CS 4510: Automata and Complexity 09/01/2022

Lecture 4: Non-Determinism

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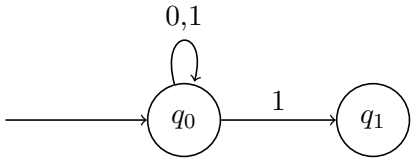
Contents

1 Non-Determinism

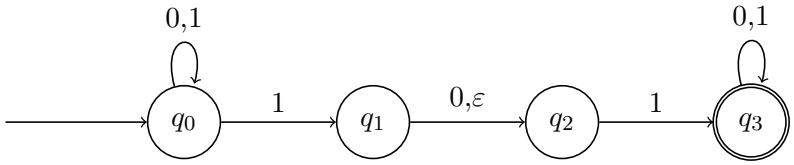
- Non-determinism looks strange and impractical, and in some sense it is, but it is very important.
- You will find out at the end of the course why the most important problem in computer science involves non-determinism and non-deterministic computation.
- Our definition of finite automaton so far is deterministic. This means, when we are at some state, and we see a symbol, we go to exactly one other state. Also, from every state, the transition is well defined for every symbol in the alphabet.
- 2 main differences between a non-deterministic finite automaton (NFA) and deterministic finite automaton (DFA).
  - from a state, when we see a symbol, we can go to 0 or more states (in a DFA, when we see a symbol, we go to exactly 1 state)
  - we have "ε"-transitions, which are transitions that we can take without seeing any symbol.
- The starting, accepting, and states all work the same. The big difference is in δ (transition function).

Example 1. Consider the following NFA:



- If we are at q_0 and see a 1, we do not have to go to q_1.
- Rather, think of it like we are in many states at once. (referred to as "guessing")
- In other words, we say that from q_0, we can go to several states.
- The automaton is a genius and knows where to go. It has the foresight to think, "I see 1 several times, I stay at q_0, but I will transition to q_1 at just the right time."
- The choice of the 1 to transition is because the automaton can see the future.
- Note: this is abstract and not practical

Example 2. Consider the following NFA:



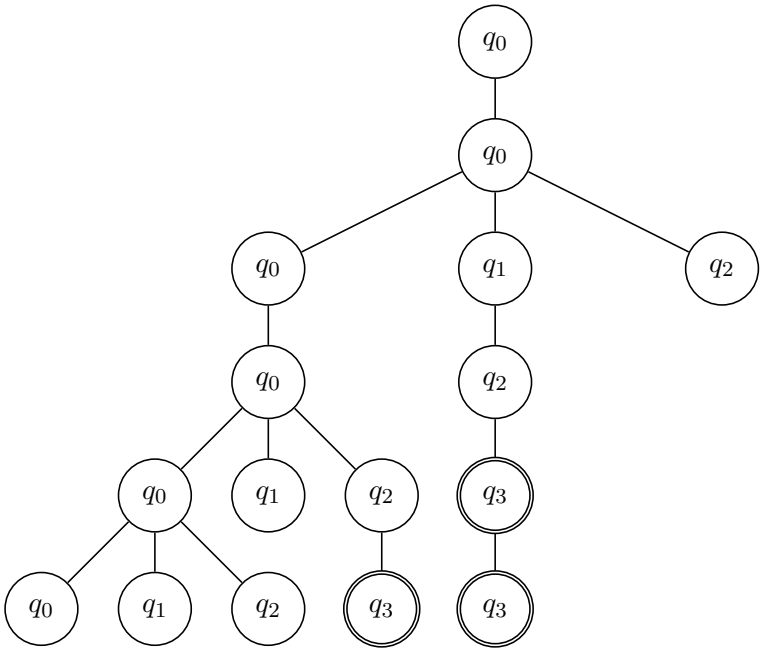
Suppose at q_1 and we see a 1. There is no transition with 1, so we "abort". At q_1 we see ε and move to q_2. Note: we always see ε, so without doing anything at q_1, we "slide" to q_2.

Below is a transition table describing the NFA.

Q \ symbol	0	1	ε
$\rightarrow q_0$	q_0	q_0, q_1	-
q_1	q_2	-	q_2
q_2	-	q_3	-
q_3	q_3	q_3	-

Think of split timelines every time we have to go to multiple states. From q_0 seeing 1, we can go back to q_0 or go to q_1. Note that we accept the input if at least one of the timelines ends at an accepting state.

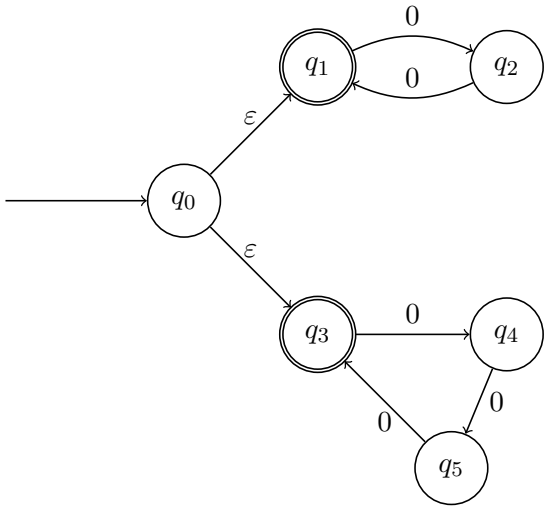
Below is a decision parse tree for input $x = 01011$



The nodes at each layer represent the states we are in at that time (given the input). At q_0, we see 0 and can only go back to q_0 At q_1, we see 1 and can go to q_0, q_1, or slide with ε to q_2. And repeat for the remaining binary values in the input string x. Once we finish, as long as one state in the final depth is accepting, the string x is accepted. This is the reason why we call this "guessing".

2 Unary Language

Example 3. M_1



This NFA guesses if the input string length is divisible by either 2 or 3 by guessing to go higher q_1 or lower q_3.

3 Nondeterministic Finite Automaton

3.1 Formal Definition Of A Nondeterministic Finite Automaton

Definition 1. A **nondeterministic finite automaton** is a 5-tuple $Q, \Sigma, \delta, q_0, F$ where:

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ is the transition function, where $\mathcal{P}(Q)$ is the power set of Q
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states

Note: the only change in the definition compared to a DFA is the transition function δ. δ is now a set of states.

We say M accepts w if w can be written as $w = y_1...y_n$ such that $y_i \in \Sigma_\varepsilon$ and there are states $r_0, R_1, ..., r_m$ such that $r_0 = q_0, r_m \in F,$ and $r_{i+1} \in \delta(r_i, y_{i+1})$ for $0 \leq i \leq m-1$.

3.2 Equivalence Of NFAs and DFAs

Corollary 1. A language is regular if an only if some nondeterminstic finite automaton recognizes it.

Theorem 1. Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Proof Idea. As a **corollary**, the set of languages NFAs can accept (N) is equal to the set of languages that DFAs can accept (D). The first part of the proof is to note that every DFA is an NFA, so $D \subseteq N$. Then to show that $N \subseteq D$, we say for each NFA $N_i \in N$ we can construct an equivalent DFA, implying that $N_i \in D$. Together this implies that $N = D$.

Note: in terms of power of these two computing structures, this proof shows that there is no difference. But in terms of size and simplicity, NFAs have the advantage (since an equivalent DFA must have at least 2^k states).

Theorem 2. Let L_k be the regular language over $\{0, 1\}$ which contains all strings which have a 1 as the k'th character from the right end of the string. Any DFA that recognizes L_k must contain at least 2^k states.

Proof. Suppose that D is a DFA for L_k containing strictly fewer than 2^k states. Then, by the pigeonhole principle, there must be a state q such that two different binary strings of length $k, x,$ and y, both cause the machine to end in state q. But if x and y are different, there is at least one index i such that $x[i] = 0$ and $y[i] = 1$ or vice versa. But then $x0^{i-1}$ and $y0^{i-1}$ cause the machine to end in the same state, yet one must be accepted and the other must be rejected. This is a contradiction, so our assumption that there was a DFA for L_k with strictly fewer than 2^k states is incorrect. □