Robust bounds: the simplest possible toy example

Two markets, one observed point each, slope bounds only

What we're bounding. We observe a split of a fixed total quantity Q across markets. Demand curves are unknown. Let Φ be the welfare gain from reallocating Q to equalize marginal values. We want *sharp* lower/upper bounds on Φ (*sharp* = attainable by some feasible demand curves).

1. The toy environment (minimal but nontrivial)

Two markets $i \in \{1,2\}$ with unknown inverse demands $P_i(q)$ (decreasing in q). We observe one point on each:

$$(q_1^{obs}, p_{0,1}) = (1, 10), \qquad (q_2^{obs}, p_{0,2}) = (3, 6),$$

so the observed total is fixed at

$$Q = q_1^{obs} + q_2^{obs} = 4.$$

We impose only a slope bound:

$$g_L = -4 \quad \leq \quad P_i'(q) \quad \leq \quad g_U = -1.$$

Interpretation: the curve can be "steep" (-4) or "flat" (-1), but not outside.

Punchline (we will derive): Given these restrictions, the welfare gain Φ is bounded by $\underline{\mathbf{\Phi}} = \mathbf{1}$ and $\bar{\mathbf{\Phi}} = \mathbf{4}$. Moreover, the extremizers are simple: $\bar{\Phi}$ occurs when both markets are as flat as allowed, $\underline{\Phi}$ occurs when both are as steep as allowed.

2. Step 1: slope bounds $\Rightarrow$ a wedge of feasible inverse demands

Anchoring at $(q_i^{obs}, p_{0,i})$ and bounding slopes forces $P_i(\cdot)$ to lie in a wedge: between the two rays through the anchor with slopes g_L and g_U.

3. Step 2: invert the wedge $\Rightarrow$ pointwise quantity bounds $\ell_i(p) \leq q_i(p) \leq u_i(p)$

Because each P_i is decreasing, we can talk about the (unique) "quantity at price p": the q such that $P_i(q) = p$. The wedge implies that, at each price p, the feasible quantity lies in a band:

$$\ell_i(p) \leq q_i(p) \leq u_i(p).$$

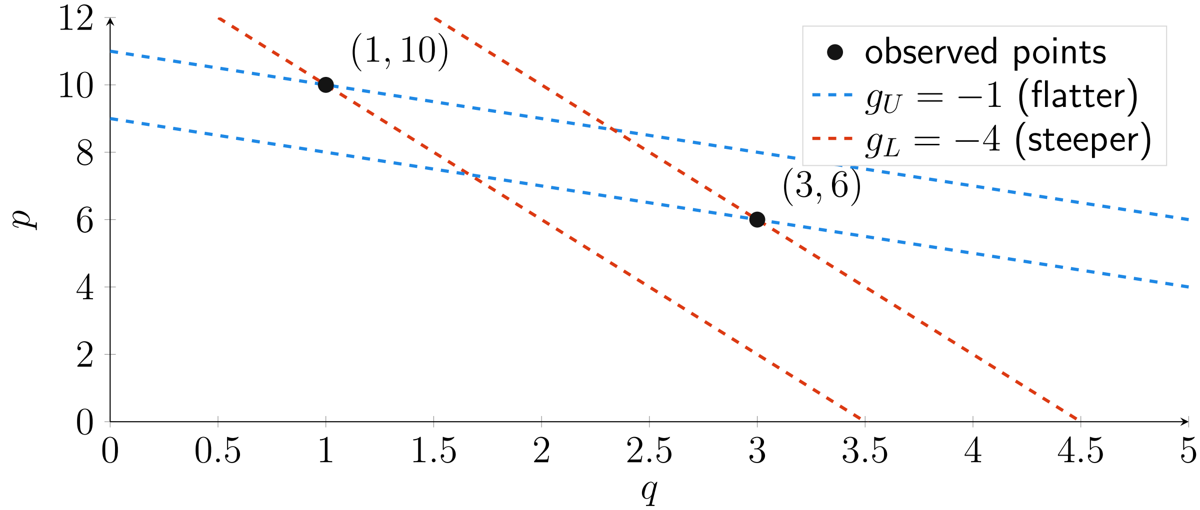

Figure 1: **Wedges.** Each inverse demand must pass through its observed point and stay between the two slope-bound rays.

In this toy example, these bounds are explicit because the wedge boundaries are straight lines. We focus on $p \in [6, 10]$, the range bracketed by the observed prices; Step 3 will show the feasible set $I \subset [6, 10]$.

Market 1 (anchor $(1, 10)$)

Solving the two boundary lines for q as a function of price p gives:

$$q = 11 - p \quad \text{(flat bound } g_U = -1\text{)}, \qquad q = 3.5 - \tfrac{p}{4} \quad \text{(steep bound } g_L = -4\text{)}.$$

Hence

$$\ell_1(p) = 3.5 - \tfrac{p}{4}, \qquad u_1(p) = 11 - p \quad \text{for } p \in [6, 10].$$

Market 2 (anchor $(3, 6)$)

Similarly:

$$q = 9 - p \quad (g_U = -1), \qquad q = 4.5 - \tfrac{p}{4} \quad (g_L = -4),$$

so

$$\ell_2(p) = 9 - p, \qquad u_2(p) = 4.5 - \tfrac{p}{4} \quad \text{for } p \in [6, 10].$$

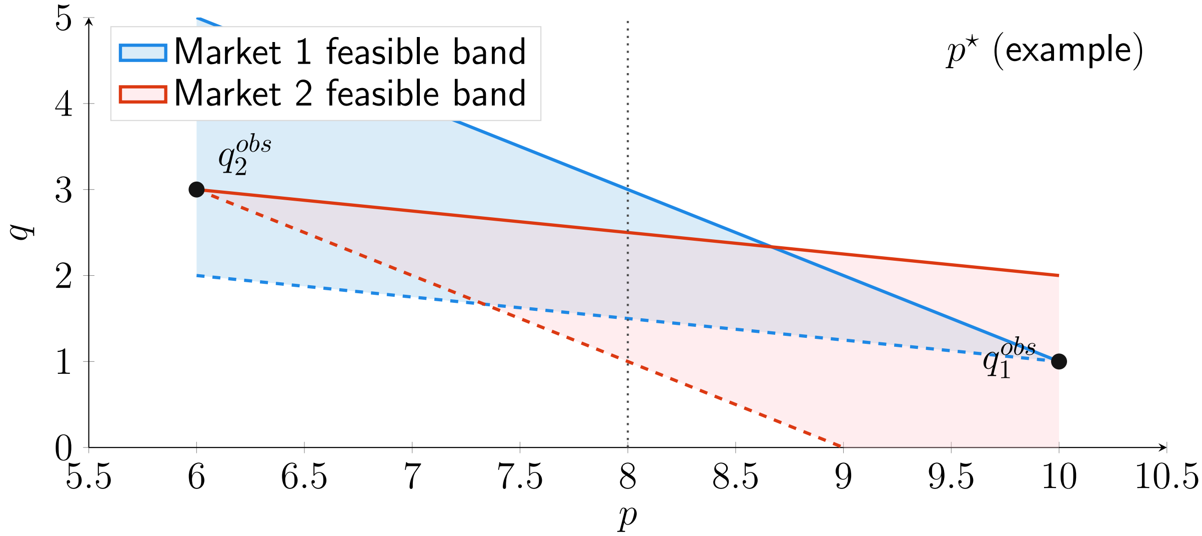

Figure 2: **Bands in (p, q) space.** The wedge becomes simple pointwise bounds $\ell_i(p) \le q_i(p) \le u_i(p)$.

4. Step 3: feasible shadow prices I

At the equal-shadow optimum there is a common shadow price $p^\star$ such that

$$q_1(p^\star) + q_2(p^\star) = Q.$$

But you only know $q_i(p^\star) \in [\ell_i(p^\star), u_i(p^\star)]$. Define

$$L(p) = \ell_1(p) + \ell_2(p), \qquad U(p) = u_1(p) + u_2(p).$$

Here, $L(p)$ is the *smallest* total quantity you can generate at price p, and $U(p)$ is the *largest*. Then the feasible shadow-price set is

$$I = \{p : \; L(p) \le Q \le U(p)\}.$$

In this toy example,

$$L(p) = (3.5 - \tfrac{p}{4}) + (9 - p) = 12.5 - 1.25p, \qquad U(p) = (11 - p) + (4.5 - \tfrac{p}{4}) = 15.5 - 1.25p,$$

so $I = [6.8, 9.2]$.

5. Step 4: finishing the toy example (compute $\underline{\Phi}$ and $\bar{\Phi}$)

Definition of Φ: welfare gain from reallocation (misallocation loss)

Let $(q_1^\star, q_2^\star)$ be the equal-shadow reallocation (so $q_1^\star + q_2^\star = Q$ and $P_1(q_1^\star) = P_2(q_2^\star) = p^\star$). Define the welfare gain (relative to the observed split) as

$$\Phi \;=\; \int_{q_1^{obs}}^{q_1^\star} \big(P_1(q) - p^\star\big)\, dq \;+\; \int_{q_2^{obs}}^{q_2^\star} \big(P_2(q) - p^\star\big)\, dq.$$

Because $q_2^\star < q_2^{obs}$ here, the second term is equivalently $\int_{q_2^\star}^{q_2^{obs}} (p^\star - P_2(q))\, dq$: area *above* P_2 and *below* $p^\star$.

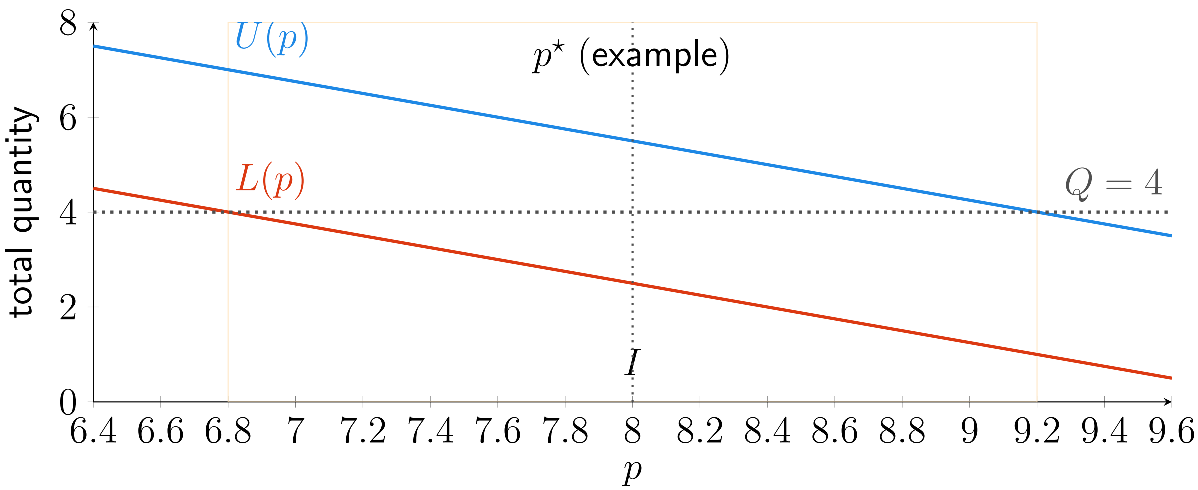

Figure 3: **Feasible shadow prices.** I is where the total Q can be matched given the pointwise bands.

Upper bound $\bar{\Phi}$: make both demands as flat as allowed

Take the flattest allowed slopes (-1) through the observed points:

$$\boldsymbol{Upper system:} \qquad P_1^U(q) = 11 - q, \qquad P_2^U(q) = 9 - q.$$

Equalize marginal values under the total constraint $q_1 + q_2 = 4$:

$$11 - q_1^{\star} = 9 - (4 - q_1^{\star}) \;\Rightarrow\; q_1^{\star} = 3,\; q_2^{\star} = 1,\; p^{\star} = 8.$$

Compute Φ:

$$\int_1^3 \big((11 - q) - 8\big)\, dq \;=\; \int_1^3 (3 - q)\, dq \;=\; 2,$$
$$\int_1^3 \big(8 - (9 - q)\big)\, dq \;=\; \int_1^3 (q - 1)\, dq \;=\; 2.$$

So $\bar{\boldsymbol{\Phi}} = \mathbf{4}$.

Lower bound $\underline{\Phi}$: make both demands as steep as allowed

Take the steepest allowed slopes (-4) through the observed points:

$$\boldsymbol{Lower system:} \qquad P_1^L(q) = 14 - 4q, \qquad P_2^L(q) = 18 - 4q.$$

Equalize marginal values:

$$14 - 4q_1^{\star} = 18 - 4(4 - q_1^{\star}) \;\Rightarrow\; q_1^{\star} = 1.5,\; q_2^{\star} = 2.5,\; p^{\star} = 8.$$

Compute Φ (each term is a small triangle):

$$\int_1^{1.5} \big((14 - 4q) - 8\big)\, dq = \int_1^{1.5} (6 - 4q)\, dq = 0.5,$$

$$\int_{2.5}^{3} \big(8 - (18 - 4q)\big)\, dq = \int_{2.5}^{3} (4q - 10)\, dq = 0.5.$$

So $\underline{\boldsymbol{\Phi}} = \mathbf{1}$.

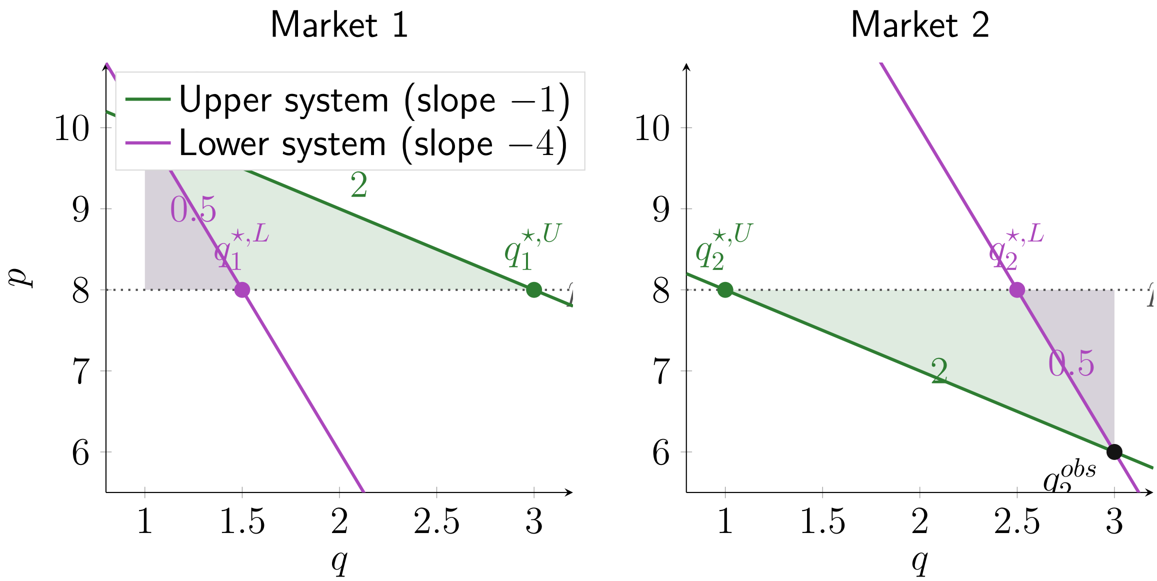

Figure 4: **Finished toy example.** In each market, the shaded area between the demand curve and the common price $p^{\star}$ is that market's contribution to Φ. The flat (green) system sums to $2+2=4$. The steep (violet) system sums to $0.5+0.5=1$.

6. What this toy example teaches (in one paragraph)

In the smallest nontrivial setting, "robust bounds" reduces to: (i) define the feasible set of demand systems via local shape restrictions (the wedge), (ii) convert that to pointwise feasible bands for $q_i(p)$, (iii) restrict the common shadow price to I, and (iv) identify extremizers at the boundary of the feasible set (here, simply the extreme slopes -1 and -4). In larger applications, you cannot guess the extremal curves market-by-market: the shadow price and feasibility must be coordinated across many markets. The machinery is what ensures the bounds are *valid* (over all admissible demands) and *sharp* (attained by a constructed extremal system).