1.4 Predicates and Quantifiers

Predicate Logic

Predicate logic is an extension of Propositional logic. It adds the concept of predicates and quantifiers to better capture the meaning of statements that cannot be adequately expressed by propositional logic.

Predicates

- Statements involving variables are neither true nor false.
- E.g. "x > 3", "x = y + 3", "x + y = z"

- "x is greater than 3"
  - "x": subject of the statement
  - "is greater than 3": the ***predicate***

We can denote the statement "x is greater than 3" by *P(x),* where *P* denotes the predicate and *x* is the variable.

Once a value is assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.

- Example: Let *P(x)* denote the statement **"x > 3."** What are the truth values of *P*(4) and *P*(2)?

  Solution: *P*(4) – "4 > 3", *true*
  P(2) – "2 > 3", *false*

- Example: Let *Q*(x,y) denote the statement "x = y + 3." What are the truth values of the propositions *Q*(1,2) and *Q*(3,0)?

  Solution: *Q*(1,2) – "1 = 2 + 3" , *false*
  Q(3,0) – "3 = 0 + 3", *true*

- Example: Let *A(c,n)* denote the statement "Computer *c* is connected to network *n*", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

  Solution: *A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false*

  A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

- A statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a ***n*-place predicate** or a ***n*-ary predicate**.

Quantifiers

- **Quantification**: express the extent to which a predicate is true over a range of elements.

- **Universal quantification**: a predicate holds for every element under consideration.

- **Existential quantification**: a predicate holds for one or more element under consideration.

➢ A domain must be specified.

Universal Quantifier

DEFINITION 1

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **Universal Quantifier**. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$." An element for which $P(x)$ is false is called a **counterexample** of $xP(x)$.

Example: Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: *Because $P(x)$ is true for all real numbers, the quantification is true.*

- A statement $\forall x P(x)$ is false, if and only if $P(x)$ is not always true where x is in the domain. One way to show that is to **find a counterexample** to the statement $\forall x P(x)$.

- Example: Let $Q(x)$ be the statement "x < 2". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

  Solution: *$Q(x)$ is not true for every real numbers, e.g. $Q(3)$ is false. $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus the quantification is false.*

- When the elements of the domain are $x_1, x_2 \ldots\ldots x_n$

  $\forall x P(x)$ is the same as the conjunction

  $$P(x_1) \land P(x_2) \land \ldots. \land P(x_n).$$

- **Example:** What does the statement $\forall xN(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: *"Every computer on campus is connected to the network."*

Existential Quantifier

DEFINITION 2

The *existential quantification* of $P(x)$ is the statement

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **Existential Quantifier**.

- The existential quantification $\exists x P(x)$ is read as

  "There is an x such that $P(x)$," or

  "There is at least one x such that $P(x)$," or

  "For some x, $P(x)$."

- Example: Let *P(x)* denote the statement "*x* > 3". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

  Solution: *"x > 3" is sometimes true – for instance when x = 4. The existential quantification is true.*

- $\exists x P(x)$ is false if and only if *P*(x) is false for every element of the domain.
- Example: Let *Q*(x) denote the statement "x = x + 1". What is the true value of the quantification $\exists x Q(x)$, where the domain consists for all real numbers?

  Solution: *Q(x) is false for every real number. The existential quantification is false.*

- If the domain is empty, $\exists x Q(x)$ is false because there can be no element in the domain for which $Q(x)$ is true.
- When the elements of the domain are $x_1, x_2 \ldots\ldots x_n$ existential quantification $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$.

Quantifiers		
Statement	***When True?***	***When False?***
$\forall x P(x)$	*P(x) is true for every x.*	*There is an x for which P(x) is false.*
$\exists x P(x)$	*There is an x for which P(x) is true.*	*P(x) is false for every x.*

a pet.

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$ b) $P(1)$ c) $P(2)$

d) $P(-1)$ e) $\exists x P(x)$ f) $\forall x P(x)$

a) $P(0)$: $0 = 0^2$ True

b) $P(1)$: $1 = 1^2$ True

c) $P(2)$: $2 = 2^2$ False

d) $P(-1)$: $-1 = (-1)^2$ False

e) $\because$ $P(0)$ is true $\Rightarrow \exists x P(x)$ is true

f) $\because$ $P(-1)$ is false $\Rightarrow \forall x P(x)$ is false

12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$

d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$

g) $\forall x \neg Q(x)$ $= \forall x\,(x+1 \le 2x)$ is false. $\because \neg Q(0)$ is false.

Answer –

a) $Q(0)$: $1 > 0$ True

b) $Q(-1)$: $0 > -2$ True

c) $Q(1)$: $2 > 2$ False.

d) $\exists x\, Q(x)$ is true, $\because Q(0)$ is true

e) $\forall x\, Q(x)$ is false, $\because Q(1)$ is false

f) $\exists x \neg Q(x) = \exists x\,(x+1 \le 2x)$ is True. $\because \neg Q(2)$ is true

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n+1>n)$ T

b) $\exists n(2n=3n)$ T

c) $\exists n(n=-n)$ T

d) $\forall n(3n \leq 4n)$ F

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3=-1)$ T

b) $\exists x(x^4<x^2)$ T $\frac{1}{2^4} < \frac{1}{2^2}$

c) $\forall x((-x)^2=x^2)$ T

d) $\forall x(2x>x)$ F

15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ T

b) $\exists n(n^2=2)$ F

c) $\forall n(n^2 \geq n)$ T

d) $\exists n(n^2<0)$ F

16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x(x^2=2)$ T

b) $\exists x(x^2=-1)$ F

c) $\forall x(x^2+2 \geq 1)$ T

d) $\forall x(x^2 \neq x)$ F

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers $-2, -1, 0, 1$, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ **b)** $\forall x P(x)$ **c)** $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ **e)** $\neg \exists x P(x)$ **f)** $\neg \forall x P(x)$

$D = \{-2, -1, 0, 1, 2\}$

a) $\exists x P(x) \equiv P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)$

b) $\forall x P(x) \equiv P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)$

c) $\exists x \neg P(x) = \neg P(-2) \vee \neg P(-1) \vee \neg P(0) \vee \neg P(1) \vee \neg P(2)$

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$ **b)** $\forall x N(x)$ **c)** $\neg \exists x N(x)$
d) $\exists x \neg N(x)$ **e)** $\neg \forall x N(x)$ **f)** $\forall x \neg N(x)$

D= all students in your school.

Answer:

a) Some student in the school has visited North Dakota.

b) Every student in the school has visited North Dakota.

c) No student in the school has visited North Dakota.

d) Some student in the school has not visited North Dakota.

e) Not all students in the school have visited North Dakota.

f) All students in the school have not visited North Dakota.

➓ Uniqueness quantifier $\exists !$ or $\exists_1$

- $\exists !xP(x)$ or $\exists_1 P(x)$ states

  "There exists a unique x such that *P*(x) is true."

Example: □ !x(x − 1 = 0), where the domain is the set of real numbers, states that there is a unique real number x such that x − 1 = 0.

This is a true statement, as x = 1 is the unique real number such that x − 1 = 0.

52. As mentioned in the text, the notation $\exists! x\, P(x)$ denotes

"There exists a unique x such that $P(x)$ is true."

If the domain consists of all integers, what are the truth values of these statements?

a) $\exists! x(x > 1)$

b) $\exists! x(x^2 = 1)$

c) $\exists! x(x + 3 = 2x)$

d) $\exists! x(x = x + 1)$

Answer:

a) This is false, since there are many values of x that make $x > 1$ true.

b) This is false, since there are two values of x that make $x^2 = 1$ true.

c) This is true, we see that the unique solution to the equation is $x = 3$.

d) This is false, since there are no values of x that make $x = x + 1$ true.

Quantifiers with restricted domains

Example: What do the following statements mean? The domain in each case consists of real numbers.

- $\forall x < 0\ (x^2 > 0)$: For every real number x with $x < 0$, $x^2 > 0$. "The square of a negative real number is positive." It's the same as $\forall x(x < 0 \rightarrow x^2 > 0)$

- $\forall y \neq 0\ (y^3 \neq 0)$: For every real number y with $y \neq 0$, $y^3 \neq 0$. "The cube of every non-zero real number is non-zero." It's the same as $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

- $\exists z > 0\ (z^2 = 2)$: There exists a real number z with $z > 0$, such that $z^2 = 2$. "There is a positive square root of 2." It's the same as $\exists z(z > 0 \land z^2 = 2)$.

➓ Precedence of Quantifiers

- $\forall$ and $\exists$ have higher precedence than all logical operators.
- E.g. $\forall x P(x) \lor Q(x)$ is the same as $(\forall x P(x)) \lor Q(x)$.

$\neq \forall x (P(x) \lor Q(x))$

➓ De Morgan's laws for Quantifiers

- $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

① Show that $\neg\left(\forall x P(x)\right) \equiv \exists x\left(\neg P(x)\right)$

$\neg\left(\forall x P(x)\right)$ is true

$\iff \forall x P(x)$ is false.

$\iff$ there exists x so that $P(x)$ is false.

$\iff$ " " " " $\neg P(x)$ is true

$\iff \exists x \neg P(x)$ is true.

$\Rightarrow \neg\left(\forall x P(x)\right) \equiv \exists x\left(\neg P(x)\right)$

② Show that —

$$\neg(\exists x\, P(x)) \equiv \forall x\, (\neg P(x))$$

$\neg(\exists x\, P(x))$ is true.

$\iff \exists x\, P(x)$ is false.

$\iff P(x)$ is false for all x.

$\iff \neg P(x)$ is true for all x.

$\iff \forall x\, \neg P(x)$ is true.

EXAMPLE 21 What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

① $\neg \forall x(x^2 > x)$

$\equiv \exists x \neg (x^2 > x)$

$\equiv \exists x (x^2 \leq x)$

② $\neg \exists x(x^2 = 2)$

$\equiv \forall x \neg (x^2 = 2)$

$\equiv \forall x (x^2 \neq 2)$

EXAMPLE 22 Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

$$\neg \forall x (P(x) \to Q(x))$$

$$\equiv \exists x \neg (P(x) \to Q(x))$$

$$\equiv \exists x \underbrace{\neg(\neg P(x) \vee Q(x))}$$

$$\equiv \exists x (\neg\neg P(x) \wedge \neg Q(x))$$

$$\equiv \exists x (P(x) \wedge \neg Q(x))$$

$p \to q$

$\equiv \neg p \vee q$

43. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

Let D be domain.

$a, b \in D$

s.t. $P(a) — T$, $P(b) — F$, $\underset{— F}{Q(a)}$

$\Rightarrow \underset{F}{\forall x P(x)} \longrightarrow \forall x Q(x)$ is True.

$P(a) \longrightarrow Q(a)$ is false.

$\Rightarrow \forall x (\overset{T}{P(x)} \rightarrow \overset{F}{Q(x)})$ is false.

$\Rightarrow$ not logically equivalent.

Translating from English into Logical Expressions

⓾ Example: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution:

If the domain consists of students in the class –

$$\forall x C(x)$$

where C(x) is the statement "x has studied calculus.

If the domain consists of all people –

$$\forall x(S(x) \rightarrow C(x))$$

where S(x) represents that person x is in this class.

24. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

D_1 = all students
D_2 = all people.

a) Everyone in your class has a cellular phone.

I $\forall x \, P(x)$ $x \in D_1$

where $P(x)$: x has a cellu. phone

II $\forall x \, (C(x) \rightarrow P(x))$ $x \in D_2$

$C(x)$: x is in your class.

b) Somebody in your class has seen a foreign movie.

$F(x)$: x has seen a foreign movie.

I $\exists x\, F(x)$ $\quad x \in D_1$

II $\exists x\,(C(x) \wedge F(x))$ $\quad x \in D_2$

c) There is a person in your class who cannot swim.

$S(x)$: x can swim.

I $\exists x\, \neg S(x)$ $\quad x \in D_1$

II $\exists x\,(C(x) \wedge \neg S(x))$ $\quad x \in D_2$

d) All students in your class can solve quadratic equations.

$Q(x)$: x can solve qua-eqn.

I $\quad \forall x\, Q(x) \qquad x \in D_1$

II $\quad \forall x\,(C(x) \longrightarrow Q(x)) \qquad x \in D_2$

e) Some student in your class does not want to be rich.

$R(x)$: x want to be rich.

I $\quad \exists x\, \neg R(x) \qquad x \in D_1$

II $\quad \exists x\,(C(x) \wedge \neg R(x)) \qquad x \in D_2$

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret, but not a dog.

d) No student in your class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

a $\exists x(C(x) \land D(x) \land F(x))$

b $\forall x(C(x) \lor D(x) \lor F(x))$

c $\exists x(C(x) \land F(x) \land \neg D(x))$

d $\neg(\exists x(C(x) \land D(x) \land F(x)))$

e $\exists x C(x) \land \exists y D(y) \land \exists z F(z)$

28. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.

b) All tools are in the correct place and are in excellent condition.

c) Everything is in the correct place and in excellent condition.

d) Nothing is in the correct place and is in excellent condition.

e) One of your tools is not in the correct place, but it is in excellent condition.

D = all things.

$C(x)$: x is in correct place.

$E(x)$: x is in excellent condition

$T(x)$: x is a Tool.

a) $\exists x \neg C(x)$

b) $\forall x (T(x) \longrightarrow (C(x) \land E(x)))$

c) $\forall x (C(x) \land E(x))$

d) $\forall x \neg (C(x) \land E(x))$

e

$$\exists! x\,(T(x) \wedge \neg C(x) \wedge E(x))$$

EXAMPLE 20 What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

1. D = set of all politician

$H(x)$: x is honest

$\neg H(x)$: x is dishonest

$\exists x\, H(x)$

$\neg \exists x (H(x)) \equiv \forall x\, \neg (H(x))$

every politician is dishonest.

All Americans eats cheesburgers.

$D =$ Americans People

$C(x)$: x eat cheesburgers

$\forall x \, C(x)$

$\neg(\forall x \, C(x)) \equiv \exists x \neg C(x)$

some American does not eat cheesburgers

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ **b)** $\forall x P(x)$

c) $\neg\exists x P(x)$ **d)** $\neg\forall x P(x)$

e) $\underline{\forall x((x \neq 3) \rightarrow P(x))} \vee \exists x \neg P(x)$

$D = \{1, 2, 3, 4, 5\}$

a) $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

b) $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

c) $\neg(P(1) \vee P(2)$ ______ $\vee P(5))$

d) $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5))$

e) $(P(1) \wedge P(2) \wedge P(4) \wedge P(5)) \vee$
$(\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$

b) $\forall x P(x)$

c) $\forall x ((x \neq 1) \to P(x))$

d) $\exists x ((x \geq 0) \wedge P(x))$

e) $\exists x (\neg P(x)) \wedge \forall x ((x < 0) \to P(x))$

$D = \{-5, -3, -1, 1, 3, 5\}$

a) $P(-5) \vee P(-3) \vee P(-1) \vee P(1) \vee P(3) \vee P(5)$

b) $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(1) \wedge P(3) \wedge P(5)$

c) $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(3) \wedge P(5)$

d) $P(1) \vee P(3) \vee P(5)$

e) $(\neg(P(-5)) \vee \neg P(-3) \ldots \vee \neg P(5)) \wedge$

$(P(-5) \wedge P(-3) \wedge P(-1))$

25. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

D = all people
P(x): x is perfect

a) No one is perfect.

$\forall x \neg P(x)$

b) Not everyone is perfect.

$\neg(\forall x P(x))$

$M(x)$: x is your friend

c) All your friends are perfect.

$$\forall x\,(M(x) \rightarrow P(x))$$

d) At least one of your friends is perfect.

$$\exists x\,(M(x) \land P(x))$$

e) Everyone is your friend and is perfect.

$$\forall x\,(M(x) \wedge P(x))$$

f) Not everybody is your friend or someone is not perfect.

$$(\neg \forall x\, M(x)) \vee (\exists x\, \neg P(x)))$$

22. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone speaks Hindi.
b) There is someone older than 21 years.
c) Every two people have the same first name.
d) Someone knows more than two other people.

a) D_T = Some particular families in Lucknow

D_F = Resident from America

b) D_T = all people

D_F = students in 1st grade.

(c) D_T = particular collection of people having same 1^{st} name.

D_F = all people

(d) D_T = all people

D_F = babies born in last 10 minutes.