Chapter 1

Keeping track of stuff in boxes

1.1 Introduction

Before understanding must come labeling and accounting (budgets). Mass and momentum and energy (“stuff”) are conserved by closed systems, but we care about different neighborhoods (“boxes”) within the world, so we must account for transport. We also care about different *categories* of stuff (air vs. water mass, heat vs. chemical energy, motions of different scales) so we must account for exchanges between categories. Finally, there are a few physical laws that must be expressed in budget terms.

We measure space, time, and mass in *Systeme Internationale* (SI) units based fundamentally on our ten fingers, Earth, and Water.[a] Space is in *meters* (m), devised as the Earth’s equator to pole distance divided by 10^7 to match the bodily needs and marketplace convenience of creatures our size. A *kilogram* is 1/1000 the mass of a cubic meter of water, again scaled for our bodies (that volume of water is also called the *liter*). The Earth gives us *days* (and nights), so these could be divided by 10 in a rational world. Clocks with decimal face labels were manufactured in France, but never caught on. Tradition prevailed, and is actually rooted in Earthly numerology (almost-12 months and almost-360 days in a year), so SI retained the *second*: a day divided by 86400 (60 x 60 x 24). Latitude in degrees (10^7m / 90^o = 111.111 km per degree), and subdivisions like nautical miles (1/60 degree), also carry this six-related history.

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[a] Replacing fractions and royal “feet”, in the French Revolution’s anti-aristocracy project.

Precision science has quietly replaced Earth and Water with more fundamental quantum profundities at the root of SI, without most people even noticing. But our 10 fingers still rule all but the messy time domain.

1.2 Conservation of the most fundamental "stuff": mass

In a given volume of space (let's say 1 m^3 for definiteness), the enclosed mass in kilograms is customarily labeled ρ, a *mass density*. The rate of change of that mass depends on the net inflow of mass into the volume, plus any physical sources (zero in the case of mass). The flow of mass through a 2D boundary into a 3D volume is measured by a *flux*, whose units [kg m^{-2} s^{-1}] embody its meaning better than any words can elaborate. Mass flux is ρ**V**, using bold face for vector velocity **V** = **i**u + **j**v + **k**w with the unit vectors **i, j, k** for a Cartesian coordinate system.

Readers should verify that the units of ρ**V** are indeed a flux [as (kg m^{-3}) (m s^{-1}) = kg m^{-2} s^{-1}]. Notice with the same units-mind that velocity **V** is a *volume flux;* and is also a *specific momentum* (where *specific* means *per unit mass*). A feeling for different interpretations of the same quantity is essential to fully appreciating the equations for convection: you need a firm physical grip on the math, but not too tight or exclusive.

A *net inflow* to a volume is called *convergence of flux*, the negative of *divergence*[b]. Translating change = convergence into math, with subscripts denoting partial derivatives along the axes in xyzt space,

$$\begin{aligned}\rho_t &= -div(\rho\mathbf{V}) = -\boldsymbol{\nabla}\cdot(\rho\mathbf{V}) \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + \rho\ sources \\ &= -(\rho u)_x - (\rho v)_y - (\rho w)_z + 0 \qquad (1.1)\end{aligned}$$

Notice that Equation (1.1) is often read in an ambitious tone:

$$something\ I\ want\ =\ work\ I\ must\ do\ to\ get\ it$$

[b] Early meteorology texts simply used the word *vergence*, letting the sign reflect its sense.

In this case, we want (predictive) power over the future. Let's call it a *grasping equation*, in contrast to *majestic equalities* like 0=0. Mathematically there is no difference. But in seeking our top virtue of *appreciation*, or science's touchstone of *prediction*[c], we can get a boost by utilizing special brain hardware, through cultivating the "brainfeel"[d] of equations. To feel a difference, transform (1.1) into a majestic equality:

$$0 = (\rho u)_x + (\rho v)_y + (\rho w)_z + (\rho \dot{t})_t \quad (1.2)$$

In the first 3 terms, velocities in space ($u = \dot{x}$, $v = \dot{y}$, $w = \dot{z}$) measure the futile-looking (zero-equated) journey of moving matter, in units of meters traversed per second of elapsed time. Here an analogous quantity $\dot{t}$ measures the journey of matter through time, in seconds traversed or endured per second of elapsed time. Since both are seconds, $\dot{t}$ is unitless, and in fact is unity (neglecting relativistic effects). But was no appreciation gained by crystallizing that invisible $\dot{t} = 1$ in (1.2) compared to (1.1)? Your brainfeel may vary, but this book may best satisfy readers who notice and appreciate a difference.

1.3 Conservation of *specific* stuff: momentum

Having established (1.1), other budgets follow trivially. To get the budget of momentum, simply multiply the *mass flux* by *momentum per unit mass*, which (as we noticed above) is velocity. In the **i** direction,

$$\begin{aligned}(u\rho)_t &= -\nabla \cdot (u\rho\mathbf{V}) \\ &= -(u\rho u)_x - (u\rho v)_y - (u\rho w)_z + u\,sources \quad (1.3)\end{aligned}$$

The difference from (1.2) is *sources* on the right. Newton taught us that $\mathbf{F} = m\mathbf{a}$: In other words, local momentum sources are called "forces" **F**. The compound SI unit for force is named in his honor (N = kg m s^{-2}). Is force a flux of some definable "stuff"? Not really: 1N = 1 (kg m^3 s) m^{-2} s^-

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[c] *Prediction* perhaps in a logical sense (if X then Y) rather than a temporal forecast sense.

[d] This term is motivated by "mouthfeel" as a food descriptor distinct from taste or nutrition.

[1], and the implied unit of stuff (kg m^3 s) has no intuitive heft or sense, at least to this author's brainfeel.

Only two real forces are needed to appreciate convection:

(1) Gravity -**k**ρg

(2) The gradient of pressure **∇**p

But what is this *pressure* p? Physics class taught us *force per unit area* with units N/m^2 =Pa (Pascals). The gradient operator reflects a *difference* of forces (or net force) per unit distance along each coordinate axis. But pressure has a clearer interpretation as a *momentum flux*, as we shall see.

In a rotating coordinate system, where "motionless" air is nonetheless accelerating, we must add a "fictitious" third "force" per unit mass (2**Ω x V** where **Ω** is the coordinate rotation vector) on the right, as derived in every dynamical meteorology book. As is customary, we retain only the horizontal Coriolis force in our Earth-tangent Cartesian plane xyzt coordinates, burying tiny vertical fictitious forces in g under the enormous "gravity" force -**k**ρg.

Gathering our stuff, to the mass budget (*continuity*) we can now add momentum budgets for a unit box of space containing mass ρ. This leaves us with 4 equations in 5 unknowns (u,v,w,p,ρ):

$$\begin{aligned}
\rho_t &= -(\rho u)_x - (\rho v)_y - (\rho w)_z \quad + 0 && (1.4a)\\
(u\rho)_t &= -(u\rho u)_x - (u\rho v)_y - (u\rho w)_z \quad -p_x + \rho f v && (1.4b)\\
(v\rho)_t &= -(v\rho u)_x - (v\rho v)_y - (v\rho w)_z \quad -p_y - \rho f u && (1.4c)\\
(w\rho)_t &= -(w\rho u)_x - (w\rho v)_y - (w\rho w)_z \quad -p_z - \rho g && (1.4d)
\end{aligned}$$

Specifying ρ is called an *equation of state.* The nature of pressure can be learned from the simplest case, ρ = constant (Problem 1.1). Readers should understand the resulting lesson well: *Continuity is the Law, Pressure is the Enforcer, F=ma is its Mechanism*. But without density variations, mighty gravity is powerless and there is no *convection* in the sense of our title. We simply must allow ρ to vary a tiny bit in space and time, so that ρg is substantial, which drags us into thermodynamics (Chapter 2).

Since each p in (1.4 b-d) could be moved inside the parentheses of the flux terms, *pressure is just another kind of momentum flux*, with units (kg m/s) m^{-2} s^{-1}. The reader should verify that these units are also *force per unit area* in the more common mnemonic above. But pressure's momentum flux is transmitted across the box's border by elastic collisions of the molecules outside the border with those inside it.

This elastic-collision momentum flux is new physics, qualitatively different in character from pieces of matter carrying their properties across the border. That border crossing by molecules is merely the smallest-scale category of flux terms like $(u\rho w)$. Traditionally that molecule-borne flux is interpreted or treated by assumption as a *diffusion* process: a flux of stuff that is proportional to the gradient of that stuff, as measured on a spatial scale long enough that the continuum approximation is valid. That diffusive treatment of momentum flux is often moved to the "sources" end of the RHS of budget (1.3), where its convergence is called the *viscosity* force. But since that is merely an *interpretation of one sub-category* of momentum-weighted mass flux $(u\rho w)$, that complication can be incurred later, when and if desired.

We will also separate fluid-body momentum transport $(u\rho w)$ into a flux of "small" fluid blobs across a border (sometimes treated again as a viscosity) vs. "large" fluid motions (often reframed as *advection*, see problem 1.2). In that scale separation, *small* and *large* are defined by *what scales you care about* (invariably the larger). Again, since scale separation is just a tool for our grasping, let's leave all of that for later, after we simplify the budgets by taming ρ.

1.4 Conservation of *specific* other stuff: heat and humidity

Specific momentum as a name for velocity helped make the budget constructs (1.4 b-d) more intuitive[e]. To close our four-equation but five-variable set by relating ρ to p, we shall see that it entangles temperature T, a measure of *heat.* Thus we will soon have to consider the words *specific heat*. For moist convection, *specific humidity q* will be a convenient variable – or should we use *water vapor mixing ratio*? Both have the mass of water vapor in the numerator, but mixing ratio has the mass of *dry air* rather than *total mass* in the denominator. To get detailed, one must decide if (1.1-1.2) are budgets for dry air or for total gaseous matter. Or perhaps it seems tempting to view ρ as *total mass*, but then the velocity w is hard to define: in rain it would have to be a mass-weighted mean of the wind velocity of the gases and the fall velocity of particles. The complications explode.

If our goal was to frame a numerical model that will be integrated over time, with a need to obey large-scale conservation laws punctiliously even as the various "stuff" of physics is shuttled among many small boxes and forms, such decisions must be strictly defined and adhered to, causing subscript labels and small terms to proliferate. If such a model must assimilate or otherwise quantitatively confront absolutely calibrated observations, it must further use *accurate as well as precise and conservative* versions of thermodynamics, coordinates, and conservation laws. The *fundamentals* of flowing air and water are tedious!

For our goal of *appreciation*, however, such details are a distraction, or even undesirable. Rather than re-derive the fundamentals, perhaps arriving at approximations that retain most of the essence at the end, we seek *essentials* sufficient to tackle the book's phenomenological scope. The difference lies in definitional ambiguities like the above for plain symbols like ρ. Our plain usage is not wrong, but rather crafted to use tacit

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[e] *Specific volume* $\alpha = 1/\rho$ is sometimes invoked for verbal convenience, but to think of volume as being somehow carried along with mass as it moves in and out of a volume in space has an unhelpfully strange brainfeel.

definitions that hide real, fundamental distinctions which don't make an essential difference. Our standard is thus "good-enough" dynamics and thermodynamics (Chapter 2). The well-rounded student of atmospheric convection should drill down to fundamentals at some stage, but plenty of books can take you there.

1.5 Problems

1. Set density to a constant ρ_0 and simplify the set (1.3) maximally in that case.

What phenomena could this *incompressible fluid* equation set describe? In other words, how can an incompressible fluid move, and why would it? What could drive motion, at what scales, and how could that motion decay?

What is the speed of compression (sound) waves in such a medium? That is, if boundary conditions jiggle the fluid on one edge of an incompressible body, how soon is fluid motion transmitted to the other side?

2. Using this simplified set, transform the *majestic equality* form of the mass continuity equation into a *grasping* equation for pressure of the form

What I want (pressure) = Work I must do to build it

Hint: differentiate mass conservation in time, momentum conservation equations in space, and substitute the latter into the former. Subscripts for partial differentiation will save many hand motions in this process! You may use the symbolic but fictitious inverse ∇^{-2} of the Laplacian operator ∇^2 in the final answer, even if solving it is a computational job (Chapter 3).

Interpret the result in your own words, elaborating on this summary: *Continuity is the Law, Pressure is the Enforcer, F = ma is the Mechanism.*

3. Using your simplified set with $\rho = \rho_0$ again, show that the bodily flux convergence terms can be wrangled into *advection* form. In other words, show from Eqs. (1.4) that the flux divergence terms can be rewritten as $-\mathrm{u}\boldsymbol{V}_x - \mathrm{v}\boldsymbol{V}_y - \mathrm{w}\boldsymbol{V}_z = -(\mathbf{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}$.

1.6 Solutions

1. ***Set density to constant ρ_0 and simplify the set (1.3) maximally.***

Substituting ρ=ρ_0, dividing by ρ_0, and negating the sign in 1.4a yields

$$
\begin{aligned}
0 &= u_x + v_y + w_z \\
u_t &= -(uu)_x - (uv)_y - (uw)_z \quad -\pi_x + fv \\
v_t &= -(vu)_x - (vv)_y - (vw)_z \quad -\pi_y - fu \\
w_t &= -(wu)_x - (wv)_y - (ww)_z \ -\pi_z - g
\end{aligned}
$$

Here we have introduced the *Exner function* $\pi = p/\rho_0$.

What phenomena could this incompressible fluid equation set describe?
If a body of such fluid were initially at rest, the only forces that could drive flow within it are pressure or viscous forces (molecular momentum flux in the parenthetical terms) applied at a boundary. Gravity is powerless without density variations, so nothing worth the name "convection" can occur. Pressure would drive divergent flows, like the motions inside a water balloon when massaged. Viscous forces could create shears, which could become shear instabilities, leading to fluid motions that would flux momentum deeper into the fluid. Eventually the fluid could contain all sorts of turbulent motions. The energy of such motions could decay into heat (molecular motions) by diffusion internally.

What is the speed of compression (sound) waves in such a medium?
Infinite, since $\rho_t = 0$ makes the medium infinitely stiff.

2. ***Using this simplified set, transform the majestic equality form of the mass continuity equation into a grasping equation for pressure***

Differentiating the u equation in x, the v equation in y, and the w equation in z, and summing them,

$$\begin{aligned} &[u_{tx} = -(uu)_{xx} - (uv)_{yx} - (uw)_{zx} \quad -\pi_{xx} + fv_x] \\ +&[v_{ty} = -(vu)_{xy} - (vv)_{yy} - (vw)_{zy} \quad -\pi_{yy} - fu_y - uf_y] \\ +&[w_{tz} = -(wu)_{xz} - (wv)_{yz} - (ww)_{zz} \ -\pi_{zz}] \end{aligned}$$

we arrive (using mass continuity to see that the left side is zero) at:

$$0 = -\boldsymbol{\nabla} \cdot [(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}] \ -\boldsymbol{\nabla}^2\pi \quad + f\zeta - u\beta$$

where z is the vertical component of relative vorticity, b is fy, and the result of the problem below has been used to express momentum flux convergence in advective form, with parentheses used to make the result depend on no notation beyond the familiar vector dot product and the vector differentiation operator $\boldsymbol{\nabla} = \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$.

Hungrily solving for π,

$$\pi = \boldsymbol{\nabla}^{-2}[-\boldsymbol{\nabla} \cdot [(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}] + f\zeta - u\beta]$$

Continuity is the Law expressed by the equation for pressure. Since flow divergence is zero to maintain the (assumed) $\rho=\rho_0$ forever ($\rho_t=0$), pressure must intimately cancel the divergent component of any other field of force (in this case only the inward- or outward-directed Coriolis force on horizontally swirling flow, or the divergent Coriolis force on zonal flow on a sphere, or the "inertial force" implied by the transport of momentum). *Pressure is the Enforcer* of mass continuity. Pressure does its work in the momentum equations, so *F = ma is the Mechanism* of that enforcement. The common exercise in dynamics courses of studying flows with the pressure field taken as a given is not very sensible: pressure is a cleanup force, the *last* force logically, the one that intimately responds to the divergence of all the others at the speed of sound. That speed is infinite, which is unrealistic, but it is still faster than all the other information-transmitting waves in more realistic fluids and equation systems so the point remains relevant: *Continuity is the Law, Pressure is the Enforcer,* ***F*** *= m**a** is the Mechanism.*

3. Using your simplified set with $\rho = \rho_0$ again, show that the bodily flux convergence terms can be wrangled into *advection* form. In other words, show from Eqs. (1.4) that the flux divergence terms can be rewritten as $-\mathrm{u}\boldsymbol{V}_x - \mathrm{v}\boldsymbol{V}_y - \mathrm{w}\boldsymbol{V}_z = -(\mathbf{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}$.

Taking just the u component for illustration,

$$-(uu)_x - (uv)_y - (uw)_z =$$
$$-uu_x - vu_y - wu_z - u(u_x + v_y + w_z)$$

The parenthetical term is zero by mass continuity, so *momentum flux convergence is precisely equal to the advection of momentum in an incompressible fluid.*