"Transport" of conserved stuff

(Eulerian $\partial/\partial t$) vs (Lagrangian d/dt) views.

For definiteness, consider water vapor mass concentration (or specific humidity) q $\left(\frac{kg_v}{kg_{air}}\right)$

advective transport

$\Downarrow$ WH pgs_6 Ch 1

$$\frac{\partial q}{\partial t} = \frac{dq}{dt} \underbrace{- u\frac{\partial q}{\partial x} - v\frac{\partial q}{\partial y} - w\frac{\partial q}{\partial z}}$$

advection: $-\vec{V}\cdot\vec{\nabla} q$

flux and its convergence

- flux of vapor = flux of mass $* \, q$ $(kg \, m^{-2} \, s^{-1})$

$$\boxed{(\rho \vec{V}) q} \text{ vapor flux}$$

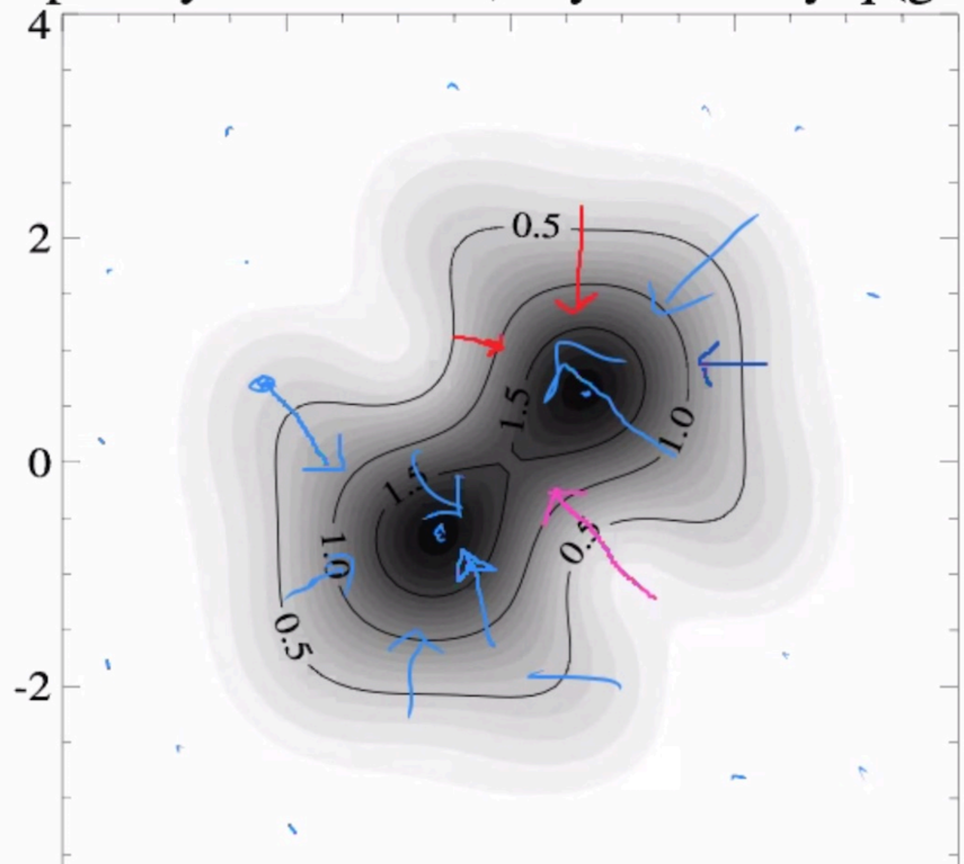
- convergence of a flux = $-$divergence " " "

$$= -\vec{\nabla}\cdot(\rho q \vec{V}) \text{ flux}$$

"convergence of"

"Advection" $= -\vec{V} \cdot \underbrace{\vec{\nabla} q}_{\text{gradient of } q}$

A patchy scalar field, say humidity q (g/kg)



$\vec{\nabla} q = \hat{i}\boxed{\frac{\partial}{\partial x} q} + \hat{j}\frac{\partial}{\partial y} q$

points uphill

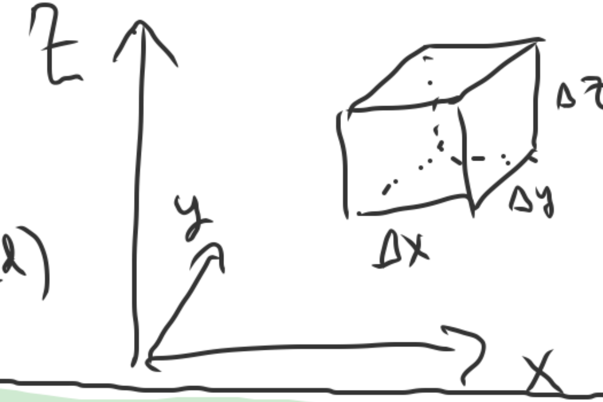
(toward higher values)

$$\left.\frac{\partial \rho}{\partial t}\right|_{box} = \left(\frac{\partial m}{\partial t}\right) \Big/ \left(\Delta x \Delta y \Delta z\right)$$

mass in box

volume of box (fixed)

change with time



a fixed box

to be divided out

$\frac{\partial m}{\partial t}$ = net flow of mass into box per unit time.

(flow in) − (flow out)

(because mass is utterly conserved)

Consider x-direction first. West face "in", east face "out".

$$\left.\frac{\partial m}{\partial t}\right|_{E-W} = \left(\text{flow in west face}\right) - \left(\text{flow out east face}\right)$$

$$= \left((\rho u)\,\Delta y\,\Delta z\right) - \left(\left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y\,\Delta z\right) = \underbrace{-\frac{\partial}{\partial x}(\rho u)}_{\text{KEY}}\,\boxed{\Delta x \Delta y \Delta z}$$

area of face

flux (flow per unit area)

area

flux

volume

volume

Gathering the E-W, N-S, and up-down directions,

$$\frac{\partial m}{\partial t} = \left[-\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) \right] \Delta x \Delta y \Delta z$$

so

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w)}$$

If $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is the "del" operator ("nabla"),

and $\vec{V} = \hat{i}\,u + \hat{j}\,v + \hat{k}\,w$ is velocity $\vec{V}_{3d}$

then $\boxed{-\vec{\nabla}\cdot(\rho\vec{V}) = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = \frac{\partial \rho}{\partial t}}$

you know how to do a dot product!