Horizontal vorticity and PV, as explanations for cyclones & anticyclones

In the Beginning...

...were the frictionless u and v momentum equations, using p as the vertical coordinate so that the PGF is simply the gradient of geopotential:

$$du/dt = -\partial\Phi/\partial x + fv$$

$$dv/dt = -\partial\Phi/\partial y - fu$$

Remember, d/dt is the total derivative, the *partial or local time tendency* minus the tendency due to *advection*. **From here on out, subscripts mean partial derivatives.**

$$du/dt = u_t + uu_x + vu_y + \omega u_p$$

To eliminate Φ, we cross-differentiate the equations, and subtract the first equation from the second, to get an equation for the new quantity, ***vorticity.***

$$\zeta \equiv v_y - u_x$$

Strictly, ζ is the vertical component of vector vorticity. It is the thing we care about: a measure of the actual swirling motion of the horizontal ground-relative winds *u* and *v*.

Relative vorticity ξ is conserved, ... almost ...

That operation gets us an equation we like, d/dt(something) = 0, *plus complications*.

$$d/dt(\zeta) = 0$$

$$+vf_y - (\zeta + f)(u_x + v_y)$$

Absolute **vorticity** $\boldsymbol{\xi}$ **is** ***more*** **conserved ...**

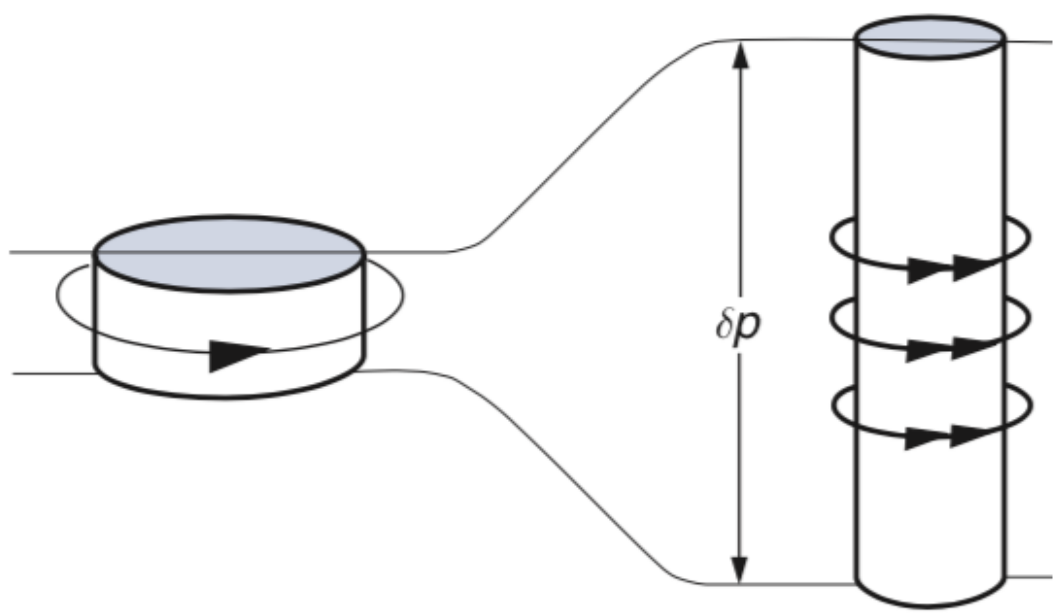
Recognizing that the first term after the zero is df/dt, we can move it to the left side and have a nicer conservation equation for (ζ+f):

$$d/dt(\zeta + f) = 0$$

$$-(\zeta + f)(u_x + v_y)$$

Still there is that pesky complication term: absolute vorticity times convergence. This is sometimes called the **stretching of vortex tubes**, since horizontal convergence would (by mass continuity) lengthen a little cylinder of spinning air of fixed identity.

Potential **vorticity is** ***even more*** **conserved**



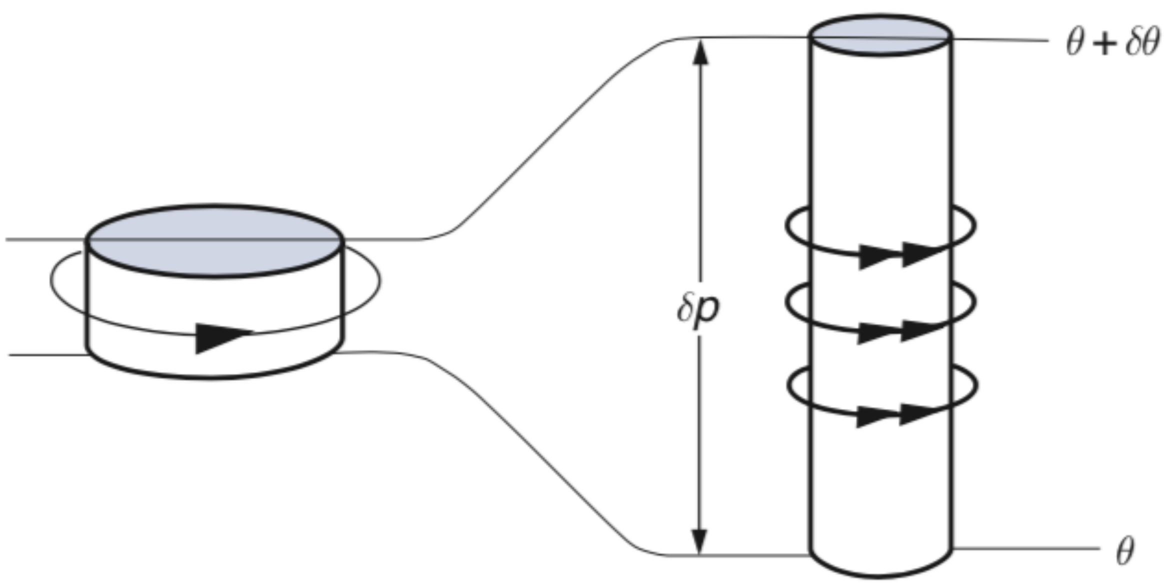
In fact, horizontal convergence *lengthens the tube (increasing* δp) *at exactly the same rate at which it makes the air spin faster*. Based on this insight, the *ratio* of (ζ+f) to the length of the tube is exactly conserved.

$$d/dt(\frac{\zeta + f}{\delta p}) = 0$$

This defines the concept of *potential vorticity*. But how do we get a handle on the ends of the cylinder, so that we can measure the length of this little tube of air? The "end caps" (gray disks in the image) are made of the **same air** before and after the stretching. So they can be defined as iso-surfaces of some other conserved quantity that retains its identity during such a stretching or unstretching process. Traditionally we use potential temperature θ, which obeys the First Law of thermodynamics in the form:

$$d/dt(\theta) = 0 + \dot{\theta}_{diabatic}$$

For adiabatic flow, the term on the right is zero and θ is conserved.



If our "length" δp (measured in hydrostatic pressure coordinates) is divided by g, it is the *mass* of this disc or layer of air between isentropic surfaces at θ and θ+$\delta\theta$. So for *a given value of* $\delta\theta$, we define a potential vorticity as

$$PV per \delta\theta = \frac{\zeta + f}{\delta p/g} = g\frac{\zeta + f}{\delta p}$$

It is more general to normalize out the specific value of $\delta\theta$. For small enough air tubes, we can take the limit $\delta\theta$→0, which gives us this continuous, local quantity called PV.

$$PV = lim_{\delta\to 0}(g\delta\theta\frac{\zeta+f}{\delta p}) = g(\zeta+f)\frac{\partial\theta}{\partial p}$$

PV is the *most* conserved!

We buried both the non-conservation complications of the z equation, making a quantity that is really conserved, for frictionless adiabatic flow.

$$d/dt(PV) = 0$$

Now we can write the next layer of complications, but they are more profound: this is the deep "why" of potential vorticity, the end of our search for reasons behind cyclones and anticyclones.

$$d/dt(PV) = 0 + (\zeta+f)\frac{\partial}{\partial p}\dot{\theta}_{diabatic} + curl(friction)\frac{\partial\theta}{\partial p}$$

Now we are ready to look at *diabatic heating profiles* with hungry eyes.