ATM651 - Homework 1

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1 Flux questions

We saw that the flux of any intensive "stuff" has units of $\frac{\text{stuff}}{m^2 s}$. Using this definition (essentially) of stuff:

a) What is another name for volume flux? Construct the units and say what the resulting units imply:

$$\frac{V}{m^2 s} = \frac{m^3}{m^2 s} = \frac{m}{s} \Rightarrow \text{velocity}$$

b) What familiar quantity's flux can be boiled down to units of kgs^{-3}?

$$\frac{\text{stuff}}{m^2 s} \Rightarrow \frac{kg}{s^3}$$

$$\frac{1}{m^2 s} \cdot \frac{kgm^2}{s^2} = \frac{kg}{s^3}$$

$$\frac{kgm^2}{s^2} = \text{energy}$$

c) What are the units of $\rho\vec{V}$? What is it a flux of?

$$\rho\vec{V} = \frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg}{m^2 \cdot s} \Rightarrow \text{mass flux}$$

d) Specific humidity (q) has units of $\frac{kg_{\text{water}}}{kg_{\text{air}}}$. What is $q\rho\vec{V}$? A flux of what?

$$q\rho\vec{V} = \frac{kg_{\text{water}}}{kg_{\text{air}}} \cdot \frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg_{\text{water}}}{kg_{\text{air}}} \frac{kg}{m^2 \cdot s} \Rightarrow \text{(specific humidity mass) flux}$$

$$q\rho\vec{V} \Rightarrow \text{water vapor (mass) flux}$$

e) Atmospheric rivers are defined by vertically integrated "IWT" (integrated water transport), units ($kg_{\text{water}} m^{-1} s^{-1}$). What is the "m" in the denominator? Meters in what direction?

Integration is ~~of the mass~~ in the vertical direction ~~over the width of the atmospheric river~~ $\Rightarrow$ meters is in the horizontal direction, perpendicular to the flow.

$$IWT(x,y,t) = IWT = \int_0^\infty (q\rho V)\, dz$$

f) Flux convergence is written $(-\vec{\nabla}\cdot\vec{F})$ for a flux field $\vec{F}(x,y,z,t)$. Expand this intro xyz components. What are the units of $-\vec{\nabla}\cdot(q\rho\vec{V})$? What does this mean?

Writing $\vec{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3$

and $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

then

$$-\vec{\nabla}\cdot\vec{F} = -\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \cancel{+\frac{dF}{dt}}$$

Straight forward "dot product"

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

all others zero

$$\frac{1}{m}\cdot\frac{kg}{m^2 s} \Rightarrow \frac{kg}{m^3 s}$$

$\Rightarrow$ the rate of change of water vapor mass per unit volume per unit time due to convergence.

2 Application of flux reasoning and Continuity

a) River 1 has a total mass flux integrated over plane A, of $10^5 kg/s$. What is its velocity?

$$10^5 kgs^{-1} = \int \frac{kg}{m^2 s} dA$$

$$\text{mass flux} = \frac{10^5 kg/s}{10m^2} = 10^4 \frac{kg}{m^2 s}$$

$$\text{velocity} = \frac{flux}{density} = \frac{10^4 \frac{kg}{m^2 s}}{10^3 \frac{kg}{m^3}} = 10ms^{-1}$$

✓

b) Outflow river speeds are $1ms^{-1}$. What is their IWF, ~~and~~ the total flux out of the lake?

I meant to say "IWT" from page 1, but failed, so

$$\text{IWF} = 1\frac{m}{s}\cdot 10^3\frac{kg}{m^3}\cdot 0.5m^2 = 500\frac{kg}{s}$$

("total, net, integrated")

all the words for all the sums

$$\text{IWF}_{\text{total}} = 2\cdot 500\frac{kg}{s} = 10^3\frac{kg}{s}$$

✓

c) What is the vertical water flux at the lakes surface? If its area is $1km^2$, what is the vertical velocity, w, of the surface?

$$10^5 kg/s - 10^3 kg/s \approx 10^5 kg/s$$

$$\frac{10^5 kg/s}{10^6 m^2} = 10^{-1}\frac{kg}{m^2 s}$$

$$w = \frac{10^{-1}\frac{kg}{m^2 s}}{10^3\frac{kg}{m^3}} = 10^{-4}ms^{-1}$$

$$\boxed{w = 10^{-4} m\cancel{s^{-1}}}$$ ✓ $\times \frac{86400\cancel{s}}{1 day} = 8.64 \frac{mm}{day}$

✓

d) A mine leak causes $q = \frac{1 g_{\text{salt}}}{1 kg_{\text{water}}}$ to be the salt concentration of River 1. What is $\frac{dq}{dt}$ averaged over the lake volume, if its depth is $10m$ and no salt initially reaches rivers 2 and 3?

$$\text{total mass flux} = \frac{dm}{dt} = 10^5 \frac{kg}{s} \qquad q = \frac{1 g_s}{1 kg_w} \qquad \text{lake volume} = 10^7 m^3$$

$$\frac{dm_{salt}}{dt} = q\frac{dm}{dt} = 1\frac{g_s}{kg_w} \cdot 10^5 \frac{kg}{s} = 10^5 \frac{g}{s} \text{ salt into the lake}$$

divide by the mass of water (itself slightly rising with time!)

$$\left(\frac{dq}{dt}\right)_{\text{lake}} = \frac{q}{\rho V}\frac{dm}{dt} = \frac{10^5 \frac{g}{s}}{10^3 \frac{kg}{m^3} 10^7 m^3} = 10^{-5} \frac{g}{kg \cdot s}$$

$$\boxed{\frac{dq}{dt} = 10^{-5} \frac{g}{kg \cdot s}}$$

correct to 5 decimal places!

$$\left(\frac{10^{+5} g \text{ salt after } 1s}{10^{10} + (10^5 - 10^3) \text{ water after } 1s}\right)$$

tiny, ha ha, negligible compared to 10^{10}

3 Leaky Bucket problem

A faucet dumps $1kg/s$ into a bucket of area $1m^2$, with a 1cm^2 hole in the side. The outflow at the leak is proportional to pressure, which is the weight of water above the leak,

$$v_{\text{leak}} = C \cdot \rho g \Delta H$$

a) Write an equation for ΔH. Does it have a steady-state solution? How deep?

$$\left(\frac{dm}{dt}\right)_{\text{tot}} = \left(\frac{dm}{dt}\right)_{\text{in}} - \left(\frac{dm}{dt}\right)_{\text{out}} = \left(\frac{dm}{dt}\right)_{\text{in}} - \rho \int v_{\text{leak}} dA_{\text{leak}}$$

$$\rho \int v_{\text{leak}} dA_{\text{leak}} = \rho (C \cdot \rho g \Delta H) A_{\text{leak}}$$

$$\left(\frac{dm}{dt}\right)_{\text{tot}} = \left(\frac{dm}{dt}\right)_{\text{in}} - C \cdot \rho^2 g \Delta H A_{\text{leak}}$$

$$\boxed{\Delta H = -\frac{1}{C \cdot \rho^2 g A_{\text{leak}}} \left[\left(\frac{dm}{dt}\right)_{\text{tot}} - \left(\frac{dm}{dt}\right)_{\text{in}}\right]}$$

$$\text{Steady-state} \Rightarrow \left(\frac{dm}{dt}\right)_{\text{tot}} = 0 \implies \left(\frac{dm}{dt}\right)_{\text{in}} = \left(\frac{dm}{dt}\right)_{\text{out}}$$

$$\Delta H = -\frac{1}{C \cdot \rho^2 g A_{\text{leak}}} \left[\cancelto{0}{\left(\frac{dm}{dt}\right)_{\text{tot}}} - \left(\frac{dm}{dt}\right)_{\text{in}}\right]$$

$$\Delta H_{\text{steady}} = \frac{1}{C \cdot \rho^2 g A_{\text{leak}}} \left(\frac{dm}{dt}\right)_{\text{in}} = \frac{1}{C \cdot (10^3)^2 (10)(10^{-2})^2}(1) = 10^{-3} C^{-1} m \text{ ?}$$

$$\boxed{\Delta H_{\text{steady}} = 10^{-3} C^{-1} m} \text{ ?}$$

$\left(\frac{\text{Spigot Source}}{C}\right)$ is key dependency

b) What is $\frac{d}{dC}(\Delta H_{\text{steady}})$? How much does depth H change if the leak area is reduced by half?

$$\frac{d}{dC}(\Delta H_{\text{steady}}) = \frac{d}{dC}\left[\frac{1}{C \cdot \rho^2 g A_{\text{leak}}}\left(\frac{dm}{dt}\right)_{\text{in}}\right] = -\frac{1}{C^2 \cdot \rho^2 g A_{\text{leak}}}\left(\frac{dm}{dt}\right)_{\text{in}}$$

✓

$$\boxed{\frac{d}{dC}(\Delta H_{\text{steady}}) = -10^{-3} C^{-2}\frac{m^2 s^2}{kg}}$$

ΔH is halved if

$$A_{\text{leak}} \rightarrow \frac{1}{2}A_{\text{leak}} \text{ or } C \rightarrow \frac{1}{2}C$$

✓

$$\boxed{\Delta H_{\text{steady}} = \frac{1}{C \cdot \rho^2 g\left(\frac{1}{2}A_{\text{leak}}\right)}\left(\frac{dm}{dt}\right)_{\text{in}} = \frac{1}{\left(\frac{1}{2}C\right) \cdot \rho^2 g A_{\text{leak}}}\left(\frac{dm}{dt}\right)_{\text{in}} = 2 * 10^{-3} C^{-1} m}$$

c) How does H change if the spigot input is halved? Is this the same or different from b? Explain why.

If the spigot output is halved, the height in the bucket will be reduced by half. This is because a lower pressure (lower water height) is required to balance the outflow with the inflow. ✓

d) Is this a stable system? Discuss, explain. Is there some dependence of outflow on H that would make it interestingly more or less stable/unstable?

This is stable system due to the linear dependence between the height, outflow and inflow. If there was some kind of non-linear component to our coefficient C, we could end up with a height that oscillates around the steady-state solution, or collapses/explodes causing the bucket to overflow or drain completely (theoretically speaking). ✓

some very special kind! great.

e) What is the net convergence of water flux for a cube of space submerged in the water?

The flux of water into a theoretical cube would be equal to the flux of water out of the cube (water is essentially incompressible, especially in this regime), so the convergence is zero.

✓ trick passed w/ flying colors!

d. $$\frac{d\,\Delta H}{dt} = \text{Inflow} - \text{Outflow} = \dot{V}_{in} - aC\sqrt{2g\,\Delta H}$$

The system is stable because the outflow increases with increasing ΔH creating negative feedback loop that brings water level back to equilibrium when perturbed. Unless the inflow is changed.

Good! clear

Outflow depends on H. Since $H = \Delta H + H_{leak}$ and according to the equation above, it's proportional to $\sqrt{\Delta H}$. ~~So,~~ That makes if ΔH increase, the pressure at leak increase, and leads to faster outflow according to Toricelli's law.

e. net convergence of water flux for submersed cubes of space in the water?

Assume the water flux $\vec{F}$, from the continuity equation fluid like water (incompressible) that $\nabla \cdot \vec{F} = 0$.

If we apply divergence theorem, which:

$$\oint_{\partial V} \vec{F} \cdot dA = \int_V (\nabla \cdot \vec{F})\,dV$$ nice!

∂V = surface of the cube

V = volume of the cube.

$$\int_V (\nabla \cdot \vec{F})\,dV = 0$$ nice!

and net convergence flux is given by the flux through the surface of the cube

$$\oint \vec{F} \cdot dA = 0$$ nice!

that implies the total flux entering the cube equal to total flux leaving the cube.

conceptually obvious, mathematically valuable to write out so well!