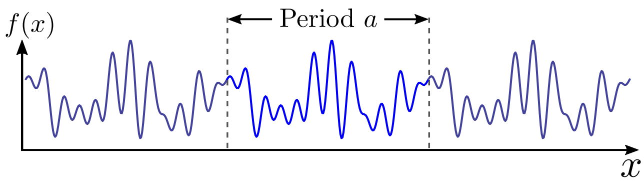
9. Fourier Series and Fourier Transforms

The **Fourier transform** is one of the most important tools for analyzing functions. The basic underlying idea is that a function $f(x)$ can be expressed as a linear combination of elementary functions (specifically, sinusoidal waves). The coefficients in this linear combination can be regarded as a counterpart function, $F(k)$, that is defined in a wave-number domain $k \in \mathbb{R}$. This is helpful because certain mathematical problems, such as differential equations, are easier to solve in terms of $F(k)$ rather than directly in terms of $f(x)$.

9.1 Fourier series

We begin by discussing the **Fourier series**, which is used to analyze functions which are periodic in their inputs. A **periodic function** $f(x)$ is a function of a real variable x that repeats itself every time x changes by a, as in the figure below:

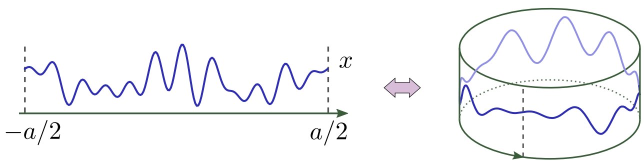


The constant a is called the **period**. Mathematically, we write this condition as

$$f(x+a) = f(x), \quad \forall\, x \in \mathbb{R}. \tag{1}$$

In a physical context, the value of $f(x)$ can be either real or complex. We will assume that the input variable x is real, and that it refers to a spatial coordinate. (Most of the following discussion can also apply to functions of time, with minor differences in convention that we'll discuss later.)

We can also think of a periodic function as being defined over a domain of length a, say $-a/2 \le x < a/2$. The periodicity condition is equivalent to joining the edges of the domain to form a ring of circumference a, as shown in the figure below. Then the position along the circumference of the ring serves as the x coordinate.



Consider what it means to *specify* an arbitrary periodic function $f(x)$. One way to specify the function is to state its value for every $-a/2 \le x < a/2$. But that's an uncountably infinite set of numbers, which is cumbersome to deal with. A better alternative is to express the function as a linear combination of simpler periodic functions, such as sines and cosines:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2\pi n x}{a}\right) + \sum_{m=0}^{\infty} \beta_m \cos\left(\frac{2\pi m x}{a}\right). \tag{2}$$

This is called a **Fourier series**. If $f(x)$ can be expressed as such a series, then it can be fully specified by the set of numbers $\{\alpha_n, \beta_m\}$, which are called the **Fourier coefficients**. These coefficients are real if $f(x)$ is a real function, or complex if $f(x)$ is complex-valued. (Note that the sum over n starts from 1, but the sum over m starts from 0; that's because the sine term with $n = 0$ is zero for all x, so it's redundant.) The justification for the Fourier series formula is that these sine and cosine functions are themselves periodic, with period a:

$$\sin\left(\frac{2\pi n(x+a)}{a}\right) = \sin\left(\frac{2\pi nx}{a} + 2\pi n\right) = \sin\left(\frac{2\pi nx}{a}\right) \tag{3}$$

$$\cos\left(\frac{2\pi n(x+a)}{a}\right) = \cos\left(\frac{2\pi nx}{a} + 2\pi n\right) = \cos\left(\frac{2\pi nx}{a}\right) \tag{4}$$

Hence, any such linear combination satisfies the periodicity condition $f(x+a) = f(x)$ automatically.

The Fourier series is a nice way to specify periodic functions, because we only need to supply the Fourier coefficients $\{\alpha_n, \beta_m\}$, which are a discrete set of numbers; then the value of $f(x)$ is completely determined for all x. Although the set of Fourier coefficients is formally infinite, in many cases the Fourier coefficients are negligible for large m and n (corresponding to very rapidly-oscillating sine and cosine waves), so we only need to keep track of a small number of low-order coefficients.

9.1.1 Square-integrable functions

For a function $f(x)$ to be expressible as a Fourier series, the series needs converge to $f(x)$ as we sum to infinity. Under what circumstances does convergence occur? The full answer to this question turns out to be long and difficult, and we will not go into the details. Luckily, most periodic functions encountered in physical contexts do have convergent Fourier series. In fact, they are usually part of a class of functions called **square-integrable functions**, which are *guaranteed* to have convergent Fourier series.

Square-integrable functions are those for which the integral

$$\int_{-a/2}^{a/2} dx\, \big|\, f(x)\, \big|^2 \tag{5}$$

exists and is finite. From now on, we will simply assume that we're dealing with functions of this sort, and not worry about the issue of convergence.

9.1.2 Complex Fourier series and inverse relations

Using Euler's formula, we can re-write the Fourier series as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x/a}\, f_n. \tag{6}$$

Instead of separate sums over sine and cosine functions, we sum over complex exponential functions. We have a new set of Fourier coefficients, f_n, and the sum includes negative integers n. This form of the Fourier series is a lot more convenient to work with, since we now only have to keep track of a single sum rather than separate sums for the sine and cosine terms. (As an exercise, try working out the explicit relationship between the old and new coefficients.)

The above Fourier series formula tells us that if the Fourier coefficients $\{f_n\}$ are known, then $f(x)$ can be determined. The converse is also true: if we are given $f(x)$, it is possible for us to determine the Fourier coefficients. To see how this is done, first observe that

$$\int_{-a/2}^{a/2} dx\, e^{-2\pi i m x/a}\, e^{2\pi i n x/a} = a\, \delta_{mn} \quad \text{for } m, n \in \mathbb{Z}, \tag{7}$$

where δ_{mn} is the Kronecker delta, defined as:

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases} \tag{8}$$

Due to this property, the set of functions $\exp(2\pi inx/a)$, with integer values of n, are said to be **orthogonal** functions. (We won't go into the details now, but the term "orthogonality" is used here with the same meaning as in vector algebra, where a set of vectors $\vec{v}_1, \vec{v}_2, \dots$ is said to be "orthogonal" if $\vec{v}_m \cdot \vec{v}_n = 0$ for $m \neq n$.) Using this result, we can show that

$$\begin{aligned}
\int_{-a/2}^{a/2} dx\, e^{-2\pi imx/a}\, f(x) &= \int_{-a/2}^{a/2} dx\, e^{-2\pi imx/a} \left[\sum_{n=-\infty}^{\infty} e^{2\pi inx/a}\, f_n\right] && (9)\\
&= \sum_{n=-\infty}^{\infty} \int_{-a/2}^{a/2} dx\, e^{-2\pi imx/a}\, e^{2\pi inx/a}\, f_n && (10)\\
&= \sum_{n=-\infty}^{\infty} a\, \delta_{mn}\, f_n && (11)\\
&= a\, f_m. && (12)
\end{aligned}$$

The procedure of multiplying by $\exp(-2\pi imx/a)$ and integrating over x acts as a kind of "sieve", filtering out all other Fourier components of $f(x)$ and keeping only the one with the matching index m. Hence, we arrive at a pair of relations expressing $f(x)$ in terms of its Fourier components, and vice versa:

$$\left\{\begin{array}{rcl} f(x) &=& \displaystyle\sum_{n=-\infty}^{\infty} e^{ik_n x}\, f_n \\ f_n &=& \displaystyle\frac{1}{a}\int_{-a/2}^{a/2} dx\, e^{-ik_n x}\, f(x) \end{array}\right\} \quad \text{where } k_n \equiv \frac{2\pi n}{a} \tag{13}$$

Here, the real numbers k_n are called **wave-numbers**. They form a discrete set, with one for each Fourier component. In physics jargon, we say that the wave-numbers are "quantized" to integer multiples of

$$\Delta k \equiv \frac{2\pi}{a}. \tag{14}$$

9.1.3 Example: Fourier series of a square wave

To get a feel for how the Fourier series expansion works, let's look at the square wave, which is a waveform that takes only two values $+1$ or -1, jumping discontinuously between those two values at periodic intervals. Within one period, the function is

$$f(x) = \left\{\begin{array}{ll} -1, & -a/2 \le x < 0 \\ +1, & 0 \le x < a/2. \end{array}\right. \tag{15}$$

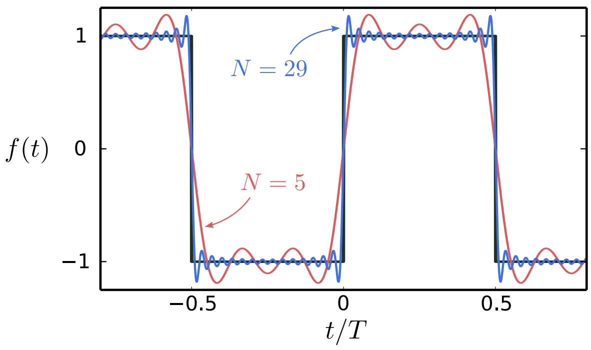
Plugging this into the Fourier relation, and doing the straightforward integrals, gives the Fourier coefficients

$$\begin{aligned} f_n &= -i\, \frac{\left[\sin\left(n\pi/2\right)\right]^2}{n\pi/2} \\ &= \left\{\begin{array}{ll} -2i/n\pi, & n \text{ odd} \\ 0, & n \text{ even.} \end{array}\right. \end{aligned} \tag{16}$$

As can be seen, the high-frequency Fourier components are less important, since the Fourier coefficients go to zero for large n. We can write the Fourier series as

$$f(x) \;\leftrightarrow\; \sum_{n=1,3,5,\dots} \frac{4\sin(2\pi nx/a)}{n\pi}. \tag{17}$$

If this infinite series is truncated to a finite number of terms, we get an approximation to $f(x)$. The approximation becomes better and better as more terms are included. This is illustrated in the following figure, where we show the Fourier series truncated up to $N = 5$, and the Fourier series truncated up to $N = 29$:



One amusing consequence of this result is that it can be used as a series expansion for the mathematical constant π. If we set $x = a/4$, then

$$f(a/4) = 1 = \frac{4}{\pi}\left[\sin(\pi/2) + \frac{1}{3}\sin(3\pi/2) + \frac{1}{5}\sin(5\pi/2) + \cdots\right], \tag{18}$$

and hence

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \tag{19}$$

9.2 Fourier transforms

The Fourier series discussed in the previous section applies to periodic functions, $f(x)$, defined over the interval $-a/2 \le x < a/2$. This concept can be generalized to functions defined over the entire real line, $x \in \mathbb{R}$, by carefully taking the limit $a \to \infty$.

Consider what happens when we apply the $a \to \infty$ limit to the right-hand side of the complex Fourier series formula from Section 9.1.2:

$$f(x) = \lim_{a\to\infty}\left(\sum_{n=-\infty}^{\infty} e^{ik_n x} f_n\right), \quad \text{where } k_n = n\Delta k, \;\; \Delta k = \frac{2\pi n}{a}. \tag{20}$$

As $a \to \infty$, the wave-number quantum Δk goes to zero. Hence, the set of values of k_n turns into a continuum, and we can replace the discrete sum with an integral over the values of k_n. To do this, we multiply the summand by a factor of $(\Delta k/2\pi)/(\Delta k/2\pi) = 1$:

$$f(x) = \lim_{a\to\infty}\left[\sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi}\, e^{ik_n x}\left(\frac{2\pi f_n}{\Delta k}\right)\right]. \tag{21}$$

If we now define

$$F(k_n) \equiv \frac{2\pi}{\Delta k}\, f_n, \tag{22}$$

then the sum becomes

$$f(x) = \lim_{a\to\infty}\left(\sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi}\, e^{ik_n x}\, F(k_n)\right). \tag{23}$$

This limiting expression matches the basic definition of an integral:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, F(k). \tag{24}$$

In case you're wondering, the factor of 2π is essentially arbitrary, and is just a matter of how we chose to define $F(k_n)$. Our choice corresponds to the standard definition of the Fourier transform.

9.2.1 The Fourier relations

The function $F(k)$ defined in the previous section is called the **Fourier transform** of $f(x)$. Just as we have expressed $f(x)$ in terms of $F(k)$, we can also express $F(k)$ in terms of $f(x)$. To do this, we apply the $a \to \infty$ limit to the inverse relation for the Fourier series (see Section 9.1.2):

$$\begin{aligned} F(k_n) &= \lim_{a\to\infty} \frac{2\pi}{\Delta k} f_n && (25) \\ &= \lim_{a\to\infty} \frac{2\pi}{2\pi/a} \left(\frac{1}{a}\int_{-a/2}^{a/2} dx\, e^{-ik_n x}\right) && (26) \\ &= \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). && (27) \end{aligned}$$

Hence, we arrive at a pair of equations called the **Fourier relations**:

$$\left\{ \begin{aligned} F(k) &= \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \\ f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} F(k) \end{aligned} \right\} \qquad (28)$$

The first equation is the Fourier transform, and the second equation is called the **inverse Fourier transform**. These relations state that if we have a function $f(x)$ defined over $x \in \mathbb{R}$, then there is a unique counterpart function $F(k)$ defined over $k \in \mathbb{R}$, and vice versa. The Fourier transform converts $f(x)$ to $F(k)$, and the inverse Fourier transform does the reverse.

It is important to note the small differences between the two formulas. Firstly, there is a factor of $1/2\pi$ that "tags along" with dk, but not with dx; this is a matter of convention, tied to our definition of $F(k)$, as mentioned before. Secondly, the integral over x contains a factor of e^{-ikx} but the integral over k contains a factor of e^{ikx}. One way to remember this is to think of the integral over k, in the inverse Fourier transform equation, as the continuum limit of a sum over complex waves, with $F(k)$ playing the role of the series coefficients; by convention, these complex waves have the form $\exp(ikx)$.

In our definition of the Fourier transform, it is clear that the Fourier series needs to remain convergent as we take the $a \to \infty$ limit. Based on our earlier discussion of square-integrable functions (Section 9.1.1), this means we always deal with functions such that

$$\int_{-\infty}^{\infty} dx\, \big|\, f(x)\,\big|^2 \qquad (29)$$

exists and is finite.

9.2.2 A simple example

Consider the function

$$f(x) = \left\{ \begin{array}{ll} e^{-\eta x}, & x \ge 0 \\ 0, & x < 0, \end{array} \right. \qquad \eta \in \mathbb{R}^+. \qquad (30)$$

For $x < 0$, this is an exponentially-decaying function, and for $x < 0$ it is identically zero. The real parameter η is called the decay constant; for $\eta > 0$, the function $f(x)$ vanishes as $x \to +\infty$ and can thus be shown to be square-integrable, and larger values of η correspond to faster exponential decay.

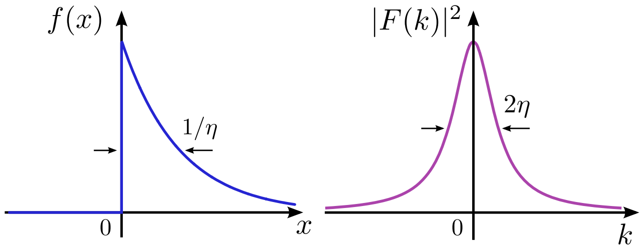
The Fourier transform can be found by directly calculating the Fourier integral:

$$F(k) \;=\; \int_{0}^{\infty} dx\, e^{-ikx}\, e^{-\kappa x} \;=\; \frac{-i}{k - i\eta}. \qquad (31)$$

It is useful to plot the squared magnitude of the Fourier transform, $|F(k)|^2$, against k. This is called the **Fourier spectrum** of $f(x)$. In this case,

$$\left|\,F(k)\,\right|^2 = \frac{1}{k^2 + \eta^2}. \tag{32}$$

This is plotted in the right-hand figure below:



We call such a graph a **Lorentzian** curve. It consists of a peak centered at $k = 0$, whose height and width are dependent on the decay constant η. For small η, i.e. weakly-decaying $f(x)$, the peak is high and narrow. For large η, i.e. rapidly-decaying $f(x)$, the peak is low and broad. This kind of relationship between the decay rate and the Fourier spectrum peak width is very common.

We can quantify the width of the Lorentzian curve by defining the **full-width at half-maximum** (FWHM), which is the width of the curve at half the value of its maximum. In this case, the maximum of the Lorentzian curve occurs at $k = 0$ and has the value of $1/\eta^2$. The half-maximum, $1/2\eta^2$, occurs when $\delta k = \pm\eta$. Hence, the original function's decay constant, η, is directly proportional to the FWHM of the Fourier spectrum, which is 2η.

Note also that this relationship is dimensionally consistent. In $f(x)$, the exponent ηx needs to be dimensionless, so the decay constant has unit of $[1/x]$. This has the same units as the wave-number variable k, which is the horizontal axis for the Fourier spectrum.

To wrap up this example, let's evaluate the inverse Fourier transform:

$$f(x) \;=\; -i \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{e^{ikx}}{k - i\eta}. \tag{33}$$

This can be done by contour integration (Chapter 8). The analytic continuation of the integrand has one simple pole, at $k = i\eta$. For $x < 0$, the numerator $\exp(ikx)$ blows up far from the origin in the upper half of the complex plane, and vanishes far from the origin in the lower half-plane; hence we close the contour in the lower half-plane. This encloses no pole, so the integral is zero. For $x > 0$, the numerator vanishes far from the origin in the upper half-plane, so we close the contour in the upper half-plane (i.e., the contour is counter-clockwise). Hence,

$$f(x) = \left(\frac{-i}{2\pi}\right)(2\pi i)\;\mathrm{Res}\left[\frac{e^{ikx}}{k - i\eta}\right]_{k=i\eta} = e^{-\eta x} \qquad (x > 0) \tag{34}$$

which is indeed the function that we started out with.

9.2.3 Fourier transforms for time-domain functions

Thus far, we have been dealing with functions of a spatial coordinate x. Of course, these mathematical relations don't care about the physical meaning of the variables, so the Fourier transform concept is also applicable to functions of time t. However, there is a vexatious difference in convention that needs to be observed: when dealing with functions of the time coordinate t, it is customary to use a different sign convention in the Fourier relations!

The Fourier relations for a function of time, $f(t)$, are:

$$\left\{ \begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} dt \; e^{i\omega t} \, f(t) \\ f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \; e^{-i\omega t} \, F(\omega). \end{aligned} \right\} \tag{35}$$

These relations differ in one notable way from the Fourier relations between $f(x)$ and $F(k)$ discussed in Section 9.2.1: the signs of the $\pm i\omega t$ exponents are flipped.

There's a good reason for this difference in sign convention: it arises from the need to describe propagating waves, which vary with both space *and* time. As we discussed in Chapter 5, a propagating plane wave can be described by a wavefunction

$$f(x,t) = Ae^{i(kx-\omega t)}, \tag{36}$$

where k is the wave-number and ω is the frequency. We write the plane wave function this way so that positive k indicates forward propagation in space (i.e., in the $+x$ direction), and positive ω indicates forward propagation in time (i.e., in the $+t$ direction). This requires the kx and ωt terms in the exponent to have opposite signs, so that when t increases by a certain amount, a corresponding *increase* in x leaves the total exponent unchanged.

Now, as we have seen, the inverse Fourier transform relation describes how a wave-form is broken up into a superposition of elementary waves. In the case of a wavefunction $f(x,t)$, the superposition is given in terms of plane waves:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \;\; e^{i(kx-\omega t)} \, F(k,\omega). \tag{37}$$

To be consistent with this, we need to treat space and time variables with oppositely-signed exponents:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \; e^{ikx} \, F(k) \tag{38}$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \; e^{-i\omega t} \, F(\omega). \tag{39}$$

The other set of Fourier relations follow from this choice.

9.2.4 Basic properties of the Fourier transform

The Fourier transform has several properties that are useful to remember. These can all be directly proven using the definition of the Fourier transform. The proofs are left as exercises.

(i) The Fourier transform is linear: if we have two functions $f(x)$ and $g(x)$, whose Fourier transforms are $F(k)$ and $G(k)$ respectively, then for any constants $a, b \in \mathbb{C}$,

$$af(x) + bg(x) \quad \xrightarrow{\text{FT}} \quad aF(k) + bG(k). \tag{40}$$

(ii) Performing a coordinate translation on a function causes its Fourier transform to be multiplied by a "phase factor":

$$f(x+b) \quad \xrightarrow{\text{FT}} \quad e^{ikb} \, F(k). \tag{41}$$

As a consequence, translations leave the Fourier spectrum $|F(k)|^2$ unchanged.

(iii) If the Fourier transform of $f(x)$ is $F(k)$, then

$$f^*(x) \quad \overset{\text{FT}}{\longrightarrow} \quad F^*(-k). \tag{42}$$

As a consequence, the Fourier transform of a real function must satisfy the symmetry relation $F(k) = F^*(-k)$, meaning that the Fourier spectrum is symmetric about the origin in k-space: $\big|\,F(k)\,\big|^2 = \big|\,F(-k)\,\big|^2$.

(iv) When you take the derivative of a function, that is equivalent to multiplying its Fourier transform by a factor of ik:

$$\frac{d}{dx}f(x) \quad \overset{\text{FT}}{\longrightarrow} \quad ikF(k). \tag{43}$$

For functions of time, because of the difference in sign convention (Section 9.2.3), there is an extra minus sign:

$$\frac{d}{dt}f(t) \quad \overset{\text{FT}}{\longrightarrow} \quad -i\omega F(\omega). \tag{44}$$

9.2.5 Fourier transforms of differential equations

The Fourier transform can be a very useful tool for solving differential equations. As an example, consider a damped harmonic oscillator that is subjected to an additional driving force $f(t)$. This force has an arbitrary time dependence, and is not necessarily harmonic. The equation of motion is

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = \frac{f(t)}{m}. \tag{45}$$

To solve for $x(t)$, we first take the Fourier transform of both sides of the above equation. The result is:

$$-\,\omega^2 X(\omega) - 2i\gamma\omega X(\omega) + \omega_0^2 X(\omega) = \frac{F(\omega)}{m}, \tag{46}$$

where $X(\omega)$ and $F(\omega)$ are the Fourier transforms of $x(t)$ and $f(t)$ respectively. To obtain the left-hand side of this equation, we made use of the properties of the Fourier transform described in Section 9.2.4, specifically linearity (i) and Fourier transformations of derivatives (iv). Note also that we have used the sign convention for time-domain functions discussed in Section 9.2.3.

The Fourier transform has turned our ordinary differential equation into an algebraic equation. This equation can be easily solved:

$$X(\omega) = \frac{F(\omega)/m}{-\omega^2 - 2i\gamma\omega + \omega_0^2} \tag{47}$$

Knowing $X(\omega)$, we can use the inverse Fourier transform to obtain $x(t)$.

To summarize, the solution procedure for the driven harmonic oscillator equation consists of (i) using the Fourier transform on $f(t)$ to obtain $F(\omega)$, (ii) using the above equation to find $X(\omega)$ algebraically, and (iii) performing an inverse Fourier transform to obtain $x(t)$. This will be the basis for the Green's function method, a method for systematically solving differential equations that will be discussed later.

9.3 Common Fourier transforms

To accumulate more intuition about Fourier transforms, we will now study the Fourier transforms of a few interesting functions. We will simply state the results, leaving the actual calculations of the Fourier transforms as exercises.

9.3.1 Damped waves

In Section 9.2.2, we saw that an exponentially decay function with decay constant $\eta \in \mathbb{R}^+$ has the following Fourier transform:

$$f(x) = \left\{ \begin{array}{cl} e^{-\eta x}, & x \geq 0 \\ 0, & x < 0, \end{array} \right. \quad \xrightarrow{\text{FT}} \quad F(k) = \frac{-i}{k - i\eta}. \tag{48}$$

Observe that $F(k)$ is given by a simple algebraic formula. If we "extend" the domain of k to complex values, $F(k)$ corresponds to an analytic function with a simple pole in the upper half of the complex plane, at $k = i\eta$.

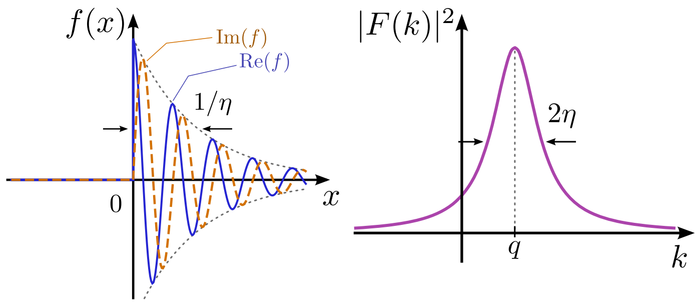
Next, consider a decaying wave with wave-number $q \in \mathbb{R}$ and decay constant $\eta \in \mathbb{R}^+$. The Fourier transform is a function with a simple pole at $q + i\eta$:

$$f(x) = \left\{ \begin{array}{cl} e^{i(q+i\eta)x}, & x \geq 0 \\ 0, & x < 0. \end{array} \right. \quad \xrightarrow{\text{FT}} \quad F(k) = \frac{-i}{k - (q + i\eta)}. \tag{49}$$

Hence, the Fourier spectrum is

$$|F(k)|^2 = \frac{1}{(k-q)^2 + \eta^2}. \tag{50}$$

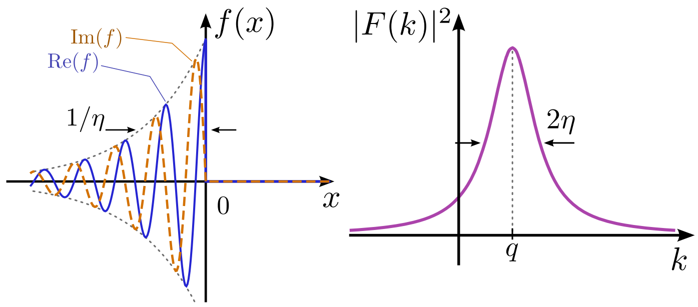
This is a Lorentzian peaked at $k = q$ and having width 2η, as shown below:



On the other hand, consider a wave that grows exponentially with x for $x < 0$, and is zero for $x > 0$. The Fourier transform is a function with a simple pole in the lower half-plane:

$$f(x) = \left\{ \begin{array}{cl} 0, & x \geq 0 \\ e^{i(q-i\eta)x}, & x < 0. \end{array} \right. \quad \xrightarrow{\text{FT}} \quad F(k) = \frac{i}{k - (q - i\eta)}. \tag{51}$$

The Fourier spectrum is, likewise, a Lorentzian centered at $k = q$.



From these examples, we see that oscillations and amplification/decay in $f(x)$ are related to the existence of poles in the algebraic expression for $F(k)$. The real part of the pole position gives the wave-number of the oscillation, and the distance from the pole to the real axis gives the amplification or decay constant. A decaying signal produces a pole in the upper half-plane, while a signal that is increasing exponentially with x produces a pole in the lower half-plane. In both cases, if we plot the Fourier spectrum of $|F(k)|^2$ versus real k, the result is a Lorentzian curve centered at $k = q$, with width 2η.

9.3.2 Gaussian wave-packets

It is also interesting to look at the Fourier transform of a function that decays faster than an exponential. In particular, let's consider a function with a decay "envelope" given by a Gaussian function:

$$f(x) = e^{iqx}\, e^{-\gamma x^2}, \quad \text{where } q \in \mathbb{C},\ \gamma \in \mathbb{R}. \tag{52}$$

Such a function is called a **Gaussian wave-packet**. The width of the Gaussian envelope can be characterized by the Gaussian function's "standard deviation", which is where the curve reaches $e^{-1/2}$ times its peak value. In this case, the standard deviation is $\Delta x = 1/\sqrt{2\gamma}$.

It can be shown that $f(x)$ has the following Fourier transform:

$$F(k) = \sqrt{\frac{\pi}{\gamma}}\, e^{-\frac{(k-q)^2}{4\gamma}}. \tag{53}$$

To derive this result, we perform the Fourier integral as follows:

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(x) &(54)\\ &= \int_{-\infty}^{\infty} dx\, \exp\left\{-i(k-q)x - \gamma x^2\right\}. &(55)\end{aligned}$$

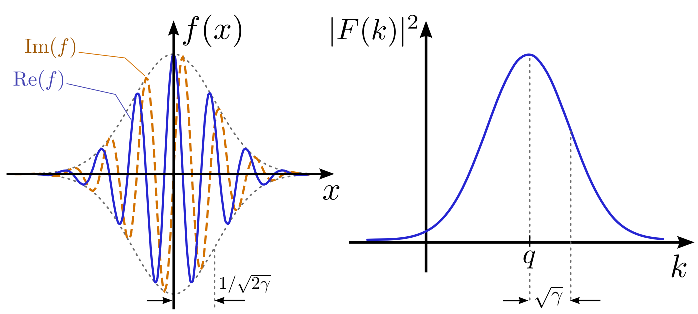
In the integrand, the expression inside the exponential is quadratic in x. We complete the square:

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} dx\, \exp\left\{-\gamma\left(x + \frac{i(k-q)}{2\gamma}\right)^2 + \gamma\left(\frac{i(k-q)}{2\gamma}\right)^2\right\} &(56)\\ &= \exp\left\{-\frac{(k-q)^2}{4\gamma}\right\} \int_{-\infty}^{\infty} dx\, \exp\left\{-\gamma\left(x + \frac{i(k-q)}{2\gamma}\right)^2\right\}. &(57)\end{aligned}$$

The remaining integral is simply the Gaussian integral (see Chapter 2), with a constant shift in x which can be eliminated by a change of variables. This yields the result stated above.

The Fourier spectrum, $|F(k)|^2$, is a Gaussian function with standard deviation

$$\Delta k = \frac{1}{\sqrt{2(1/2\gamma)}} = \sqrt{\gamma}. \tag{58}$$



Thus, we again see that the Fourier spectrum is peaked at a value of k corresponding to the wave-number of the underlying sinusoidal wave in $f(x)$. Moreover, a stronger (weaker) decay in $f(x)$ leads to a broader (narrower) Fourier spectrum.

9.4 The delta function

What happens when we feed the Fourier relations into one another? Plugging the Fourier transform into the inverse Fourier transform, we get

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} F(k) && (59)\\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \int_{-\infty}^{\infty} dx' e^{-ikx'} f(x') && (60)\\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-ikx'} f(x') && (61)\\ &= \int_{-\infty}^{\infty} dx'\; \delta(x-x')\, f(x'). && (62) \end{aligned}$$

In the last step, we have introduced

$$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}, \tag{63}$$

which is called the **delta function**. According to the above equations, the delta function acts as a kind of filter: when we multiply it by any function $f(x')$ and integrate over x', the result is the value of that function at a particular point x.

But here's a problem: the above integral definition of the delta function is non-convergent; in particular, the integrand does not vanish at $\pm\infty$. We can get around this by thinking of the delta function as a limiting case of a convergent integral. Specifically, let's take

$$\delta(x-x') = \lim_{\gamma\rightarrow 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\gamma k^2}. \tag{64}$$

For $\gamma \rightarrow 0$, the "regulator" $\exp(-\gamma k^2)$ which we have inserted into the integrand goes to one, so that the integrand goes back to what we had before; on the other hand, for $\gamma > 0$ the regulator ensures that the integrand vanishes at the end-points so that the integral is well-defined. But the expression on the right is the Fourier transform for a Gaussian wave-packet (see Section 9.3.2). Using that result, we get

$$\delta(x-x') = \lim_{\gamma\rightarrow 0} \frac{1}{\sqrt{4\pi\gamma}}\, e^{-\frac{(x-x')^2}{4\gamma}}. \tag{65}$$

This is a Gaussian function, with standard deviation $\sqrt{2\gamma}$ and area 1. Hence, the delta function can be regarded as the limit of a Gaussian function as its width goes to zero while keeping the area under the curve fixed at unity (which means the height of the peak goes to infinity).

The most important feature of the delta function is it acts as a "filter". Whenever it shows up in an integral, it picks out the value of the rest of the integrand evaluated where the delta function is centered:

$$\int_{-\infty}^{\infty} dx\; \delta(x-x_0)\, f(x) = f(x_0). \tag{66}$$

Intuitively, we can understand this behavior from the above definition of the delta function as the zero-width limit of a Gaussian. When we multiply a function $f(x)$ with a narrow Gaussian centered at x_0, the product will approach zero almost everywhere, because the Gaussian goes to zero. The product is non-zero only in the vicinity of $x = x_0$, where the Gaussian peaks. And because the area under the delta function is unity, integrating that product over all x simply gives the value of the other function at the point x_0.

9.5 Multi-dimensional Fourier transforms

When studying problems such as wave propagation, we will often have to deal with Fourier transforms acting on several variables simultaneously. This is conceptually straightforward. For a function $f(x_1, x_2, \ldots, x_d)$ which depends on d independent spatial coordinates $x_1, x_2, \ldots x_d$, we can simply perform a Fourier transform on each coordinate individually:

$$F(k_1, k_2, \ldots, k_d) = \int_{-\infty}^{\infty} dx_1\, e^{-ik_1x_1} \int_{-\infty}^{\infty} dx_2\, e^{-ik_2x_2} \cdots \int_{-\infty}^{\infty} dx_d\, e^{-ik_dx_d}\, f(x_1, x_2, \ldots, x_N) \tag{67}$$

Note that each coordinate gets Fourier-transformed into its own independent k variable, so that the result is still a function of d independent variables.

We can compactly express such a "multi-dimensional Fourier transform" via vector notation. Let $\vec{x} = [x_1, x_2, \ldots, x_d]$ be a d-dimensional coordinate vector. The Fourier-transformed coordinates can be written as $\vec{k} = [k_1, k_2, \ldots, k_d]$, and the Fourier transform is

$$F(\vec{k}) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_d\; e^{-i\,\vec{k}\cdot\vec{x}}\, f(\vec{x}), \tag{68}$$

where $\vec{k}\cdot\vec{x} \equiv k_1x_1 + k_2x_2 + \cdots + k_dx_d$ is the usual dot product of two vectors.

The inverse Fourier transform is

$$f(\vec{x}) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dk_d}{2\pi}\; e^{i\,\vec{k}\cdot\vec{x}}\, F(\vec{k}). \tag{69}$$

A multi-dimensional delta function can be defined as the Fourier transform of a plane wave:

$$\delta^d(\vec{x} - \vec{x}') = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dk_d}{2\pi}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}. \tag{70}$$

Note that δ^d has the dimensions of $[x]^{-d}$. The multi-dimensional delta function has a "filtering" property similar to the one-dimensional delta function. For any $f(x_1, \ldots, x_d)$,

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_d\; \delta^d(\vec{x} - \vec{x}')\, f(\vec{x}) = f(\vec{x}'). \tag{71}$$

Finally, if we have a mix of spatial and temporal coordinates, then the usual sign conventions (see Section 9.2.3) apply to each individual coordinate. For example, if $f(x,t)$ is a function of one spatial coordinate and one temporal coordinate, the Fourier relations are

$$F(k,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\; e^{-i(kx-\omega t)}\, f(x,t) \tag{72}$$

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; e^{i(kx-\omega t)}\, F(k,\omega). \tag{73}$$

9.6 Exercises

1. Find the relationship between the coefficients $\{\alpha_n, \beta_m\}$ in the sine/cosine Fourier series and the coefficients f_n in the complex exponential Fourier series:

$$f(t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2\pi nx}{a}\right) + \sum_{m=0}^{\infty} \beta_m \cos\left(\frac{2\pi mx}{a}\right) \tag{74}$$

$$= \sum_{n=-\infty}^{\infty} f_n \exp\left(\frac{2\pi inx}{a}\right). \tag{75}$$

2. Find the Fourier series expansion of the triangular wave

$$f(x) = \begin{cases} -cx, & -a/2 \le x < 0, \\ cx, & 0 \le x < a/2 \end{cases} \tag{76}$$

where c is some real constant. Using the result, derive a series expansion for π^2.

3. Prove the properties of the Fourier transform listed in Section 9.2.4.

4. Find the Fourier transform of $f(x) = \sin(\kappa x)/x$.

5. Prove that if $f(x)$ is a real function, then its Fourier transform satisfies $F(k) = F(-k)^*$.