ATM651 Wallace and Hobbs 7.25 & 7.26(a) Solution:

7.25

Prove that the thermal wind equation can also be expressed in the forms

$$\frac{\partial \mathbf{V}_g}{\partial p} = -\frac{R}{fp} \times \nabla T$$

and

$$\frac{\partial \mathbf{V}_g}{\partial z} = \frac{g}{fT} \times \nabla T + \frac{1}{T}\frac{\partial T}{\partial z}\mathbf{V}_g$$

(from Wallace and Hobbs, Ch7)

The second form, $\frac{\partial V_g}{\partial z}$, can be derived from the definition of geostrophic wind in the z-coordinate $V_g = \frac{1}{f}\, k \times \frac{1}{\rho}\nabla_h P$. Note that ∇_h is the horizontal gradient only, so the "del" in this question should be really ∇_h.

Geostrophic wind: $V_g = \frac{1}{f}\, k \times \frac{1}{\rho}\nabla_h P$, and taking the z derivative on both sides we have

$$\frac{\partial V_g}{\partial z} = \frac{1}{f}\, k \times \frac{\partial}{\partial z}\left(\frac{1}{\rho}\nabla_h P\right)$$

$$= \frac{1}{f}\, k \times \left[\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\nabla_h P + \frac{1}{\rho}\frac{\partial}{\partial z}(\nabla_h P)\right]$$

$$= \frac{1}{f}\, k \times \left[\frac{\partial \alpha}{\partial z}\nabla_h P + \alpha\frac{\partial}{\partial z}(\nabla_h P)\right]$$

Using chain rule

$$= \frac{1}{f}\, k \times \left[\frac{R}{P}\frac{\partial T}{\partial z}\nabla_h P + \frac{\alpha}{P}\frac{\partial P}{\partial z}(\nabla_h P) + \alpha\frac{\partial}{\partial z}(\nabla_h P)\right]$$

$$= \frac{1}{f}\, k \times \left[\frac{R}{P}\frac{\partial T}{\partial z}\nabla_h P + \frac{\alpha}{P}\frac{\partial P}{\partial z}(\nabla_h P) - \alpha(\nabla_h \rho g)\right]$$

Recall that thermowind consists of geostrophic wind balance and also "hydrostatic balance" which implies $\rho = \rho(z)$. Thus, the last term in the bracket becomes zero. Now you can see the importance of correct notations when deriving equations.

$$= \frac{1}{f}\, k \times \left[\frac{R}{P}\frac{\partial T}{\partial z}\nabla_h P + \frac{\alpha}{P}\frac{\partial P}{\partial z}(\nabla_h P)\right]$$

$$= \frac{1}{f}\, k \times \left[\frac{1}{\rho T}\frac{\partial T}{\partial z}\nabla_h P + \frac{g}{P}(\nabla_h P)\right]$$

$$= \frac{1}{f}\, k \times \left[\frac{1}{T}\frac{\partial T}{\partial z}\frac{1}{\rho}\nabla_h P + \frac{g}{P}(\nabla_h P)\right]$$

$$= \frac{1}{f}\, k \times [\frac{1}{T}\frac{\partial T}{\partial z}\frac{1}{\rho}\nabla_h P + \frac{Rg}{P}(T\nabla_h \rho + \rho \nabla_h T)]$$

$$= \frac{1}{f}\, k \times [\frac{1}{T}\frac{\partial T}{\partial z}(\frac{1}{\rho}\nabla_h P) + \frac{Rg}{P}(\rho\nabla_h T)]$$

$$= \frac{1}{f}\, k \times [\frac{1}{T}\frac{\partial T}{\partial z}(\frac{1}{\rho}\nabla_h P) + \frac{g}{T}(\nabla_h T)]$$

$$= \frac{1}{T}\frac{\partial T}{\partial z}V_g + \frac{g}{fT}\, k \times \nabla_h T$$

7.26 (a)

(a) Prove that the geostrophic temperature advection in a thin layer of the atmosphere (i.e., the rate of change of temperature due to the horizontal advection of temperature) is given by

$$\frac{f}{R \ln (p_B/p_T)} V_{gB} V_{gT} \sin\theta$$

where the subscripts B and T refer to conditions at the bottom and top of the layer, respectively, and θ is the angle between the geostrophic wind at the two levels, defined as positive if the geostrophic wind veers with increasing height. (from Wallace and Hobbs, Ch7)

According to the thermowind balance equation (eq. 7.20), we have its vector form as

$$\overrightarrow{V_T} = \frac{R}{f}\ln(\frac{P_b}{P_t}) k \times \nabla T$$

and rearrange this equation by moving k to the LHS to get the temperature gradient, ∇T,

$$\nabla T = \frac{-1}{\frac{R}{f}\ln(\frac{P_b}{P_t})} k \times \overrightarrow{V_T} = \frac{-1}{\frac{R}{f}\ln(\frac{P_b}{P_t})} k \times (\overrightarrow{V_{gt}} - \overrightarrow{V_{gb}})$$

The assumption of "thin layer" implies that the temperature varies slightly with height, so the temperature gradient ∇T is identical within this layer. Now we calculate the geostrophic temperature advection at both top and bottom levels.

At the top level:

$$\overrightarrow{V_{gt}} \cdot \nabla T = \frac{-1}{\frac{R}{f}\ln(\frac{P_b}{P_t})} \overrightarrow{V_{gt}} \cdot \left[k \times \left(\overrightarrow{V_{gt}} - \overrightarrow{V_{gb}}\right)\right]$$

Using the vector identity to swap k and Vgt

$$= \frac{-1}{\frac{R}{f}\ln(\frac{P_b}{P_t})} k \cdot \left[\overrightarrow{V_{gt}} \times \left(\overrightarrow{V_{gt}} - \overrightarrow{V_{gb}}\right)\right]$$

$$= \frac{-1}{\frac{R}{f}\ln(\frac{P_b}{P_t})} k \cdot (-\overrightarrow{V_{gt}} \times \overrightarrow{V_{gb}})$$

$$= \frac{f}{R\ln(\frac{P_b}{P_t})} |V_{gt}||V_{gb}|sin\theta$$

Following the same approach, we have $\overrightarrow{V_{gb}} \cdot \nabla T = \frac{f}{R\ln(\frac{P_b}{P_t})} |V_{gt}||V_{gb}|sin\theta$ at the bottom level.

Therefore, the geostrophic temperature advection at these two levels is the same in this case.