The Boussinesq Equations and the Kinetic Energy Budget in Flux Form

1. Governing Equations

We begin from the three-dimensional, incompressible Boussinesq equations on Cartesian coordinates (x, y, z) with velocity components (u, v, w) :

$$\begin{aligned}
\frac{\partial u}{\partial t} + (\mathbf{V} \cdot \nabla)u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_u, \\
\frac{\partial v}{\partial t} + (\mathbf{V} \cdot \nabla)v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_v, \\
\frac{\partial w}{\partial t} + (\mathbf{V} \cdot \nabla)w &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b + F_w, \\
\nabla \cdot \mathbf{V} &= 0.
\end{aligned}$$

Here: - $\mathbf{V} = (u, v, w)$ is the velocity vector, - p is the pressure perturbation, - ρ_0 is a constant reference density, - b is the buoyancy, typically defined as $b = g(\theta'/\theta_0)$ where θ' is potential temperature perturbation, - $F = (F_u, F_v, F_w)$ is a generalized frictional or drag force.

These equations represent momentum conservation under the Boussinesq approximation (see Vallis 2017; Holton & Hakim 2013).

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2. Kinetic Energy Equation in Flux Form

The kinetic energy density is

$$KE = \tfrac{1}{2}(u^2 + v^2 + w^2).$$

Multiply each momentum equation by the corresponding velocity component and sum:

$$\frac{\partial KE}{\partial t} + \nabla \cdot (KE\mathbf{V} + \tfrac{p}{\rho_0}\mathbf{V}) = bw + F \cdot \mathbf{V}.$$

This is the flux-form kinetic energy budget: - The **left-hand side** contains storage and transport of kinetic energy. - The **right-hand side** contains sources and sinks: conversion from potential energy via bw , and dissipation via $F \cdot \mathbf{V}$.

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3. Domain-Integrated Budget

Define the domain integral (mass integral) of any field q as

$$[q] = \iiint q\, dV.$$

Integrating the kinetic energy equation over the volume and assuming impermeable, periodic, or otherwise energy-conserving boundary conditions so that the flux divergence term integrates to zero, we obtain:

$$\frac{d[KE]}{dt} = [bw] + [F \cdot \mathbf{V}].$$

Thus: - **Source:** buoyancy flux $[bw]$, converting potential energy to kinetic. - **Sink:** frictional dissipation $[F \cdot \mathbf{V}]$.

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4. Modal Decomposition and Orthogonality

Consider expanding the velocity field in an orthonormal basis of modes $\phi_n(\mathbf{x})$, e.g. Fourier modes in x, y, z or spherical harmonics in a planetary atmosphere:

$$\mathbf{V}(\mathbf{x}, t) = \sum_n \mathbf{a}_n(t) \phi_n(\mathbf{x}).$$

Orthogonality ensures that the total kinetic energy splits additively:

$$[KE] = \sum_n [KE_n],$$

where $[KE_n] = \frac{1}{2} |\mathbf{a}_n|^2$.

Because the governing equations are quadratic and preserve inner products under orthonormal projections, the same balance applies mode by mode:

$$\frac{d[KE_n]}{dt} = [bw]_n + [F \cdot \mathbf{V}]_n + T_n,$$

where T_n denotes transfers between modes. Crucially, the source $[bw]$ and sink $[F \cdot \mathbf{V}]$ still apply separately to each mode.

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5. Transport Terms and Scale Interactions

Returning to the flux divergence term in advective form:

$$\nabla \cdot (KE\mathbf{V}) = \mathbf{V} \cdot \nabla KE + KE\,(\nabla \cdot \mathbf{V}).$$

Using incompressibility $(\nabla \cdot \mathbf{V} = 0)$, the advective transport reduces to redistribution of kinetic energy across space, not net creation or destruction.

When decomposed into wavenumber space, these nonlinear transport terms mediate **energy transfers across scales**: - **Low to high wavenumbers (direct cascade):** processes like frontogenesis sharpen gradients, concentrating energy at smaller scales (higher wavenumbers). This reflects the physical creation of fine-scale structures (Hoskins 1982). - **High to low wavenumbers (inverse cascade):** in two-dimensional turbulence, nonlinear triad interactions can transfer energy upscale. - **Resonant triads:** in wave turbulence theory, nonlinear terms allow three Fourier modes k_1, k_2, k_3 satisfying resonance conditions ($\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$, and frequency matching) to exchange energy efficiently (Phillips 1960; Nazarenko 2011).

A Note on "Wavenumber" vs. "Scale"

- **Wavenumber** is the spectral representation (inverse length scale).
- **Scale** is the physical size of a structure in real space. High wavenumbers correspond to small scales, while low wavenumbers correspond to planetary or synoptic scales.

Thus, the advective transport term represents a conservative redistribution of energy among modes, governing the spectral cascade central to turbulence and atmospheric dynamics.

References

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