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* UNIT 3 : Sets Relations and Algebraic Structures *

* **Set :-** Any collection of well defined objects is called a set. The objects in a set can be numbers or people or letters etc.

Any object belong to a set is called a member or Element of the set

Eg) The set of vowels in the alphabet

The set of all positive integers less than 10

collection of Rocks etc.

→ In general a set is denoted by a capital Letter and elements inside set seperated by comma and those are enclosed in curly braces "{ }"

Eg) { 5, 7, 8, 9, 10 }

* **Representation of Sets :-** There are mainly 3 ways to represent a set

1. Statement Form (Descriptive form)
2. Roasted Form (Tabular form)
3. Set Builder Form

- **Statement Form**

A single statement describe all the elements inside a set.

Eg) B = Set of All vowels in alphabets

- **Roasted Form**

In this form all the members of given set are enlisted within a pair of braces { } seperated by "," (comma)

Eg) The set of All even numbers in whole numbers between 1 to 10

$\Rightarrow E = \{2, 4, 6, 8\}$

- **Set Builder Form**

A property is stated that must be common to all elements of that particular set.

Eg) $N = \{x : x$ is a positive integer between 1 to 20$\}$

* Types of sets

Empty set

A set which does not contain any element is called an empty or null set denoted by "$\emptyset$"

Eg) $A = \{x : 2 < x < 3, x \text{ is a natural number}\}$

Single ton set

A set which contain only one element is called Singleton set

Eg) $A = \{2\}$

$B = \{x : x \text{ is either prime or composite number}\}$

Finite set

A set which contains definite number of elements is called Finite set

Eg) The set of All colors in Rainbow

Infinite set

A set which contain uncountable members is called infinite set.

Eg) Set of all points in a plane,

$A = \{x : x \in N, x > 1\}$

* Subset.

If A and B are two sets and every element of set A is also an element of set B then A is called subset of B and we write it as $A \subset B$

Eg) $A = \{2, 3, 4, 7\}$

$B = \{3, 2, 7, 4\}$

$\therefore A \subset B$

- Every set is a subset of itself
- Empty set is a subset of everyset
- "⊂" is used to denote that "is a subset of" or "is contained in"
- A is subset of B means "$A \subset B$"

  A is contained in B

*** Proper subset**

A proper subset of a set A is a subset of A that is not equal to A. In other words if B is a proper subset of A then all elements of B are in A but A contains atleast one element that is not in B.

Eg) $A = \{1, 3, 5\}$

$B = \{1, 5\}$ $\quad B \subset A$

*** Super Set**

A set "A" is a superset of another set B if all elements of the set B are elements of A set

- The representation of a super set can be written as $A \supset B$

Eg) $A = \{1, 3, 5\}$ $\quad A \supset B$

$B = \{1, 5\}$

* power set

The collection of all subsets of set A is called the power set of A. It is denoted by P(A). In P(A), every element is a set

. Number of elements in powerset $= 2^n$

Eg) If $A = \{p, q\}$ then the subsets of A will be $P(A) = \{\emptyset, \{p\}, \{q\}, \{p,q\}\}$

* Equal Sets

Equal sets have the exact same elements in them, even though they are unordered.

Eg) $A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$, $C = \{3, 1, 2\}$

* Equivalent sets :

Equivalent sets have different elements but they have the number of elements same. The symbol for denoting an equivalent set is " $\leftrightarrow$ "

Eg) $A = \{1, 2, 3\}$ Here $n(A) = 3$ and

$B = \{p, q, r\}$ Here $n(B) = 3$ $\quad \therefore A \leftrightarrow B$

* **universal Set :**

A set which contains all the elements of other given sets is called a universal set. The symbol for denoting universal set is "U"

Eg) If $A = \{1, 2, 3\}$ $B = \{2, 3, 4\}$ $C = \{3, 5, 7\}$

then $U = \{1, 2, 3, 4, 5, 7\}$

Here $A \subseteq U$, $B \subseteq U$, $C \subseteq U$ and $U \supseteq A$;
$U \supseteq B$, $U \supseteq C$

* **Disjoint sets :**

The two sets A and B are said to be disjoint if the set does not contain any common element.

Eg) set $A = \{1, 2, 3, 4\}$ set $B = \{5, 6, 7, 8\}$
are disjoint sets because there is no common element between.

* **Set operations :**

Let $U = \{1, 2, 3, 4, 5, 6\}$ and within that set $A = \{1, 2\}$ and $B = \{2, 3, 4\}$

• **union :**

The union of two sets is the set of elements that belong to one

OR both of the two sets

The union of A, B is $A \cup B = \{1, 2, 3, 4\}$

- Intersection

The intersection of two sets is the set of elements that are common to both sets

The intersection of A, B is $A \cap B = \{2\}$

- complement

The complement of an event is the set of all elements in the universal space but not in the event

Eg) $U = \{1, 3, 4, 5, 6, 7\}$ and

$A = \{1, 4, 6\}$ then $A' = \{3, 5, 7\}$

- Set Difference

Difference between sets is denoted by "A − B", is the set containing elements of set A but not in B i.e., all elements of A except the elements of B.

Eg) $A = \{1, 2, 3\}$ $B = \{3, 4\}$

$A - B = \{1, 2\}$

If set A and set B are two sets then the cartesian product of set A and set B is a set of all ordered pairs (a,b) such that a is an element of A and b is an element of B.

- It is denoted by "A x B"

Eg) set $A = \{1, 2, 3\}$ set $B = \{A, B\}$ then

$$A \times B = \{(1, A), (1, B), (2, A), (2, B), (3, A), (3, B)\}$$

* Set Formulas

For any three sets A, B and C

* $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

* If $A \cap B = \emptyset$ then $n(A \cup B) = n(A) + n(B)$

* $n(A - B) + n(A \cap B) = n(A)$

* $n(B - A) + n(A \cap B) = n(B)$

* $n(A - B) + n(A \cap B) + n(B - A) = n(A \cup B)$

* $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

* Basic Laws of Sets :

1. Commutative Law :

For any two finite sets A and B

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

2. Associative Law :

For any three finite sets A, B and C

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Thus union and intersection are commutative and associative

3. Idempotent Law :

For any finite set A

(i) $A \cup A = A$

(ii) $A \cap A = A$

4. Distributive Laws :

For any three finite sets A, B and C

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

* **More Laws of algebra of sets :**

For any two finite sets A and B

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \emptyset$

(iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \emptyset$

(vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

For any three finite sets A, B and C

(i) $A - (B \cap C) = (A - B) \cap (A - C)$

(ii) $A - (B \cup C) = (A - B) \cup (A - C)$

(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

* Principle of Inclusion and Exclusion

How many elements are there in two finite sets when two tasks can be done at the same time, to count the number of ways in which any one of the task can be done we add the number of ways doing each of the two tasks and then subtract the number of ways of doing both the tasks, These techniques are called principle of Inclusion and Exclusion

- Statement

Let A and B be any 2 finite sets then the number of elements in the union of 2 sets A and B is the sum of the number of elements in the sets − the number of elements in the intersection i.e,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

- Among 60 students in a class 45 passed in 1st sem exams and 30 passed in 2nd sem exams. If 12 students passed in neither semister. How many passed both the sems.

Let A represent the set of students who passed in the first semister examination and B represents the set of students who passed in the second semister examination. Then

$|A| = 45$

$n(B) = 30$

Given 12 students did not passed in both the semisters.

We can write $|A' \cap B'| = 12$

Now the number of students passed in any one of the semister $= |A \cup B|$

$= n - |A' \cap B'|$

$= 60 - 12$

$= 48$

$\therefore$ By the principle of inclusion and exclusion we know that.

$|A \cup B| = |A| + |B| - |A \cap B|$

Substitute $|A|$, $|B|$, $|A \cup B|$ values in above eq, we get

$48 = 45 + 30 - |A \cap B|$

$\Rightarrow |A \cap B| = 27$

Hence the number of students passed in both sem $= 27$

- How many positive integers not exceeding 2000 are divisible by 7 or 11.

Let A be the set of positive integers not exceeding 2000 that are divisible by 7 & B be the set of positive integers not exceeding 2000 that are divisible by 11

$A \cap B$ is the number of positive integers not exceeding 2000 divisible by 7 and 11.

$$\therefore |A| = \left| \frac{2000}{7} \right| = 285 \cdot 7142 \simeq 286$$

$$|B| = \left| \frac{2000}{11} \right| = 181 \cdot 8182 \simeq 182$$

$$|A \cap B| = \left| \frac{2000}{7 \times 11} \right| = 25 \cdot 97 \simeq 26$$

$\therefore$ By the principle of inclusion and exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Here $|A| = 286$, $|B| = 182$, $|A \cap B| = 26$.

$$\therefore |A \cup B| = 286 + 182 - 26$$

$$= 442$$

Hence the number of positive integers not exceeding 2000 that are divisible by 7 or 11 are 442

. In a class 50 Students, 20 students play football and 16 students play Hockey. It is found that 10 students play both the games. Find the number of students that play neither.

Let A represents the set of students playing football and B represents the set of students playing Hockey and $A \cap B$ represents the set of students playing both the games.

$|A| = 20$

$|B| = 16$

$|A \cap B| = 10$

By principle of inclusion and exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 20 + 16 - 10$

$|A \cup B| = 26$

Thus 26 students play either football or Hockey. Then the number of students who play neither hockey nor football

$= n - |A \cup B|$

$= 50 - 26$

$= 24$

• If A, B, C are any 3 finite sets then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

(∴ Distributive law)

$$= |A| + |B| + |C| - |B \cap C| - \Big[|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|\Big]$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

• General form of principle of inclusion and exclusion

If $A_1, A_2, \dots A_n$ are finite sets then

$$|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_i| =$$

$$\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le 1} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le 1} |A_i \cap A_j \cap A_k|$$

$$\dots\dots + (-1)^n \,|(A_1 \cap A_2 \cap \dots \cap A_n|$$

. Give a formula for the number of elements in a union of four sets

$n(A \cup B \cup C \cup D)$

$[= n((A \cup B) \cup (C \cup D))$

$= n(A \cup B) + n(C \cup D) - n((A \cup B) \cap (C \cup D))$

$= n(A) + n(B) - n(A \cap B) + n(C) + n(D) -$

$n(C \cap D) - n((A \cup B) \cap (C \cup D)) \cdots]$

$= n(A) + n(B) - n(A \cap B) + n(C) + n(D) - n(C \cap D)$

$- n(A \cap B \cap C \cap D)$

$- n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D)$

$+ n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D)$

$+ n(B \cap C \cap D)$

A survey of 260 television viewers produced the following information:

64 watch sports

58 watch films

94 watch news,

26 watch both sports and news

28 watch both sports and films

22 watch both films and news items and

14 are interested in all the 3 types. How many people in the survey are not interested in any one of the 3 types. How many are in the survey are interested in seeing only news items

Let x represents the set of people who watch sports, L represents the set of people who see films, N represents the people who see news items then

$n(X) = 64$

$n(L) = 58$

$n(N) = 94$

$n(X \cap L) = 28$

$n(L \cap N) = 22$

$n(X \cap N) = 26$

$n(X \cap L \cap N) = 14$

Number of people interested in any one of 3 types $= n(X \cup L \cup N)$

By principle of inclusion and exclusion

$n(X \cup L \cup N) = n(X) + n(L) + n(N) - n(L \cap N)$

$- n(X \cap N) - n(X \cap L) + n(X \cap L \cap N)$

$= 64 + 58 + 94 - 22 - 26 - 28 + 14$

$= 154$

Hence the number of people in the survey not interested in any one of these 3 types is $n(X' \cap L' \cap N') = n - n(X \cup L \cup N)$

$= 260 - 154$

$= 106$

Number of people interested in seeing only news items and films

$= n(L \cap N) - n(X \cap L \cap N)$

$= 22 - 14$

$= 8$

Number of people interested only sports and news items

$= n(X \cap N) - n(X \cap L \cap N)$

$= 26 - 14$

$= 12$

$\therefore$ Number of people intersted in seeing only news items

$= n(N) - 12 - 14 - 8 = 60$ people

- Determine the number of positive integers n where $1 \leqslant n \leqslant 2000$ and n is not divisible by 2, 3 or 5, but is divisible by 7

- Let P_1 be the property that n is divisible by 2
- P_2 be the property that n is divisible by 3
- P_3 be the property that n is divisible by 5
- P_4 be the property that n is divisible by 7

$$n(P_1) = \left| \frac{2000}{2} \right| = 1000$$

$$n(P_2) = \left| \frac{2000}{3} \right| = 666.6 \simeq 667$$

$$n(P_3) = \left| \frac{2000}{5} \right| = 400$$

$$n(P_4) = \left| \frac{2000}{7} \right| = 285.71 \simeq 286$$

$$n(P_1 \cap P_2) = \left| \frac{2000}{2 \times 3} \right| = 333.33 \simeq 333$$

$$n(P_1 \cap P_2 \cap P_3 = \left| \frac{2000}{2 \times 3 \times 5} \right| = 66.66 \simeq 67$$

$n(P_1 \cap P_3) = \left| \frac{2000}{2 \times 5} \right| = 200$

$n(P_2 \cap P_3) = \left| \frac{2000}{3 \times 5} \right| = 133.33 \simeq 133$

$\therefore$ n is divisible by 2, 3, 5 is

$n(P_1 \cup P_2 \cup P_3)$

$= n(P_1) + n(P_2) + n(P_3) - n(P_1 \cap P_2) - n(P_2 \cap P_3)$

$- n(P_1 \cap P_3) + n(P_1 \cap P_2 \cap P_3)$

$= 1000 + 667 + 400 - 333 - 133 - 200 + 67$

$= 1468$

$\therefore$ The number of numbers not divisible by 2, 3, 5 is

$n(P_1' \cap P_2' \cap P_3') = N - n(P_1 \cup P_2 \cup P_3)$

$= 2000 - 1468$

$= 532$

$\therefore$ The numbers are not divisible by 2, 3 or 5 and divisible by 7

$= \left| \frac{532}{7} \right| = 76$

- Determine the number of primes not exceeding 100 that are not divisible by 2, 3, 5 or 7

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Let P_1 be the property that an integer is divisible by 2

P_2 be the property that an integer is divisible by 3

P_3 be the property that an integer is divisible by 5

P_4 be the property that an integer is divisible by 7

Thus the number of positive integers not exceeding 100 that are not divisible by 2, 3, 5 or 7 is

$$N(P_1' \cap P_2' \cap P_3' \cap P_4') = N - N(P_1 \cup P_2 \cup P_3 \cup P_4)$$

$$= 100 - [N(P_1) + N(P_2) + N(P_3) + N(P_4)$$
$$- N(P_1 \cap P_2) - N(P_1 \cap P_3) - N(P_1 \cap P_4)$$
$$- N(P_2 \cap P_3) - N(P_2 \cap P_4)$$
$$- N(P_3 \cap P_4) + N(P_1 \cap P_2 \cap P_3) +$$
$$N(P_1 \cap P_2 \cap P_4) + N(P_2 \cap P_3 \cap P_4) +$$
$$N(P_1 \cap P_3 \cap P_4)$$
$$- N(P_1 \cap P_2 \cap P_3 \cap P_4)]$$

$$= 100 - \left[\left|\frac{100}{2}\right| + \left|\frac{100}{3}\right| + \left|\frac{100}{5}\right| + \left|\frac{100}{7}\right| \right.$$
$$- \left|\frac{100}{2 \times 3}\right| - \left|\frac{100}{2 \times 5}\right| - \left|\frac{100}{2 \times 7}\right|$$

$$- \left\lfloor \frac{100}{3\times5} \right\rfloor - \left\lfloor \frac{100}{3\times7} \right\rfloor$$

$$- \left\lfloor \frac{100}{5\times7} \right\rfloor + \left\lfloor \frac{100}{2\times3\times5} \right\rfloor + \left\lfloor \frac{100}{2\times3\times7} \right\rfloor$$

$$+ \left\lfloor \frac{100}{3\times5\times7} \right\rfloor + \left\lfloor \frac{100}{2\times5\times7} \right\rfloor - \left\lfloor \frac{100}{2\times3\times5\times7} \right\rfloor \Bigg]$$

$$= 100 - [50 + 33 + 20 + 14 - 17 - 10 - 7 - 7 - 5 - 3 + 3 + 2 + 1 + 1 - 0]$$

$$= 100 - 75$$

$$= 25$$

Hence the number of values not exceeding 100 not divisible by 2, 3, 5 or 7 = 25

∴ Number of primes not exceeding 100 that are not divisible by 2, 3, 5 or 7

$= 25 - 4$

$= 21$

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RELATIONS

* Relation

: Given any 2 non empty sets A, B a Relation R from A to B is a subset of cartesian product (A×B) and is derived by describing a relation between the first element say, x and the other element say y of the order pairs in A and B.

- If a Relation exists between two sets then it is called as a Binary relation

- Representation of Relations

Relations can be represented as four forms

1. Roasted Form
2. Set Builder Form
3. Matrix Form
4. Di graph Form

- consider an example of 2 sets

$A = \{9, 16, 25\}$ and $B = \{5, 4, 3, -3, -4, -5\}$

The Relation is the elements of A are the squares of the elements of B

- In set Builder Form

$R = \{(x, y) : x \text{ is a square of } y\}$

$R = \{(9,3), (9,-3), (16,4), (16,-4), (25,5), (25,-5)\}$

In Roaster Form

consider an example of 2 sets:

$A = \{2, 5, 7, 8, 9, 10, 13\}$

$B = \{1, 2, 3, 4, 5\}$

The cartesian product of $A \times B$ has 35 order pairs such as

$A \times B = \{(2,1), (2,2), (2,3), (2,4), (2,5), (5,1), (5,2), (5,3), (5,4) \ldots\ldots (13,5)\}$

From this we can obtain the subsets of $A \times B$ by introducing a relation R between the first element and second elements of the order pairs (x, y) as

Set Builder Form

$R = \{(x, y) : x = 4y - 3, x \in A \text{ and } y \in B\}$

Roaster Form

$R = \{(5,2), (9,3), (13,4)\}$

- Representing relation using matrices:

A relation between finite sets can be represented using a zero-one matrix

- Suppose R is a relation from $A = \{a_1, a_2, \ldots a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$

The Relation R is represented by the matrix M_R where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Eg) Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$ and $a > b$

What is matrix Representing R

$R = \{(2,1), (3,1), (3,2)\}$

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

- Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, \ldots$

Which ordered pairs are in the Relation R represented by the matrix

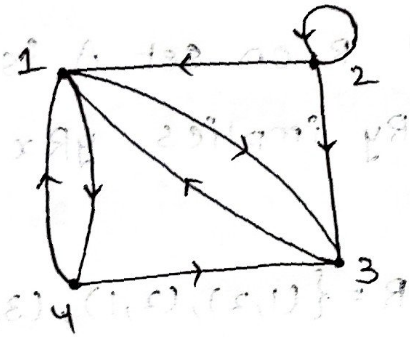
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

- Representing relation using Digraphs

A directed graph or digraph consists of a set V of vertices and set E of ordered pairs (formed from elements of called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and the vertex b is called the terminal vertex of this edge. We can use arrows to display graphs.

- What are the ordered pairs in the relation represented by this digraph.



The ordered pairs in the relation are

(1,3), (1,4) (2,1), (2,2), (2,3), (3,1), (4,1) and (4,3)

* Properties of Binary Relation

- Reflexive

A relation R on a set A is called Reflexive if $\forall a \in A$ is related to a (aRa holds)

eg) The Relation $R = \{(a,a), (b,b)\}$ on set $X = \{a,b\}$ is reflexive

- Irreflexive

A relation R on set A is called Irreflexive if no $a \in A$ is related to a (aRa does not hold)

eg) The Relation $R = \{(a,b), (b,a)\}$ on set $X = \{a,b\}$ is irreflexive

- Symmetric

A relation R on set A is called symmetric if xRy implies yRx, $\forall x \in A$ and $\forall y \in A$.

eg) The Relation $R = \{(1,2), (2,1), (3,2), (2,3)\}$ on set $A = \{1,2,3\}$ is symmetric

- Anti-Symmetric

A relation R on set A is called Anti-Symmetric if xRy and yRx hold only when $x = y$, $\forall x \in A$ and $\forall y \in A$.

eg) The Relation

$$R = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

R is not anti symmetric here because of $(1,2) \in R$ and $(2,1) \in R$ but $1 \neq 2$

- Assymmetric

A relation R on set A is called assymmetric if $(x,y) \in R$ implies $(y,x) \notin R$, $\forall x \in A$ and $\forall y \in A$

eg) The relation "Is less than" is an assymmetric such as $7<15$ but 15 is not less than 7

- Transitive

A relation R on set A is called Transitive if xRy and yRz implies xRz, $\forall x,y,z \in A$.

eg). The Relation $R = \{(1,2), (2,3), (1,3)\}$ on set $A = \{1,2,3\}$ is transitive.

* Matrices of Relation's over properties

- A relation R is reflexive, if all the elements on the main diagonal of M_R equal to 1
- A relation R is irreflexive if all the elements on the main diagonal of M_R equal to 0
- A relation R is a Symmetric if $M_{ij} = 1$ then $M_{ji} = 1$ and vice versa for 0.

- A Relation R is an anti symmetric if and only if $M_{ij} = 0$ then $M_{ji} = 1$ when $i \neq j$ or if $M_{ji} = 0$ then $M_{ij} = 1$ when $i \neq j$

- A Relation R is a transitive if $M_{ij} = 1$ and $M_{jk} = 1$ then $M_{ik} = 1$ and vice versa for 0

Suppose $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ a relation R

Here for $M_R = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ ∴ reflexive

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$M_{12} = M_{21} = 1$

$M_{13} = M_{31} = 0$

$M_{23} = M_{32} = 1$

∴ anti-symmetric

$(M_R)^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ∴ Symmetric

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· Equivalence Relation

A relation is an equivalence relation if it is reflexive, symmetric and transitive

Eg) $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$ on

$A = \{1,2,3\}$ is an equivalence relation since it is reflexive, symmetric and transitive

· partial ordering

A Binary relation R in a set P is called partial order relation or partial ordering on P if and only if R is reflexive, anti symmetric and transitive

· Partially ordered SET [POSET]

A set P together with A partial ordering R is called a partially ordered set or POSET.

· The Relation R is often denoted by the symbol $\leq$ which is differ from usual less than or equal to symbol. Thus if $\leq$ is a partial order in P then the order pair (P, R) or $(P, \leq)$ is called POSET

• Hasse Diagram

A partial order $\leqslant$ on a set P can be represented by means of a digraph known as Hasse Diagram of $(P, \leqslant)$

• In such a diagram

i) Each element is represented by a small circle or dot

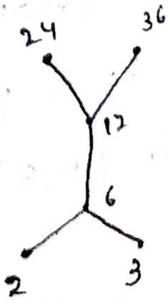
ii) The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$ and a line is drawn between x and y. if y covers x

iii) If $x < y$ but y does not cover x then x and y are not connected by a single Line

Eg of Hasse diagram } $P = \{1, 2, 3, 4\}$, $\leqslant$ is the relation then Hasse diagram of $(P, \leqslant)$ is

• $P = \{2, 3, 6, 12, 24, 36\}$, $\leqslant$: divides

Hasse Diagram of $(P, \leqslant)$ is



. Draw the Hasse Diagram for i) $(S_{24}, /)$

ii) $(S_6, /)$

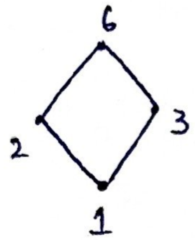
iii) $(S_{30}, /)$

iv) $(S_{60}, /)$

i) previous question

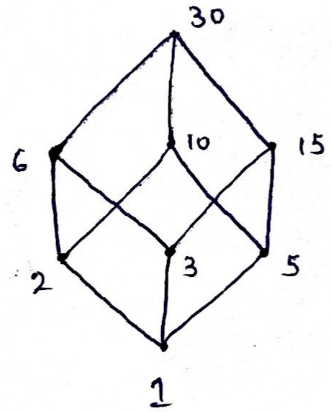
(ii) $(S_6, /)$

$S_6 = \{1, 2, 3, 6\}$



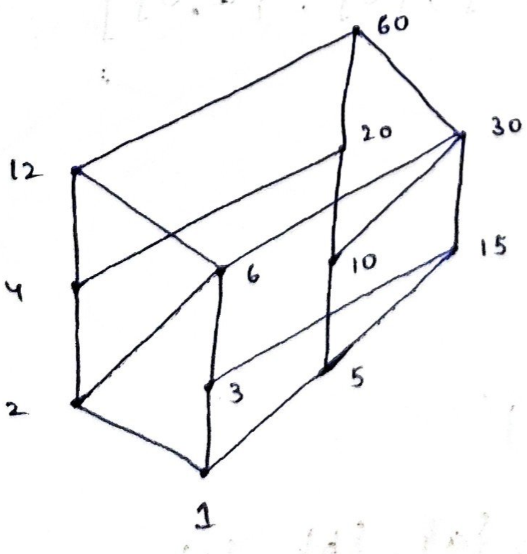
(iii) $(S_{30}, /)$

$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



(iv) $(S_{60}, /)$

$S_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$



- Let A be given finite set and P(A) be its power set, Let $\leq$ be the inclusion relation on the element of P(A). Draw Hasse diagrams of $(P(A), \leq)$ for :

$A = \{a\}$;

$A = \{a, b\}$;

$A = \{a, b, c\}$;

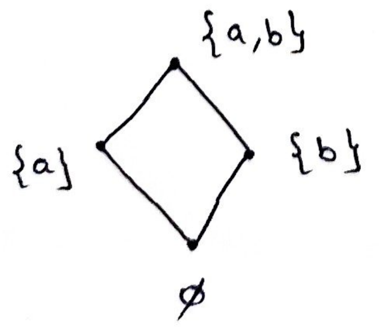
$A = \{a, b, c, d\}$

(a) $A = \{a\}$

$P(A) = \{\phi, \{a\}\}$

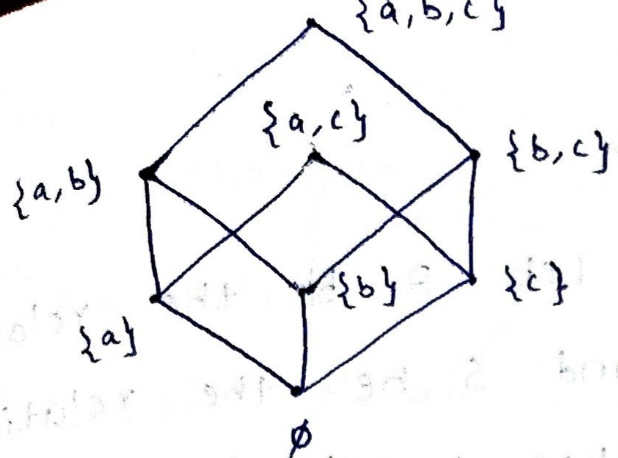
(b) $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



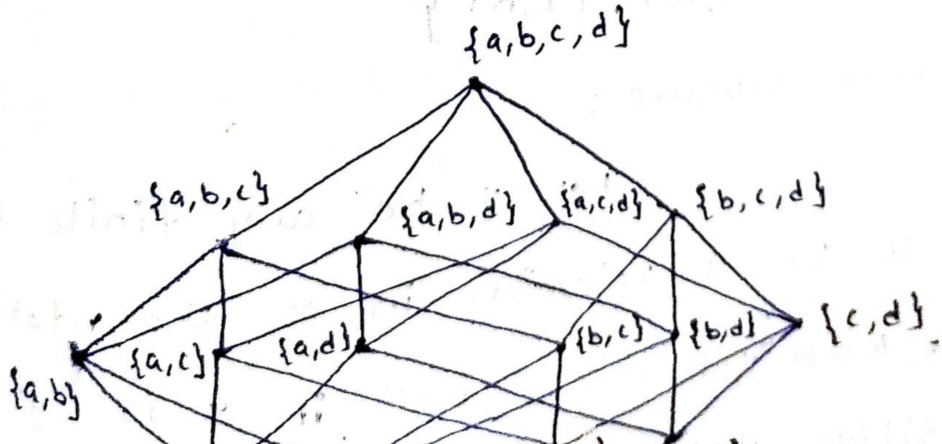
(c) $A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\},$
$\{a, c\}, \{a, b, c\}\}$



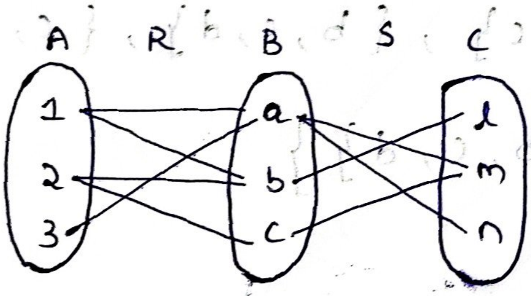
(d) $A = \{a, b, c, d\}$

$P(A) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, \{c,d,a\}, \{d,a,b\}, \{a,b,c,d\} \}$



- **Composite Relation**

Let R be the relation from x to y and S be the relation from y to z then a relation written ROS is called composite relation of R and S where $ROS = \{(x,z) / x \in X, z \in Z\}$ and there exist $y \in Y$ with $(x,y) \in R$ and $(y,z) \in S$



$ROS = \{(1,m), (1,n), (1,l), (2,l), (2,m), (3,m), (2,n)\}$

- **Transitive closure :**

Let X be any finite set and R be a relation in X the relation $R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$ in X is called transitive closure of R

- Let $R = \{(1,2), (2,3), (3,3)\}$ on the set $\{1,2,3\}$ what is the transitive closure of R

given

$R = \{(1,2), (2,3), (3,3)\}$

$X = \{1,2,3\}$

Transitive closure $R^+ = R \cup R^2 \cup \cdots \cup R^3$

$R^2 = R \circ R$

$= \{(1,2), (2,3), (3,3)\} \circ \{(1,2), (2,3), (3,3)\}$

$= \{(1,3), (2,3), (3,3)\}$

$R^3 = R^2 \circ R$

$= \{(1,3), (2,3), (3,3)\} \circ \{(1,2), (2,3), (3,3)\}$

$= \{(1,3), (2,3), (3,3)\}$

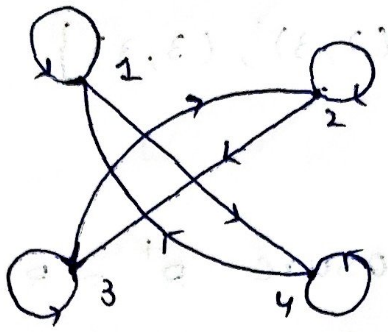
$\therefore R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

$= \{(1,3), (2,3), (3,3)\}$

- Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4),$ $(4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$ Write the matrix form of R and sketch it's graph

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

graph :



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* Functions *

A function is a special case of relation.

- Let X and Y be any two non empty sets, a relation "f" from X to Y is called a function if for every $x \in X$, there is a unique element $y \in Y$.

Eg) Let $X = \{1, 2, 3\}$, $Y = \{p, q, r\}$ and

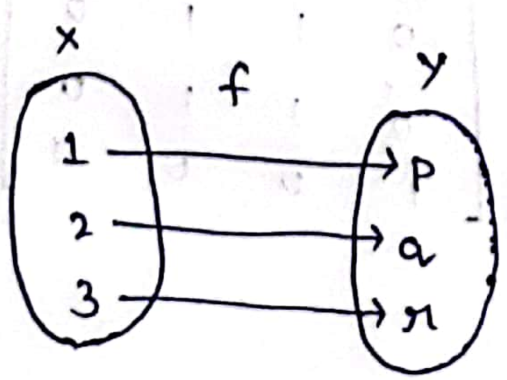
$f = \{(1,p), (2,q), (3,r)\}$ then

$f(1) = p$

$f(2) = q$

$f(3) = r$.

clearly f is a function from X to Y



Domain and Range of a function

If $f: X \rightarrow Y$ is a function then X is called the domain of f and the set Y is called the co-domain of f.

- The Range of f is defined as the set of all images under f. It is denoted by $f(x) = \{ y / \text{ for some } x \text{ in } X, f(x) = y \}$ and is called the image of X in Y
- The Range of f is also denoted by "R_f".

Eg) If the function f is defined by $f(x) = x^2 + 1$ on the set $\{-2, -1, 0, 1, 2\}$ find the Range of f:

$f(-2) = (-2)^2 + 1 = 5$

$f(-1) = (-1)^2 + 1 = 2$

$f(0) = (0)^2 + 1 = 1$

$f(1) = (1)^2 + 1 = 2$

$f(2) = (2)^2 + 1 = 5$

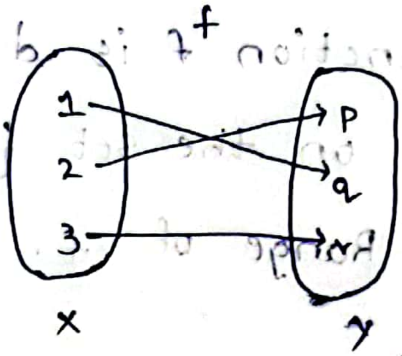
$\therefore$ The Range of f

$R_f = \{1, 2, 5\}$

Types of functions

① One - to - One (Injection)

A mapping $f: X \rightarrow Y$ is called one - to - one if distinct elements of X are mapped into distinct elements of Y i.e, f is one - to - one if $x_1 \neq x_2$ i.e, $f(x_1) \neq f(x_2)$. (or) equivalently $x_1 = x_2$ $\Rightarrow f(x_1) = f(x_2)$.



eg) $f: R \rightarrow R$ defined by $f(x) = 3x$ $\forall$ $x \in R$ is one - to - one since

$$f(x_1) = f(x_2)$$
$$3x_1 = 3x_2$$
$$x_1 = x_2$$

- Determine whether $f: Z \rightarrow Z$, given by $f(x) = x^2$, $x \in Z$ is a one - to - one function.

The function $f: Z \rightarrow Z$ given by

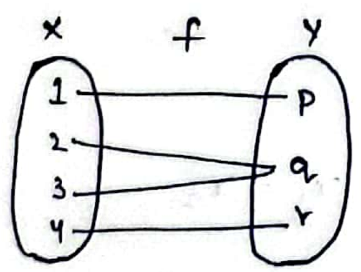
$f(x) = x^2$, $x \in Z$ is not a one-to-one function because both 1 and -1 have 1 as their image which is against the definition of a one-to-one function.

② Onto function (Surjection)

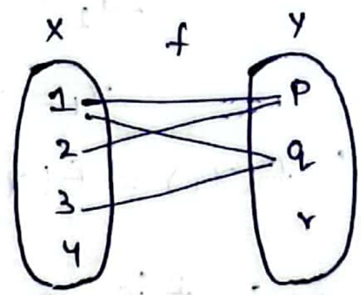
A mapping $f: X \to Y$ is called onto if the Range of set $R_f = Y$ if $f: X \to Y$ is onto then each element of Y is f image of atleast one element of X i.e,

$\{ f(x) : x \in X \} = Y$.

If f is not onto then it is said to be Into function



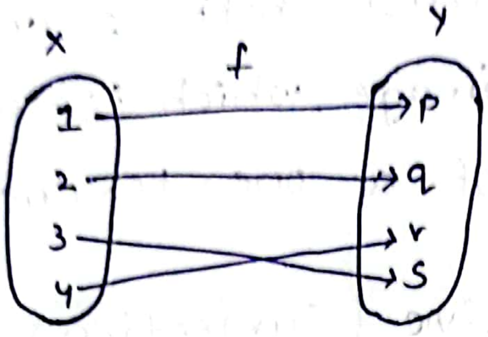
Surjective



Not surjective

③ Bijection function

A mapping $f: X \to Y$ is called one-to-one, onto or Bijective if it is satisfying both one-to-one and Onto condition's. Such a mapping is also called a one-to-one correspondance between X and Y



Bijective

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- Show that a mapping $f: R \to R$ defined by $f(x) = 2x+1$ for $x \in R$ is a bijective map from R to R

* To prove one to one

$$f(x_1) = f(x_2)$$

$$2x_1 + 1 = 2x_2 + 1$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one - one

* To prove onto :

$$f(x) = y$$

$$2x+1 = y$$

$$2x = y-1$$

$$x = \frac{y-1}{2} \quad \forall x \in R$$

$\therefore f(x)$ is onto

$\therefore$ f is satisfying one - one and onto. Hence f is bijective

Identity function :

Let X be any set and f be a function such that $f : X \to X$ is defined by $f(x) = x$ $\forall$ $x \in X$ then f is called Identity function or Identity transformation on X. It can be denoted by I or I_X. The identity function is both one to one and onto

composition of functions

Let $f : X \to Y$ and $g : Y \to Z$ be 2 functions, then the composition of f and g denoted by gof is the function from X to Z defined as $(gof)(x) = g(f(x))$ $\forall$ $x \in X$.

- Let $X = \{1, 2, 3\}$ and f, g, h and s be the functions from X to X given by

$f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,3)\}$

$h = \{(1,1), (2,2), (3,1)\}$

$s = \{(1,1), (2,2), (3,3)\}$

Find fof ; gof ; $fohog$; sog ; gos ; sos ; fos

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$(f \circ f)(x)$

$\therefore x = \{1, 2, 3\}$

$(f \circ f)(1) = f(f(1)) = f(2) = 3$

$(f \circ f)(2) = f(f(2)) = f(3) = 1$

$(f \circ f)(3) = f(f(3)) = f(1) = 2$

$(f \circ h \circ g)(x)$

$(f \circ h \circ g)(1) = f[h(g(1))] = f[h(2)] = f(2)$
$= 3$

$(f \circ h \circ g)(2) = f[h(g(2))] = f[h(1)] = f(1)$
$= 2$

$(f \circ h \circ g)(3) = f(h(g(3))) = f[h(3)] = f(1)$
$= 2$

$(g \circ f)(x)$

• What is composition of function. Let f and
(1A) g be functions from R to R where R is a set of Real numbers defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find fof ; fog ; gof

• Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$
(1B) $\forall x \in R$ where R is the set of all real numbers. Find gof ; fog ; fof ; gog ; foh ; hog ; hof and fohog

(A) $f : R \longrightarrow R$ then $f(x) = x^2 + 3x + 1$

$g : R \longrightarrow R$ then $g(x) = 2x - 3$

$$(fof)x = f(f(x))$$
$$= f(x^2 + 3x + 1)$$
$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$

$$(fog)x = f(g(x))$$
$$= f(2x - 3)$$
$$= (2x - 3)^2 + 3(2x - 3) + 1$$
$$= 4x^2 - 6x + 1$$

$$(gof)x = g(f(x))$$
$$= g(x^2 + 3x + 1)$$
$$= 2(x^2 + 3x + 1) - 3$$
$$= 2x^2 + 6x - 1$$

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Inverse of Function

A function $f: x \to y$ is said to be invertible of its inverse function f^{-1} is also a function from the Range of f into x.

A function f is mapping $f: x \to y$ is invertible if and only if f is one-to-one and onto.

Eg) Let $x = \{A, B, C, D\}$ and $y = \{1, 2, 3, 4\}$ and let $f: x \to y$ be given by $f = \{(A,1), (B,2), (C,2), (D,3)\}$. Is f^{-1} a function

$f^{-1} = \{(1,A), (2,B), (2,C), (3,D)\}$

Here 2 has 2 distinct images B and C $\therefore$ f^{-1} is not a function.

- Let R be the set of Real numbers and $f: R \to R$ be given by $f = \{(x, x^2) / x \in R\}$

Is f^{-1} a function

The inverse of a given function is defined as $f^{-1} = \{(x^2, x) / x \in R\}$

$\therefore$ It is not a function because it is not satisfying one-to-one condition.

- If $f: X \to Y$ and $g: Y \to X$ be such that $g \circ f = I_x$ and $f \circ g = I_x$ then f and g are both invertible furthur more.

$$f^{-1} = g \text{ and } g^{-1} = f$$

- Let $X = \{1, 2, 3, 4\}$ and f, g be functions from X to X given by $f = \{(1,4), (2,1), (3,2), (4,3)\}$ and $g = \{(1,2), (2,3), (3,4), (4,1)\}$ prove that f and g are inverse of each other.

given $f: X \to X$

$X = \{1,2,3,4\}$

$f = \{(1,4), (2,1), (3,2), (4,3)\}$

$g = \{(1,2), (2,3), (3,4), (4,1)\}$

$(g \circ f)(x) = g(f(x))$

$(g \circ f)(1) = g(4) = 1$

$(g \circ f)(2) = g(1) = 2$

$(g \circ f)(3) = g(2) = 3$

$(g \circ f)(4) = g(3) = 4$

$(f \circ g)(x) = f(g(x))$

$(f \circ g)(1) = f(2) = 1$

$(fog)(2) = f(3) = 2$

$(fog)(3) = f(4) = 3$

$(fog)(4) = f(1) = 4$

$(gof)(x) = I_x$

$(fog)(x) = I_x$

$\therefore$ f and g are invertible

$f = g^{-1}$

$g = f^{-1}$

- show that the function $f(x) = x^3$ and $g(x) = x^{1/3}$ $\forall$ $x \in R$ are inverse of one another

given $f(x) = x^3$; $f: R \rightarrow R$

$g(x) = x^{1/3}$; $g: R \rightarrow R$

$(gof)(x) = g[f(x)]$

$= g(x^3)$

$= (x^3)^{1/3}$

$= x$

$(fog)(x) = f(g(x))$

$= f(x^{1/3})$

$= (x^{1/3})^3$

$= x$.

$\therefore fog = gof$ $\therefore$ f and g are inverse of each other

• If $f: R \to R$ is defined by $f(x) = Ax + B$, for $A, B \in R$ and $A \neq 0$, show that f is invertible and find the inverse of f.

given

$f: R \to R$ is defined by $f(x) = ax + b$

To prove one - one

$$f(x_1) = f(x_2)$$

$$ax_1 + b = ax_2 + b$$

$$x_1 = x_2$$

$$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

$\therefore$ f is one - to - one

To prove onto

$$f(x) = y$$

$$ax + b = y$$

$$ax = y - b$$

$$x = \frac{y - b}{a} \in R$$

$\therefore$ f is onto

$\because$ f is both one-to-one and onto

$\exists$ inverse for f.

The inverse function is

$$f^{-1} = \frac{x - b}{a}$$

• If $f: R \to R$ be given by $f(x) = x^3 - 2$. Find f^{-1}

$f(x) = x^3 - 2$; $f: R \to R$

To prove one-to-one

$f(x_1) = f(x_2)$

$x_1^3 - 2 = x_2^3 - 2$

$x_1^3 = x_2^3$

$x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one-to-one

To prove onto

$f(x) = y$

$x^3 - 2 = y$

$x^3 = y + 2$

$x = (y+2)^{1/3} \in R$

$\therefore$ f is onto

$\because$ f is both one-to-one and onto

f^{-1} exists

The inverse function is

$f^{-1} = (x+2)^{1/3}$

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* Algebraic structures *

A system consisting of a non empty set and one or more N-array operations on the set is called an algebraic system or algebraic structure

An Algebraic structure is denoted by $\{S, f_1, f_2, f_3, \ldots f_N\}$ where S is a non empty set & $f_1, f_2, \ldots f_N$ are N-array operations on S.

We will mostly deal with Algebraic structures with N = 0, 1 and 2 containing 1 or 2 operations only

* General properties of Algebraic structures

Let $(S, +, *)$ be an algebraic system where $*$, $+$ are binary operations on S.

closure property

For any 2 elements $a, b \in S$

$a * b \in S$ and $a + b \in S$

Eg) If $a, b \in Z$; $a+b \in Z$ and $a*b \in Z$ where the operations + and * are the operations of addition and multiplication

- Associative property

For any $a, b \in S$ then

$$a * (b * c) = (a * b) * c$$

$$a + (b + c) = (a + b) + c$$

Eg) If $a, b, c \in Z$ then it is satisfying associative property

- commutative property

For any $a, b \in S$ then

$a + b = b + a$ and $a * b = b * a$

- Identity element

There exist an element e, $e \in S$ such that for any element 'a', $a \in S$ then $a * e = e * a = a$

$$a + e = e + a = a$$

The element e is called identity element of S with respect to the operations $*$ and $+$

Eg) 0 and 1 are the identity elements of Z with respect to the operations of addition and multiplication respectively since for any element a, $a \in Z$

$$a + 0 = 0 + a = a \,;$$

$$a * 1 = 1 * a = a$$

• Inverse element

For each element a, $a \in S$ there exist an element "a^{-1}", $a^{-1} \in S$ such that

$a * a^{-1} = a^{-1} * a = e$

The element a^{-1} is called the inverse of a under the operation * and e is called identity element with respect to the operation *.

Eg) For each $a \in Z$, $-a$ is the inverse of a under the operation addition.

Since $a + (-a) = 0$

where 0 is the identity element of Z under addition.

• Distributive property

For any 3 elements, $a, b, c \in S$

$a * (b + c) = (a * b) + (a * c)$

In this case the operation * is said to be distributive over the operation +

eg) For any 3 elements $a, b, c \in Z$

$a * (b + c) = (a * b) + (a * c)$

$a + (b * c) = (a + b) * (a + c)$

- Cancellation property

For any 3 elements $a, b, c \in S$ and $a \neq 0$.

$a * b = a * c \Rightarrow b = c$

$b * a = c * a \Rightarrow b = c$

- Idempotent property

An element a, $a \in S$ is called an idempotent element with respect to the operation $*$, if $a * a = a$

eg) $0 \in Z$ is an idempotent element under addition $0 + 0 = 0$

$0, 1 \in Z$ are idempotent elements under multiplication $0 * 0 = 0$; $1 * 1 = 1$

* Semi Groups

Let S be a non-empty set and $*$ be a binary operation on S, then algebraic system $(S, *)$ is called a semi group if and only if it satisfies the following properties

(i) <u>closure property</u> :-

For any 2 elements a, b where $a, b \in S$ then $a * b \in S$

(ii) <u>Associative property</u> :-

For any 2 elements

a, b, c where $a, b, c \in S$ then

$$a * (b * c) = (a * b) * c$$

eg) $(N, +)$, $(N, *)$ are semi groups where

$N = \{1, 2, 3, \ldots\}$

$(E, +)$ and $(E, *)$ are semi groups where

$E = \{0, \pm 2, \pm 4, \ldots\}$ (even Number set)

* Monoid

Let M be a non empty set and $*$ be a binary operation on S, then the algebraic system $(M, *)$ is called a Monoid iff it satisfies the following properties

(i) closure property:

For any 2 elements $a, b \in S$ such that $a * b \in M$ where $a, b \in M$

(ii) Associative property:

For any 3 elements $a, b, c \in M$ such that $a * (b * c) = (a * b) * c$

(iii) Identity property:

For all $a \in M$, there exist $e \in M$ Such that $a * e = e * a = a$

where "e" is identity element

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eg) ① $(N,+)$ is a monoid with 1 as the identity element

② $Z_+ = \{0,1,2,3,\ldots\}$ is the set of all non negative integers $(Z_+,+)$ and $(Z_+,*)$ are monoids with 0 and 1 as identity elements

* Group

Let G be the non empty set and $*$ is a binary operation of G then the algebraic system $(G,*)$ is called a group iff it satisfies the following properties

(i) Closure property:

For any 2 elements a,b such that $a*b \in G$, where $a,b \in G$

(ii) Associative property:

For any 3 elements a,b,c; Such that $a*(b*c) = (a*b)*c$

(iii) Identity property:

For all $a \in G$ there exists $a \in G$ Such that $a*e = e*a = a$

where e is an identity element

(iv) Inverse property

For any element a , $a \in G$ there exist an element a^{-1} , $a^{-1} \in G$ such that

$$a * a^{-1} = a^{-1} * a = e$$

(or)

$$a + (-a) = (-a) + a = e$$

where e is identity element

eg) (I, +) is a group,

where I is set of integers

$$I = \{-\infty, ----, -3, -2, -1, 0, 1, 2, 3, ---, \infty\}$$

* order of a group

The Number of elements in a group G is called the order of the group and is denoted by $O(G)$ or $|G|$

* Abelian Group :

A group $(G, *)$ is called an abelian group or commutative group if $a * b = b * a$ for all $a, b \in G$

1/2/22

· Show that $(Z, *)$ is a Abelian Group where $*$ is defined by $a * b = a + b + 1$

Given $a * b = a + b + 1$

closure property :

Let $a, b \in Z$

$a*b = a+b+1 \in Z$

let $a=2, b=3$ | $a=-1, b=-2$

$a*b = 6 \in Z$ | $a*b = -2 \in Z$

Hence closure property is satisfied

Associative property

Let $a, b, c \in Z$

$a*(b*c) = (a*b)*c$

$LHS = a*(b*c)$

$= a*(b+c+1)$

$= a+(b+c+1)+1 = a+b+c+2$

$RHS = (a*b)*c$

$= (a+b+1)*c$

$= (a+b+1)+c+1 = a+b+c+2$

Hence Associative property is satisfied

Let $a=1, b=-1, c=3$

$1*(-1*3) = 1*(-1+3+1)$

$= 1*3 = 1+3+1 = 5$

$(1*-1)*3 = (1+1-1)*3$

$= 1*3 = 5$

Existence of Identity element

Let $a \in Z$

$a*e = e*a = a$

consider $a*e = a$

$a+e+1 = a$

$e = -1 \in z$

$e*a = a$

$e+a+1 = a \Rightarrow e = -1 \in z$

Hence Identity property is satisfied

Inverse property

Let $a \in z$, $b \in z$

// $a * a^{-1} = a^{-1} * a = e$

$a*b = b*a = e$ } Inverse property

consider $a*b = e$

$a+b+1 = -1$

$a+b = -2$

$b = -2-a$

Now $a*b = b*a = e$

$a*(-2-a) = (-2-a)*a$

$= (a+(-2-a)+1)$ | $= -2-a+a+1$

$= -1 \in z = e$ | $= -1 = e$

Hence Inverse property is satisfied

Commutative property

Let $a, b \in z$

$a*b = b*a$

LHS $= a+b+1$

RHS $= b+a+1 =$ LHS

Hence commutative property is satisfied

$\therefore (z, *)$ is an Abelian group where $*$ is defined by $a+b+1$

• verify that the set z of all integers with binary operation $*$ defined by $a*b = a+b+3$ $\forall$ $a,b \in z$ is an Abelian group

given $a*b = a+b+3$

closure property :

Let $a,b \in z$

$a*b = a+b+3 \in z$

let $a=2$, $b=3$ | let $a=-3$, $b=3$

$a*b = 8 \in z$ | $a*b = 3 \in z$

Hence closure property is satisfied

Associative property

Let $a,b \in z$ and $c \in z$

$a*(b*c) = (a*b)*c$

LHS $= a*(b*c)$

$= a*(b+c+3)$

$= a+b+c+3+3 = a+b+c+6$

RHS $= (a*b)*c$

$= (a+b+3)*c$

$= a+b+3+c+3$

$= a+b+c+6$

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Hence Associative property is satisfied

Existence of Identity element

Let $a \in Z$

$a * e = e * a = a$

consider $a * e = a$

$a + e + 3 = a$

$e = -3 \in Z$

$e * a = a$

$e + a + 3 = a \Rightarrow e = -3 \in Z$

Hence Identity property is satisfied

Inverse property

Let $a \in Z$, $b \in Z$

$a * b = b * a = e$

consider $a * b = e$

$a + b + 3 = -3$

$a + b = -6 \Rightarrow b = -6 - a$

Now $a * b = b * a = e$

$a * (-6 - a) = (-6 - a) * a$

$a - 6 - a + 3 = -6 - a + a + 3$

$= -3 = e$

Hence Inverse property is satisfied

commutative property

Let $a, b \in Z$

$a * b = b * a$

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$LHS = a+b+3$

$RHS = b+a+3$

$\therefore$ commutative property is satisfied

$\therefore$ $(z, *)$ is an abelian group

Page Turn over

2/2/2

. prove that

(i) $G = \{1, -1, i, -i\}$

(ii) $G = \{1, \omega, \omega^2\}$ are abelian groups

under multiplication

(i) given set $G = \{1, -1, i, -i\}$

construct the composition table for

set G by using multiplication operation

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(i) closure property

∵ All the elements of composition table are elements of G

For eg let $1, -1 \in G \Rightarrow 1 * -1 = -1 \in G$

Hence $(G, *)$ is satisfying closure property

(ii) Associative property

Let $1, -1, i \in G$

LHS} $1 * (-1 * i) = 1 * -i = -i$

RHS} $(1 * -1) * i = -1 * i = -i$

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(iii) Existence of identity element

$\because 1 \in G$

$a * e = e * a = a$

$\Rightarrow 1 * e = 1$

$\Rightarrow e = 1 \in G$

(iv) Existence of inverse element

Let elements are

	1	-1	i	$-i$
inverse	1	-1	$-i$	i

$\because$ every element of G there exist a inverse

(v) commutative property

Let $i, -i \in G$

$a * b = i * -i = -1 \in G$

$b * a = -i * i = -1 \in G$

$\Rightarrow a * b = b * a$

Hence $(G, *)$ is satisfying commutative property

Hence $(G, *)$ is satisfying all 5 properties

$\therefore (G, *)$ is an abelian group

3/2/2

- Show that the set $G = \{0,1,2,3,4\}$ is an abelian group with respect to addition modulo 5 (or)

Show that $(Z_5, +_5)$ is an abelian group

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) closure property

$0, 1 \in G$

$0 +_5 1 = 1 \in G$

$2, 4 \in G$

$2 +_5 4 = 1 \in G$

(ii) Associative property

$2, 3, 4 \in G$

$2 +_5 (3 +_5 4) = 2 +_5 (2) = 4$

$(2 +_5 3) +_5 4 = 0 +_5 4 = 4$

$\therefore a + (b + c) = (a + b) + c$

(iii) Identity property

$a \in G$

$a +_5 e = a$

let $2 \in G$

$2 +_5 e = 2$

$\rightarrow e = 0$

(iv) Inverse element

	0	1	2	3	4 are elements
Inverse	0	4	3	2	1

$\therefore$ for every element of G

there exist a inverse

(v) Commutative property

Let $1, 4 \in G$

$1 +_5 4 = 0$

$4 +_5 1 = 0$

let $2, 0 \in G$

$0 +_5 2 = 2$

$2 +_5 0 = 2$

$\therefore$ "$a+b$" $= b+a$

Hence $(Z_5, +_5)$ is an abelian group

• show that the set $G = \{1, 2, 3, 4, 5, 6\}$ is an abelian group with respect to multiplication modulo 7 (or)

Show that $(Z_7, *_7)$ is an abelian group

$*_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	6	1

(i) closure property

$1, 2 \in G$

$1 *_7 2 = 2 \in G$

$2, 6 \in G$

$2 *_7 6 = 5 \in G$

(ii) Associative property

$2, 4, 6 \in G$

$2 *_7 (4 *_7 6) = 2 *_7 (3) = 6$

$(2 *_7 4) *_7 6 = 1 *_7 6 = 6$

$\therefore a + (b + c) = (a + b) + c$

(iii) Identity property

Let $a \in G$

$a *_7 e = a$

Let $5 \in G$

$2 *_7 e = a$

$\Rightarrow 2 *_7 e = 2$

$\therefore e = 1$

(iv) Inverse element

Let the elements are

	1	2	3	4	5	6
inverse :	1	4	5	2	3	6

$\therefore$ for every element of G

there exist inverse

(v) commutative property

let $3, 5 \in G$

$3 *_7 5 = 1$

$5 *_7 3 = 1$

$2, 3 \in G$

$2 *_7 3 = 6$

$3 *_7 2 = 6$

$\therefore a+b = b+a$

$\therefore (Z_7, *_7)$ is an abelian group

4/2/22

- Sub group

Let $(G,*)$ be a group if H be a finite sub set of group G then H is a sub group of G if and only if satisfy the following properties with respect to the operation $*$

(i) closure property:

If $a, b \in H$ then $a*b \in H$

(ii) Associative property:

If $a, b, c \in H$ then $a*(b*c) = (a*b)*c$

(iii) Existence of identity:

Let $a \in H$ then there exist special element $e \in H$ then $a*e = e*a = a$

(iv) Inverse property:

Let $a \in H$ there exist inverse $a^{-1} \in H$ such that $a*a^{-1} = a^{-1}*a = e$

eg) Let is a sub group of $(Q,+)$

$(Q,+)$ is a subgroup of $(R,+)$

- Lagrange's Theorem

Let G is a finite group and H is a sub group of G then $|H|$ divides $|G|$ [O_H divides O_G]

eg) Let $G = \{1, -1, i, -i\}$ $O_G = 4$

$H_1 = \{1, -1\}$ $O_{H_1} = 2$

$H_2 = \{1\}$ $|H_2| = 1$

$H_3 = \{-1, i\}$ $|H_3| = 2$

$H_4 = \{1, i, -i\}$ $|H_4| = 3$

Here H_1, H_2, H_3 satisfies Lagrange's Theorem but H_4 does not satisfy Lagrange's theorem