Homework 0: Solutions

Due Date: Tuesday, February 1st.

(points in parenthesis)

Vector

1. (5) Find the resultant winds:

a. $\mathbf{V}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k},\ \mathbf{V}_2 = 4\mathbf{i} - \mathbf{j} - \mathbf{k},\ \mathbf{V}_3 = -7\mathbf{i} - \mathbf{j}$

$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

b. $\mathbf{V}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \mathbf{V}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},\ \mathbf{V}_3 = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 0\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$

2. (5) Let $\mathbf{G} = 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{H} = \mathbf{i} - \mathbf{j}$ be two forces. Determine a force $\mathbf{F}$ such that $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are in equilibrium.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then for all the forces to balance we need $\mathbf{F} + \mathbf{G} + \mathbf{H} = \mathbf{0}$:

$(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) + 3\mathbf{j} - 4\mathbf{k} + \mathbf{i} - \mathbf{j} = \mathbf{0} \Rightarrow (F_1 + 1)\mathbf{i} + (F_2 + 3 - 1)\mathbf{j} + (F_3 - 4)\mathbf{k} = \mathbf{0}$

$F_1 + 1 = 0,\ F_2 + 2 = 0,\ F_3 - 4 = 0 \Rightarrow \mathbf{F} = -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

Dot Product

3. (5) Find the work W done by a force $\mathbf{F}$ acting on a point P that is displaced from A to B when:

a. $\mathbf{F} = 2\mathbf{i} + \mathbf{j},\ A:(0,0,0),\ B:(0,1,0)$

The distance from A to B is $\mathbf{d} = (0,1,0)$ while the force is $\mathbf{F} = (2,1,0)$. Thus the work is: $W = \mathbf{F} \cdot \mathbf{d} = 1$

b. $\mathbf{F} = \mathbf{i} + 2\mathbf{j},\ A:(4,-7,3),\ B:(4,-7,8)$

similarly, $W = 0$,

c. $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ A:(1,-1,2),\ B:(2,1,3)$

and $W = 4$.

4. (7) Let $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 4\mathbf{j}$. Find all the unit vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ such that the angle between $\mathbf{a}$ and $\mathbf{b}$ is equal to the angle between $\mathbf{a}$ and $\mathbf{c}$.

Let γ be the angle of $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}, \mathbf{c})$:

$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|} \Rightarrow \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|}$$

but from the data of the problem we have:

$|\mathbf{b}| = \sqrt{5}, |\mathbf{c}| = 2\sqrt{5}, \mathbf{a}\cdot\mathbf{b} = 2a_1 + a_2$, and $\mathbf{a}\cdot\mathbf{c} = 2a_1 + 4a_2$. Thus from the above equation we find: $a_1 = a_2$.

We also know that $\mathbf{a}$ is a unit vector:

$$a_1^{\,2} + a_2^{\,2} + a_3^{\,3} = 1 \Rightarrow 2a_2^{\,2} + a_3^{\,2} = 1 \Rightarrow a_3 = \pm(1 - 2a_2^{\,2})^{1/2}$$

Finally, if we set $a_2 = \alpha$ we find that $\mathbf{a}$ has the form:

$$\mathbf{a} = \alpha\mathbf{i} + \alpha\mathbf{j} \pm \sqrt{1-2\alpha^2}\,\mathbf{k}\,.$$

Cross Product

5. **(7)** Calculate the so-called triple product: $(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

and similarly:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot\left(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}\right) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

6. **(5)** Let $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{b} = 0$. Does that imply that $\mathbf{a} = \mathbf{0} = \mathbf{b}$?

Yes, since the two vector are both normal ($\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a}\perp\mathbf{b}$) and parallel ($\mathbf{a}\times\mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a}\,||\,\mathbf{b}$) to each other. The only vector like that is the zero vector.

7. **(5)** Complete the following identities:

   a. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

   b. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a}$

8. **(7)** Show that if $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ and $\mathbf{b} = \mathbf{a}\times\mathbf{c}$ then $\mathbf{b} = \mathbf{0} = \mathbf{c}$.

$\mathbf{c} = \mathbf{a}\times\mathbf{b} = \mathbf{a}\times(\mathbf{a}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{a} - |\mathbf{a}|^2\mathbf{c} \Rightarrow \mathbf{c} = \dfrac{\mathbf{a}\cdot\mathbf{c}}{1+|\mathbf{a}|^2}\mathbf{a}$, thus the vectors $\mathbf{a},\mathbf{c}$ are parallel to each other. Similarly for the vectors $\mathbf{a},\mathbf{b}$. So, $\mathbf{a}\times\mathbf{c} = \mathbf{0}$ and $\mathbf{a}\times\mathbf{b} = \mathbf{0}$. So finally $\mathbf{b} = \mathbf{0} = \mathbf{c}$.

9. **(5)** A force $\mathbf{F}$ acts on a line through a point A. Find the moment vector $\mathbf{m}$ and the moment $m = |\mathbf{m}|$ of $\mathbf{F}$ about a point Q where:

a. $\mathbf{F} = \mathbf{i} - \mathbf{j},\ A:(1,1,1),\ Q:(2,-1,3)$

$\mathbf{r} = \overrightarrow{QA} = (-1,+2,-2)$

$$\mathbf{m} = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -2 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k},\quad |\mathbf{m}| = 3$$

b. $\mathbf{F} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k},\ A:(4,2,-1),\ Q:(0,1,2)$

$\mathbf{r} = \overrightarrow{QA} = (4,1,-3)$

$$\mathbf{m} = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & -3 \\ 2 & 4 & 1 \end{vmatrix} = 13\mathbf{i} - 10\mathbf{j} + 14\mathbf{k},\quad |\mathbf{m}| = \sqrt{465}$$

10. **(5)** Find the following:

a. $(\mathbf{i}\times\mathbf{j})\times\mathbf{k} = (\mathbf{i}\cdot\mathbf{j})\times\mathbf{j} - (\mathbf{k}\cdot\mathbf{j})\times\mathbf{i} = \mathbf{0}$

b. $\mathbf{i}\times(\mathbf{j}\times\mathbf{k}) = (\mathbf{i}\cdot\mathbf{k})\times\mathbf{j} - (\mathbf{i}\cdot\mathbf{j})\times\mathbf{k} = \mathbf{0}$

c. $(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k} = \mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = (\mathbf{i}\ \mathbf{j}\ \mathbf{k}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

d. $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = \mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = 1$

where $\mathbf{i},\mathbf{j},\mathbf{k}$ are the standard unit vectors.

Vector Calculus

11. **(5)** Find the first partial differential derivatives with respect to (x,y,z)

a. $\mathbf{V}(x,y,z) = x^2\mathbf{i} - z^2\mathbf{j} + y^2\mathbf{k}$

$$\frac{\partial \mathbf{V}}{\partial x} = 2x\mathbf{i}, \frac{\partial \mathbf{V}}{\partial y} = 2y\mathbf{k}, \frac{\partial \mathbf{V}}{\partial z} = -2z\mathbf{j}$$

b. $\mathbf{V}(x, y, z) = e^y\mathbf{i} - e^{-z}\mathbf{j} + e^{2x}\mathbf{k}$

$$\frac{\partial \mathbf{V}}{\partial x} = 2e^{2x}\mathbf{k}, \frac{\partial \mathbf{V}}{\partial y} = e^y\mathbf{i}, \frac{\partial \mathbf{V}}{\partial z} = e^{-z}\mathbf{j}$$

12. **(7)** Show that if $\mathbf{V}(t) = u(t)\mathbf{i} + v(t)\mathbf{j} + w(t)\mathbf{k}$ has length 1, and its derivative $\mathbf{V}'(t) = u'(t)\mathbf{i} + v'(t)\mathbf{j} + w'(t)\mathbf{k}$ is not zero, then $\mathbf{V}$ and $\mathbf{V}'$ are orthogonal.

For the vectors $\mathbf{V}$ and $\mathbf{V}'$ to be orthogonal their dot product needs to be zero:

$$\mathbf{V} \cdot \mathbf{V}' = 0 \Leftrightarrow uu' + vv' + ww' = 0$$

We also have, $|\mathbf{V}(t)| = 1 \Rightarrow u^2 + v^2 + w^2 = 1 \Rightarrow u = \pm\left(1 - (v^2 + w^2)\right)^{1/2}$.

With no loss of generality, let's take the positive root for u. Then we have:

$$u' = \frac{du}{dt} = \frac{1}{2}\frac{-2vv' - 2ww'}{\left(1 - (v^2 + w^2)\right)^{1/2}} \Rightarrow u' = -\frac{vv' + ww'}{u} \Rightarrow uu' = -(vv' + ww').$$

Velocity and Acceleration

13. **(5)** Find the centripetal acceleration of the moon towards the earth. The moon orbits the earth at a distance $R = 239,000$ miles. It takes 27.3 earth days for one complete revolution.

First we change units to SI:

Radius: $R = 239,000 \text{ miles} = 239,000 \text{ miles} * 1,609 \dfrac{\text{m}}{\text{miles}} = 3.845 \times 10^8 \text{m}$.

Period: $T = 27.3 \text{ days} = 27.3 \text{days} \times 24 \text{hrs} \times 60 \min \times 60 \sec = 2.358 \times 10^6 \text{s}$.

Frequency: $\omega = \dfrac{2\pi}{T} = 2.664 \times 10^{-6} \text{s}$.

The magnitude of the centripetal acceleration is: $|\mathbf{a}| = \omega^2 R = 2.729 \times 10^{-3} \text{ m s}^{-2}$.

The Chain Rule

14. **(5)** Find $\dfrac{df}{dt}$ where $f(x, y, z) = x^2 + y^2 + z^2$, and $x(t) = e^t \cos t,\ y(t) = e^t \sin t,\ z(t) = e^t$.

$\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$, where we have:

$\dfrac{\partial f}{\partial x} = 2x, \dfrac{\partial f}{\partial y} = 2y, \dfrac{\partial f}{\partial z} = 2z$ and

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \frac{dy}{dt} = e^t(\cos t + \sin t), \frac{dz}{dt} = e^t .$$

Putting everything together:

$$\frac{df}{dt} = 2e^t \cos t \cdot e^t(\cos t - \sin t) + 2e^t \sin t \cdot e^t(\cos t + \sin t) + 2e^t = 4e^{2t}$$

15. **(5)** Let $f(x, y, z) = \frac{c}{r}$ denote the gravitational potential, where $c = GMm$ is a constant, and $\mathbf{r} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is the distance vector, with magnitude r. Show that: $\nabla^2 f = 0$.

First we have: $\frac{\partial r}{\partial x} = \frac{x - x_0}{r}$.

Then as we mentioned in class:

$$\frac{\partial f}{\partial x} = c\frac{\partial(1/r)}{\partial x} = -c\frac{x - x_0}{r^3},\ \frac{\partial f}{\partial y} = -c\frac{y - y_0}{r^3},\ \frac{\partial f}{\partial x} = -c\frac{z - z_0}{r^3}$$

$$\frac{\partial^2 f}{\partial x^2} = -c\frac{\partial}{\partial x}(\frac{x - x_0}{r^3}) = -c\frac{r^3 - 3(x - x_0)^2 r}{r^6} = -c(\frac{1}{r^3} - \frac{3(x - x_0)^2}{r^5})$$

Similarly for y, z: $\frac{\partial^2 f}{\partial y^2} = -c(\frac{1}{r^3} - \frac{3(y - y_0)^2}{r^5})$ and $\frac{\partial^2 f}{\partial y^2} = -c(\frac{1}{r^3} - \frac{3(z - z_0)^2}{r^5})$.

Putting everything together we have:

$$\nabla^2 f = -c(\frac{1}{r^3} - \frac{3(x - x_0)^2}{r^5} + \frac{1}{r^3} - \frac{3(y - y_0)^2}{r^5} + \frac{1}{r^3} - \frac{3(z - z_0)^2}{r^5})$$

$$= -c(\frac{3}{r^3} - \frac{3\left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)}{r^5}) = 0$$

16. **(5)** Find the gradient of the following functions:

a. $f(x, y) = e^x \sin y$

$$\frac{\partial f}{\partial x} = e^x \sin y, \frac{\partial f}{\partial y} = e^x \cos y \Rightarrow \nabla f = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$$

b. $f(x, y) = \frac{1}{2}\ln(x^2 + y^2)$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2} \Rightarrow \nabla f = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$$

17. **(5)** Find the divergence and curl of the following vectors $\mathbf{V}$:

a. $\mathbf{V} = 2x\mathbf{i} + 2y\mathbf{j}$

Divergence: $\nabla \cdot \mathbf{V} = 2 + 2 = 4$, and curl: $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = \mathbf{0}$.

b. $\mathbf{V} = -2y\mathbf{i} + 2x\mathbf{j}$

Divergence: $\nabla \cdot \mathbf{V} = 0$, and curl: $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & 2x & 0 \end{vmatrix} = 4\mathbf{k}$.

c. $\mathbf{V} = y^2\mathbf{i}$

Divergence: $\nabla \cdot \mathbf{V} = 0$, and curl: $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 0 & 0 \end{vmatrix} = -2y\mathbf{k}$.

18. **(5)** Show that $\nabla \times \nabla f = \mathbf{0}$ for a function $f = f(x, y, z)$.

Let $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$, then the curl of $\mathbf{g}$ is

$$\nabla \times \mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= (\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z})\mathbf{i} - (\frac{\partial g_3}{\partial z} - \frac{\partial g_1}{\partial x})\mathbf{j} + (\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y})\mathbf{k}$$

$$= (\frac{\partial\left(\frac{\partial f}{\partial z}\right)}{\partial y} - \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial z})\mathbf{i} - (\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial z} - \frac{\partial\left(\frac{\partial f}{\partial z}\right)}{\partial x})\mathbf{j} + (\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} - \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y})\mathbf{k}$$

$$= \mathbf{0}$$