Conservation equations: we really really love ‘em!

- For momentum: lots of massaging to beat RHS terms down

- For ‘warmth’: have to account for compression & expansion (adiabatic Temperature warming/cooling in the absence of actual diabatic or latent *heating*)

"You know, Sid, I really like bananas... I mean, I know that's not profound or nothin'... Heck! We ALL do... But for me, I think it goes much more beyond that."

We really really like $\frac{d}{dt}(?) = 0$ + complications (or $\frac{\partial}{\partial t}(\,) = 0$)

- **Diagnostic eqs** (like hydrostatic)
  - balance
  - equilibrium
  - steady state

$$\frac{dw}{dt} = g - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$ (almost 99.9%)

- **Prognostic eqs**

horizontal wind $\vec{V}$

$$\frac{d\vec{V}}{dt} = P + C + F$$

P: pressure (PGF); C: coriolis; F: friction

PGF, coriolis

$$\frac{d\vec{V}}{dt} = -\vec{\nabla}\Phi - f\hat{k}\times\vec{V} + F$$

ignore

1 tactic: take curl $\boxed{\vec{\nabla}\times\vec{\nabla}\Phi = 0}$

$$\frac{d\zeta}{dt} = 0!!! + (\zeta + f)\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\nabla\cdot V} + v\frac{\partial f}{\partial y}$$

- Relative vort cons.:

$$\frac{\partial \zeta}{\partial t} = -(\vec{V}\cdot\vec{\nabla})\zeta$$ advection only!

- Absolute vort. cons.

$$\frac{d(\zeta + f)}{dt} = 0!!! \quad -(\zeta + f)(\nabla\cdot V)$$

$$\zeta_{abs} = (\zeta_{rel} + f)$$

Advection plus Rossby waves

Prognostic equation for "heat" or "warmth"

$$\frac{d}{dt}(\text{warmth}) = \underset{\text{adiabatic}}{0!!!\ldots} + \underset{\text{latent heating}}{L(c-e)} + Q_R + Q_{\substack{\text{conduction,}\\ \text{mixing}}}$$

2 tactics here

1. potential temperatures (log-related to <u>entropy</u>)

$$\theta = T\underbrace{\left(\frac{P_0}{P}\right)^{R/c_p}}_{\left(\frac{\theta}{T}\right)}$$

as warmth conserved but not for condensation, rad, mixing...

2. static energies

$$s = \underbrace{C_pT}_{\text{heat energy}} + \underbrace{gz}_{\text{height energy}}$$

"dry" static energy

$$h = s + Lq \simeq \underbrace{C_pT}_{\text{heat}} + \underbrace{gz}_{\text{height}} + \underbrace{Lq_v}_{\text{latent}}$$

mixed layer $\frac{\partial s}{\partial z} = 0$

$$\boxed{C_p\frac{\partial T}{\partial z} + g = 0}$$

Example: dry adiabatic lapse rate.

unsaturated parcel converts gz to C_pT

$$\frac{ds}{dt} = 0 \Rightarrow \frac{dT}{dt} = \frac{g}{c_p}\underbrace{\frac{dz}{dt}}_{w}$$

Momentum: not conserved enough

$$\frac{D}{Dt}\vec{V}_h = -f\hat{k}\times\vec{V}_h - \vec{\nabla}_p\Phi$$

- To predict vector momentum $\mathbf{V_h}$, need Φ
- But that drags thermo into our equation set
    - must *predict* T for thickness for heights Z

- Let’s avoid that with *vorticity*

Holy grail of dynamics: get div & ω

$$\frac{D}{Dt}\vec{V}_h = -f\hat{k} \times \vec{V}_h - \vec{\nabla}_p \Phi$$

Gotta avoid dragging thermo into this via Φ.

Get rid of Φ at any cost. **Curl to the rescue!**

$$\nabla \times \left(\frac{D}{Dt}\vec{V}_h\right) = \nabla \times \left(-f\hat{k} \times \vec{V}_h\right) - \cancel{\nabla \times \left(\vec{\nabla}_p \Phi\right)}$$

Ker-CHING!

We are Masters of the Universe with our sexy vector identities!

The grail is in the bag!

Heh heh ... did I say "any cost"... ? gulp

$$\frac{\partial}{\partial x}[\text{y-component momentum equation}] - \frac{\partial}{\partial y}[\text{x-component momentum equation}] =$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial t} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} + w\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial x} + f\frac{\partial u}{\partial x} + u\cancel{\frac{\partial f}{\partial x}} = -\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}} + \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right)$$

$$- \quad \frac{\partial}{\partial y}\frac{\partial u}{\partial t} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + w\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y} = -\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}} + \frac{1}{\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} + \boxed{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} + \boxed{v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} + \boxed{w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} + \boxed{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \boxed{f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

$$+ \boxed{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)} + \boxed{v\frac{\partial f}{\partial y}} = \boxed{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

$$\boxed{\frac{df}{dt} = \cancel{\frac{\partial f}{\partial t}} + u\cancel{\frac{\partial f}{\partial x}} + v\frac{\partial f}{\partial y} + w\cancel{\frac{\partial f}{\partial z}}}$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\boxed{\frac{d}{dt}(\zeta + f) = -(\zeta + f)\boxed{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

vorticity equation

Wait a sec, what's this??

Can we scrape back some of these cobwebs?

Simplified Vorticity Equation

Thus the vorticity equation can be simplified to

$$\frac{d}{dt}(\zeta + f) \cong -(\zeta + f)\nabla_p \cdot V$$

The rate of change of absolute vorticity of particular portions of fluid is equal to minus the absolute vorticity multiplied by the divergence.

$$\frac{\partial \zeta}{\partial t} = -\vec{V} \bullet \nabla(\zeta + f) - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Advection of absolute vorticity

Divergence term

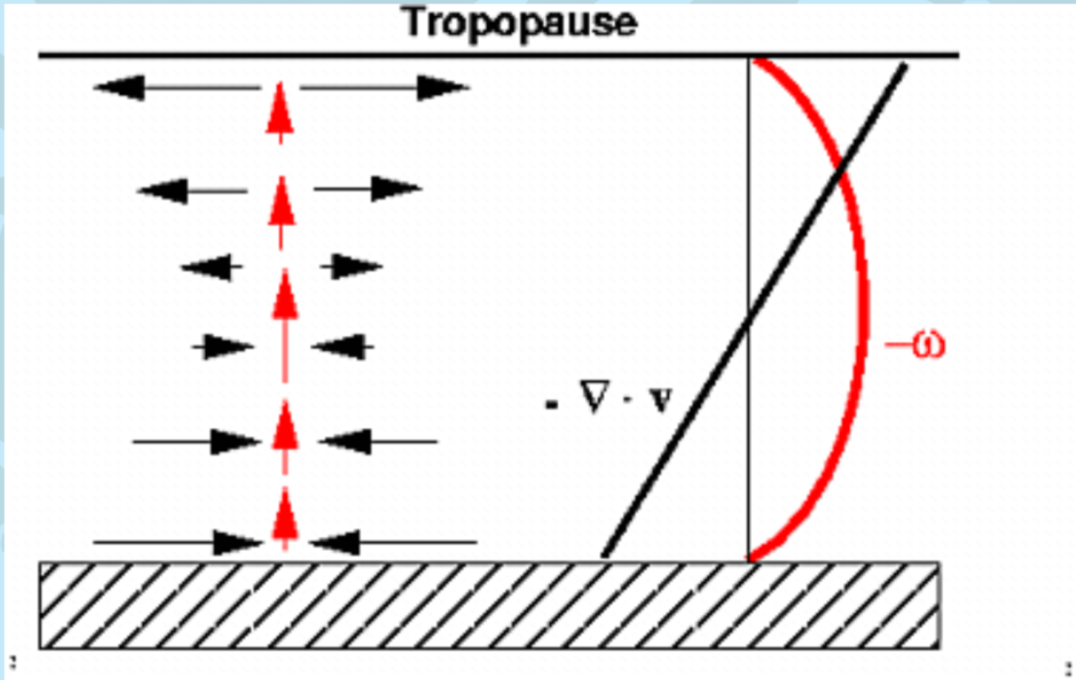
←Same equation in Eulerian (*local* rate of change with time) form.

Lackmann Eq. 1.51

Absolute Vorticity is Conserved

almost,

at midlevels especially



If we look back to our simplified model of upper-level and lower-level divergence, at mid levels the divergence must approach zero. At that level, absolute vorticity is conserved.

$$\frac{d}{dt}(\zeta + f) \cong 0 \Rightarrow \zeta + f \cong \mathit{Constant}$$

Ignoring the divergence term... for now...

- We will return to it when we define "potential vorticity" ...

$$\boxed{\frac{D\zeta_{abs}}{Dt} \simeq 0}$$

Same eqn.
←(BVE)→

$$\boxed{\frac{\partial\zeta_{abs}}{\partial t} \simeq -\vec{V} \cdot \vec{\nabla}_p \zeta_{abs}}$$

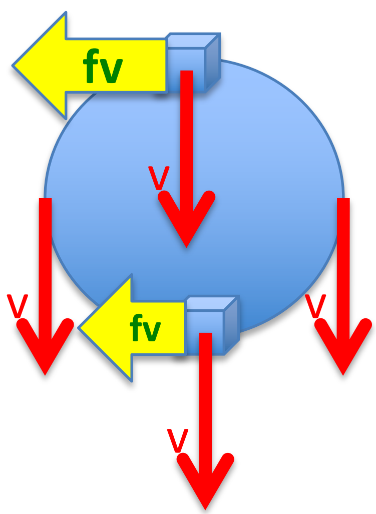
Conservation of absolute vorticity: how?

When air moves north or south, it tends to *conserve* its absolute vorticity -- or in other words it *converts planetary to relative vorticity*. How?

- Consider equatorward motion of a disc of fluid.
- The initially high f = ζ_{plan} is converted to ζ_{rel}. How?

- Physically, the Coriolis **force** *fv* is a little bit stronger on its poleward edge (where f is bigger) than on its equatorward edge. This imparts a cyclonic *torque* on the disk of fluid as it moves southward.

All v winds the same, as the disk moves southward as a unit →



(an eastward PGF must balance the *mean* Coriolis force, to keep the air flowing southward)

Relative, planetary, and absolute vorticity

- absolute = (relative + planetary): $\zeta_{abs} = \zeta_{rel} + \zeta_{plan}$
- planetary vorticity $\zeta_{plan} = f$ (aka the "Coriolis parameter")
- $f = 2\Omega \cos(\lambda\alpha\tau\iota\tau\upsilon\delta\varepsilon)$
- Ω = (2π radians)/(1 day) since the Earth turns once a day
- At 40N, $f = 10^{-4}\ s^{-1}$

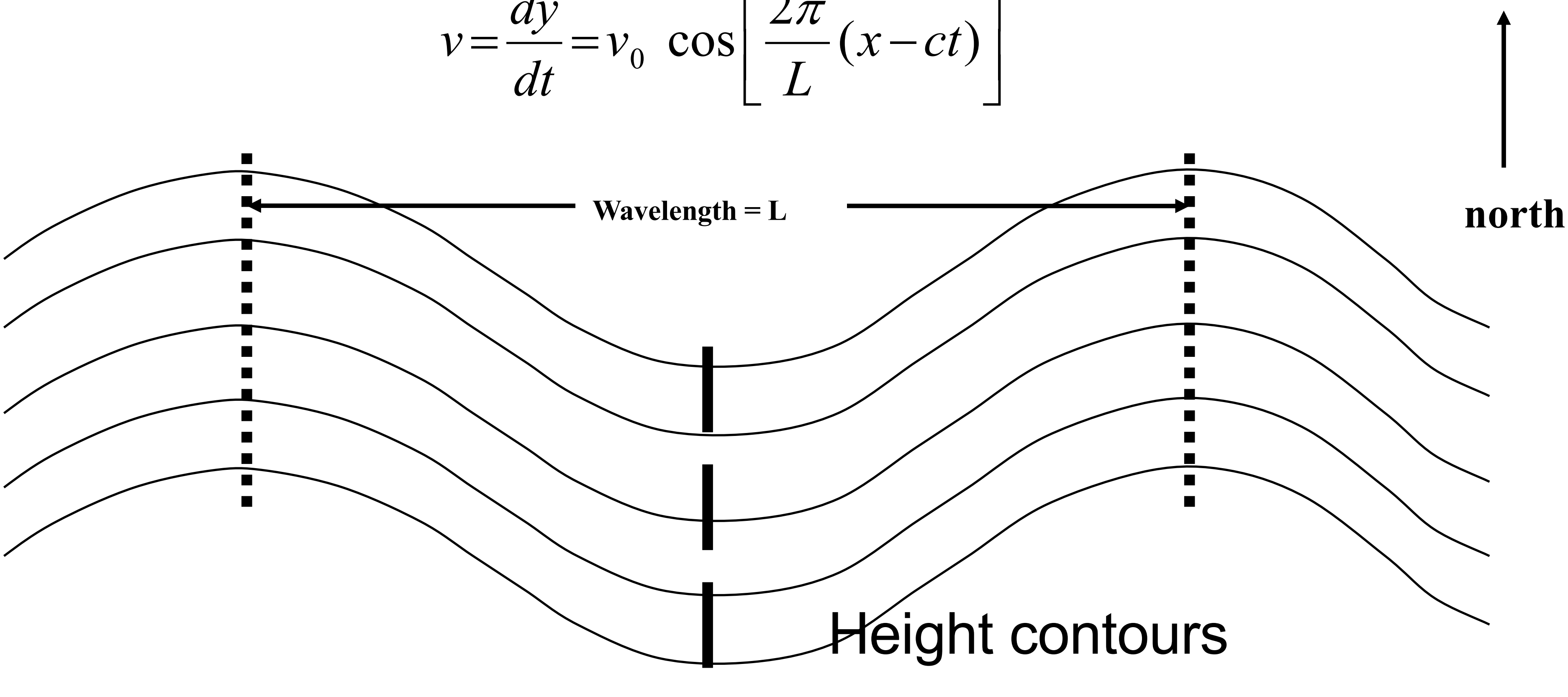
Rossby waves: derive their speed c

Consider a simple westerly flow (constant value of U)

Sinusoidal wave pattern in v is superimposed

Wave assumed to propagate eastward at *phase speed* c

$$v = \frac{dy}{dt} = v_0 \cos\left[\frac{2\pi}{L}(x - ct)\right]$$



Compute the vorticity for this flow… only v varies, and only in x

$$v = \frac{dy}{dt} = v_0 \cos\left[\frac{2\pi}{L}(x-ct)\right]$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}; \qquad -\frac{\partial u}{\partial y} = 0 \quad and \quad \frac{\partial v}{\partial x} = -\frac{2\pi}{L} v_0 \sin\left[\frac{2\pi}{L}(x-ct)\right]$$

Now, take time derivative of the vorticity….

$$\frac{\mathrm{d}(\zeta + \mathrm{f})}{\mathrm{dt}} = \frac{\mathrm{df}}{\mathrm{dt}} - \left(\frac{2\pi}{\mathrm{L}}\right)^2 \left\{\mathrm{v}_0 \cos\left[\frac{2\pi}{\mathrm{L}}(\mathrm{x}-\mathrm{ct})\right]\right\} \left(\frac{\mathrm{dx}}{\mathrm{dt}} - \mathrm{c}\frac{\mathrm{dt}}{\mathrm{dt}}\right) = 0$$

Note that $\dfrac{df}{dt} = v\dfrac{\partial f}{\partial y} = v\beta$

$$\frac{d(\zeta + f)}{dt} = \beta\, v_0 \cancel{\cos\left[\frac{2\pi}{L}(x-ct)\right]} - \left(\frac{2\pi}{L}\right)^2 \left[v_0 \cancel{\cos\left[\frac{2\pi}{L}(x-ct)\right]}\right](U-c) = 0$$

Cancel common terms, and we have

$$\frac{\beta L^2}{4\pi^2} - (U-c) = 0,$$

Rearrange to solve for c…

$$\boxed{c = U - \frac{\beta L^2}{4\pi^2}}$$

Rossby Waves: phase vs. group velocity

$$c = U - \frac{\beta L^2}{4\pi^2}$$

this c is *phase velocity*

- Rossby *wave packet energy* moves at the GROUP velocity: eastward instead of westward, for Rossby waves.

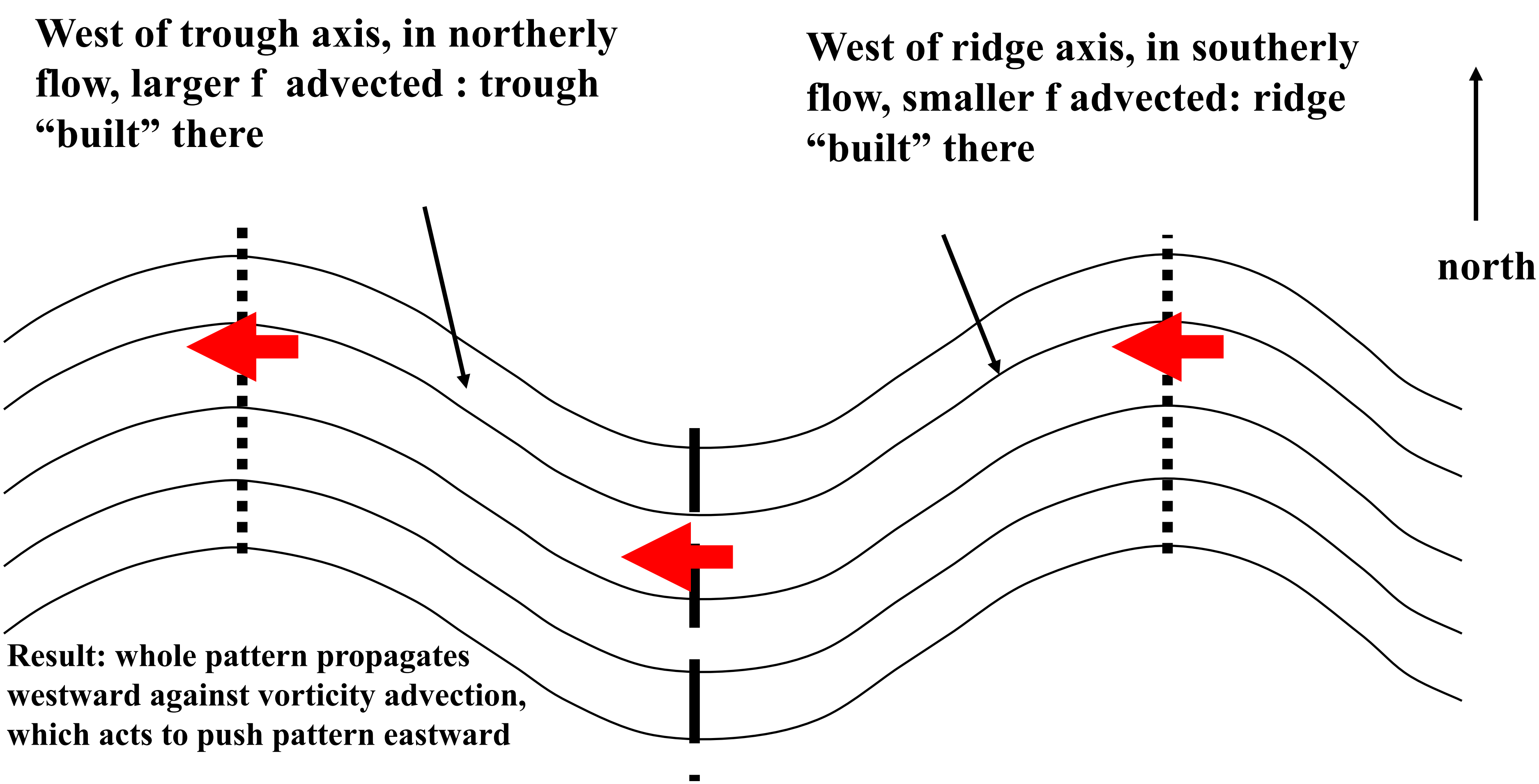
$$c_g = U + \frac{\beta L^2}{4\pi^2}$$

Note!!

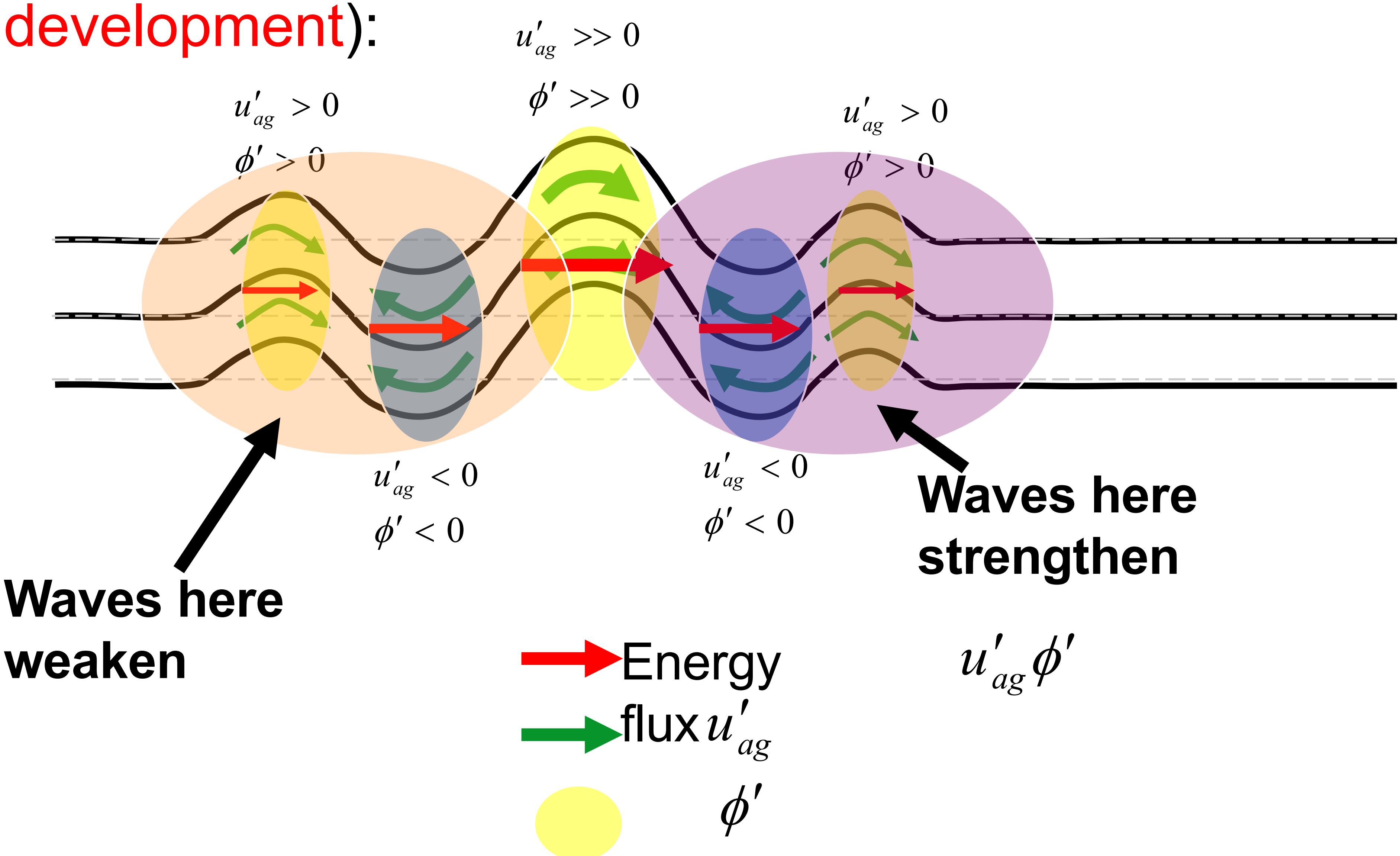
Rossby Wave Recap

Physically, how can we understand tendency for Rossby waves to propagate ***westward*** *relative to the flow*?

This is related to the "beta-effect", motion related to the *advection of planetary vorticity* (the fact that f changes with latitude)



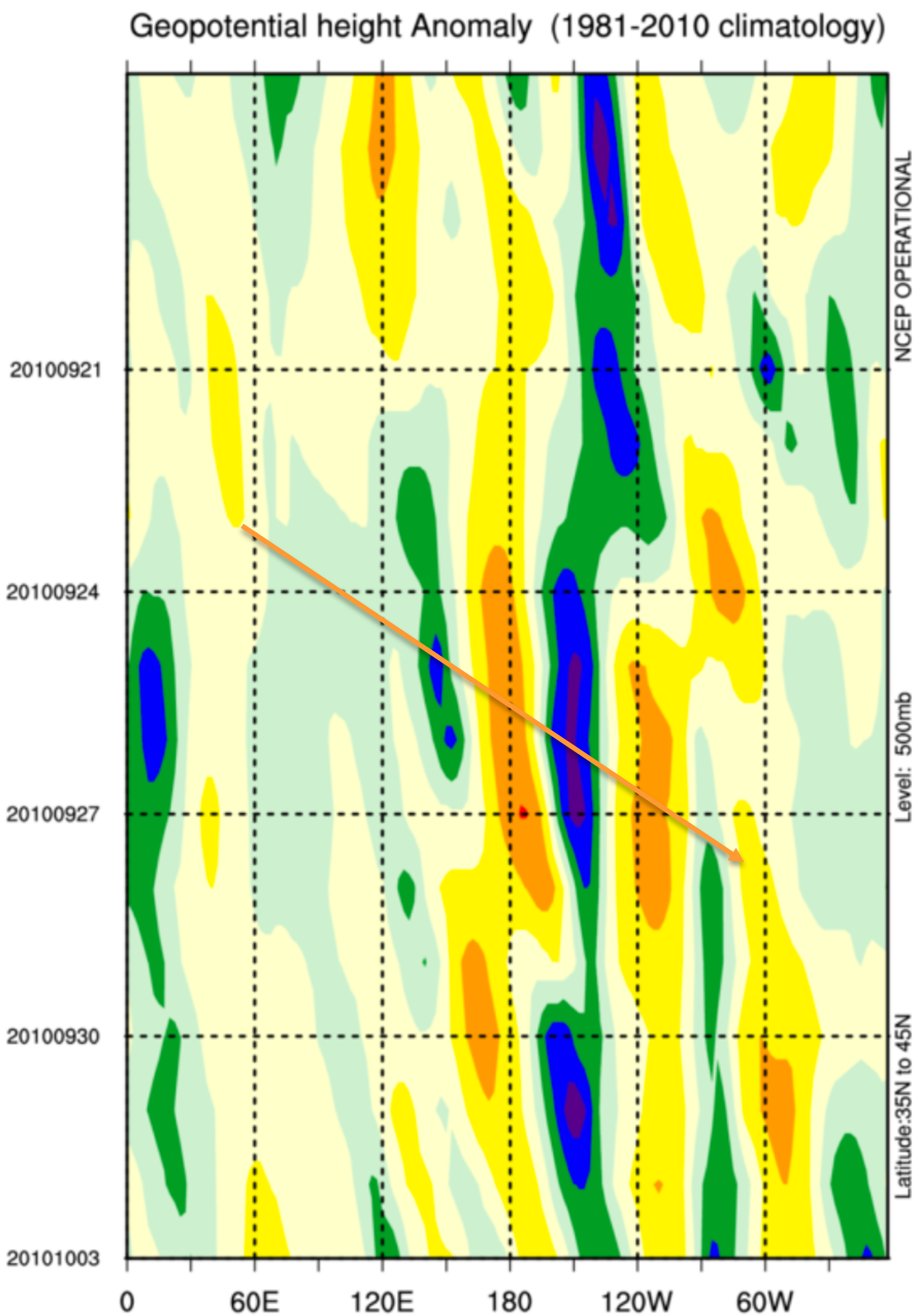
Rossby wave group velocity energetics (downstream development):



$$\left(\frac{\partial}{\partial t} + U_g \frac{\partial}{\partial x}\right)\frac{v_g'^2}{2} = -\frac{\partial}{\partial x}\left(u'_{ag}\phi'\right) - \frac{\partial}{\partial z}\left(w'\phi'\right) - w'b'$$

Example:
Z'(lon, time) diagram

time→



phase velocity near zero (almost stationary waves,) like in the river

group or energy speed: fast eastward

midlatitude Rossby waves review

- Wirth, V., M. Riemer, E.K. Chang, and O. Martius, 2018: Rossby Wave Packets on the Midlatitude Waveguide—A Review.
- *Mon. Wea. Rev.,* **146**, 1965–2001
- https://doi.org/10.1175/MWR-D-16-0483.1