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UNIT-II
Recurrence Relations

Recurrence Relation: An equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2 \dots a_{n-1}$ $\forall$ integers n with $n \geq n_0$ where n_0 is a non negative integer is called a recurrence relation of the sequence $\{a_n\}$.

Note: If the terms of the sequence satisfies the recurrence relation then the sequence is called solution of the recurrence solution

Let sequence $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{(n-1)} + 3a_{(n-2)}$ for $n = 2, 3, 4 \dots$ unlet $a_0 = 1$ & $a_1 = 2$. What are the values of a_2 and a_3

Sol: The given recurrence relation is $a_n = a_{n-1} + 3a_{n-2}$, $n = 2, 3, 4 \dots$.

Given that $a_0 = 1$ & $a_1 = 2$

Find a_2: Take $n = 2$ in eq^n ① we get

$$a_2 = a_1 + 3a_0$$
$$= 2 + 3(1) = 2 + 3 = 5 \qquad a_2 = 5$$

Find a_3: Take $n = 3$ in eq^n ① we get

$$a_3 = a_2 + 3a_1$$
$$= 5 + 3(2) = 5 + 6 = 11$$
$$\therefore a_3 = 11$$

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Find the first 5 terms of the sequence defined by each of the following recurrence relation and initial conditions

(i) $a_n = a_{n-1}^2$, $a_1 = 2$

(ii) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

(iii) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

Sol (i) Given recurrence relation $a_n = a_{n-1}^2$, $a_1 = 2$

Find $a_2 = a_1^2 = (2)^2 = 4$ $\boxed{a_2 = 4}$

$a_3 = a_2^2 = (4)^2 = 16$ $\boxed{a_3 = 16}$

$a_4 = a_3^2 = (16)^2 = 256$ $\boxed{a_4 = 256}$

$a_5 = a_4^2 = (256)^2 = 65536$ $\boxed{a_5 = 65536}$.

(ii) Given recurrence relation $a_n = na_{n-1} + n^2 a_{n-2}$

$a_0 = 1$ $a_1 = 1$

$a_2 = 2(1) + 2^2(1) = 2 + 4 = 6$ $\boxed{a_2 = 6}$

$a_3 = 3(6) + 3^2(1) = 18 + 9 = 27$. $\boxed{a_3 = 27}$

$a_4 = 4(27) + 4^2(6) = 204$ $\boxed{a_4 = 204}$

(iii) Given recurrence relation $a_{n-1} + a_{n-3}$

$a_0 = 1$ $a_1 = 2$ $a_2 = 0$

$a_3 = 0 + 1 = 1$ $\boxed{a_3 = 1}$

$a_4 = 1 + 2 = 3$ $\boxed{a_4 = 3}$

Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$, if

i) $a_n = -n + 2$ (ii) $a_n = 3(-1)^n + 2^n - n + 2$

Sol The given recurrence relation is

$$a_n = a_{n-1} + 2a_{n-2} + 2n - 9 \longrightarrow (1)$$

(i) $a_n = -n + 2$

$a_{n-1} = -(n-1) + 2 = -n + 3$

$a_{n-2} = -(n-2) + 2 = -n + 4$

Now Consider RHS of eqn (1)

$$a_{n-1} + 2a_{n-2} + 2n - 9$$
$$= (-n+3) + 2(-n+4) + 2n - 9$$
$$= -n + 3 - \cancel{2n} + 8 + \cancel{2n} - 9$$
$$= -n + 2$$
$$= a_n$$

$\therefore a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ is the solution of the recurrence solution as it satisfies the given recurrence relation

(ii) $a_n = 3(-1)^n + 2^n - n + 2$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 =$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2 =$

Now consider the RHS of eqn (1)

$$a_{n-1} + 2a_{n-2} + 2n - 9$$
$$= 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2]$$
$$+ 2n - 9$$
$$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 2[3(-1)^{n-2} + 2^{n-2} - n + 4] + 2n - 9$$
$$= \left[\frac{3(-1)^n}{-1} + \frac{2^n}{2} - n + 3\right] + 2\left[\frac{3(-1)^n}{(-1)^2} + \frac{2^n}{2^2} - n + 4\right] 2n - 9$$
$$= 3(-1)^n(-1 + 2) + 2^n\left[\frac{1}{2} + \frac{1}{2}\right] - n + 3 - \cancel{2n} + 8 + \cancel{2n} - 9$$
$$= 3(-1)^n + 2^n - n + 2$$
$$= a_n$$

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Let $a_n = 2^n + 5(3^n)$ for $n = 0, 1, 2, 3 \ldots$

i) find a_0, a_1, a_2, a_3 & a_4

(ii) Show that $a_2 = 5a_1 - 6a_0$; $a_3 = 5a_2 - 6a_1$
$a_4 = 5a_3 - 6a_2$

(iii) show that $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$

i) $a_n = 2^n + 5(3^n)$

$a_0 = 2^0 + 5(3^0) = 1+5 = 6$

$a_1 = 2^1 + 5(3^1) = 2+15 = 17$

$a_2 = 2^2 + 5(3^2) = 4+45 = 49$

$a_3 = 2^3 + 5(3^3) = 8+135 = 143$

$a_4 = 2^4 + 5(3^4) = 421$

ii) $5a_1 - 6a_0 = 5(17) - 6(6)$

$= 49 = a_2 \quad \therefore a_2 = 5a_1 - 6a_0$

$5a_2 - 6a_1 = 5(49) - 6(17)$

$= 143 \quad \therefore a_3 = 5a_2 - 6a_1$

$5a_3 - 6a_2 = 5(143) - 6(49)$

$= 421 \quad \therefore a_4 = 5a_3 - 6a_2$

iii) Consider RHS

$5a_{n-1} - 6a_{n-2}$

$= 5(2^{n-1} + 5(3^{n-1})) - 6(2^{n-2} + 5(3^{n-2}))$

$= 2^{n-2}(5 \times 2 - 6 \times 1) + 3^{n-2}(5 \times 5 \times 3 - 6 \times 5)$

$= 2^{n-2}(4) + 3^{n-2}(75 - 30)$

$= 2^{n-2}(4) + 3^{n-2}(45)$

$= 2^{n-2}(4) + 3^{n-2}(5 \times 9)$

$= \frac{2^n}{\cancel{2^2}} \cancel{(4)} + \frac{3^n}{\cancel{3^2}} 5 \times \cancel{9}$

$= 2^n + 5(3^n)$

$2^{n-1}(5 \times 2 - 6 \times 1) + 3^{n-1}(5 \times 5 \times 3 - 6 \times 5)$

Solution of linear homogeneous recurrence relations with constant coefficients

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k}$ ① where $c_1, c_2, c_3 \dots c_k$ and $c_k \neq 0$ are constants, is called a linear homogeneous recurrence relation with constant coefficients of having the degree 'k'

i) The recurrence relation given in equation ① is called linear since each a_i has the power '1' and has no terms of the type $a_i . a_j$ occured.

ii) The degree of the recurrence relation is k, since a_n is expressed in terms of the previous k terms of the sequence. i.e, degree is the difference between greatest and lowest subscripts of the members of the sequence occuring in the recurrence relation.

iii) The coefficients of the terms of the sequence are all constants. They are not functions of n

iv) If the recurrence relation $F(n)=0$ Then it is called homogeneous otherwise the recurrence relation is called non-homogeneous.

Determine whether the following recurrence relations are linear homogeneous recurrence relation with constant coefficients

i) $a_n = 2a_{n-1} + a^2_{n-3}$ The recurrence relation is not a linear

ii) $H_n = 2H_{n-1} + 2$ The recurrence relation is non homogeneous

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iii) $B_n = nB_{n-1}$ It does not have Constant coefficients

Determine which of the following recurrence relations are linear homogeneous recurrence relations with constant coefficients and also find their degrees.

i) $a_n = 3a_{n-1} + 4a^2_{n-2} + 5a_{n-3}$

It is clearly not a linear

ii) $a_n = 2na_{n-1} + a_{n-2}$

It is linear but it does not have constant coefficient

iii) $a_n = a_{n-1} + a_{n-4}$

It is clearly a linear homogeneous recurrence relation with constant coefficients with degree 4

iv) $a_n = a_{n-2}$

It is clearly a linear homogeneous recurrence relation with constant coefficients with degree 2.

v) $a_n = a_{n-1} + n$

It is non homogeneous.

Characteristic equation:

Let us consider the recurrence relation

$a_n = C_1a_{n-1} + C_2a_{n-2} + C_3a_{n-3} \cdots$ (1)

The characteristic equation is

$$\left.\begin{aligned} r - C_1r^2 - C_2r^1 - C_3r^0 = 0 \\ r^3 - C_1r^2 - C_2r - C_3 = 0 \end{aligned}\right\} \text{—— (2)}$$

The roots of the characteristic equation are called characteristic roots.

Note: Let r_1, r_2, r_3 be the roots of the characteristic eq^n (2) then the solutions of recurrence relation given below.

Case (i) Suppose that the three roots r_1, r_2, r_3 are real and different then the solution of the recurrence relation is $a_n = \alpha_1 (r_1^n) + \alpha_2 (r_2^n) + \alpha_3 (r_3^n)$ where $\alpha_1, \alpha_2, \alpha_3$ are arbitary constants

Case (ii) Suppose 2 roots are real and equal and third root is real and different then the solution of the recurrence relation is
$a_n = (\alpha_1 + \alpha_2 n)(r_1^n) + \alpha_3 (r_3^n)$

Ex: Three roots are real and equal then the solution of the recurrence relation is
$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2)(r_1^n)$

Case (iii) If the roots are complex conjugate (i.e, $\alpha_1 \pm i\alpha_2$) then the solution of the recurrence relation is $a_n = r^n (\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Sol Given recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2}$ (1)
Clearly eq^n (1) is linear homogeneous recurrence relation with constant coefficients having the degree 2.

The characteristic eq (1) is

$$r^2 - 5r^{2-1} + 6r^{2-2} = 0$$
$$\Rightarrow r^2 - 5r + 6 = 0$$
$$\Rightarrow (r-3)(r-2) = 0$$
$$\Rightarrow r-3 = 0 \ , \ r-2 = 0$$

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$r = 2, 3$

$\therefore$ Clearly the two roots are real and different.

$\therefore$ The solution of the recurrence relation is

$a_n = \alpha(r_1^n) + \alpha_2(r_2^n)$ where α_1, α_2 are arbitary constants.

$\Rightarrow a_n = \alpha_1(2^n) + \alpha_2(3^n)$ —— eqn (2)

The initial values are $a_0 = 1, a_1 = 0$

Let us take $n = 0$ in eqn (2)

$$a_0 = \alpha_1(2^0) + \alpha_2(3^0)$$

$$1 = \alpha_1 + \alpha_2 \text{ —— eqn (3)}$$

Let us take $n = 1$ in eqn (2)

$$a_0 = \alpha_1(2^1) + \alpha_2(3^1)$$

$$0 = 2\alpha_1 + 3\alpha_2 \text{ —— eqn (4)}$$

Solving eqn (3) & (4)

$$2\alpha_1 + 2\alpha_2 = 2$$

$$2\alpha_1 + 3\alpha_2 = 0$$

$$-\alpha_2 = +2$$

$$\alpha_2 = -2$$

Substitute $\alpha_2 = -2$ in eqn (3), we get

$$1 = \alpha_1 - 2$$

$$\alpha_1 = 3$$

Substitute α_1 & α_2 values in eqn (2)

Then $a_n = 3(2^n) + (-2)(3^n)$, $n \geq 2$

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Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with the initial values $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

Given recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ ①

Clearly equation ① is linear homogeneous recurrence relation with constant coefficients having degree 3

The characteristic eq^n is

$$r^3 - 2r^{3-1} - r^{3-2} + 2r^{3-3} = 0$$

$$\Rightarrow r^3 - 2r^2 - r + 2 = 0$$

$$\Rightarrow r = 1, \quad r^2 - r - 2 = 0$$

$$\Rightarrow r = 1 \quad (r-2)(r+1) = 0$$

$$r = 2 \quad r = -1$$

1	1	-2	1	2
	0	1	-1	-2
	1	-1	-2	0

$\therefore r_1 = -1 \quad r_2 = 1 \quad r_3 = 2$

$\because$ Clearly the roots are real and distinct then the solution of the recurrence relation is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitary constants

$$\Rightarrow a_n = \alpha_1 (-1)^n + \alpha_2 (1)^n + \alpha_3 (2)^n \quad \text{—— eq}^n \text{②}$$

The initial values are $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

Let us take $n = 0$ in eq^n ②

$$3 = \alpha_1 (-1)^0 + \alpha_2 (1)^0 + \alpha_3 (2)^0$$

$$3 = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{——— ③}$$

Let us take $n = 1$ in eq^n ②

$$6 = \alpha_1 (-1)^1 + \alpha_2 (1)^1 + \alpha_3 (2)^1$$

$$6 = -\alpha_1 + \alpha_2 + 2\alpha_3 \quad \text{——— ④}$$

Let us take $n = 2$ in eq^n ②

$$0 = \alpha_1 (-1)^2 + \alpha_2 (1)^2 + \alpha_3 (2)^2$$

$$0 = \alpha_1 + \alpha_2 + 4\alpha_3 \quad \text{——— ⑤}$$

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Solving eq^n ③, ④, ⑤

$\cancel{\alpha_1} + \alpha_2 + \alpha_3 = 3$

$-\cancel{\alpha_1} + \alpha_2 + 2\alpha_3 = 6$

$2\alpha_2 + 3\alpha_3 = 9$ ——— ⑥

$\cancel{\alpha_1} + \alpha_2 + 4\alpha_3 = 0$

$-\cancel{\alpha_1} + \alpha_2 + 2\alpha_3 = 6$

$2\alpha_2 + 6\alpha_3 = 6$ ——— ⑦

Solving eq^n ⑥ & ⑦

$\cancel{2\alpha_2} + 3\alpha_3 = 9$

$(-) \cancel{2\alpha_2} + 6\alpha_3 = 6$

$-3\alpha_3 = +3$

$\boxed{\alpha_3 = -1}$

Substitute α_3 value in eq^n ⑦

$2\alpha_2 + 6\alpha_3 = 6$

$2\alpha_2 + 6(-1) = 6$

$2\alpha_2 - 6 = 6$

$\boxed{\alpha_2 = 6}$

Substitute α_2, α_3 value in eq^n ③

$3 = \alpha_1 + 6 + (-1)$

$\boxed{\alpha_1 = -2}$

Find an explicit formula for the fibanocci numbers

The fibanocci numbers are 0, 1, 1, 2, 3, 5, 8, 13 - - -

The recurrence relation of the fibanocci sequence is $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$, $a_0 = 0$, $a_1 = 1$ ——— ①

Clearly eq^n ① is linear homogeneous recurrence relation with constant coefficients having degree 2

The characteristic eq^n is

$r^2 - r - 1$

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$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{1-(-4)}}{2a} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1-\sqrt{5}}{2} \quad r_2 = \frac{1+\sqrt{5}}{2}$$

clearly the roots are ~~complex conjugate~~ real and distinct

The solution of the recurrence relation is

$\times$ ($a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$ —— eq^n 2

The initial values are $a_0 = 0$, $a_1 = 1$

Let us take $n=0$ in eq^n 2 we get

$0 = r^0(\alpha_1 \cos 0 + \alpha_2 \sin 0)$

$0 = (\alpha_1)$ $\times$

$$a_n = \alpha_1(r_1^n) + \alpha_2(r_2^n)$$

$$a_n = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Let us take $n=0$, we get

$$a_0 = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right)^0 + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)^0$$

$$0 = \alpha_1 + \alpha_2 \text{ —— } 3$$

Let us take $n=1$, we get

$$a_1 = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)$$

$$1 = \frac{\alpha_1}{2}(1-\sqrt{5}) + \frac{\alpha_2}{2}(1+\sqrt{5})$$

Solving eq^n 3 & 4

$$\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \cancel{\alpha_2\left(\frac{1+\sqrt{5}}{2}\right)} = 0$$

$$\alpha_1\left(\frac{1-\sqrt{5}}{2}\right) + \cancel{\alpha_2\left(\frac{1+\sqrt{5}}{2}\right)} = 1$$

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$\alpha_2 = \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$

$\alpha_2 = \left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)$ $\alpha_2 = \left(\frac{2\sqrt{5}}{2}\right) = 1$

$\alpha_2(\sqrt{5}) = 1$

$\alpha_2 = \frac{1}{\sqrt{5}}$

Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}, n \geq 2$
$a_0 = 1, a_1 = 2$

Given the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$ — (1)
clearly the eq^n (1) is L.H.R.R with constant coefficient having the degree '2'.
The characteristic eq^n is

$r^2 - 2r + 2 = 0$

$a = 1 \quad b = -2 \quad c = 2$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i\sqrt{4}}{2} = \frac{2 \pm 2i}{2}$

$z = 1 \pm i$

The roots are $1+i, 1-i$
clearly the roots are complex conjugate
The solution of the recurrence relation is

$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$ — (2)

$z = 1 \pm i \implies a = 1 \quad b = 1$

$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1} 1 = \tan^{-1}\left(\tan \frac{\pi}{4}\right)$

$= \frac{\pi}{4}$

Let us take $n=0$, we get

$$1 = (\sqrt{2})^0 \left(\alpha_1 \cos(0)\frac{\pi}{4} + \alpha_2 \sin(0)\frac{\pi}{4}\right)$$

$$1 = 1(1+0) \qquad 1 = \alpha_1 \text{ —— eq}^n \text{ (3)}$$

Let us take $n=1$, we get

$$2 = (\sqrt{2})^1 \left(\alpha_1 \cos(1)\frac{\pi}{4} + \alpha_2 \sin(1)\frac{\pi}{4}\right)$$

$$2 = (\sqrt{2}) \left(\alpha_1 \frac{1}{\sqrt{2}} + \alpha_2\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$2 = \cancel{\sqrt{2}}\left(\frac{1}{\cancel{\sqrt{2}}}\right) \alpha_1 + \alpha_2$$

$$2 = \alpha_1 + \alpha_2 \text{ —— (4)}$$

Substitute α_1 value in eq^n (3)

$$2 = 1 + \alpha_2$$

$$\alpha_2 = 1$$

Substitute α_1, α_2 value in eq^n (2)

$$a_n = (\sqrt{2})^n \left(\cos n\frac{\pi}{4} + \sin n\frac{\pi}{4}\right)$$

Solution of non-homogeneous recurrence relation

A linear non-homogeneous recurrence relation with constant coefficients of degree 'k' is a recurrence relation of the form $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_n a_{n-k} + G(n)$ where $C_1, C_2 \dots C_n$ are real numbers and $G(n)$ is a function not identically '0' depending only on 'n'.

→ We have to solve the non-homogeneous recurrence relation by using following steps.

Step 1: We obtain the homogeneous solution

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First we write the associated homogeneous recurrence relation by taking $G(n) = 0$. Then we find its general solution which is called the homogeneous solution.

Step 2: We obtain the particular solution. There is no general procedure for finding the particular solution of a recurrence relation. However if $G(n)$ has any one of the following forms (i) polynomial in n
(ii) A constant
(iii) Powers of constant

Particular Solution for given $G(n)$

S.No	$G(n)$	Form of particular sol^n
1.	A constant c	d
2.	A linear function $c_0 + c_1 n$	$d_0 + d_1 k$
3.	n^2	$d_0 + d_1 k + d_2 k^2$
4.	$r^n, r \in R$	$d r^n$

Step 3: We substitute the guess from Step 2 into the recurrence relation. If the guess is correct then we can determine the unknown coefficient of the guess. If we are unable to determine the constants then our guess is wrong and hence we go to step 2.

Step 4: The general solution of the recurrence relation is sum of the homogeneous and particular solution, i.e, $a_n = a_n^{(H)} + a_n^{(P)}$
If no initial conditions are given then Step

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& will give the soln.
If the initial values are given then we find the arbitary constant values, then we get a complete solution.

Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, for $a_0 = 1$

Sol The given recurrence relation is $a_n = 3a_{n-1} + 2^n$ —— ①
Clearly eq^n ① is non-homogeneous recurrence relation. $a_n - 3a_{n-1} = 2^n$

Step 1: we have to find Homogeneous Solution by taking $G(n) = 2^n = 0$

Then eq^n ① becomes as

$$a_n - 3a_{n-1} = 0$$

The characterstic eq^n is $r - 3 = 0$

$$r = 3$$

clearly it is real root

∴ The solution of the homogeneous recurrence relation is $a_n^{(H)} = c_1 r^n$

$$a_n^{(H)} = c_1 3^n$$

Step 2: To find particular Solution of the non-homogeneous recurrence relation. $G(n) = 2^n$

Now let us choose the particular solution as $a_n = d r^n = d 2^n$ —— eq^n ②

Now substitute $a_n = d2^n$ in eq^n ①

$$d2^n - 3d2^{n-1} = 2^n$$

$$\Rightarrow d2^n - \tfrac{3}{2} d 2^n = 2^n$$

$$\Rightarrow 2^n \left[d - \frac{3}{2} d\right] = 2^n$$

$$\Rightarrow \left(\frac{-1}{2}\right) d = 1 \quad \Rightarrow d = -2$$

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Now substitute $d = -2$ in eq^n (2) we get particular solution of recurrence relation.

$$a_n^{(P)} = (-2) \cdot 2^n$$

The general solution of the recurrence relation is

$$a_n = a_n^{(H)} + a_{n-1}^{(P)}$$

$$\boxed{a_n = c_1 3^n + (-2) 2^n} \quad \text{eq}^n \ (3)$$

Take $n = 0$ in eq^n (3), we get

$$a_0 = c_1 3^0 + (-2) 2^0$$

$$1 = c_1 - 2$$

$$\boxed{c_1 = 3}$$

$$\therefore a_n = 3 \, 3^n - 2 \cdot 2^n$$

$$a_n = 3^{n+1} - 2^{n+1}$$

Solve the recurrence relation
$a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$, $a_0 = 1$ & $a_1 = 1$

The given recurrence relation is
$a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$ —— (1)

Clearly the eq^n (1) is non homogeneous recurrence relation

$$a_n - 4a_{n-1} + 4a_{n-2} + \cancel{3n} = 2^n + 3n$$

Step 1: we have to ~~take~~ find homogeneous solution by taking $G(n) = 2^n + 3n = 0$

Then eq^n (1) becomes as

$$a_n - 4a_{n-1} + 4a_{n-2} = \cancel{3n} = 0$$

The characteristic eq^n is

$$\cancel{r^3 - 4r^2 + 4r - 3} = \cancel{0}$$

$$r^2 - 4r + 4 = 0.$$

$$r^2 - 2r - 2r + 4 = 0$$

$$r(r-2) - 2(r-2) = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2, 2$$

Clearly they are real roots and equal roots

∴ The solution of the homogeneous recurrence relation is $\cancel{a_n^{(H)} = C_1 2^n}$ $a_n^{(H)} = (\alpha_1 + \alpha_2 n)r^n$

$$\therefore a_n^{(H)} = (\alpha_1 + \alpha_2 n)2^n$$

Step 2: To find particular solution

$$G(n) = 3n + 2^n$$

$$a_n^{(P)} = a_n^{(P_1)} + a_n^{(P_2)}$$

part - 1: $a_n^{(P_2)}$

Choose $a_n = d_0 + d_1 n$

$$(d_0 + d_1 n) - 4[d_0 + d_1(n-1)] + 4[d_0 + d_1(n-2)] = 3n$$

$$\Rightarrow (d_0 - 4d_0 + 4d_0) + (d_1 n - 4d_1 n + 4d_1 + 4d_1 n - 8d_1) = 3n$$

$\Rightarrow d_0 + (d_1 n - 4d_1) = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1 n = 3n$

Compare the coefficients

$d_0 - 4d_1 = 0 \qquad d_1 = 3$

$d_0 - 12 = 0 \qquad d_0 = 12.$

$a_n^{(P_1)} = 12 + 3n$

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i) $a_n = 2a_{n-1} + 3a_{n-2}$ ~~$+4n$~~ for $n \geq 2$, $a_0 = 2$, $a_1 = 2$

ii) $a_n = 5a_{n-1} + 6a_{n-2} + 4n$ $n \geq 2$

iii) $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ $a_0 = 0$, $a_1 = 1$

iv) $a_n = 3a_{n-1} + 4a_{n-2}$, $n \geq 2$ $a_0 = a_1 = 1$

i) The given recurrence relation is $a_n = 2a_{n-1} + 3a_{n-2}$ — ①

Clearly the eqⁿ ① is homogeneous linear equation

The characteristic eqⁿ is

$$r^2 - 2r - 3 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(-3)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \cancel{\frac{2 \pm 3\sqrt{2}}{2}}$$

$$= \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \frac{6}{2}, \frac{-2}{2}$$

$$r = -1, 3.$$

Clearly the roots are real and different

Hence $a_n^{(H)} = \alpha_1 (r_1^n) + \alpha_1 (r_n^n)$

$$a_n^{(H)} = \alpha_1 (-1)^n + \alpha_2 (3^n) \quad \text{— ②}$$

The initial values are $a_0 = 2$, $a_1 = 2$

Now take $n = 0$ in eqⁿ ①, we get

$$2 = \alpha_1 + \alpha_2 \quad \text{— ③}$$

Now take $n = 1$ in eqⁿ ②

$$2 = -\alpha_1 + 3\alpha_2 \quad \text{— ④}$$

Solve ② & ④

$$\alpha_1 + \alpha_2 = 2$$
$$-\alpha_1 + 3\alpha_2 = 2$$
$$4\alpha_2 = 4$$
$$\boxed{\alpha_2 = 1}$$

$\alpha_1 = 1$

The solution is $a_n^{(H)} = (-1)^n + (3)^n$

ii) Given recurrance relation is

$5a_{n-1} + 6a_{n-2} + 4n$ ——— ①

we have to find the homogeneous solution by taking $G(n) = 4n = 0$.

eqⁿ ① becomes as $5a_{n-1} + 6a_{n-2} = 0$

The characterstic eqⁿ is

$$r^2 - 5r - 6 = 0$$

$$r(r-6) + 1(r-6)$$

$$r = -1 \quad r = 6$$

$$r = -1, 6$$

The obtained roots are real and different

∴ The solution of the homogeneous RR is

$$a_n^{(H)} = \alpha_1 (r_1^n) + \alpha_2 (r_2^n)$$

$$a_n^{(H)} = \alpha_1 (-1) + \alpha_2 (-6)$$

Step 2:- we have to find the particular solution

clearly $G(n)$ is of the form $G(n) = 4n$

$a_n = d_0 + d_1 n$ Substitute in eqⁿ①

$\rightarrow d_0 + d_1 n - 5(d_0 + d_1(n-1)) - 6(d_0 + d_1(n-2)) = 4n$

$\Rightarrow d_0 + d_1 n - 5d_0 + 5d_1 n + 5d_1 - 6d_0 - 6d_1 n$

$- \cancel{6}^{12} d_1 = 4n$

$\Rightarrow d_0 - 5d_0 - 6d_0 + d_1 n - d_1 5n + 5d_1 - d_1 6n$

$+ 12 d_1 = 4n$

$= -10d_0 + [-10dn + 17d_1] = 4n$

$17 d_1 - 10 d_0 = 0, \quad -10 d_1 n = 4n$

$d_1 = 4/-10 = -2/5$

$d_1 = -2/5$

$17\left(\frac{-2}{5}\right) - 10 d_0 = 0 \Rightarrow -\frac{34}{5} - 10 d_0 = 0$

$\Rightarrow -\frac{34}{5} = 10 d_0 \Rightarrow d_0 = \frac{-34}{50} = \frac{-17}{25}$

$$a_n = \frac{-17}{25} - \frac{2}{5}n \Rightarrow -\left(\frac{17}{25} + \frac{2}{5}\right)$$

The GS is $a_n = \alpha_1(-1)^n + \alpha_2(6^n) - \left(\frac{17}{25} + \frac{2}{5}n\right)$

3. The given R.R is $F_n = F_{n-1} + F_{n-2}$ —— ①

Clearly eqⁿ ① is linear homogeneous

The characteristic eqⁿ is

$$r^2 - r - 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2}$$

$$r = \frac{1+\sqrt{5}}{2}, \quad r = \frac{1-\sqrt{5}}{2}$$

Clearly the roots are real and distinct

Solution is $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$F_n = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)^n \text{ —— ②}$$

$$n = 0 \Rightarrow F_0 = \alpha_1 + \alpha_2$$

$$0 = \alpha_1 + \alpha_2 \text{ —— ③}$$

$$n = 1 \Rightarrow F_1 = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)$$

$$1 = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right) \text{ —— ④}$$

Solve ③ and ④ $\Rightarrow (0 = \alpha_1 + \alpha_2) \times \left(\frac{1+\sqrt{5}}{2}\right)$

$$1 = \alpha_1\left(\frac{1-\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)$$

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$$\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right) = 0$$

$$\underset{(-)}{\alpha_1}\left(\frac{1-\sqrt{5}}{2}\right) \underset{(-)}{+} \alpha_2\left(\frac{1+\sqrt{5}}{2}\right) \underset{(-)}{=} 1$$

$$-1 = \alpha_2\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)$$

$$\alpha_2 = -1/\sqrt{5}$$

$$\alpha_1 = 1/\sqrt{5}$$

Sub α_1, α_2 in ②

$$F_n = \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n, \; n \geq 2 \text{ u}$$

The Solution is R.R

4. Given eq^n is Linear homogeneous RR with constant Coefficient of degree 2.

The characteristic eq^n is

$$r^2 - 3r - 4 = 0$$

$$r(r-4) + 1(r-4) = 0$$

$$(r-4)(r+1) = 0$$

$$r = 4, -1$$

Clearly the roots are real and distinct

The homogeneous solution is $a_n = \alpha_1(r_1^n) + \alpha_2(r_2^n)$

$$a_n = \alpha_1(4^n) + \alpha_2(-1)^n \quad \text{— ②}$$

$$n = 0 \Rightarrow a_0 = \alpha_1(4^0) + \alpha_2(-1^0)$$

$$1 = \alpha_1 + \alpha_2 \quad \text{— ③}$$

$$n = 1 \Rightarrow a_1 = \alpha_1(4) + \alpha_2(-1)$$

$$1 = 4\alpha_1 - \alpha_2 \quad \text{— ④}$$

Solve eq^n ③ & ④

$$1 = \alpha_1 + \alpha_2$$
$$1 = 4\alpha_1 - \alpha_2$$
$$5\alpha_1 = 2$$
$$\alpha_1 = 2/5$$

$$1 = \frac{2}{5} + \alpha_2$$

$$\alpha_2 = 1 - 2/5$$
$$\alpha_2 = 3/5$$

Substitute α_1, α_2 in ②

$$a_n = \frac{2}{5}(4)^n + \frac{3}{5}(-1)^n$$