

# Robust Bounds on Misallocation Under a Price Ceiling

wedge → band →  $\mathcal{I}$  → 1D objective

## What we're doing

**Observed:** anchor points  $(q_i^{obs}, p_{0,i})$  and total  $Q = \sum_i q_i^{obs}$ .  
**Unknown:** full demand curves. **Output:** tight upper/lower bounds on misallocation loss using only slope bounds and a choke-price bound.

- ▶ Step 1: **Wedge** of admissible  $P_i(q)$  around  $(q_i^{obs}, p_{0,i})$
- ▶ Step 2: **Band** of feasible  $q_i(p)$ :  $\ell_i(p) \leq q_i(p) \leq u_i(p)$
- ▶ Step 3: **Feasible** common shadow prices  $\mathcal{I} = \{p : L(p) \leq Q \leq U(p)\}$
- ▶ Step 4: **Compute**  $\Phi^U = \max_{p \in \mathcal{I}} \bar{\Phi}(p)$ ,  $\Phi^L = \min_{p \in \mathcal{I}} \underline{\Phi}(p)$

## Setup + visual dictionary

- ▶ Markets  $i = 1, \dots, n$ ; inverse demand  $P_i(q)$  on  $[0, q_i^{\max}]$ , nonincreasing.
- ▶ Observe  $q_i^{obs}$  and total  $Q = \sum_i q_i^{obs}$ .
- ▶ Anchor:  $P_i(q_i^{obs}) = p_{0,i}$  (often  $p_{0,i} = \bar{p} - b_i$ ).
- ▶ Slope bounds:  $g_{i,L} \leq P'_i(q) \leq g_{i,U} < 0$ .
- ▶ Choke bound:  $P_i(0) \leq M_i$  (upper bound on willingness-to-pay at  $q = 0$ ).

**Everything is geometry + a 1D search.**

### Color key

red =  $g_{i,L}$  (steep)

blue =  $g_{i,U}$  (flat)

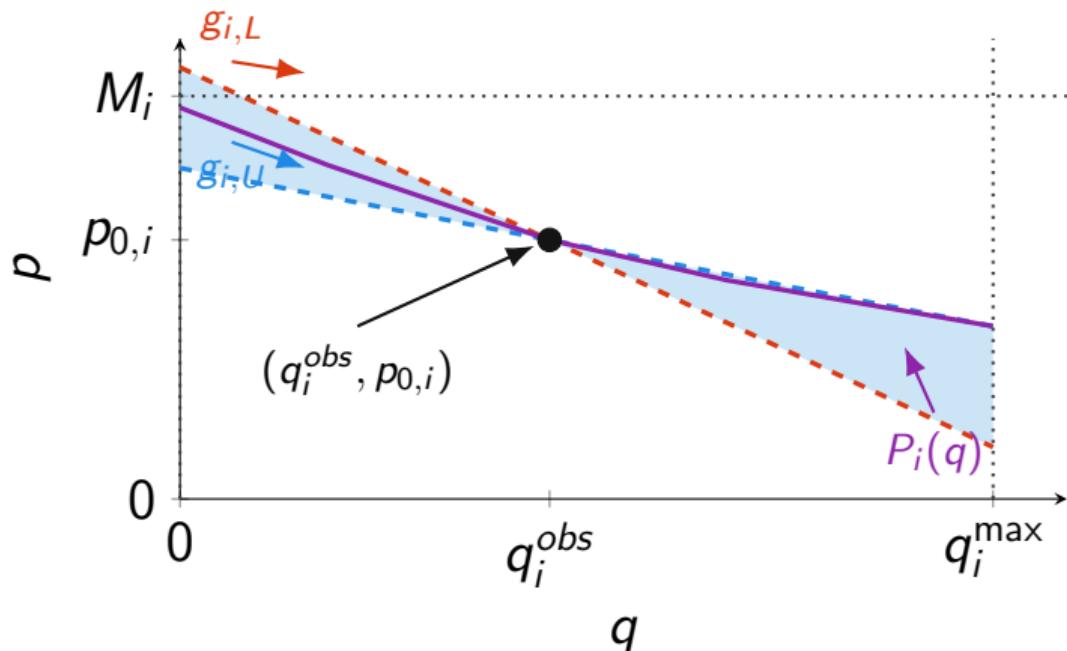
shaded = feasible wedge/band

teal =  $L(p)$ , orange =  $U(p)$

orange fill =  $\mathcal{I}$

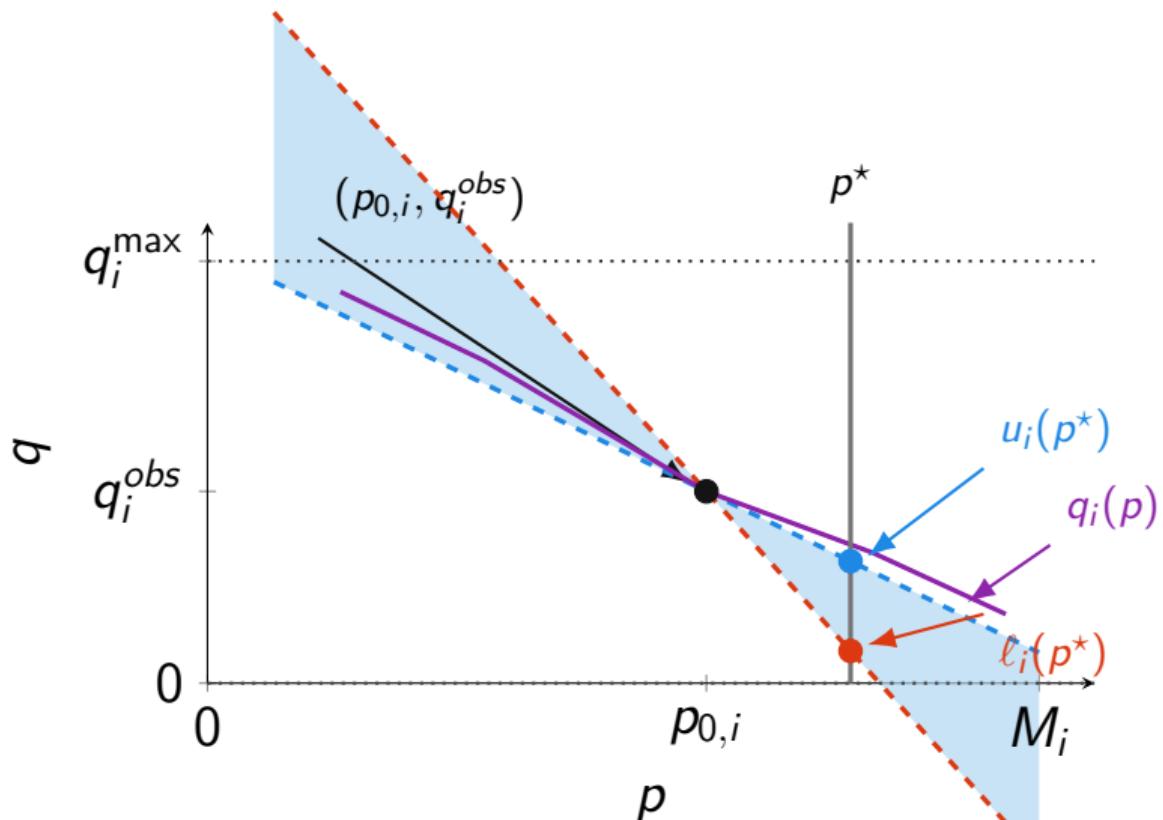
## Step 1: wedge in $(q, p)$ from slope bounds

- ▶ **Definition:** admissible  $P_i(\cdot)$  pass through  $(q_i^{obs}, p_{0,i})$ .
- ▶ **Geometry:** bounded slope forces  $P_i$  to stay between rays with slopes  $g_{i,L}$  and  $g_{i,U}$ .
- ▶ **Payoff:** replaces unknown curvature with a simple “wedge” constraint.



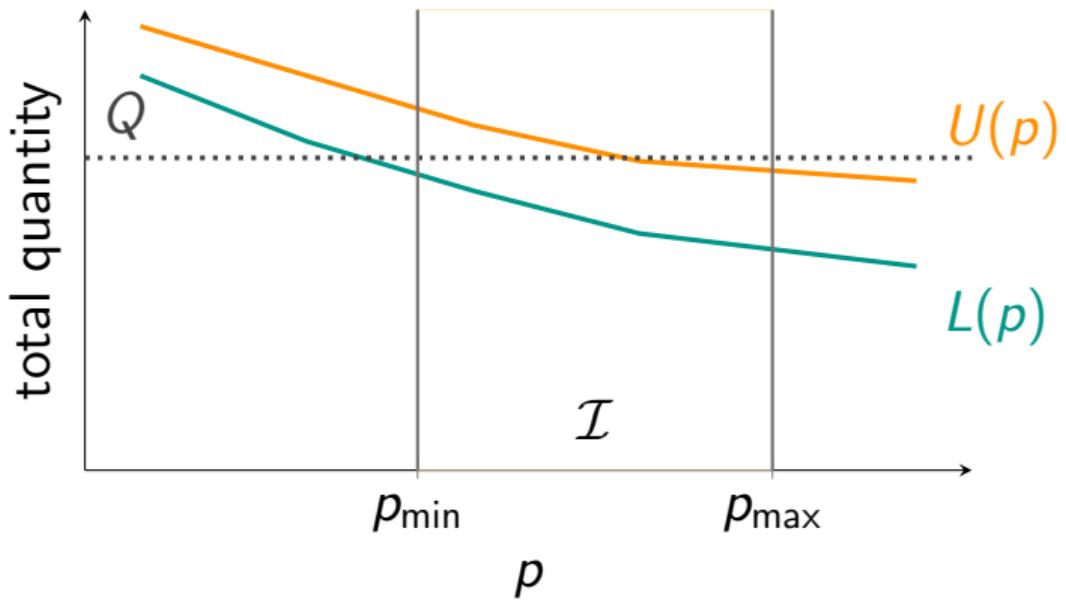
## Step 2: invert wedge $\Rightarrow$ band for $q_i(p)$

- ▶ **Definition:**  $q_i(p) =$  generalized inverse (“quantity at price  $p$ ”).
- ▶ **Geometry:** wedge in  $(q, p)$  becomes band:  
 $\ell_i(p) \leq q_i(p) \leq u_i(p)$ .
- ▶ **Payoff:** pointwise box constraints on  $q_i(p)$ .



## Step 3: aggregate $\Rightarrow$ feasible shadow prices $\mathcal{I}$

- ▶ **Definition:**  
 $L(p) = \sum_i \ell_i(p)$ ,  
 $U(p) = \sum_i u_i(p)$ .
- ▶ **Geometry:**  $p$  feasible iff  $L(p) \leq Q \leq U(p)$ .
- ▶ **Payoff:** reduces everything to  $p \in \mathcal{I}$ .



## Step 4: welfare identity $\Rightarrow$ 1D optimization

**One scalar decision: pick a common marginal value (shadow price)  $p \in \mathcal{I}$ . Everything else is endpoint choices inside  $[\ell_i(p), u_i(p)]$ .**

- ▶ **Welfare gap (picture):** misallocation loss is the *area* between  $P_i(\cdot)$  and the common shadow price when moving from  $q_i^{obs}$  to the equal-shadow allocation.

- ▶ **Key identity (price space):**

$$\Phi(P) = Q p^* - \sum_i q_i^{obs} p_{0,i} - \sum_i \int_{p_{0,i}}^{p^*} q_i(p) dp.$$

- ▶ **Reduction:** the objective is linear in the unknown functions  $q_i(\cdot)$ , and constraints are just *boxes*  $q_i(p) \in [\ell_i(p), u_i(p)]$ . So worst/best cases sit on the envelopes (except possibly one market to hit  $\sum_i q_i(p) = Q$ ).  $\Rightarrow \bar{\Phi}(p)$  and  $\underline{\Phi}(p)$  are *scalar* functions of  $p \in \mathcal{I}$ .

## Objective plot: compute $\bar{\Phi}(p)$ and $\underline{\Phi}(p)$ on $\mathcal{I}$

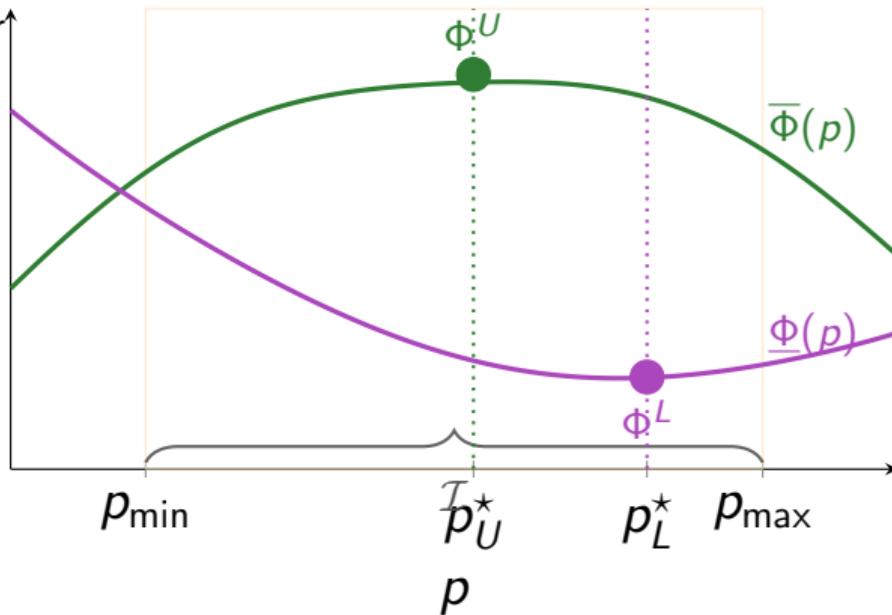
- ▶ Compute on a grid over  $\mathcal{I} = [p_{\min}, p_{\max}]$ .

- ▶ Report:

$$\Phi^U = \max_{p \in \mathcal{I}} \bar{\Phi}(p)$$

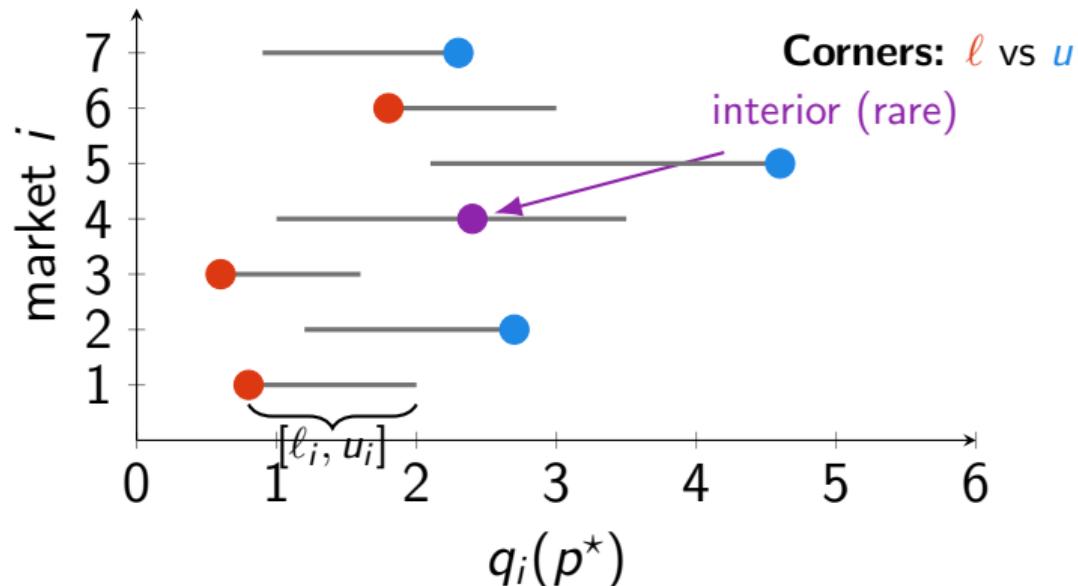
$$\Phi^L = \min_{p \in \mathcal{I}} \underline{\Phi}(p)$$

- ▶ Also report the argmax/argmin  $p_U^*, p_L^*$ .



## Endpoint saturation at fixed $p^*$ : why corners dominate

- ▶ Fix feasible  $p^* \in \mathcal{I}$ .
- ▶ Each market: interval  $[\ell_i(p^*), u_i(p^*)]$ .
- ▶ **Linearity:** linear objective + box constraints  $\Rightarrow$  endpoints (“corners”) dominate.
- ▶ At most one interior choice to satisfy  $\sum_i q_i(p^*) = Q$ .



## What you compute in practice

**Implementation: build  $\ell/u$ , find  $\mathcal{I}$ , then do a 1D search over  $p \in \mathcal{I}$ .**

1. Construct  $\ell_i(p)$  and  $u_i(p)$  from anchor + slope bounds (solve boundary lines for  $q$ , then clip).
2. Form  $L(p) = \sum_i \ell_i(p)$ ,  $U(p) = \sum_i u_i(p)$ ; compute  $\mathcal{I} = \{p : L(p) \leq Q \leq U(p)\}$ .
3. Evaluate  $\bar{\Phi}(p)$  and  $\underline{\Phi}(p)$  on a grid of  $p \in \mathcal{I}$ ; take max/min and argmax/argmin.
4. Plot the objective curves and report  $\Phi^U, \Phi^L, p_U^*, p_L^*$ .

## Appendix: overlay computed objectives from CSV

Export a table with columns p, upperObj, lowerObj. Then:

```
\addplot table[x=p,y=upperObj,col sep=comma] {objective.csv};  
\addplot table[x=p,y=lowerObj,col sep=comma] {objective.csv};
```

Shade  $[p_{\min}, p_{\max}]$  as the set  $\mathcal{I}$ .