**CS 4510: Automata and Complexity** 08/23/2022-09/15/2022

# Lecture EXAM 1: Compiled Notes (Lecture 1-8)

*Lecturer: Zvi Galil* *Author: Austin Peng*

## Contents

# 1 Lecture 1: Introduction

## 1.1 Topics To Be Familiar With:

### 1.1.1 Sets

- sets are elements drawn from some universe:
  - $\mathbb{Z}$ (integers)
  - $\mathbb{N}$ (natural numbers)
  - $\mathbb{R}$ (real numbers)
  - $\mathbb{C}$ (complex numbers)
- set operations:
  - union: $A \cup B$
  - intersection: $A \cap B$
  - complement: $\overline{A}$
  - Cartesian product: $A \times b$ (pairs)
  - power: $A^k$ (k-tuples)

### 1.1.2 Functions

- functions are mappings from one set to another:
  - $f : D \to R$
    $f(x) = x^2$
  - $f : Z \times Z \to Z$
    $f(x, y) = x + y$

### 1.1.3 Relations

- relation $R$ can be written as $xRy$
  - **reflexive**: $xRx$
  - **symmetric**: $xRy \implies yRx$
  - **transitive**: $xRy$ and $yRz \implies xRz$
  - **equivalence**: if relation R is reflexive, symmetric, and transitive

### 1.1.4 Graphs

- a graph $G = (V, E)$ has vertices $V$ and edges $E$
- a graph may be directed or undirected

### 1.1.5 Strings

- "abc" is a string
- a string is a sequence of symbols from an alphabet $\Sigma$
- $\Sigma^*$ is the set of all strings over $\Sigma$
- $\Sigma^*$ contains the empty string $\varepsilon$, all strings of length 1, all strings of length 2, etc.

### 1.1.6 Boolean Logic

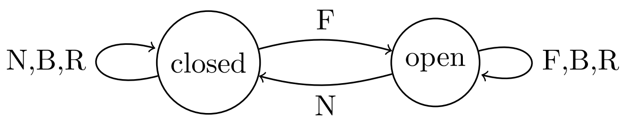
- 3 main operations: "and" ($\wedge$), "or" ($\vee$), "not" ($\neg$)
- these operations are performed on variables and constants (true and false)

### 1.1.7 Proofs

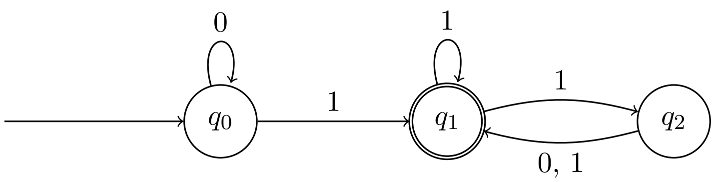
- mathematics consists of definitions, statements/lemmas/theorems, and proofs
- 3 common types of proofs:
  - proof by construction
  - proof by contradiction
  - proof by induction
- these operations are performed on variables and constants (true and false)

## 1.2 Finite Automata and (Regular Languages)

Consider an automatic door with the inputs $\{\mathrm{F(ront)}, \mathrm{N(either)}, \mathrm{B(oth)}, \mathrm{R(ear)}\}$. The door has 2 states: open and closed. The door will only open if there is someone in front and not in the rear (door swings inward). If the door is open, but nobody is there, the door will close.




**Example 1.** $M_1$




- $q_0$ is the starting state
- $q_1$ is the accepting statements

- all transitions ($\delta = 0, 1$) are defined for all states
- let x be a binary string 1101, $M_1$ will accept $x$ because it ends at $q_1$
- $L(M_1) =$ all $x$ such s.t $M_1$ accepts $x$ if $x$ has a 1 AND either ends with a 1 or with an even number of 0s

#### 1.2.1 Formal Definition Of A Deterministic Finite Automaton

**Definition 1.** A deterministic finite automaton is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where:

- $Q$ is a finite set called the states
- $\Sigma$ is a finite set called the alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0$ is the initial state
- $F \subseteq Q$ is the accepting states

Note: $\delta(q, a) = p$ means that if DFA is in state $q$ and sees $a$, it goes to state $p$

### 1.2.2 Formal Definition Of Computation

**Definition 2.** A **computation** of a deterministic finite automaton (DFA) $M$ on an input string $x = x_1, ..., x_n \in \Sigma^*$ is a sequence of states $q_0, ..., q_n \in Q$ s.t. for $i = 0, ..., n-1$ $\delta(q_i, x_{i+1}) = q_{i+1}$. We say that $M$ accepts $x$ if $q_n \in F$. If $q_n \notin F$, then we say $M$ rejects $x$.

### 1.2.3 Formal Definition Of A Language

**Definition 3.** The **language** of $M$ is defined as the set $\{x \in \Sigma^* : M \text{ accepts } x\}$. A language is a set of strings.

**Example 2.** $\emptyset, \{\varepsilon\}$ are both langauges.

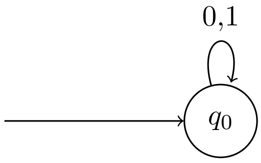
**Definition 4.** A language $L$ is called **regular** if there is a DFA accepting $L$.

**Example 3.** $L_w = \{w\}$ is regular. $L'_w = \{\text{all strings that end in } w\}$ is also regular.

# 2 Lecture 2: Deterministic Finite Automata

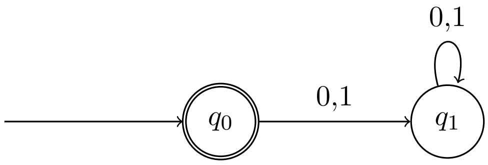
## 2.1 DFA Examples

**Example 4.** $M_2$



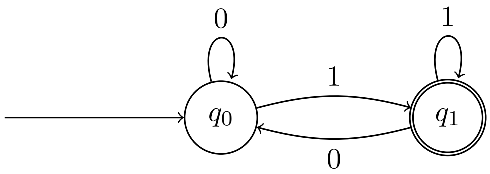

$L(M_1) = \varphi$. The language is empty because there are no accepting states.

**Example 5.** $M_2$



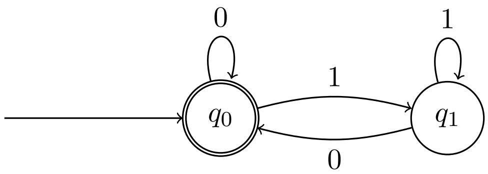

$L(M_2) = \{\varepsilon\}$.
Note the difference between $M_1$ and $M_2$. They recognize different langauges.

**Example 6.** $M_3$



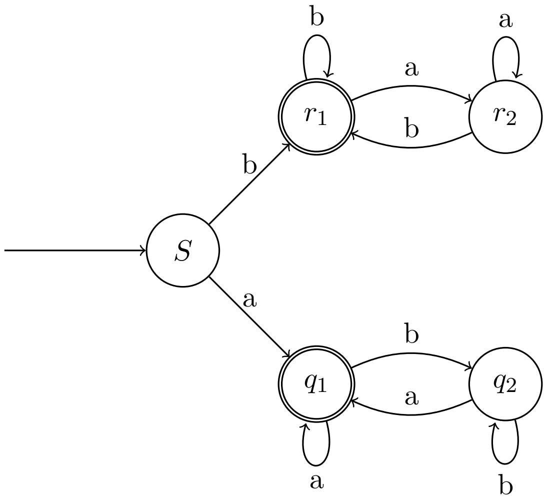

$L(M_3) = \{w|w \text{ ends in a } 1\}$

**Example 7.** $M_4$




$L(M_4) = \{\varepsilon \cup \text{ strings ending with } 0\}$. Note this is the complement of $L(M_3)$.

**Example 8.** $M_5$




$L(M_5)$ is the set of all strings that start and end with the same character. Note: $\Sigma = \{a, b\}$

## 2.2 Applications Of DFA

### 2.2.1 Modular Arithmetic

Let $w \in \{0,1\}^*$ (aka any binary string). We define $\overline{w}$ to be the value of the string as a binary number. Then, for $w \in \{0,1\}^*$ and $a \in \{0,1\}$, we have the following properties:

- $\overline{a} = a$
- $\overline{wa} = 2\overline{w} + a$

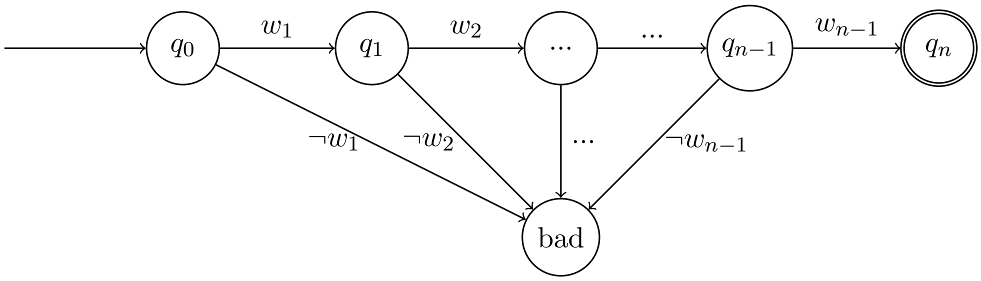
We can use a DFA to recognize modular arithmetic. For the following example, we will consider the following transition table of $\overline{w}$ mod 3. Note that the start state of our transition table is marked with an arrow.

| input $a$ / $\overline{w}$ mod 3 | $\overline{w0}$ mod 3 | $\overline{w1}$ mod 3 | state |
|---|---|---|---|
| 0 (state $q_0$) | 0 | 1 | $\rightarrow q_0$ |
| 1 (state $q_1$) | 2 | 0 | $q_1$ |
| 2 (state $q_2$) | 1 | 2 | $q_2$ |

If we set the accepting state to be $q_1$ then this DFA will accept exactly those strings which are $\equiv 1$ modulo 3 (aka congruent to 1 modulo 3).

### 2.2.2 String Matching: Recognizing A Single String

For a string $w$, we can create a DFA for the language $L_w\{w\}$ as follows:


$q_0$
$w_1$
$q_1$
$w_2$
...
...
$q_{n-1}$
$w_{n-1}$
$q_n$
$\neg w_1$
$\neg w_2$
...
$\neg w_{n-1}$
bad

### 2.2.3 String Matching: Recognizing A Suffix

Let $L'_w$ be the set of strings that end in $w$. An example string from this language is $1101001 \in L'_{001}$, because it ends in 001. We can use the following transition table:

| $Q$ \ input | 0 | 1 |
|---|---|---|
| $\rightarrow$ bad | $q_0$ | bad |
| $q_0$ | $q_{00}$ | bad |
| $q_{00}$ | $q_{00}$ | $q_{001}$ |
| $q_{001}$ | $q_0$ | bad |

We define $q_{001}$ to be our only accepting state.

In the general case, we need to keep track of the longest suffix seen so far. We will use the states $\{\text{bad}, q_0, ..., q_n\}$.

The DFA will be in state $q_i$ if $w_1...w_i$ is the longest suffix of the input seen so far that is a prefix of $w$. If we are in state $q_i$, then we have to see $n - i$ more symbols until we find the string. The transition function is defined as follows:

- $\delta(q_{i-1}, w_i) = q_i$
- $\delta(q_{i-1}, a \neq w_i) = q_j$, where $w_1 w_2 ... w_j$ is the largest prefix of $w$ that is a suffix of the current input (including $a$).

# 3 Lecture 3: Operations On Languages

## 3.1 Operations On Languages

### 3.1.1 The Regular Operations: Union, Concatenation, Kleene Star

**Definition 5.** Let $A$ and $B$ be languages. We define the regular operations union, concatenation, and Kleene star as follows:

- **union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
- **Kleene star**: $A^* = \{x_1 x_2 ... x_n \mid k \geq 0 \text{ and each } x_i \in A\}$
  - $A^* = \bigcup_{k \geq 0} A^k$

Note: $A^+ = A^* - \{\varepsilon\}$.

**Example 9.**

$$A = \{\text{good}, \text{bad}\}, B = \{\text{boy}, \text{girl}\}$$

$$A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$$

$$A \circ B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

$$A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badgood}, \text{badbad}, ...\}$$

**Theorem 1.** The class of regular languages is closed under the union operation. That is, if $A$ and $B$ are regular languages, so is $A \cup B$.

*Proof.* Let $M_1$ recognize $A_1$, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2$ recognize $A_2$, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct $M$ to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q, F)$.

1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$
   - this set $Q$ is essentially $Q_1 \times Q_2$, the Cartesian product of sets $Q_1$ and $Q_2$
   - it is the set of all pairs of states: the first from $Q_1$ and the second from $Q_2$
2. $\Sigma = \Sigma$, the same as in $M_1$ and $M_2$
   - for simplicity, assume $M_1$ and $M_2$ have the same alphabet $\Sigma$
   - the theorem remains true if they have different alphabets $\Sigma_1$ and $\Sigma_2$
   - then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$
3. $\delta$ (transition function) is defined as follows:
   - for each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
4. $q_0$ is the pair $(q_1, q_2)$
5. $F$ is the set of pairs in which either member is an accepting state of $M_1$ or $M_2$ as follows:
   - $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$
   - the above expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
   - Note: $F = (F_1 \times F_2)$ IS NOT CORRECT! (takes intersection instead of union)

□

## 3.2 Other Regular Operations

### 3.2.1 Complement

**Theorem 2.** The class of regular languages is closed under the complement operation.

*Proof.* Let $L$ be a regular language, then some finite automaton $M$ recognizes $L$.

Let $\overline{M}$ be the same as $M$, but with the accepting and non-accepting states interchanged. Then $\overline{M}$ accepts a string $x$ if and only if $M$ does not accept $x$. So, $L(\overline{M}) = \overline{L}$.

□

### 3.2.2 Intersection

**Theorem 3.** If $A_1$ and $A_2$ are regular languages, then so is $A_1 \cap A_2$.

*Proof.* Let $M_1 = (Q^1, \Sigma, \delta^1, q_0^1, F^1)$ decide $A_1$ and $M_2 = (Q^2, \Sigma, \delta^2, q_0^2, F^2)$ decide $A_2$.

We construct the automaton $M = (Q, \Sigma, \delta, q_0, F)$ as follows:

- $Q = Q^1 \times Q^2$ (each state in $M$ is a pair of states in $M_1$ and $M_2$)
- $\Sigma$ is the same shared alphabet as $M_1$ and $M_2$
- $\delta((r_1, r_2), x) = (\delta^1(r_1, x), \delta^2(r_2, x))$
- $q_0 = (q_0^1, q_0^2)$
- $F = F^1 \times F^2$ (both $M_1$ and $M_2$ must be in an accepting state for $M$ to accept)

□

### 3.2.3 Set Difference

**Theorem 4.** If $A$ and $B$ are regular languages, then so is $A_1 \backslash A_2 = \{x \mid x \in A \text{ and } x \notin B\}$.

*Proof.* Note: $A \backslash B = A \cap \overline{B}$

Since regular languages are closed under intersection and complement, regular languages are closed under subtraction.

□

### 3.2.4 Symmetric Difference

**Theorem 5.** If $A$ and $B$ are regular langauges, then so is $A \oplus B$.

*Proof.* Note: $A \oplus B = (A \cup B) \backslash (A \cap B)$

Since regular languages are closed under union, intersection, and subtraction, regular languages are closed under symmetric difference.

□

## 3.3 Closed Operations

A set $S$ is closed under an operation $\cdot$ if for every $a, b \in S, a \cdot b \in S$. That is, if we apply the operation to any two element in the set, we another element in the same set.

- $\mathbb{N}$ is closed under addition
- $\mathbb{N}$ is not closed under subtraction (ex. $3 - 5 = -2 \notin \mathbb{N}$)
- $\mathbb{Z}$ is closed under addition, subtraction, multiplication, but not division.

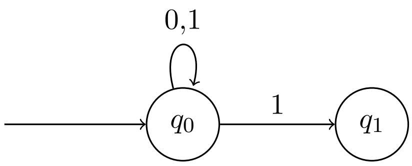
- $\mathbb{Q}$ is closed under addition, subtraction, multiplication, and $\mathbb{Q}\backslash\{0\}$ is closed under division.
- $\mathbb{R}$ is not closed under square root, but $\mathbb{R}^+$ is closed under square root.
  - $\mathbb{R}$ has negative numbers, while $\mathbb{R}^+$ does not.
- $\mathbb{C}$ is closed under square root (because it includes imaginary numbers).

# 4 Lecture 4: Non-Determinism

## 4.1 Non-Determinism

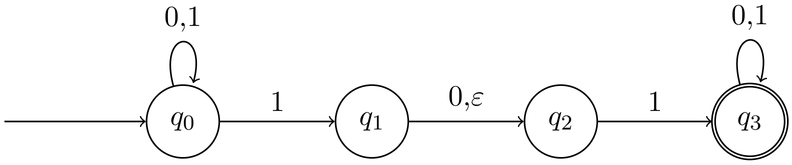
- Our definition of finite automaton so far is deterministic (the transition from each state is well-defined for every symbol in the alphabet, and every symbol goes to exactly 1 other state).
- 2 main differences between a non-deterministic finite automaton (NFA) and deterministic finite automaton (DFA).
  - from a state, when we see a symbol, we can go to 0 or more states
  - we have "$\varepsilon$"-transitions, which are transitions that we can take without seeing any symbol
- The starting, accepting, and states all work the same. The big difference is in $\delta$ (transition function).

**Example 10.** Consider the following NFA:




- If we are at $q_0$ and see a 1, we do not have to go to $q_1$.
- Rather, think of it like we are in many states at once. (referred to as "guessing")
- In other words, we say that from $q_0$, we can go to several states.

**Example 11.** Consider the following NFA:



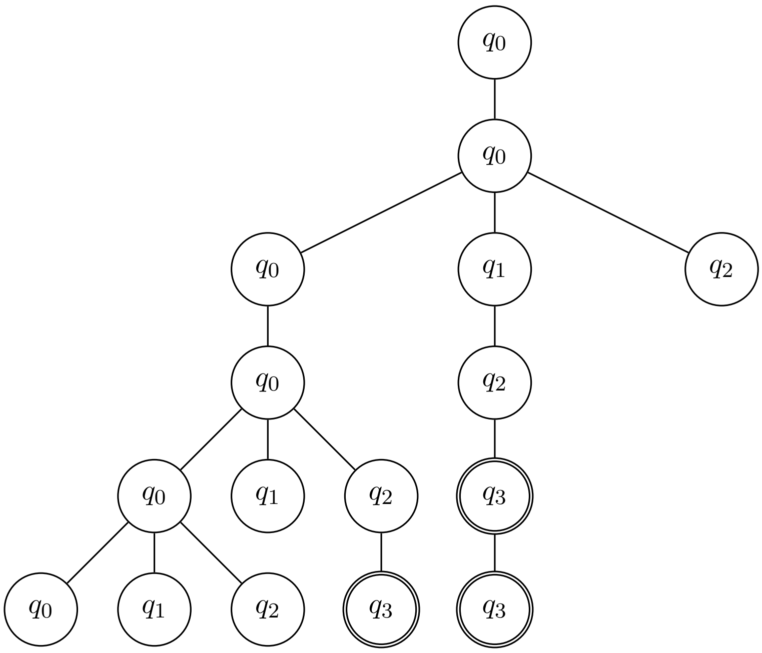

Suppose at $q_1$ and we see a 1. There is no transition with 1, so we "abort". At $q_1$ we see $\varepsilon$ and move to $q_2$. Note: we always see $\varepsilon$, so without doing anything at $q_1$, we "slide" to $q_2$.

Below is a transition table describing the NFA.

| $Q$ \ symbol | 0 | 1 | $\varepsilon$ |
|---|---|---|---|
| $\rightarrow q_0$ | $q_0$ | $q_0, q_1$ | - |
| $q_1$ | $q_2$ | - | $q_2$ |
| $q_2$ | - | $q_3$ | - |
| $q_3$ | $q_3$ | $q_3$ | - |

Think of split timelines every time we have to go to multiple states. From $q_0$ seeing 1, we can go back to $q_0$ or go to $q_1$. **Note that we accept the input if at least one of the timelines ends at an accepting state.**

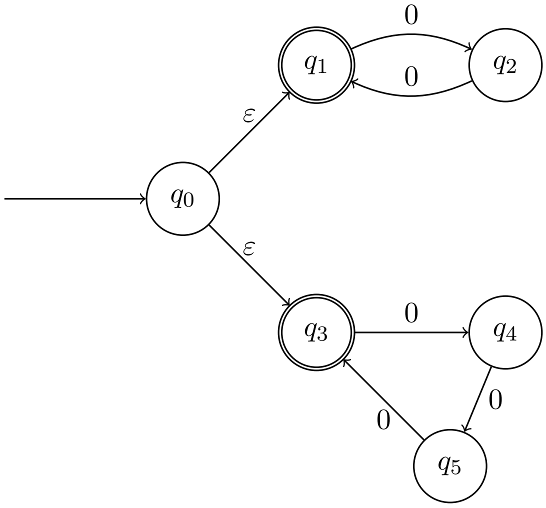
Below is a decision parse tree for input $x = 01011$



The nodes at each layer represent the states we are in at that time (given the input). At $q_0$, we see 0 and can only go back to $q_0$ At $q_1$, we see 1 and can go to $q_0$, $q_1$, or slide with $\varepsilon$ to $q_2$. And repeat for the remaining binary values in the input string $x$. Once we finish, as long as one state in the final depth is accepting, the string $x$ is accepted.

## 4.2 Unary Language

**Example 12.** $M_1$




This NFA guesses if the input string length is divisible by either 2 or 3 by guessing to go higher $q_1$ or lower $q_3$.

## 4.3 Nondeterministic Finite Automaton

### 4.3.1 Formal Definition Of A Nondeterministic Finite Automaton

**Definition 6.** A **nondeterministic finite automaton** is a 5-tuple $\{Q, \Sigma, \delta, q_0, F\}$ where:

- $Q$ is a finite set of states
- $\Sigma$ is a finite alphabet
- $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ is the transition function, where $\mathcal{P}(Q)$ is the power set of $Q$
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states

Note: the only change in the definition compared to a DFA is the transition function $\delta$. $\delta$ is now a set of states.

We say $M$ accepts $w$ if $w$ can be written as $w = y_1...y_n$ such that $y_i \in \Sigma_\varepsilon$ and there are states $r_0, R_1, ..., r_m$ such that $r_0 = q_0, r_m \in F,$ and $r_{i+1} \in \delta(r_i, y_{i+1})$ for $0 \leq i \leq m - 1$.

### 4.3.2 Equivalence Of NFAs and DFAs

**Corollary 1.** A language is regular if an only if some nondeterminstic finite automaton recognizes it.

**Theorem 6.** Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

*Proof Idea.* As a **corollary**, the set of languages NFAs can accept ($N$) is equal to the set of languages that DFAs can accept ($D$). The first part of the proof is to note that every DFA is an NFA, so $D \subseteq N$. Then to show that $N \subseteq D$, we say for each NFA $N_i \in N$ we can construct an equivalent DFA, implying that $N_i \in D$. Together this implies that $N = D$.

Note: in terms of power of these two computing structures, this proof shows that there is no difference. But in terms of size and simplicity, NFAs have the advantage (since an equivalent DFA must have at least $2^k$ states).

**Theorem 7.** Let $L_k$ be the regular language over $\{0, 1\}$ which contains all strings which have a 1 as the k'th character from the right end of the string. Any DFA that recognizes $L_k$ must contain at least $2^k$ states.

*Proof.* Suppose that $D$ is a DFA for $L_k$ containing strictly fewer than $2^k$ states. Then, by the pigeonhole principle, there must be a state $q$ such that two different binary strings of length $k, x,$ and $y$, both cause the machine to end in state $q$. But if $x$ and $y$ are different, there is at least one index i such that $x[i] = 0$ and $y[i] = 1$ or vice versa. But then $x0^{i-1}$ and $y0^{i-1}$ cause the machine to end in the same state, yet one must be accepted and the other must be rejected. This is a contradiction, so our assumption that there was a DFA for $L_k$ with strictly fewer than $2^k$ states is incorrect.

□

# 5 Lecture 5: NFA To DFA

## 5.1 Equivalence Of NFAs and DFAs (cont.)

**Theorem 8.** Every NFA has an equivalent DFA.

*Proof.* Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language $A$. We construct DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing $A$. Before doing the full construction, first consider the easier case when $N$ has no $\varepsilon$ arrows. We will take $\varepsilon$ into account later.

1. $Q' = \mathcal{P}(Q)$ (the set of subsets of $Q$)

   Every state of $M$ is a set of states of $N$.

2. $\Sigma$ (the alphabet) doesn't change

3. For $R \in Q'$, and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$

   If $R$ is a state of $M$, it is also a set of states of $N$. When $M$ reads a symbol $a$ in state $R$, it shows where $A$ takes each state in $R$. Because each state may go to a set of states, we take the union of all these sets.

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

4. $q_0' = \{q_0\}$

   $M$ starts in the state corresponding to the collection containing just the start state of $N$.

5. $F' = \{R \in Q' \mid R \text{ contains an accepting state of } N\}$

   The machine $M$ accepts if one of the possible states that $N$ could be in at this point is an accepting state.

Now consider the $\varepsilon$ arrows. For any state $R$ of $M$, we define $E(R)$ to be the collection of states that can be reached from members of $R$ by going only along $\varepsilon$ arrows, including members of $R$ themselves. Formally, for $R \subseteq Q$, let

$$E(R) = \{q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}$$

Then we modify the transition function of $M$ to place additional fingers on all states that can be reached by going along $\varepsilon$ arrows after every step. Replacing $\delta(r, a)$ by $E(\delta(r, a))$ achieves this. Finally, we need to modify the start state of $M$ to move the fingers initially to all possible states that can be reached from the start state of $N$ along the $\varepsilon$ arrows.

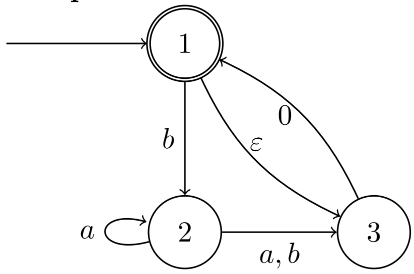
The changes mentioned above to account for $\varepsilon$ arrows are shown below:

3. $\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}$

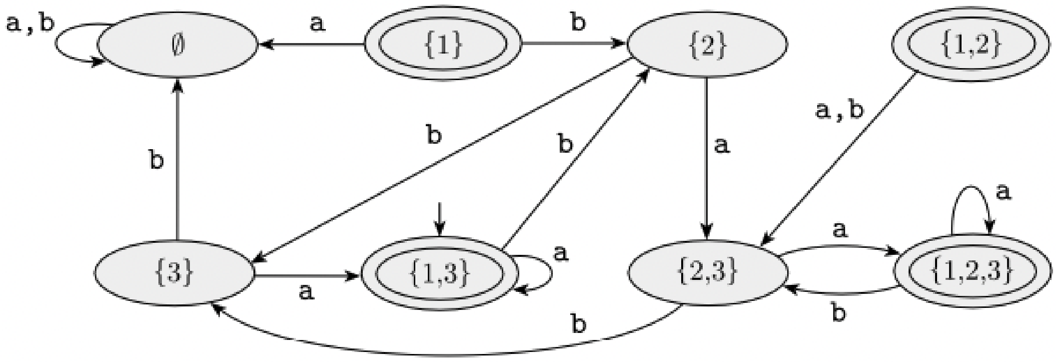
4. $q_0' = E(\{q_0\})$

$\square$

**Example 13.** Consider the following NFA $N$:




Note that the DFA will have 8 states, one for each subset of the states of $N$. The DFA and its transitions are shown below:




The NFA's start state is 1, so the DFA's start state is $E(\{1\}) = \{1, 3\}$ (the set of states reachable from 1 by travelling along $\varepsilon$ arrows and 1 itself). The NFA's accepting state is 1, so the DFA's accepting states are all sets of states that include 1: $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

As for $D$'s transition function, each of $D$'s states goes to one place on input $a$ and one place on input $b$ (by definition of DFA). We will illustrate a few.

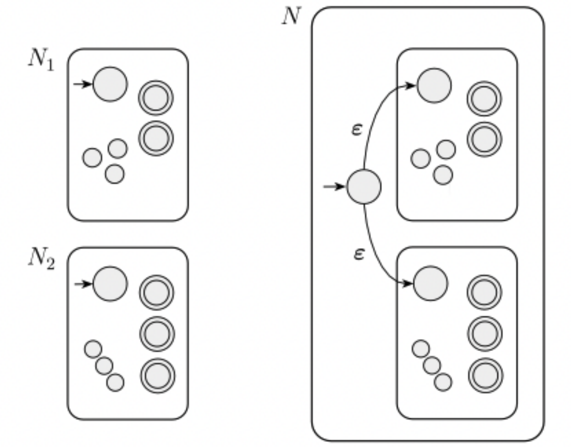
- in $D$, state $\{2\}$ goes to $\{2, 3\}$ on input $a$ because in $N$, state 2 goes to both 2 and 3 on input $a$.
- in $D$, state $\{1\}$ goes to $\emptyset$ on input $a$ because no $a$ arrows exit it.
- in $D$, state $\{1, 2\}$ goes to $\{2, 3\}$ on input $a$ because in $N$, state 1 goes nowhere on input $a$ and state 2 goes to both 2 and 3 on input $a$

NFA with $n$ states $\rightarrow$ DFA with $2^n$ states.

## 5.2 Closure Under Regular Operations

**Theorem 9.** The class of regular languages is closed under the union operation.

*Proof Idea.* We have regular languages $A_1$ and $A_2$ and want to prove that $A_1 \cup A_2$ is regular. The idea is to take 2 NFAs, $N_1$ and $N_2$, that accept $A_1$ and $A_2$ respectively, and combine them into a single new NFA $N$.




*Proof.* Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$

   The states of $N$ are all the states of $N_1$ and $N_2$ with the addition of a new start state $q_0$.

2. The state $q_0$ is the start state of $N$.

3. The set of accepting states $F = F_1 \cup F_2$.

   The accepting states of $N$ are all the accepting states of $N_1$ and $N_2$. That way, $N$ accepts if either $N_1$ or $N_2$ accepts.

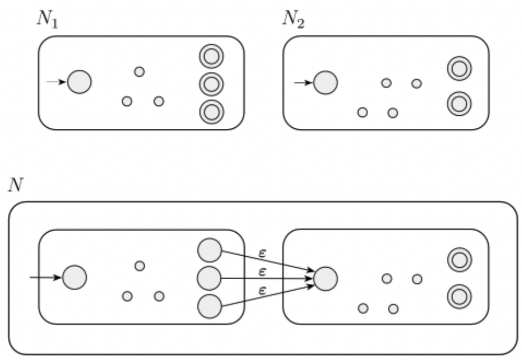
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

□

**Theorem 10.** The class of regular languages is closed under the concatenation operation.

*Proof Idea.* We have regular languages $A_1$ and $A_2$ and want to prove that $A_1 \circ A_2$ is regular. The idea is to take 2 NFAs, $N_1$ and $N_2$, and combine them into a single new NFA $N$ (like how we did in the union closure proof, but with a few changes).




*Proof.* Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \circ A_2$.

1. $Q = Q_1 \cup Q_2$

   The states of $N$ are all the states of $N_1$ and $N_2$.

2. The state $q_1$ is the start state of $N_1$.

3. The set of accepting states $F_2$ are the same as the accepting states of $N_2$.

4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

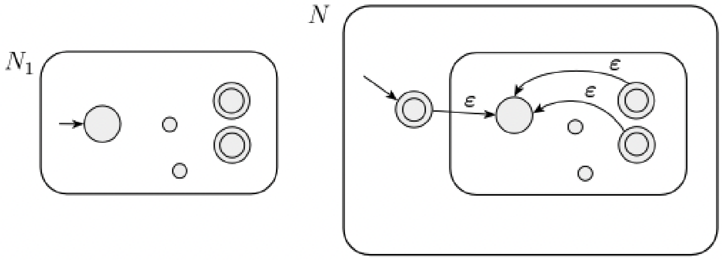
$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a \in \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

□

**Theorem 11.** The class of regular languages is closed under the Kleene star operation.

*Proof Idea.* We have a regular language $A$ and want to prove that $A_1^*$ also is regular. We take an NFA $N_1$ for $A_1$ and modify it to recognize $A_1^*$ as shown. The resulting NFA $N$ will accept its input whenever it can be broken into several pieces and $N$ accepts each piece.

We can construct $N$ like $N_1$ with additional $\varepsilon$ arrows returning to the start state from the accepting states. This way when the processing gets to the end of a piece that $N_1$ accepts, the machine $N$ has the option of jumping back to the start state to try to read another piece that $N_1$ accepts.




*Proof.* Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1^*$.

1. $Q = \{q_0\} \cup Q_1$

   The states of $N$ are the states of $N_1$ plus a new start state.

2. The state $q_0$ is the new start state.

3. $F = \{q_0\} \cup F_1$

   The accepting states are the old accepting states plus the new start state.

4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \notin \varepsilon \\ \delta_1(q, a) \cup \{q_1\} & q \in F_1 \text{ and } a \in \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a \in \varepsilon \\ \emptyset & q = q_0 \text{ and } a \notin \varepsilon \end{cases}$$

□

# 6 Lecture 6: Regular Expressions

## 6.1 Regular Expressions

We have seen two different ways to define regular languages: DFAs and NFAs. We will introduce a third way, called a **regular expression (reg exp)**, that represent the same class of language as DFAs and NFAs.

Let us consider an example of a regular expression, such as $(0 \cup 1) \circ 0^*$. This regular expression denotes a language that includes all string that start with a 0 or 1 and then followed by any number of zeros. It is important to note here that we use 0 as a shorthand for the set $\{0\}$, and the same with 1 and the set $\{1\}$, since regular expressions operate over sets of strings (languages).

More examples of regular expressions we have already seen include $\Sigma^*$ and $(0 \cup 1)^*$.
In addition, there is a hierarchy of operator precedence in regular expressions. The order is:

1. parentheses
2. Kleene star $*$
3. concatenation
4. union

## 6.2 Formal Definition Of A Regular Expression

**Definition 7.** Say that $R$ is a regular expression if $R$ is:

1. $a$ for some $a$ in the alphabet $\Sigma$
2. $\varepsilon$
3. $\emptyset$
4. $(R_1 \cup R_2)$, where $R_1$ and $R_2$ are regular expressions
5. $(R_1 \circ R_2)$, where $R_1$ and $R_2$ are regular expressions
6. $(R_1)^*$, where $R_1$ is a regular expression

In items 1 and 2, the regular expressions $a$ and $\varepsilon$ represent the languages $\{a\}$ and $\{\varepsilon\}$, respectively. In item 3, the regular expression $\emptyset$ represents the empty language. In item $4, 5$, and 6, the expressions represent the languages obtained by taking the union or concatenation of the languages $R_1$ and $R_2$, or the star of the languages $R_1$, respectively.

### 6.3 Examples: Regular Expressions

**Example 14.** Consider the regular expression: $0^*10^*$

It represents all binary strings that contains exactly one 1. This includes strings like 0010000000, 1000, and 1.

**Example 15.** Consider the regular expression: $\Sigma^* abracadabra \Sigma^*$

It represents all strings that contain the substring *abracadabra*.

**Example 16.** Consider the regular expression $(\Sigma\Sigma)^*$

It represents all strings of even length.

**Example 17.** Consider the regular expression: $(0 \cup \varepsilon)1^* = 01^* \cup 1^*$.

The expression $(0 \cup \varepsilon)$ describes the language $\{0, \varepsilon\}$, so the concatenation operation adds either 0 or $\varepsilon$ before every string in $1^*$.

**Example 18.** Consider the regular expression: $\emptyset^*$.

The Kleene star operation puts together any number of strings from the language to get a string in the result. If the language is empty, the star operation can put together 0 strings, giving only the empty string.

If you take 0 strings and concatenate them, you get $\varepsilon$. This has nothing to do with whether $X = \emptyset$ or not. The empty string $\varepsilon$ is always in $X^*$, regardless of what $X$ is.

### 6.4 Identities

- $\emptyset^* = \{\varepsilon\}$ since the Kleene star operator always contains the empty string.
- $1^*\emptyset = \emptyset$ since some number of 1s followed by an element from the empty set cannot exist
- $R \cup \emptyset = R$
- $R \circ \varepsilon = R$
- Note that $R \cup \varepsilon$ is only equal to $R$ if $R$ contains $\varepsilon$
- $R\emptyset = \emptyset$

  $R\emptyset =$ set of all strings made from choosing 1 string from $R$ and 1 string from $\emptyset$.

But there are no strings in $\emptyset$ to concatenate with a string in $R$.

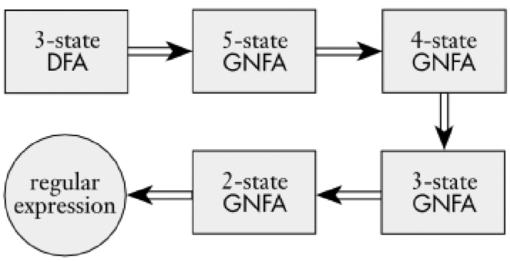
So for any $R$, $R\emptyset = \emptyset R = \emptyset$.

## 6.5 Formal Definition Of A Generalized Nondeterministic Finite Automata

For convenience, GNFAs always have a special form that meets the following conditions:

- The start state has transition arrows going to every other state but no arrows coming in from any other state.
- There is only a single accepting state, and it has arrows coming in from every other state but no arrows going to any other state. The accepting state is not the same as the start state.
- Except for the start and accepting states, one arrow goes from every state to every other state and also from each state to itself.

The following idea is used for the proof later:



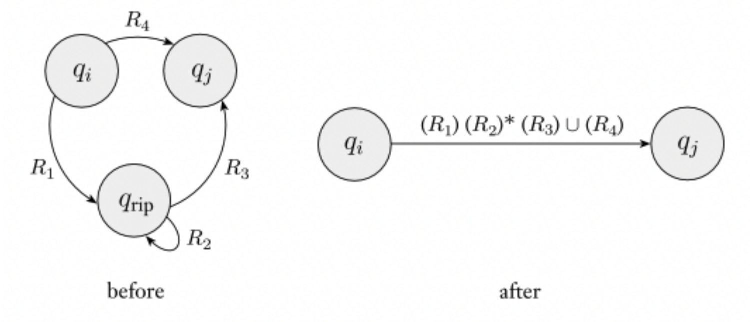

If the $GNFA$ has $k$ states, then we know that $k \geq 2$ since a GNFA must have a start and accepting state that must be different from each other.

If $k > 2$, we construct an equivalent $GNFA$ with $k-1$ states. This step can be repeated on the new $GNFA$ until it is reduced to 2 states.

If $k = 2$, the GNFA has a single arrow that goes from the start state to the accepting state. The label of this arrow is the equivalent regular expression.

The important step is constructing an equivalent GNFA with 1 fewer state when $k > 2$. This is done by selecting a state $q_{rip}$ and "ripping" it out of the machine, and repairing the remainder so that the same language is still recognized. Any state works, as long as it is not the start or accepting state.

After removing $q_{rip}$, the machine is repaired by altering the regular expressions that label each of the remaining arrows. The new labels compensate for the absence of $q_{rip}$. THe new label going from $q_i$ to $q_j$ is a regular expression that describes all strings that would take the machine from $q_i$ to $q_j$ either directly or through $q_{rip}$.




**Example 19.** Constructing an equivalent GNFA with one fewer state.

In the old machine, if

- $q_i$ goes to $q_{rip}$ with an arrow labeled $R_1$
- $q_{rip}$ goes to itself with an arrow labeled $R_2$
- $qrip$ goes to $q_j$ with an arrow labeled $R_3$
- $q_i$ goes to $q_j$ with an arrow labeled $R_4$

then in the new machine, the arrow from $q_i$ to $q_j$ gets the label

$$(R_1)(R_2)^*(R_3) \cup (R_4)$$

**Definition 8.** A generalized nondeterministic finite automaton is a 5-tuple, $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where:

1. $Q$ is the finite set of states
2. $\Sigma$ is the input alphabet
3. $\delta : (Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \to \mathcal{R}$ is the transition function
4. $q_{start}$ is the start state
5. $q_{accept}$ is the accepting state

A GNFA accepts a string $w$ in $\Sigma^*$ if $w = w_1 w_2 ... w_k$, where each $w_i$ is in the $\Sigma^*$ and a sequence of states $q_0, q_1, ..., q_k$ exists such that

- $q_0 = q_{start}$ is the start state
- $q_k = q_{accept}$ is the accepting state
- for each $i$, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$ (aka $R_i$ is the expression on the arrow from $q_{i-1}$ to $q_i$)

### 6.5.1 Langauge Is Regular IFF Some Regular Expression Describes It

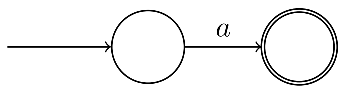
**Theorem 12.** A language is regular if an only if some regular expression describes it.

*Proof.* The proof has two directions, and we will prove both directions as separate lemmas. □

**Lemma 1.** If a language is described by a regular expression, then it is regular.

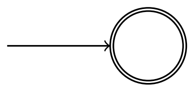
*Proof.* Let us convert $R$ into an NFA $N$. We consider the 6 cases in the formal definition of regular expression.

1. $R = a$ for some $a \in \Sigma$. Then $L(R) = \{a\}$, and the following NFA recognizes $L(R)$.

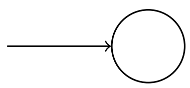
   
   

   Note that this machine fits the definition of NFA but not DFA (but we can provide an equivalent DFA).

2. $R = \varepsilon$. Then $L(R) = \{\varepsilon\}$, and the following NFA recognizes $L(R)$.

3. $R = \emptyset$. Then $L(R) = \emptyset$, and the following NFA recognizes $L(R)$.

4. $R = R_1 \cup R_2$
5. $R = R_1 \circ R_2$
6. $R = R_1^*$

Note that for items 4-6, we use the constructions given in the proofs that the class of regular languages is closed under the regular operations.

□

**Lemma 2.** If a language is regular, then it is described by a regular expression.

*Proof Idea.* We need to show that if a language $A$ is regular, a regular expression describes it. Because $A$ is regular, it is accepted by a DFA, which we can convert into an equivalent regular expression.

First we show how to convert DFAs into generalized nondeterministic finite automatons (GNFAs), and then GNFAs into regular expressions.

*Proof.* Let $M$ be the DFA for the language $A$. Then we convert $M$ to a GNFA $G$ by adding a new start state, a new accepting state, and additional transition arrows as needed. Let the proecedure $CONVERT(G)$ which takes a GNFA and returns an equivalent regular expression (using recursion where each call to itself processes a GNFA with 1 fewer state).

$CONVERT(G)$

1. Let $k$ be the number of states of $G$.
2. If $k = 2$, then $G$ must consist of a start state, an accepting state, and a single arrow connecting them and labeled with a regular expression $R$. Return the expression $R$.
3. If $k > 2$, we select any state $q_{rip} \in Q$ different from $q_{start}$ and $q_{accept}$ and let $G'$ be the GNFA $(Q', \Sigma, \delta', q_{start}, q_{accept})$, where

$$Q' = Q - \{q_{rip}\},$$

   and for any $q_i \in Q' - \{q_{accept}\}$ and any $q_j \in Q' - \{q_{start}\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$

   for $R_1 = \delta(q_i, q_{rip}), R_2 = \delta(q_{rip}, q_{rip}), R_3 = \delta(q_{rip}, q_j), R_4 = \delta(q_i, q_j)$.
4. Compute $CONVERT(G')$ and return this value.

□

**Claim 1.** For any GNFA $G$, $CONVERT(G)$ is equivalent to $G$.

*Proof.* We will prove this claim using induction.

Basis: Prove the claim is true for $k = 2$ states.
If $G$ has only 2 states, it can have only a single arrow going from start to accepting. The regular expression label on this arrow describes all the st rings that allow $G$ to get to the accepting state. Therefore, it is equivalent to $G$.

Induction step: Assume that the claim is true for $k - 1$ states and use this assumption to prove that the claim is true for $k$ states.
First show that $G$ and $G'$ recognize the same language. Suppose that $G$ accepts input $w$. Then in an accepting branch of computation, $G$ enters a sequence of states:

$$q_{start}, q_1, q_2, \cdots, q_{accept}$$

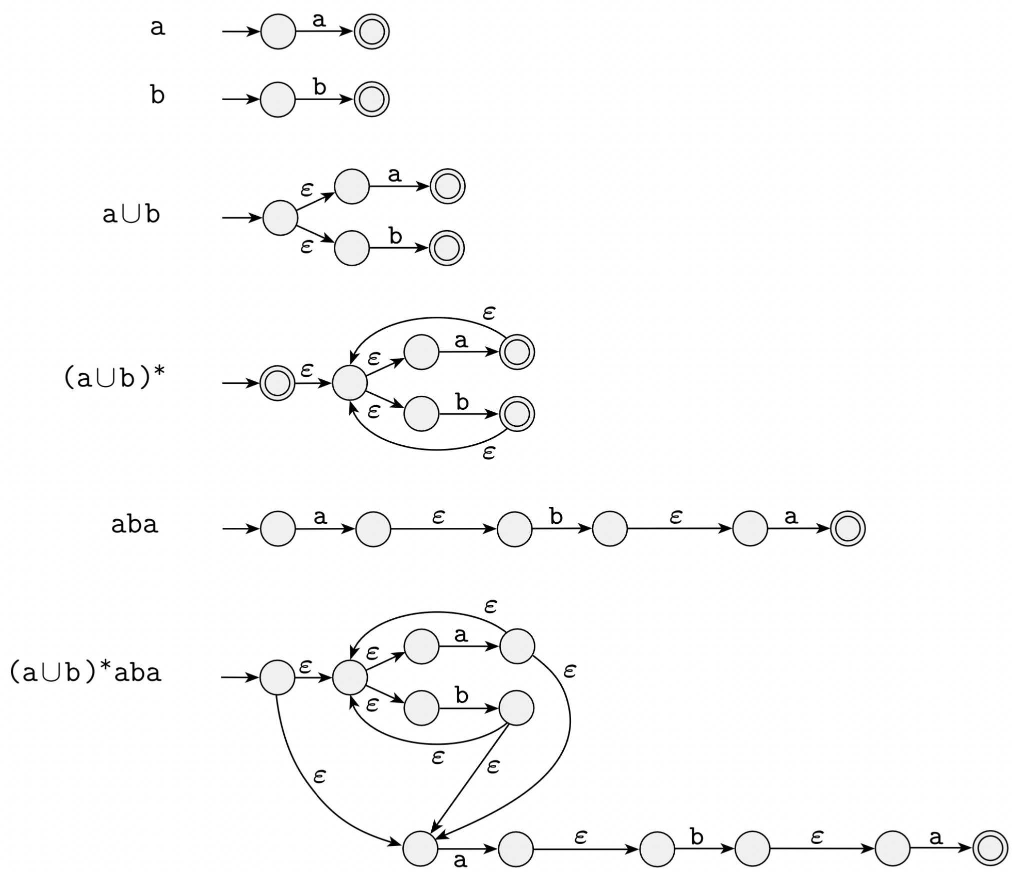
If none of them is the removed state $q_{rip}$, then $G'$ accepts $w$. This is because $G'$ contains the old regular expression as part of a union.

If $q_{rip}$ appears, removing each run of consecutive $q_{rip}$ states forms an accepting computation for $G'$. This is because the states $q_i$ and $q_j$ now have a new regular expression on the arrow between them that describes all strings taking $q_i$ to $q_j$ through $q_{rip}$ on $G$. So $G'$ accepts $w$.

Suppose that $G'$ accepts an input $w$. Since each arrow between any 2 states $q_i$ and $q_j$ in $G'$ describes the collection of strings taking $q_i$ to $q_j$ in $G$, either directly or through $q_{rip}$, $G$ must also accept $w$. $\square$

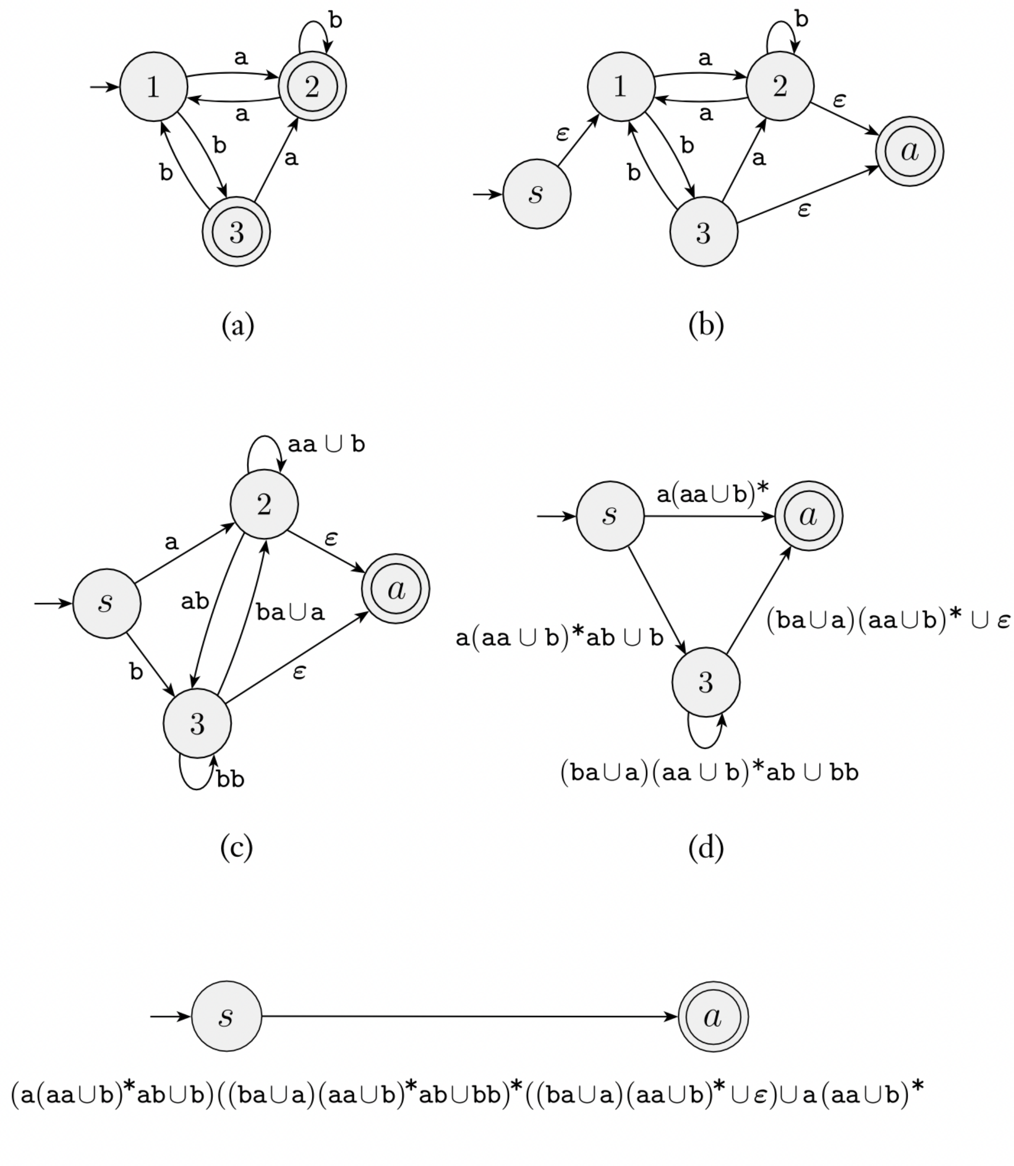
### 6.5.2 Example: Building NFA From A Regular Expression

**Example 20.** Build an NFA for the regular expression $(a \cup b)^* aba$.



### 6.5.3 Example: Building Regular Expression From A DFA

**Example 21.** Beginning with a 3-state DFA, convert the DFA to a regular expression (using GNFAs).




(a) (b) (c) (d) (e)

# 7 Lecture 7: Nonregular Languages

## 7.1 Example: Nonregular Language

**Example 22.** Consider the language $L = \{O^n 1^n \mid n \geq 0\}$.

If we attempt to find a DFA that recognizes $L$, we find that the machine needs to remember how many 0s have been read so far. Because the number of 0s is not limited, the machine will have to track an unlimited number of possibilities (but this can't be done with finite states). Therefore, $L$ is not a regular language.

**Example 23.** Consider the language $A = \{w \mid w$ has an equal number of 0s and 1s$\}$ and $B = \{w \mid w$ has an equal number of occurrences of 01 and 10 as substrings$\}$.
$A$ is not regular but $B$ is. Below, we will use the pumping lemma to show how to prove that certain languages are not regular.

## 7.2 Pumping Lemma For Regular Languages

**Theorem 13.** If $A$ is a regular language, then there is a number $p$ (pumping length) where if $s$ is any string in $A$ of length $\geq p$, then $s$ may be divided into 3 pieces, $s = xyz$, satisfying the following conditions:

- for each $i \geq 0, xy^i z \in A$
- $|y| > 0$
- $|xy| \leq p$

*Proof.* If $A$ is regular then there exists a DFA $M$ accepting $A$.

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing $A$ and $p$ be the number of states of $M$.

Let $s = s_1 s_2 ... s_n$ be a string in $A$ of length $n$, where $n \geq p$. Let $r_1, ..., r_{n+1}$ be the sequence of states that $M$ enters while processing $s$, so $r_{i+1} = \delta(r_i, s_i)$ for $1 \leq i \leq n$. This sequence has length $n + 1$, which is at least $p + 1$. Among the first $p + 1$ elements in the sequence, 2 must be the same state, by the pigeonhole principle. Call the first of these $r_j$ and the second $r_l$. Because $r_l$ occurs among the first $p+1$ places in a seuqnce starting at $r_1$, we have $l \leq p+1$.

Now let $x = s_1 \cdots s_j - 1, y = s_j \cdots s_{l-1}, z = s_l \cdots s_n$. As $x$ takes $M$ from $r_1$ to $r_j$, $y$ takes $M$ from $r_j$ to $r_j$, and $z$ take s$M$ from $r_j$ to $r_{n+1}$, which is an accepting state so $M$ must accept $xy^i z$ for $i \geq 0$. We know that $j \neq l$, so $|y| > 0$, and $l \leq p + 1$, so $|xy| \leq p$, and have satisfied all conditions of the pumping lemma.

□

### 7.2.1 Examples: Using Pumping Lemma To Prove Nonregularity

**Example 24.** Consider the language $B = \{0^n 1^n \mid n \geq 0\}$.

*Proof.* Assume that $B$ is regular.

Let $p$ be the pumping length given by the pumping lemma. Choose $s$ to be a string $0^p 1^p$. Because $s$ is a member of $B$ and $s$ has length more than $p$, the pumping lemma guarantees that $s$ can be split into 3 pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^i z$ is in $B$. Let us consider 3 cases to show that this result is impossible:

1. The string $y$ consists only of 0s. For example, $xyyz$ (when $i = 2$) has more 0s than 1s, so it is in the language $B$. This breaks condition 1 of the pumping lemma.
2. The string $y$ consists only of 1s. This also breaks condition 1.
3. The string $y$ consists of both 0s and 1s. The string $xyyz$ may have the same number of 0s and 1s, but they are out of order. So the string is not in $B$ and breaks condition 1 of the pumping lemma.

$\square$

**Example 25.** Let $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$. Prove that $C$ is not regular.

*Proof.* Assume that $C$ is regular.

Let $p$ be the pumping length given by the pumping lemma. Choose $s$ to be the string $0^p 1^p$. Because $s$ is a member of $C$ and $s$ has length more than $p$, the pumping lemma guarantees that $s$ can be split into 3 pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^i z$ is in $C$.

Let $x$ and $z$ be the empty string and $y$ be the string $0^p 1^p$. Then $xy^i z$ always has an equal number of 0s and 1s and is in $C$.

BUT THIS IS WRONG! CONDITION 3 OF THE PUMPING LEMMA COMES INTO PLAY.

By pumping $s$, it must be divided so that $|xy| \leq p$. That means the string we selected $s = 0^p 1^p$ cannot be pumped. If we select $y = 0^n, 0 < n \geq p$. Then $s = 0^{p-n} 0^n 1^p$. However, if $i > 0$, then $0^{p-n} 0^{i \cdot n} 1^p = 0^{p+n(i-1)} 1^p$. This is a contradiction since $p + n(i-1) \neq p$ so $xy^i z \notin C$.

$s$ cannot be pumped and therefore $C$ is not regular.

$\square$

**Example 26.** Consider the language $D = \{ww \mid w \in \{0,1\}^*\}$. Prove that $D$ is not regular.

*Proof.* Assume that $F$ is regular.

Let $s$ be the string $0^p10^p1$. Because $s$ is a member of $D$ and $s$ has length more than $p$, the pupming lemma guarantees that $s$ can be split into 3 pieces $s = xyz$ satisfying the pumping lemma.

Condition 3 of the pumping lemma is important again. We cannot let $x$ and $z$ be of length 0 because of condition 3. This means that $y$ must consist of only 0s, so $xyyz \notin D$.

□

**Example 27.** Consider the language $\tilde{L_3} = \{w_1w_2 \mid w_1 \neq w_2; w_1, w_2 \in \Sigma^*\}$.

Then $L_3 = (\Sigma\Sigma)^* - \tilde{L_3} = (\Sigma\Sigma)^* \cap \tilde{L_3}$

### 7.2.2 Example: Pumping Lemma On Unary Languages

**Example 28.** Consider the language $E = \{1^{n^2} \mid n \geq 0\}$. $E$ essentially contain all strings of 1s whose lengths is a perfect square. We will show that $E$ is not regular.

*Proof.* By contradiction:

Assume that $E$ is regular. Let $p$ be the pumping length given by the pumping lemma. Let $s$ be the string $1^{p^2}$. Because $s$ is a member of $E$ and $s$ has length at least $p$, the pumping lemma guarantees that $s$ can be split into $s = xyz$, where for any $i \geq 0$, the string $xy^iz$ is in $D$.

Consider the 2 strings $xyz$ and $xy^2z$. These strings differ from each other by a single repetition of $y$ and their lengths differ by the length of $y$. By condition 3 of the pumping lemma, $|xy| \leq p$ and thus $|y| \leq p$. We have $|xyz| = p^2$ and so $|xy^2z| \leq p^2 + p$. But $p^2 + p < p^2 + 2p + 1 = (p+1)^2$. Condition 2 implies that $y$ is not an empty string so $|xy^2z| > p^2$. The length of $xy^2z$ lies strictly between the consecutive perfect squares $p^2$ and $(p+1)^2$. So the length cannot be a perfect square, see that $xy^2z \notin E$, and conclude that $E$ is not regular.

□

### 7.2.3 Example: Pumping Down

**Example 29.** Consider the language $F = \{0^j 1^j \mid i > j\}$. Show that $F$ is not regular.

*Proof.* Assume that $F$ is not regular.

Let $p$ be the pumping length of $F$ given by the pumping lemma. By condition 3, $y$ consists only of 0s. Let's example the string $xyyz$ to see whether it can be in $F$. Adding an extra copy of $y$ increases the number of 0s. But $F$ contains all strings in $0^*1^*$ that have more 0s and 1s, so increasing the number of 0s will still give the string in $F$. There is no contradiction. Try the following.

The pumping lemma states that $xy^iz \in F$ even when $i = 0$, so let's consider the string $xy^0z = xz$. Removing string $y$ decreases the number of 0s in $s$. Remember than $s$ has just one more 0 than 1. Therefore, $xz$ cannot have more 0s than 1s, so it cannot be a member of $F$. This is a contradiction.

□

# 8 Lecture 8: Context-Free Languages (CFLs)

## 8.1 Context-Free Grammars (CFGs)

- A grammar consists of collection of **substitution rules**, called **productions**.
- Each rule appears as a line in the grammar, comprising a symbol and a string separated by an arrow
- The symbol is called a **variable**.
- The string consists of variables and other symbols called **terminals**.
- The variables are often represented by capital letters. The terminals are often represented by lowercase letters, numbers, or special symbols.
- One variable is designated as the **start variable**.

You use a grammar to describe a language by generating each string of that language in the following manner.

1. Write down the start variable. It is the variable on the left-hand side of the top rule, unless specified otherwise.
2. Find a variable that is written down and a rule that starts with that variable. Replace the written down variable with the right-hand side of that rule.
3. Repeat step 2 until no variables remain.

**Example 30.** Consider the grammar $G_1$:

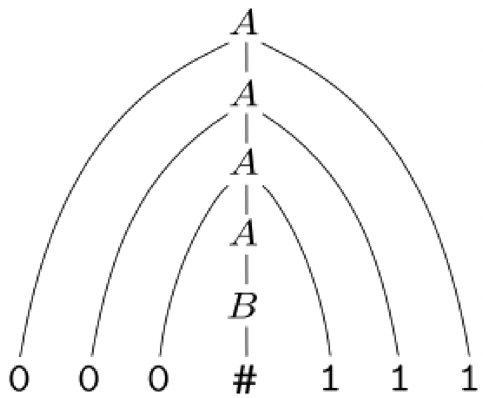
$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

- $0, 1, \#$ are terminals
- let $A$ be the start symbol
- $A \to 0A1 \to 00A11 \to 000B111 \to 000\#111$ is a derivation
- $000\#111$ is a string of terminals

We can say that $G_1$ generates $0^3\#1^3$.

Define $L(G)$ to be all strings generated by $G$. We can prove $L(G_1) = \{0^n\#1^n \mid n \geq 0\}$. Then $L(G_1)$ is a CFL because it is generated by a CFG. We say that $S = 000\#111 \in L(G_1)$.

The following is a parse tree that describes the derivation of $S$.




Pulling all the leaves from the tree (shown by thin lines), we can see the string that is generated. Every word that is generated/derived from the start state has a parse tree. Grammars and parse tress come from linguistics.

For example, English has a grammar and can be parsed.

### 8.1.1 Formal Definition Of A Context-Free Grammar

**Definition 9.** A **context-free grammar** is a 4-tuple $(V, \Sigma, R, S)$, where:

- $V$ is a finite set called the **variables**
- $\Sigma$ is a finite set, disjoint from $V$, called the **terminals**
- $R$ is a finite set of **rules**, with each rule being a variable and a string of variables and terminals, shown below

$$\text{variable} \rightarrow \text{string}$$

- $S \in V$ is the start variable

If $u, v, w$ are strings of variables and terminals, and $A \rightarrow w$ is a rule of the grammar, we say that $uAv$ yields $uwv$, written $uAv \Rightarrow uwv$. Say that $u$ derives $v$, written $u \Rightarrow v$, if $u = v$ or if a sequence $u_1, u_2, ..., u_k$ exists for $k \geq 0$ and

$$u \Rightarrow u_1 \Rightarrow ... \Rightarrow u_k \Rightarrow v$$

**Example 31.** Consider the grammar $G_2 = (\{S\}, \{a, b\}R, S)$. The set of rules $R$ is

$$S \rightarrow aSb \mid SS \mid \varepsilon$$

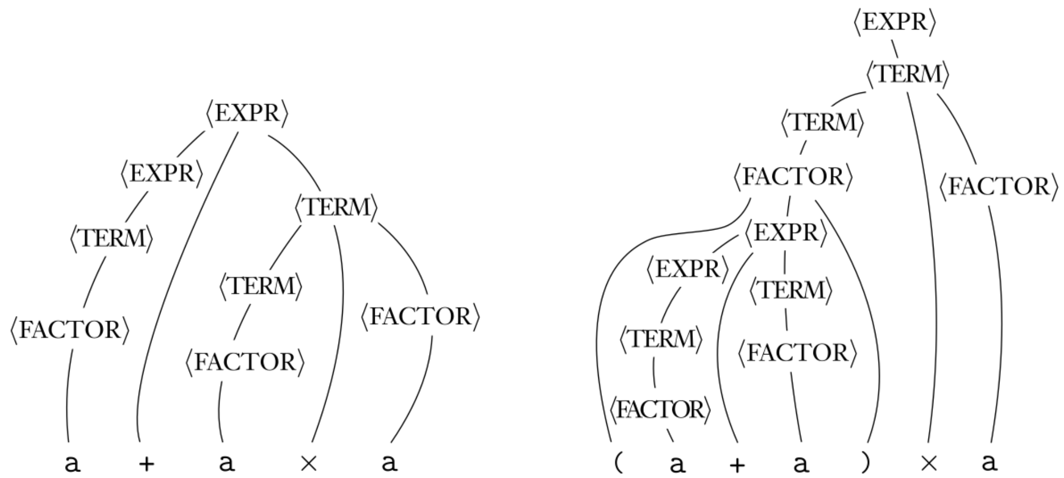
This grammar generates strings such as $abab$, $aaabbb$, and $aabaab$. Think of $a$ as '(' and $b$ as ')'. Viewed in this way, $L(G_3)$ is the language of all strings of properly nested (balanced) parentheses. Note that the language may contain the empty string $\varepsilon$.

**Example 32.** Consider the grammar $G_3 = (V, \Sigma, R, \langle EXPR\rangle)$.

$V$ is $\{\langle EXPR\rangle, \langle TERM\rangle, \langle FACTOR\rangle\}$ and $\Sigma$ is $\{a, +, \times, (, )\}$

$$\begin{aligned}\langle EXPR\rangle &\to \langle EXPR\rangle + \langle TERM\rangle \mid \langle TERM\rangle\\ \langle TERM\rangle &\to \langle TERM\rangle \times \langle FACTOR\rangle \mid \langle FACTOR\rangle\\ \langle FACTOR\rangle &\to (\langle EXPR\rangle) \mid a\end{aligned}$$

The two strings $a + a \times a$ and $(a + a) \times a$ are both generated with grammar $G_3$. The parse trees are shown below:




### 8.1.2 Designing Context-Free Grammars

Many CFLs are unions of simpler CFLs.

**Example 33.** To get a grammar for the language $\{0^n1^n \mid n \geq 0\} \cup \{1^n0^n \mid n \geq 0\}$, first construct the grammar.

$S_1 \to 0S_11 \mid \varepsilon$ for language $\{0^n1^n \mid n \geq 0\}$ $S_2 \to 1S_20 \mid \varepsilon$ for language $\{1^n0^n \mid n \geq 0\}$

Then add the rule $S \to S_1 \mid S_2$ to give the grammar:

$$\begin{aligned}S &\to S_1 \mid S_2\\ S_1 &\to 0S_11 \mid \varepsilon\\ S_2 &\to 1S_20 \mid \varepsilon\end{aligned}$$

Constructing a CFG for a language that happens to be regular is easy if you can first construct a DFA for that language. You can convert any DFA into an equivalent CFG by:

1. Make a variable $R_i$ for each state $q_i$ of the DFA.
2. Add the rule $R_i \to aR_j$ to the CFG if $\delta(q_i, a) = q_j$ is the transition in the DFA.
3. Add the rule $R_i \to \varepsilon$ if $q_i$ is an accepting state of the DFA.
4. Make $R_0$ the start variable of the grammar, where $q_0$ is the start state of the machine

Certain CFLs contain strings with two substrings that are "linked" in the sense that a machine for such a language would need to remember an unbounded amount of information about one of the substrings to verify that it corresponds properly to the other substring.

**Example 34.** Consider the language $\{0^n 1^n \mid n \geq 0\}$.

The machine would need to remember the number of 0s in order to verify that it equals the number of 1s. You can construct a CFG to handle this situation by using a rule of the form $R \to uRv$, which generates strings wherein the portion containing the $u$'s correspond to the portion containing the $v$'s.

In more complex languages, the strings may contain certain structures that appear recursively as part of other (or the same) structures. An example is Example 3, where the grammar generates arithmetic expressions. Any time a symbol $a$ appears, an entire parenthesized expression might appear recursively instead. To do this, put the variable symbol generating the structure in the location of the rules corresponding to where that structure may recursively appear.

### 8.1.3 Nondeterminism With CFGs

CFGs are in some sense nondeterministic because at each step, we can choose:

1. which rule to apply
2. which variable to apply the rule

**Example 35.** Consider the string $0AB1A0$.

We can eliminate nondeterminism for (2) with leftmost derivation (applying the rule to the leftmost variable). However we cannot entirely eliminate nondeterminism because there can be multiple choices for the rule from a variable such as $A \to w_1 \mid w_2 \mid w_3 \mid w_4$.

### 8.1.4 Ambiguity

If the grammar generates the same string in several ways, we say that the string is derived **ambiguously**. If a grammar generates some string ambiguously, we say that the grammar is **ambiguous**.

More formally, if $w \in L(G)$, then $S \overset{*}{\Rightarrow} w$. Sometimes $S \overset{*}{\Rightarrow} w$ in more than 1 way, and if this happens with $G$, then $G$ is ambiguous.

**Example 36.** Let $L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$.

It can be shown that $L$ is generated by a CFG, and $L$ is inherently ambiguous. Consider the following grammar $G_5$ for $L$:

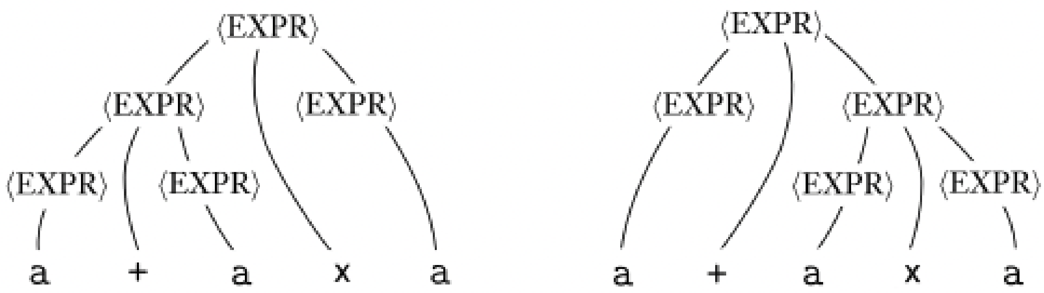
$$
\begin{aligned}
S &\to S_1 C \mid A S_2 \\
S_1 &\to a S_1 b \mid \varepsilon \\
S_2 &\to b S_2 c \mid \varepsilon \\
A &\to a A \mid \varepsilon \\
C &\to C c \mid \varepsilon
\end{aligned}
$$

A string such as $aabbcc$ can be generated through multiple different paths.

**Example 37.** Consider grammar $G_4 = (V, \Sigma, R, \langle EXPR \rangle)$.

$$\langle EXPR \rangle \to \langle EXPR \rangle + \langle EXPR \rangle \mid \langle EXPR \rangle \times \langle EXPR \rangle \mid (\langle EXPR \rangle) \mid a$$

The following is two parse trees for the string $a + a \times a$ in grammar $G_4$.




Note that grammar $G_4$ does not capture the usual precedence relations ($\times$ before $+$).