CS 4510: Automata and Complexity 08/30/2022

Lecture 3: Operations On Languages

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Contents

1 Operations On Languages **2**
  1.1 The Regular Operations: Union, Concatenation, Kleene Star 2

2 Other Regular Operations **4**
  2.1 Complement 4
  2.2 Intersection 4
  2.3 Set Difference 4
  2.4 Symmetric Difference 5

3 Closed Operations **6**

1 Operations On Languages

1.1 The Regular Operations: Union, Concatenation, Kleene Star

Definition 1. Let A and B be langauges. We define the regular operations union, concatenation, and Kleene star as follows:

- **union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
- **Kleene star**: $A^* = \{x_1 x_2 ... x_n \mid k \geq 0 \text{ and each } x_i \in A\}$
  - $A^* = \bigcup_{k \geq 0} A^k$

Note: $A^+ = A^* - \{\varepsilon\}$.

Example 1.

$$A = \{\text{good}, \text{bad}\}, B = \{\text{boy}, \text{girl}\}$$

$$A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$$

$$A \circ B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

$$A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badgood}, \text{badbad}, ...\}$$

Theorem 1. The class of regular languages is closed under the union operation. That is, if A and B are regular languages, so is $A \cup B$.

Proof. Let M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q, F)$.

1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$
   - this set Q is essentially $Q_1 \times Q_2$, the Cartesian product of sets Q_1 and Q_2
   - it is the set of all pairs of states: the first from Q_1 and the second from Q_2
2. $\Sigma = \Sigma$, the same as in M_1 and M_2
   - for simplicity, assume M_1 and M_2 have the same alphabet Σ
   - the theorem remains true if they have different alphabets Σ_1 and Σ_2
   - then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$
3. δ (transition function) is defined as follows:
   - for each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
4. q_0 is the pair (q_1, q_2)
5. F is the set of pairs in which either member is an accepting state of M_1 or M_2 as follows:
   - $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$
   - the above expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
   - Note: $F = (F_1 \times F_2)$ IS NOT CORRECT! (takes intersection instead of union)

$\square$

2 Other Regular Operations

2.1 Complement

Theorem 2. The class of regular languages is closed under the complement operation.

Proof. Let L be a regular language, then some finite automaton M recognizes L.

Let $\overline{M}$ be the same as M, but with the accepting and non-accepting states interchanged. Then $\overline{M}$ accepts a string x if and only if M does not accept x. So, $L(\overline{M}) = \overline{L}$.

□

2.2 Intersection

Theorem 3. If A_1 and A_2 are regular languages, then so is $A_1 \cap A_2$.

Proof. Let $M_1 = (Q^1, \Sigma, \delta^1, q_0^1, F^1)$ decide A_1 and $M_2 = (Q^2, \Sigma, \delta^2, q_0^2, F^2)$ decide A_2.

We construct the automaton $M = (Q, \Sigma, \delta, q_0, F)$ as follows:

- $Q = Q^1 \times Q^2$ (each state in M is a pair of states in M_1 and M_2)
- Σ is the same shared alphabet as M_1 and M_2
- $\delta((r_1, r_2), x) = (\delta^1(r_1, x), \delta^2(r_2, x))$
- $q_0 = (q_0^1, q_0^2)$
- $F = F^1 \times F^2$ (both M_1 and M_2 must be in an accepting state for M to accept)

□

2.3 Set Difference

Theorem 4. If A and B are regular languages, then so is $A_1 \backslash A_2 = \{x \mid x \in A \text{ and } x \notin B\}$.

Proof. Note: $A \backslash B = A \cap \overline{B}$

Since regular languages are closed under intersection and complement, regular languages are closed under subtraction.

□

2.4 Symmetric Difference

Theorem 5. If A and B are regular langauges, then so is $A \oplus B$.

Proof. Note: $A \oplus B = (A \cup B) \backslash (A \cap B)$

Since regular languages are closed under union, intersection, and subtraction, regular languages are closed under symmetric difference.

$\square$

3 Closed Operations

A set S is closed under an operation $\cdot$ if for every $a, b \in S, a \cdot b \in S$. That is, if we apply the operation to any two element in the set, we another element in the same set.

- $\mathbb{N}$ is closed under addition
- $\mathbb{N}$ is not closed under subtraction (ex. $3 - 5 = -2 \notin \mathbb{N}$)
- $\mathbb{Z}$ is closed under addition, subtraction, multiplication, but not division.
- $\mathbb{Q}$ is closed under addition, subtraction, multiplication, and $\mathbb{Q}\backslash\{0\}$ is closed under division.
- $\mathbb{R}$ is not closed under square root, but $\mathbb{R}^+$ is closed under square root.
  - $\mathbb{R}$ has negative numbers, while $\mathbb{R}^+$ does not.
- $\mathbb{C}$ is closed under square root (because it includes imaginary numbers).