Homework 0

Vectors: do the components math, but don't forget to use geometry (sketch graphs and remember parallel, perpendicular, cosθ, sinθ)

1. Find the resultant (net) vector $\mathbf{V}$ by adding up its 3 parts:

   a. $\mathbf{V}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{V}_2 = 4\mathbf{i} - \mathbf{j} - \mathbf{k}, \mathbf{V}_3 = -7\mathbf{i} - \mathbf{j}$

   b. $\mathbf{V}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{V}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \mathbf{V}_3 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

2. Let $\mathbf{G} = 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{H} = \mathbf{i} - \mathbf{j}$ be two forces. Determine a force $\mathbf{F}$ such that $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are in equilibrium.

Dot Product

3. Find the work W (Joules) done by a force $\mathbf{F}$ (in Newtons) acting on a 1 kg mass while it moves from A to B (measured in meters) when:

   a. $\mathbf{F} = 2\mathbf{i} + \mathbf{j}, A:(0,0,0), B:(0,1,0)$

   b. $\mathbf{F} = \mathbf{i} + 2\mathbf{j}, A:(4,-7,3), B:(4,-7,8)$

   c. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}, A:(1,-1,2), B:(2,1,3)$

4. Let $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 4\mathbf{j}$. Find all the unit vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ such that the angle between $\mathbf{a}$ and $\mathbf{b}$ is equal to the angle between $\mathbf{a}$ and $\mathbf{c}$.

Cross Product

5. Concerning the so-called triple product: show that these are equal: $(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

6. Let $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = 0$. Does that imply that $\mathbf{a} = \mathbf{0} = \mathbf{b}$?

7. Complete the following identities:

   a. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) =$

   b. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$

8. Show that if $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{b} = \mathbf{a} \times \mathbf{c}$ then $\mathbf{b} = \mathbf{0} = \mathbf{c}$.

9. Find the following:

   a. $(\mathbf{i}\times\mathbf{j})\times\mathbf{k} =$

   b. $\mathbf{i}\times(\mathbf{j}\times\mathbf{k}) =$

   c. $(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k} =$

   d. $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) =$

where $\mathbf{i},\mathbf{j},\mathbf{k}$ are the standard unit vectors.

Vector Calculus

10. Find the first partial differential derivatives with respect to (x, y, z)

    a. $\mathbf{V}(x,y,z) = x^2\mathbf{i} - z^2\mathbf{j} + y^2\mathbf{k}$

    b. $\mathbf{V}(x,y,z) = e^{y}\mathbf{i} - e^{-z}\mathbf{j} + e^{2x}\mathbf{k}$

11. Show that if $\mathbf{V}(t) = u(t)\mathbf{i} + v(t)\mathbf{j} + w(t)\mathbf{k}$ has length 1, and its derivative $\mathbf{V}'(t) = u'(t)\mathbf{i} + v'(t)\mathbf{j} + w'(t)\mathbf{k}$ is not zero, then $\mathbf{V}$ and $\mathbf{V}'$ are orthogonal.

Velocity and Acceleration

12. Find the centripetal acceleration of the moon d**V**/dt towards the earth. The moon orbits the earth at a distance $R = 239{,}000$ miles. It takes 27.3 earth days for one complete revolution.

The Chain Rule (total derivative):

13. Find $\dfrac{df}{dt}$ where $f(x,y,z) = x^2 + y^2 + z^2$, and $x(t) = e^t\cos t$, $y(t) = e^t\sin t$, $z(t) = e^t$.

14. Let $f(x,y,z) = \dfrac{c}{r}$ denote the gravitational potential, where $c = GMm$ is a constant, and $\mathbf{r} = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}$ is the distance vector, with magnitude r. Show that: $\nabla^2 f = 0$.

15. Find the gradient of the following functions.

    a. $f(x,y) = e^x \sin y$

i. Sketch contours of the field f in a in the 2D domain [-1,1] for x and [-π,π] for y. Sketch arrows showing its gradient.

b. $f(x,y)=\dfrac{1}{2}\ln(x^2+y^2)$

16. Find the divergence and curl of the following vector **fields** $\mathbf{V}$:

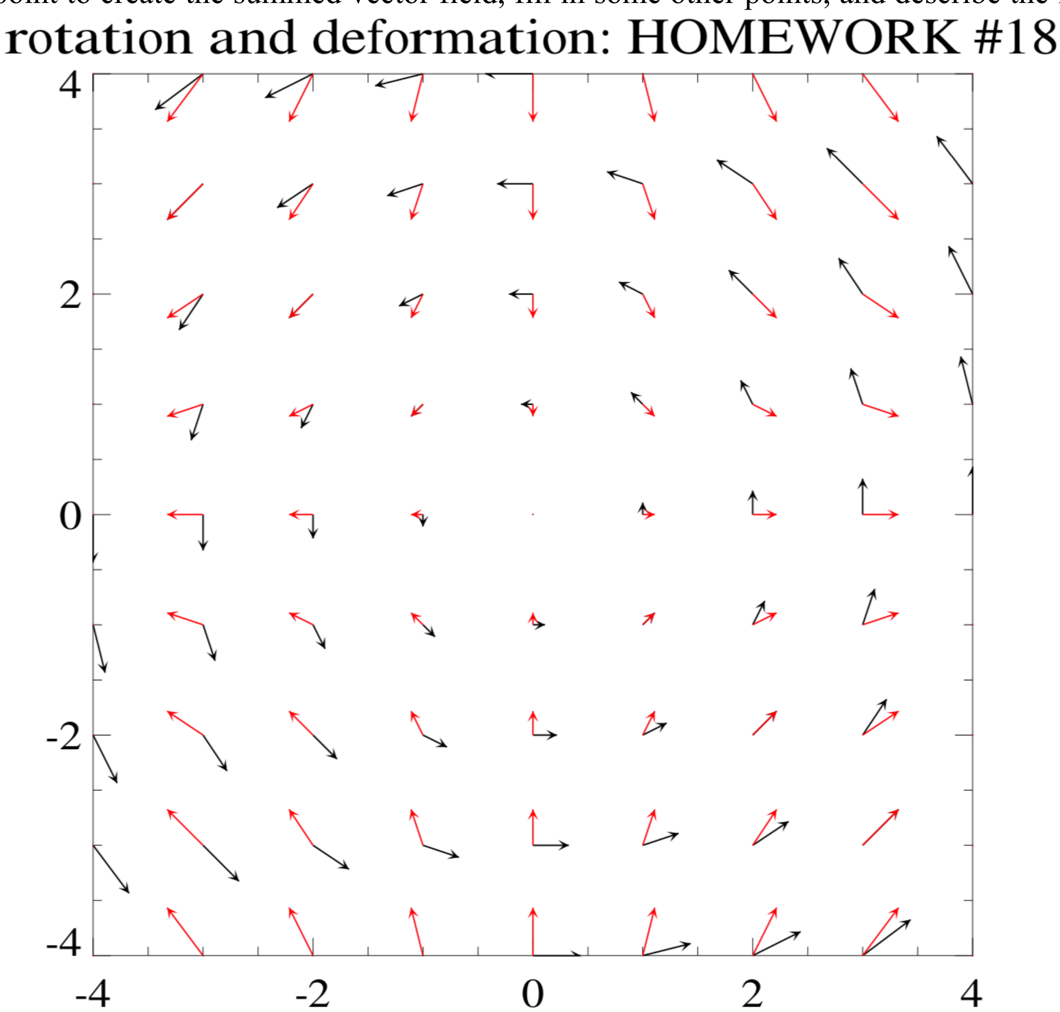
a. $\mathbf{V}=2x\mathbf{i}+2y\mathbf{j}$

b. $\mathbf{V}=-2y\mathbf{i}+2x\mathbf{j}$

c. $\mathbf{V}=y^2\mathbf{i}$

17. Show that $\nabla\times\nabla f=\mathbf{0}$ for any function $f=f(x,y,z)$.

18. Here are two vector fields: pure rotation and pure deformation. Do vector addition at each point to create the summed vector field, fill in some other points, and describe the flow.



19. Sketch the curve of a function f(t) whose *slope is positively proportional to its value*. Write a differential equation corresponding to those words. What is its solution mathematically?

20. Sketch the curve of a function f(t) whose *slope is negatively proportional to its value*. Write a differential equation corresponding to those words. What is its solution mathematically?

21. Sketch the curve of a function f(t) whose *curvature is negatively proportional to its value*. Write a differential equation corresponding to those words. What is its solution mathematically?

22. Sketch the curve of a function f(t) whose *curvature is negatively proportional to its value, with that proportionality constant decreasing with time*. Write a differential equation corresponding to those words. What is its solution mathematically?

23. Sketch the curve of a function f(t) whose *curvature is positively proportional to its value*. Write a differential equation corresponding to those words. What is its solution mathematically?