Homework 0

Vectors: do the components math, but don't forget to use geometry (sketch graphs and remember parallel, perpendicular, cosθ, sinθ)

1. Find the resultant (net) vector $\mathbf{V}$ by adding up its 3 parts:

   a. $\mathbf{V}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{V}_2 = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{V}_3 = -7\mathbf{i} - \mathbf{j}$

   b. $\mathbf{V}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{V}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{V}_3 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

2. Let $\mathbf{G} = 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{H} = \mathbf{i} - \mathbf{j}$ be two forces. Determine a force $\mathbf{F}$ such that $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are in equilibrium.

Dot Product

3. Find the work W (Joules) done by a force $\mathbf{F}$ (in Newtons) acting on a 1 kg mass while it moves from A to B (measured in meters) when:

   a. $\mathbf{F} = 2\mathbf{i} + \mathbf{j}$, $A:(0,0,0)$, $B:(0,1,0)$

   b. $\mathbf{F} = \mathbf{i} + 2\mathbf{j}$, $A:(4,-7,3)$, $B:(4,-7,8)$

   c. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $A:(1,-1,2)$, $B:(2,1,3)$

4. Let $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 4\mathbf{j}$. Find all the unit vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ such that the angle between $\mathbf{a}$ and $\mathbf{b}$ is equal to the angle between $\mathbf{a}$ and $\mathbf{c}$.

Cross Product

5. Concerning the so-called triple product: show that these are equal: $(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

6. Let $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{b} = 0$. Does that imply that $\mathbf{a} = \mathbf{0} = \mathbf{b}$?

7. Complete the following identities:

   a. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =$

   b. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} =$

8. Show that if $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ and $\mathbf{b} = \mathbf{a}\times\mathbf{c}$ then $\mathbf{b} = \mathbf{0} = \mathbf{c}$.

9. Find the following:

   a. $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k} =$

   b. $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) =$

   c. $(\mathbf{i} \times \mathbf{j}) \cdot \mathbf{k} =$

   d. $\mathbf{i} \cdot (\mathbf{j} \times \mathbf{k}) =$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors.

Vector Calculus

10. Find the first partial differential derivatives with respect to (x, y, z)

   a. $\mathbf{V}(x, y, z) = x^2\mathbf{i} - z^2\mathbf{j} + y^2\mathbf{k}$

   b. $\mathbf{V}(x, y, z) = e^y\mathbf{i} - e^{-z}\mathbf{j} + e^{2x}\mathbf{k}$

11. Show that if $\mathbf{V}(t) = u(t)\mathbf{i} + v(t)\mathbf{j} + w(t)\mathbf{k}$ has length 1, and its derivative $\mathbf{V}'(t) = u'(t)\mathbf{i} + v'(t)\mathbf{j} + w'(t)\mathbf{k}$ is not zero, then $\mathbf{V}$ and $\mathbf{V}'$ are orthogonal.

Velocity and Acceleration

12. Find the centripetal acceleration of the moon d**V**/dt towards the earth. The moon orbits the earth at a distance $R = 239{,}000$ miles. It takes 27.3 earth days for one complete revolution.

The Chain Rule (total derivative):

13. Find $\dfrac{df}{dt}$ where $f(x, y, z) = x^2 + y^2 + z^2$, and $x(t) = e^t \cos t$, $y(t) = e^t \sin t$, $z(t) = e^t$.

14. Let $f(x, y, z) = \dfrac{c}{r}$ denote the gravitational potential, where $c = GMm$ is a constant, and $\mathbf{r} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is the distance vector, with magnitude r. Show that: $\nabla^2 f = 0$.

15. Find the gradient of the following functions.

   a. $f(x, y) = e^x \sin y$

i. Sketch contours of the field f in a in the 2D domain [-1,1] for x and [-π,π] for y. Sketch arrows showing its gradient.

b. $f(x,y)=\frac{1}{2}\ln(x^2+y^2)$

16. Find the divergence and curl of the following vector **fields** $\mathbf{V}$:

a. $\mathbf{V}=2x\mathbf{i}+2y\mathbf{j}$

b. $\mathbf{V}=-2y\mathbf{i}+2x\mathbf{j}$

c. $\mathbf{V}=y^2\mathbf{i}$

17. Show that $\nabla\times\nabla f=\mathbf{0}$ for any function $f=f(x,y,z)$.

18. Here are two vector fields: pure rotation and pure deformation. Do vector addition at each point to create the summed vector field, fill in some other points, and describe the flow.