$= \lim_{s \to 0} s.K.\frac{(s + z_1)(s + z_2)(s + z_3)..}{(s + p_1)(s + p_2)(s + p_3)..}$$

$$= 0$$

$$e_{ss} = \frac{1}{K_v} = \infty$$

Type 1 System

$$K_v = \lim_{s \to 0} s.G(s).H(s)$$

$$= \lim_{s \to 0} s.K \frac{(s + z_1)(s + z_2)(s + z_3)..}{s.(s + p_1)(s + p_2)(s + p_3)..}$$

$$= K \frac{z_1.z_2.z_3 \ldots \ldots.}{p_1.p_2.p_3 \ldots ..} = constatnt$$

$$e_{ss} = \frac{1}{K_v} = constant$$

Type 2 System

$$K_v = \lim_{s \to 0} s.G(s).H(s)$$

$$= \lim_{s \to 0} s.K.\frac{(s+z_1)(s+z_2)(s+z_3)..}{s^2.(s+p_1)(s+p_2)(s+p_3)..}$$

$$= \infty$$

$$e_{ss} = \frac{1}{K_v} = 0$$

In systems with type number 2 or greater, for a unit ramp input , the value of K_v is infinity , hence steady state error becomes 0.

Steady State Error When Input is Unit Parabolic Signal

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s.R(s)}{1+G(s)H(s)}\right\}$$

For unit parabolic input, $R(s) = \frac{\mathbf{1}}{\boldsymbol{s^3}}$

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s.\frac{1}{s^3}}{1+G(s)H(s)}\right\}$$

$$= \lim_{s\to 0}\ \frac{1}{s^2[1+G(s)H(s)]}$$

$$e_{ss} = \lim_{s\to 0}\ \frac{1}{s^2+s^2.G(s)H(s)} = \frac{1}{\lim_{s\to 0}\ [s^2.G(s)H(s)]}$$

$$= \frac{1}{K_a}$$

Type 0 System

$$K_a = \lim_{s\to 0} s^2.G(s).H(s)$$

$$= \lim_{s\to 0} s^2.K.\frac{(s+z_1)(s+z_2)(s+z_3)..}{(s+p_1)(s+p_2)(s+p_3)..}$$

$$= 0$$

$$e_{ss} = \frac{1}{K_a} = \infty$$

Type 1 System

$$K_a = \underset{s \to 0}{\mathrm{lt}}\, s^2.G(s).H(s)$$

$$= \underset{s \to 0}{\mathrm{lt}}\, s^2.K\frac{(s+z_1)(s+z_2)(s+z_3)..}{s.(s+p_1)(s+p_2)(s+p_3)..} = 0$$

$$e_{ss} = \frac{1}{K_a} = \infty$$

Type 2 System

$$K_a = \underset{s \to 0}{\mathrm{lt}}\, s^2.G(s).H(s)$$

$$= \underset{s \to 0}{\mathrm{lt}}\, s^2.K\frac{(s+z_1)(s+z_2)(s+z_3)..}{s^2.(s+p_1)(s+p_2)(s+p_3)..}$$

$$K_a = K\frac{z_1.z_2.z_3 \dots \dots}{p_1.p_2.p_3 \dots ..} = constatnt$$

$$e_{ss} = \frac{1}{K_a} = constant$$

Type 3 System

$$K_a = \underset{s \to 0}{\mathrm{lt}}\, s^2.G(s).H(s)$$

$$= \underset{s \to 0}{\mathrm{lt}}\, s^2.K\frac{(s+z_1)(s+z_2)(s+z_3)..}{s^3.(s+p_1)(s+p_2)(s+p_3)..}$$

$$= \infty$$

$$e_{ss} = \frac{1}{K_a} = 0$$

In systems with type number 3 or greater, for a unit parabolic input , the value of K_a is infinity , hence steady state error becomes 0.

Static Error Constants for Various type number of systems

Error Constant	Type number of system			
	0	1	2	3
K_P	constant	∞	∞	∞
K_V	0	constant	∞	∞
K_a	0	0	constant	∞

Steady State Error for Various types of inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Q. Determine the position , velocity and acceleration error constants for a unity feedback control system whose open loop transfer function is $\frac{\mathbf{10\,(s+2)}}{\mathbf{s^2(s+1)}}$

Soln:

$$G(s) = \frac{10\,(s+2)}{s^2(s+1)}$$

$$H(s) = 1$$

Positional error constant : $K_p = \underset{s\to 0}{\mathrm{lt}}\, G(s).H(s)$

$$K_p = \underset{s\to 0}{\mathrm{lt}}\ \frac{10\,(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant : $K_v = \underset{s\to 0}{\mathrm{lt}}\, s.G(s).H(s)$

$$K_v = \underset{s\to 0}{\mathrm{lt}}\, s.\frac{10\,(s+2)}{s^2(s+1)} = \infty$$

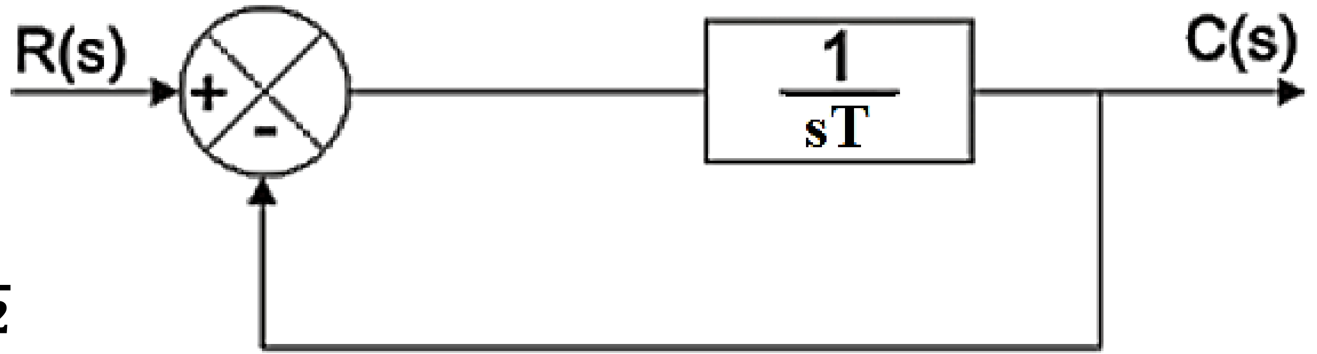
Acceleration error constant :

$$K_a = \underset{s\to 0}{\mathrm{lt}}\, s^2.G(s).H(s)$$

$$= \underset{s\to 0}{\mathrm{lt}}\, s^2\frac{10\,(s+2)}{s^2(s+1)} = 20$$

Q. Obtain the time response of a first order system to a ramp input and find the steady state error.

Soln:



$$e_{ss} = \underset{s\to 0}{\mathrm{lt}} \left\{ \frac{s.R(s)}{1 + G(s)H(s)} \right\}$$

For unit ramp input, $R(s) = \frac{\mathbf{1}}{\boldsymbol{s}^{\mathbf{2}}}$

For a first order system, $G(s) = \frac{\mathbf{1}}{\boldsymbol{sT}}$

$H(s) = 1$

$$e_{ss} = \underset{s\to 0}{\mathrm{lt}} \left\{ \frac{s.\frac{1}{s^2}}{1 + \frac{1}{sT}} \right\} = \underset{s\to 0}{\mathrm{lt}} \left\{ \frac{T}{1 + sT} \right\} = T$$

Generalized (Dynamic) Error Co-efficients

- Static error constants does not show variation of error with time and input should be standard.
- Generalized error constants gives steady state error as a function of time for any input.

$$E(s) = \frac{R(s)}{1 + G(s)H(s)} = R(s).\frac{1}{1 + G(s)H(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} = \frac{1}{k_1} + \frac{1}{k_2}.s + \frac{1}{k_3}.s^2 + \frac{1}{k_4}.s^3 + \cdots \ldots.$$

$$E(s) = \frac{1}{k_1}R(s) + \frac{1}{k_2}.sR(s) + \frac{1}{k_3}.s^2R(s) + \frac{1}{k_4}.s^3R(s) + \cdots \ldots.$$

- $k_1, k_2, k_3, \ldots..-$ generalized (dynamic) error co-efficients

$$e(t) = \frac{1}{k_1}r(t) + \frac{1}{k_2}r'(t) + \frac{1}{k_3}r''(t) + \frac{1}{k_4}r'''(t) + \cdots \ldots..$$

- Steady state error $e_{ss} = \underset{t\to\infty}{\mathrm{lt}}\, e(t)$

Q. Find the dynamic error constants and steady state error for a unity feedback control system if G(s) =$\frac{\mathbf{10}}{\boldsymbol{s}\,(\boldsymbol{s}+\mathbf{1})}$. Given, the input to the system is R(s)=$\frac{\mathbf{1}}{\boldsymbol{s}} + \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{2}}} + \frac{\mathbf{6}}{\boldsymbol{s}^{\mathbf{3}}}$

Soln:

$$r(t) = 1 + 2t + 3t^2$$

$$r'(t) = 2 + 6t$$

$$r''(t) = 6$$

$$r'''(t) = 0$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} = \frac{\boldsymbol{s}^{\mathbf{2}} + s}{\boldsymbol{s}^{\mathbf{2}} + s + 10}$$

$$= 0.1s + 0.09s^2 - 0.019s^3 + \ldots$$

$$E(s) = 0.1sR(s) + 0.09s^2R(s) - 0.019s^3R(s)\ldots$$

$$e(t) = 0.1r'(t) + 0.09\, r''(t) - 0.019\, r'''(t) + \cdots$$

Generalized (Dynamic) Error Co-efficients

- Static error constants does not show variation of error with time and input should be standard.
- Generalized error constants gives steady state error as a function of time for any input.

$$E(s) = \frac{R(s)}{1 + G(s)H(s)} = R(s).\frac{1}{1 + G(s)H(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)} = \frac{1}{k_1} + \frac{1}{k_2}.s + \frac{1}{k_3}.s^2 + \frac{1}{k_4}.s^3 + \cdots \ldots.$$

$$E(s) = \frac{1}{k_1}R(s) + \frac{1}{k_2}.sR(s) + \frac{1}{k_3}.s^2R(s) + \frac{1}{k_4}.s^3R(s) + \cdots \ldots.$$

- $k_1, k_2, k_3, \ldots.. -$ generalized (dynamic) error co-efficients

$$e(t) = \frac{1}{k_1}r(t) + \frac{1}{k_2}r'(t) + \frac{1}{k_3}r''(t) + \frac{1}{k_4}r'''(t) + \cdots \ldots..$$

$$e(t) = 0.1r'(t) + 0.09\, r''(t) - 0.019\, r'''(t) + \cdots$$

$$= 0.1\,[2 + 6t] + 0.09\,[6]$$

$$= 0.74 + 0.6t$$

Steady state error $e_{ss} = \underset{t\to\infty}{\mathrm{lt}}\, e(t)$

$$e_{ss} = \underset{t\to\infty}{\mathrm{lt}}\, e(t) = \underset{t\to\infty}{\mathrm{lt}}[0.74 + 0.6t\ ] = \infty$$

$k_1, k_2, k_3, \ldots\ldots -$ generalized (dynamic) error co-efficients

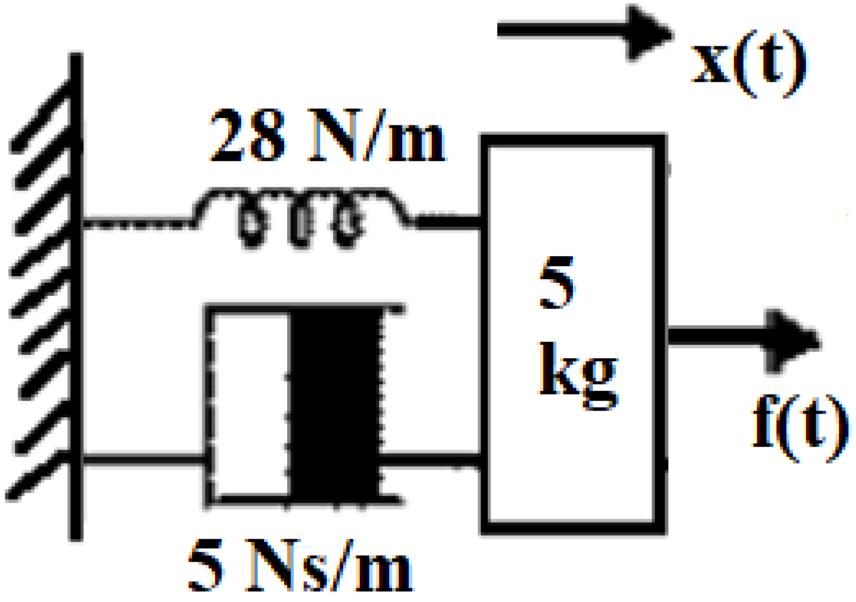
$$\frac{1}{k_1} = 0 \quad ; k_1 = \infty$$

$$\frac{1}{k_2} = 0.1 \quad ; k_2 = 10$$

$$\frac{1}{k_3} = 0.09 \quad ; k_3 = 11.1$$

Q. For the system shown find the peak time & % peak overshoot

Soln:



$$F(t) - F_M(t) - F_B(t) - F_K(t) = 0$$

$$F(t) = M\frac{d^2x(t)}{dt^2} + B.\frac{dx(t)}{dt} + Kx(t)$$

$$F(t) = 5\frac{d^2x(t)}{dt^2} + 5.\frac{dx(t)}{dt} + 28.x(t)$$

$$F(s) = 5s^2X(s) + 5.sX(s) + 28.X(s)$$

$$F(s) = X(s)[5s^2 + 5s + 28]$$

$$\frac{X(s)}{F(s)} = \frac{1}{5s^2 + 5s + 28} = \frac{1}{28}.\frac{28/5}{(s^2 + s + \frac{28}{5})}$$

$\omega_n^2 = 28/5 \therefore \boldsymbol{\omega_n} = 2.367 \text{ rad/sec}$

$2\xi\omega_n = 1$

$\therefore\ \xi = 0.211$

$$\boldsymbol{t_p} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\%M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100 =$$