AE 6210: ADVANCED DYNAMICS I

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SPACECRAFT ATTITUDE DYNAMICS: SIMULATION & VISUALIZATION

Problem Formulation:

The problem we have been presented with is simulating and visualizing the attitude dynamics of a satellite (CubeSat) with a primary focus on its 3D Orientation. The satellite has a general 3D initial angular velocity with no applied control moments. The computation of the satellite orientation needs to be simulated using 3 different approaches:

- **Using the Direction Cosine Matrix (DCM) to represent the CubeSat Orientation**
- **Using the Orientation Parameters**
- **Using the Orientation Angles**

After simulating the orientation, we need to show the following:

- **Validation of any one of the Simulations using the Conservation of Angular Momentum and the Conservation of Energy Principle**
- **Comparison of the 3 sets of Simulation Results**
- **Animation of the Satellite Orientation**

Solution:

Steps to Approach the Problem:

To approach the above ***Torque-Free Rigid Body Problem***, we have segmented the whole problem into multiple sub-problems and decided to solve them one at a time. These are as follows:

1. Defining the Geometry and Inertial Properties of the CubeSat
2. Reformulation of Euler's Dynamical Equations for the Free Rigid Body Motion
3. Representing the Satellite Orientation in terms of the Euler Angles
4. Validating the Simulation for the Euler Angles
5. Representing the Satellite Orientation in terms of Quaternions and comparing the obtained DCM with the DCM for Euler Angles.
6. Representing the Satellite Orientation in terms of Euler Parameters and comparing the obtained DCM with the DCM for Euler Angles

Step #01: Defining the Geometry and Inertial Properties of the CubeSat

We consider a standard 6U larger CubeSat with a dimension of 35x15x10 cm and a mass of m = 12 kg.

Assumptions:

- We assume that the origin of the body frame lies on the CoM of the CubeSat.
- We also assume that the body frame is a **principal axis frame** about which the Moment of Inertia (MOI) Matrix can be given as a diagonal matrix as follows:

$$[I^C] = \begin{bmatrix} I_{xx}^c & 0 & 0 \\ 0 & I_{yy}^c & 0 \\ 0 & 0 & I_{zz}^c \end{bmatrix}$$

where,

$$I_{xx}^c = \frac{m}{12} * (w^2 + t^2)$$

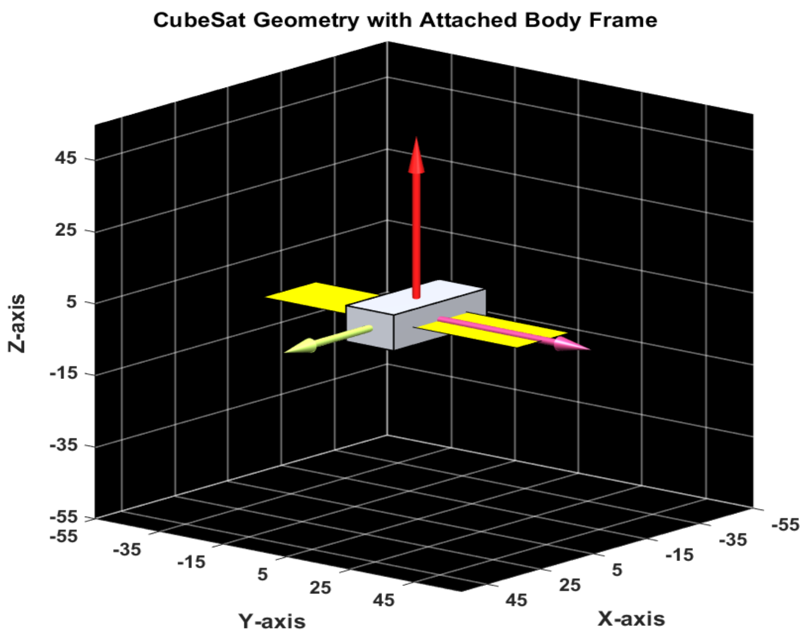
$$I_{yy}^c = \frac{m}{12} * (l^2 + t^2)$$

$$I_{zz}^c = \frac{m}{12} * (w^2 + l^2)$$

CubeSat Length, l = 0.35 m (along the X-axis)

CubeSat Width, w = 0.15 m (along the Y-axis)

CubeSat Height, t = 0.10 m (along the Z-axis)



Step #02: Reformulating Euler's Dynamical Equations for the Free Rigid Body Motion

According to Euler's 2nd Law of Rigid Body Dynamics, we know that the rotational motion of a rigid body can be defined as,

$$[I^C]_B * \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix}_B + \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}_B \text{ x } \left([I^C]_B * \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}_B \right) = \begin{Bmatrix} M_1^C \\ M_2^C \\ M_3^C \end{Bmatrix}_B$$

Assumptions:

- As our body of interest is a tumbling CubeSat, **no control moments or torques** are acting on it.
- We are only focusing on the 3D orientation, i.e., rotational motion of the satellite. Hence, we shall **not discuss** the **translational dynamics of the satellite**.
- All the **vector components** are taken in the **body frame** to analyze the rotational dynamics of the satellite.

As such, the equation of rotational dynamics for the tumbling satellite can be reformulated as,

$$[I^C]_B * \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix}_B + \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}_B \text{ x } \left([I^C]_B * \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}_B \right) = 0 \quad \text{-----------------} \quad \mathbf{(1)}$$

Expanding the above equations in the form of individual vector components, we get the **three nonlinear equations of rotational motion** (Rotational Kinetics), which relate the rate of change of angular velocities to angular accelerations.

Rotational EoMs:

$$\boxed{\dot{\omega}_1 = \left(\frac{I_{yy} - I_{zz}}{I_{xx}}\right) * \omega_2\omega_3} \quad \text{-----------------} \quad \mathbf{1(a)}$$

$$\boxed{\dot{\omega}_2 = \left(\frac{I_{zz} - I_{xx}}{I_{yy}}\right) * \omega_3\omega_1} \quad \text{-----------------} \quad \mathbf{1(b)}$$

$$\boxed{\dot{\omega}_3 = \left(\frac{I_{xx} - I_{yy}}{I_{zz}}\right) * \omega_1\omega_2} \quad \text{-----------------} \quad \mathbf{1(c)}$$

Step #03: Representing the Satellite Orientation in terms of the Euler Angles

To compute **Angular Kinematics**, i.e., to **relate the angular velocities with the rate of change of body orientations**, we have **three approaches** at hand for the accurate representation of CubeSat orientations.

Representing the orientation of the **CubeSat Body Frame {B}** relative to the **Newtonian Frame {N}** in terms of Euler Angles requires **only three angles ($\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi}$)** about the inertial frame axis system.

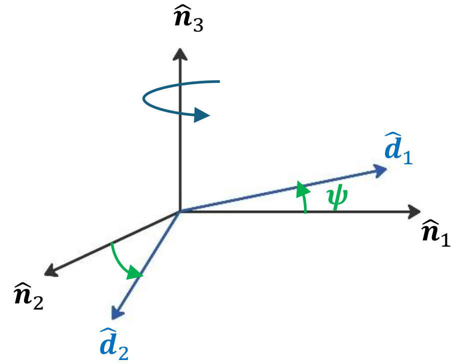
Assumptions:

- To represent 3D orientation by Euler Angles, we consider each of the rotations of the body as **a simple rotation**
- For this tumbling satellite, we are considering the **Body 3-2-1 sequence of rotations**.

Angular Kinematics in terms of Euler Angles:

Let $\{N\}$ be the Inertial Frame and $\{D\}$ be an intermediate Frame. For Body 3-2-1, we assume that initially the **$\{D\}$ frame rotates** about **$\hat{n}_3$ or $\hat{d}_3$ by an angle ψ**.

Therefore, the DCM for the rotation of the $\{D\}$ relative to $\{N\}$ about $\hat{d}_3$ can be given by,



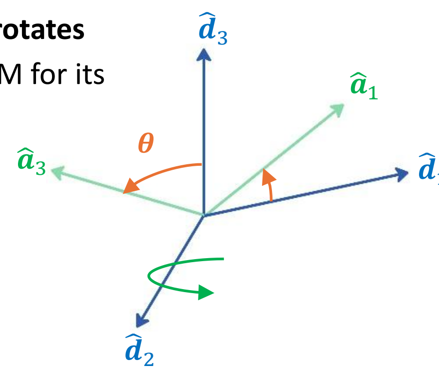
$$C_3(\psi) = [{}^N C^D] = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore,$ $\boxed{\hat{N} = [{}^N C^D].\hat{d}}$ ---------------- **2(a)**

Besides, since we assumed the relative rotation of $\{D\}$ is a simple rotation, its angular velocity relative to $\{N\}$ is equal to the rate at which the frame is rotating, i.e., $\dot{\psi}$ in $\hat{d}_3$ direction.

$\therefore,$ the angular velocity of $\{D\}$ relative to $\{N\}$ is given as, $\boxed{{}^N\vec{\omega}^D = \dot{\psi}\,\hat{d}_3}$ ---------------- **3(a)**

Similarly, we assumed another **intermediate frame $\{A\}$**, which **rotates relative to $\{D\}$ about $\hat{d}_2$ or $\hat{a}_2$ by an angle θ**. Therefore, the DCM for its rotation relative to $\{D\}$ about $\hat{a}_2$ can be given by,



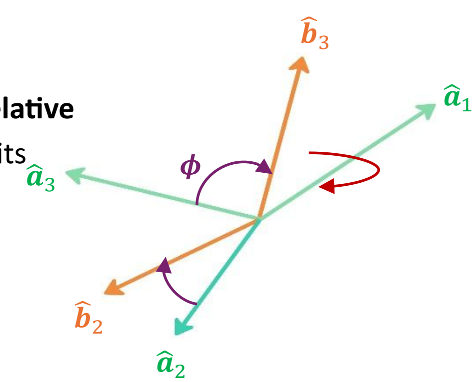
$$C_2(\theta) = [{}^D C^A] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$\therefore,$ $\boxed{\hat{d} = [{}^D C^A].\hat{a}}$ ---------------- **2(b)**

Meanwhile, the angular velocity of $\{A\}$ relative to $\{D\}$ can be given as,

$\boxed{{}^D\vec{\omega}^A = \dot{\theta}\,\hat{a}_2}$ ---------------- **3(b)**

Finally, we assume the **CubeSat body frame $\{B\}$** is **rotating relative to $\{A\}$ about $\hat{a}_1$ or $\hat{b}_1$ by an angle ϕ**. Therefore, the DCM for its rotation relative to $\{A\}$ about $\hat{b}_1$ can be given by,



$$C_1(\phi) = [{}^A C^B] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore,$ $\boxed{\hat{a} = [{}^A C^B].\hat{b}}$ ---------------- **2(c)**

Meanwhile, the angular velocity of $\{B\}$ relative to $\{A\}$ can be given as,

$\boxed{{}^A\vec{\omega}^B = \dot{\phi}\,\hat{b}_1}$ ---------------- **3(c)**

Based on the **Additive Theorem of Angular Velocity**, the **addition of equations 3(a-c)** will give us the angular velocity of the CubeSat relative to the Inertial Frame, as below,

$$N_{\vec{\omega}^B} = N_{\vec{\omega}^D} + D_{\vec{\omega}^A} + A_{\vec{\omega}^B} = \dot{\psi}\,\hat{d}_3 + \dot{\theta}\,\hat{a}_2 + \dot{\phi}\,\hat{b}_1$$

Now, representing all the unit vectors relative to the CubeSat body frame with the help of DCM will eventually give us the angular kinematics that relates the rate of change of the Euler Angles with the angular velocity components of the body. Therefore, we get,

Kinematic Equations: $$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}_B = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta * \sin\phi \\ 0 & -\sin\phi & \cos\theta * \cos\phi \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}$$ ---------------- **4(a-c)**

Meanwhile, putting the values of equation 2(c) to 2(b) and then 2(b) to 2(a), we get the **Direction Cosine Matrix (DCM)** to represent the rotation of the CubeSat Body frame relative to the Inertial Frame in terms of the Euler Angles.

$N_{C^B} = C_3(\psi) * C_2(\theta) * C_1(\phi)$

$$= \begin{pmatrix} \cos(\Psi)\cos(\Theta) & \cos(\Psi)\sin(\Phi)\sin(\Theta) - \cos(\Phi)\sin(\Psi) & \sin(\Phi)\sin(\Psi) + \cos(\Phi)\cos(\Psi)\sin(\Theta) \\ \cos(\Theta)\sin(\Psi) & \cos(\Phi)\cos(\Psi) + \sin(\Phi)\sin(\Psi)\sin(\Theta) & \cos(\Phi)\sin(\Psi)\sin(\Theta) - \cos(\Psi)\sin(\Phi) \\ -\sin(\Theta) & \cos(\Theta)\sin(\Phi) & \cos(\Phi)\cos(\Theta) \end{pmatrix}$$

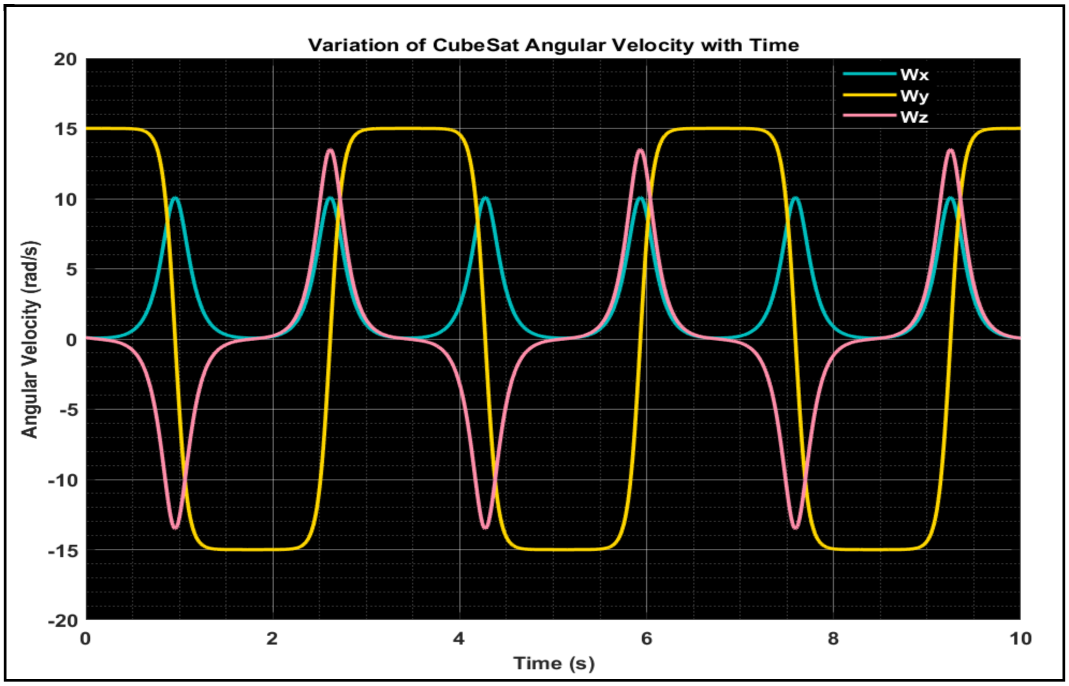
Simulating the CubeSat Rotational Dynamics in terms of Euler Angles:

After deriving the EoMs and the Kinematic Equations for the tumbling CubeSat, we obtained a system of 6 nonlinear, 1st order ordinary differential equations (ODEs), which will let us solve for the three Euler angles and the angular velocity of the satellite as a function of time.

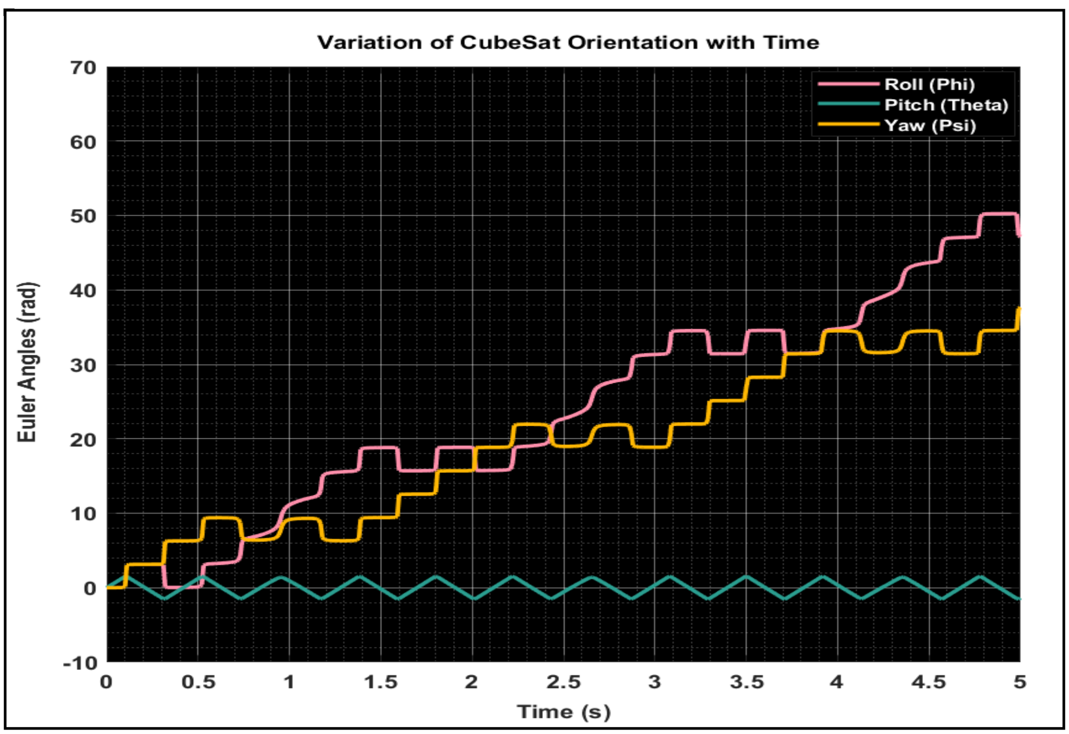
To run this simulation, we assumed the **initial conditions** at $t_0 = 0$ sec, to be as follows:

Rolling Angle, $\phi = 0^0$
Pitching Angle, $\theta = 0^0$
Yawing Angle, $\psi = 0^0$

Angular Velocity Component in $\hat{b}_1, \omega_1$= 0.1 rad/s
Angular Velocity Component in $\hat{b}_1, \omega_2$= 15 rad/s
Angular Velocity Component in $\hat{b}_1, \omega_1$= 0.1 rad/s



Applying the above initial conditions, we computed the CubeSat angular velocity and its 3D orientations over a **timespan of $t_f = 30$ secs**. Then we plotted them in a 2D space to visualize their variations over time.



Step #04: Validating the Simulation Results for the Euler Angles

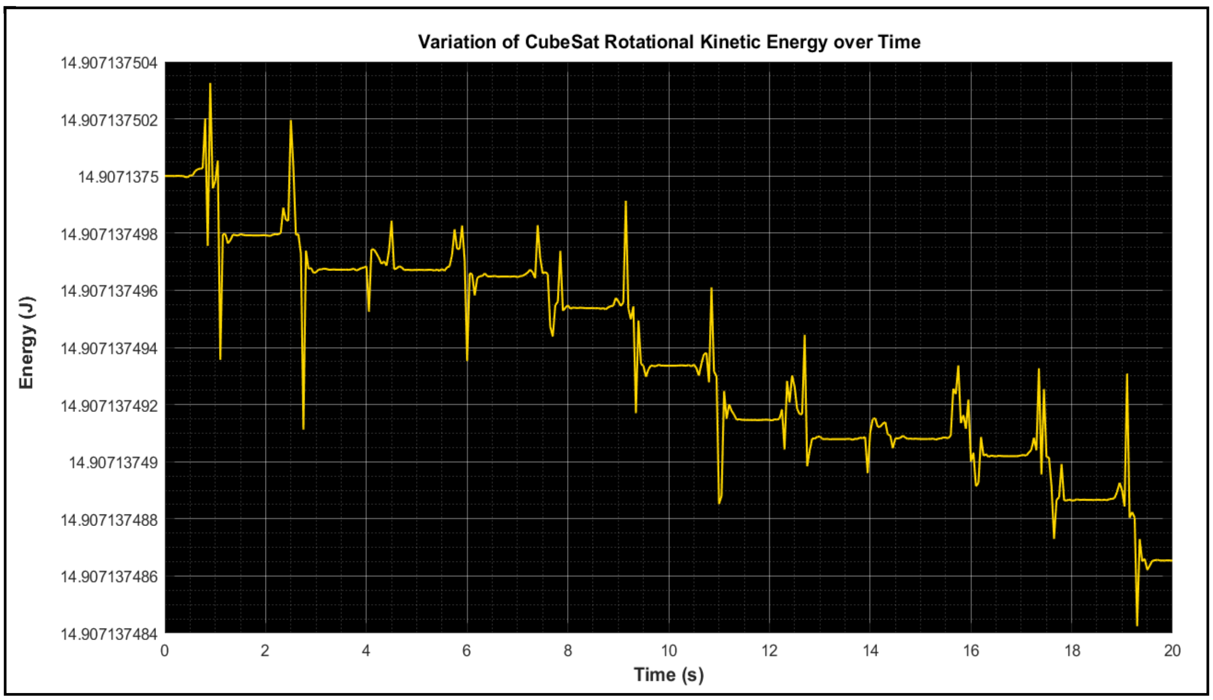
To validate the simulation results obtained from the ODE45 Solver, we have chosen to check the **principle of conservation** for **rotational kinetic energy** and **angular momentum** at every time step of the simulation. In the context of a tumbling CubeSat, ideally, its rotational kinetic energy and angular momentum remain conserved as long as no external torque or moment is applied to it. Therefore, it will be sufficient to ensure that its rotational kinetic energy and angular momentum remain constant throughout the total time window.

Verifying the Conservation Principle of Rotational Kinetic Energy :

We know that,

Rotational Kinetic Energy, $KE = \frac{1}{2} I \omega^2$

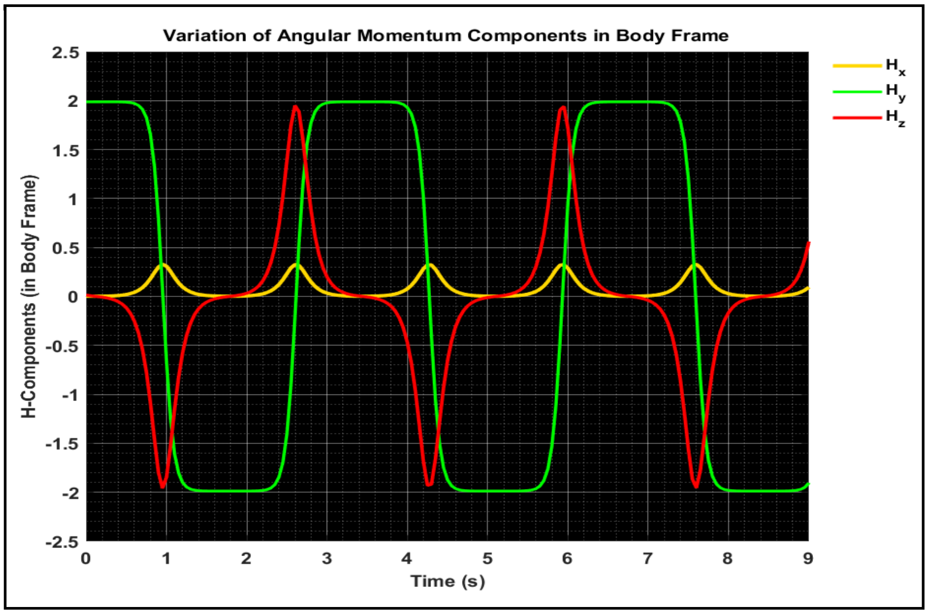
As we run a time loop to compute the rotational energy of the CubeSat for each time step, we then plot them against time in a 2D space and obtain the following graph:



From the plot, we can see that the rotational kinetic energy remains mostly **constant** for all time **up to the fifth decimal place**. However, due to the numerical integration error, small fluctuations still exist in the result, which we can overlook in this case.

Verifying the Conservation Principle of Angular Momentum Components :

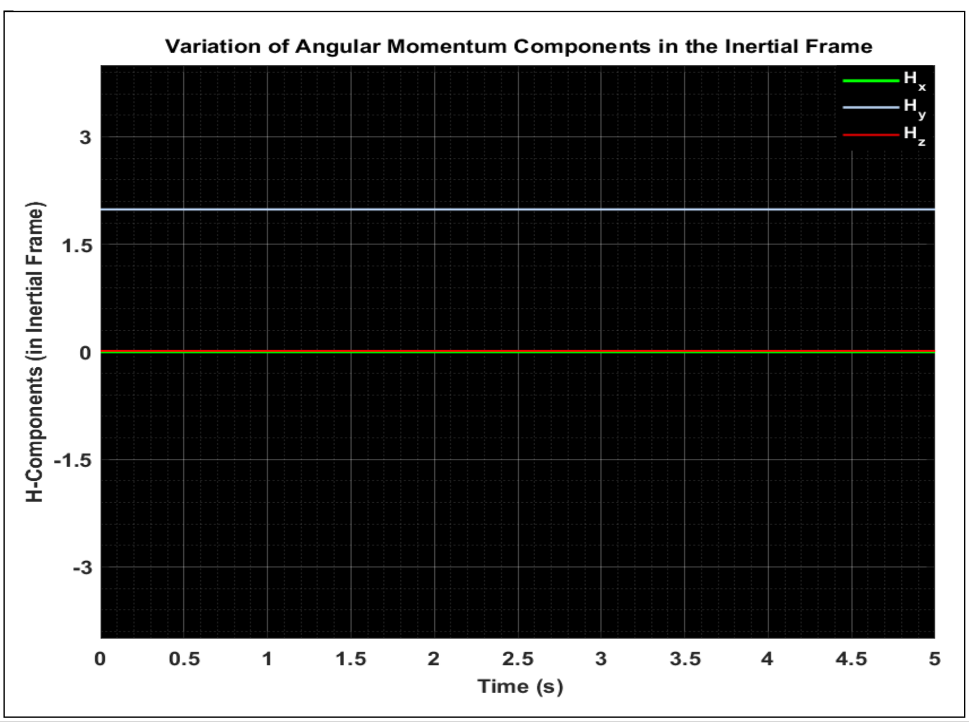
Ideally speaking, the angular momentum of a CubeSat with no external torques acting on it should remain conserved for each time step. But when we computed the components of the **angular momentum in the body frame**, they were seen to have **fluctuated** within a **range of -2 to 2**, which is very significant.



However, it is seen that transforming the **body frame components to the inertial frame** by multiplying with the DCM results in **constant angular momentum components** throughout the entire time window. Therefore, the angular momentum components in the inertial frame are obtained as follows:

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix}_N = [N_{C^B}] * \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix}_B$$

From the plot, it is seen that there is **no component** of angular momentum in the **X and Z directions** of the Inertial Frame.



Step #05: Representing the Satellite Orientation in terms of Quaternions and comparing the obtained DCM with the DCM for Euler Angles

For certain values of Euler Angles, the coefficient matrix obtained in the kinematic equation might become non-invertible, which can lead to a computational stalemate. Therefore, we can choose Euler Parameters or Quaternions to represent the 3D orientations.

Here, representing the orientations requires **4 parameters** $(\boldsymbol{q_0}, \boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3})$**,** and therefore, leads to a constraint equation.

Angular Kinematics in terms of Quaternions:

The kinematic equation in terms of quaternions can be given as follows:

$\dot{q} = \frac{1}{2} * \begin{bmatrix} 0 & -\omega^T \\ \omega & -\omega^X \end{bmatrix}$, where, ω^X is the skew-symmetric matrix, i.e., $\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \dot{q_0} \\ \dot{q_1} \\ \dot{q_2} \\ \dot{q_3} \end{bmatrix} = \frac{1}{2} * \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}_B * \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$ ---------------- **5(a-d)**

Equations 5(a-d) relate the rate of change of the quaternions to the angular velocity components in the body frame, and therefore, solving these equations along with the EoMs will allow us to compute the 3D orientations of the CubeSat in terms of the Euler Parameters.

Simulating the CubeSat Rotational Dynamics in terms of Euler Parameters:

To solve these 7 nonlinear 1st Order ODEs with the help of the ODE45 Solver, we assumed the **initial conditions** for the nominal orientation of the CubeSat to be as follows:

$q_0 = 1$	Angular Velocity Component in $\hat{b}_1, \omega_1$= 0.1 rad/s
$q_1 = 0$	Angular Velocity Component in $\hat{b}_1, \omega_2$= 15 rad/s
$q_2 = 0$	Angular Velocity Component in $\hat{b}_1, \omega_1$= 0.1 rad/s
$q_3 = 0$	

Validating the Simulation Results for the Quaternion by Comparing its DCM with the DCM of Euler Angles:

The direction cosine matrix in terms of quaternions to transform the vector components in the body frame to the inertial frame can be given as follows:

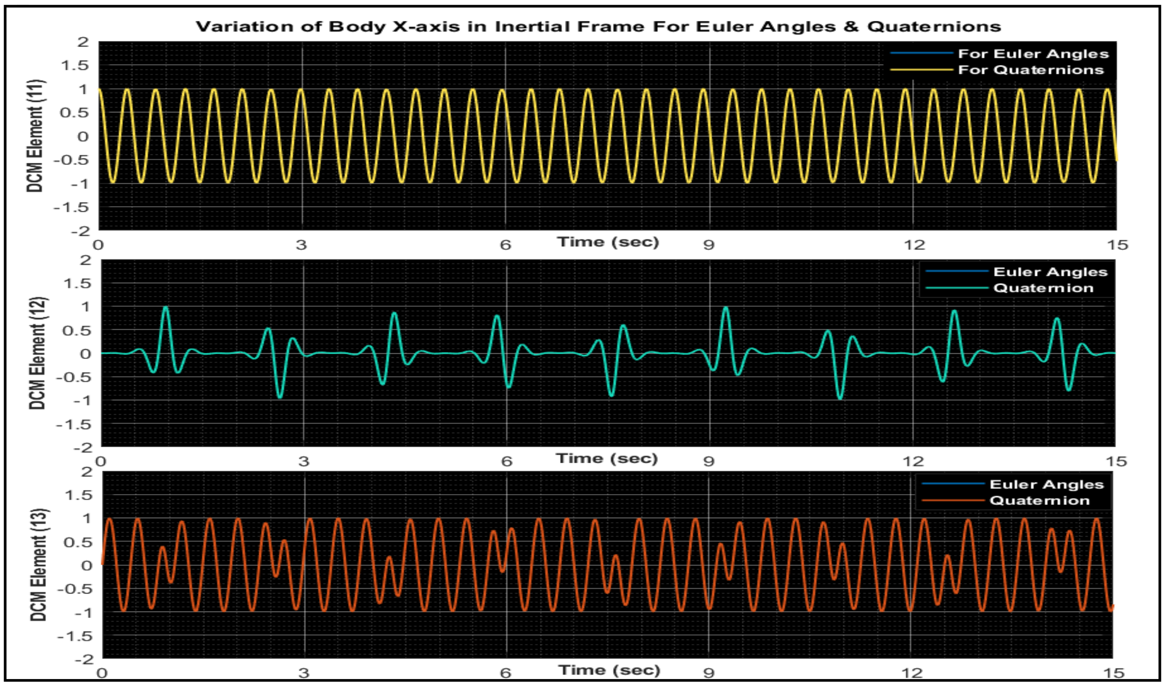
$[N_{C^B}] = (1 - 2\{\bar{q}\}^T\{\bar{q}\}) * [I] + 2 * \{\bar{q}\}\{\bar{q}\}^T + 2 * q_0 * [\bar{q}^X]$, where,

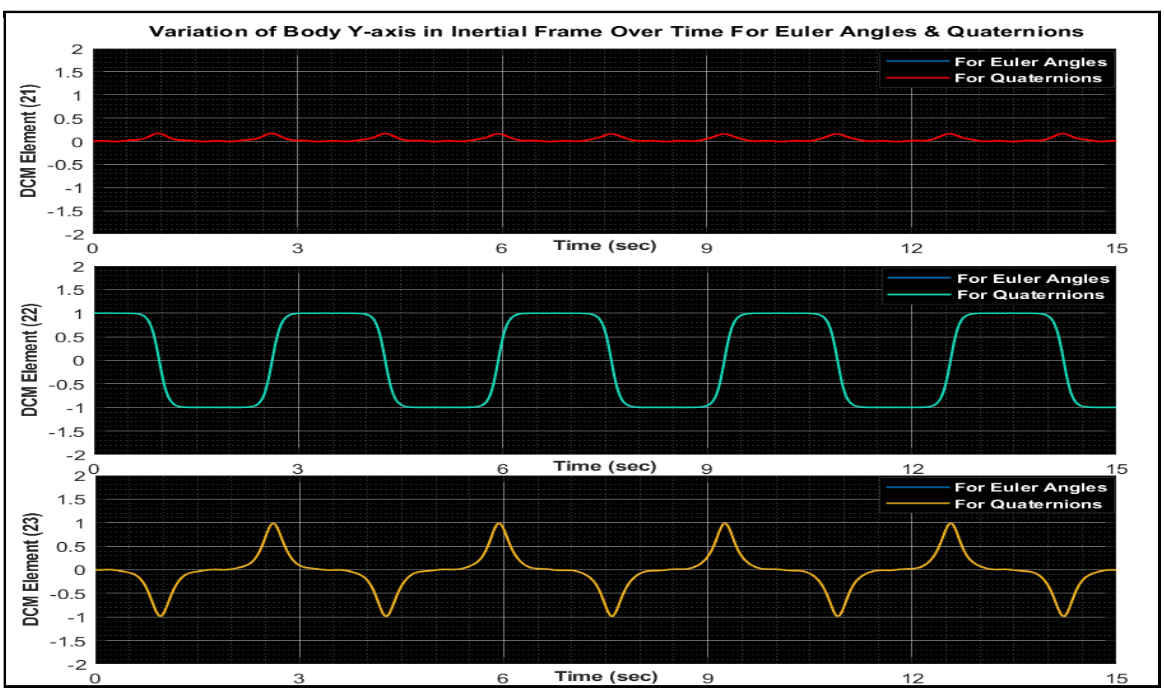
$$\{\bar{q}\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} ; [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \ [\bar{q}^X] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

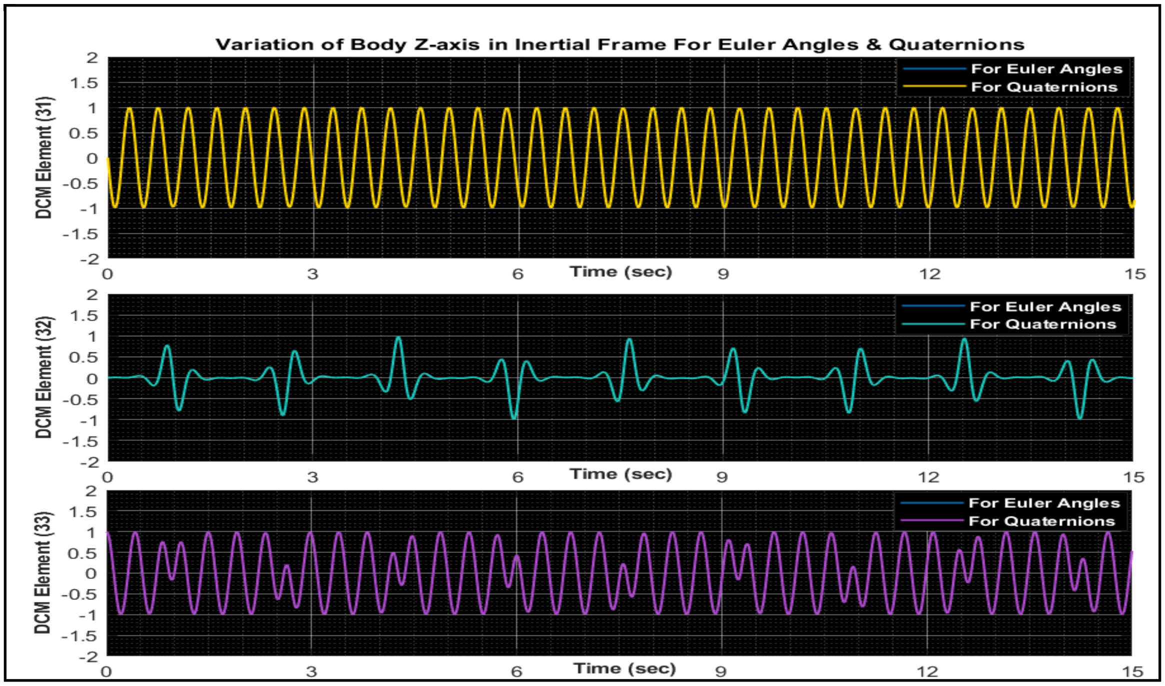
$$\Rightarrow \boxed{\left[N_{C^B}\right] = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2-q_3q_4) & 2(q_1q_3+q_2q_4) \\ 2(q_1q_2+q_3q_4) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_1q_4) \\ 2(q_1q_3-q_2q_4) & 02(q_2q_3+q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}}$$

DCM in terms of Quaternions

We computed the elements of the DCM of both Euler Angles and Quaternions at every time step. Then we plotted the two sets of DCM elements against time in a 2D space to find that both sets of DCM elements coincide with one another.







Step #06: Representing the Satellite Orientation in terms of Euler Parameters and comparing the obtained DCM with the DCM for Euler Angles

Instead of representing how Euler's Angles or Quaternions change as a function of the angular velocity of the CubeSat, in this form of representation, we are going to observe how the entire Direction Cosine Matrix of the satellite changes as a function of its angular velocity. Thus, in this approach, we will represent the satellite 3D orientations by 9 variables, which will lead to 6 constraint equations.

Angular Kinematics in terms of DCM: (Poisson's Kinematic equations)

The kinematic equation in terms of DCM can be given in the form of a matrix as follows:

$$[N_{\dot{C}^B}] = -[\omega^X] * [N_{C^B}] \quad \text{----------------} \quad \mathbf{6(a\text{-}i)}$$

Equations 6(a-i) relate the rate of change of the DCM elements to the angular velocity components in the body frame, and therefore, solving these equations along with the EoMs will allow us to compute the 3D orientations of the CubeSat in terms of the DCM.

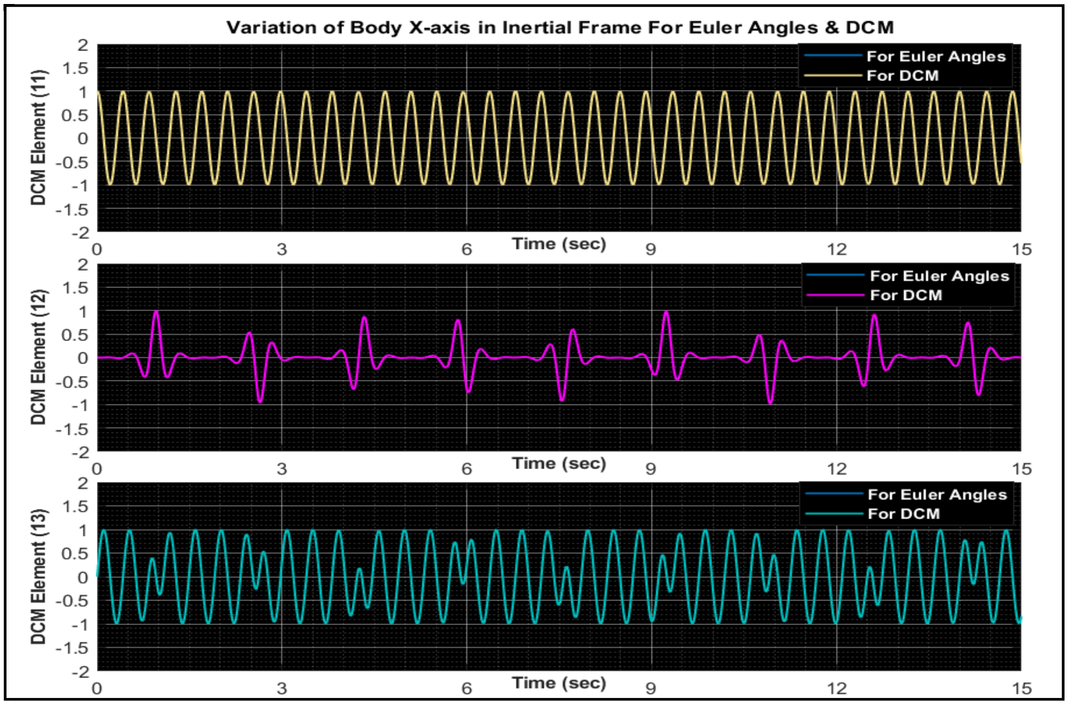
Simulating the CubeSat Rotational Dynamics in terms of DCM:

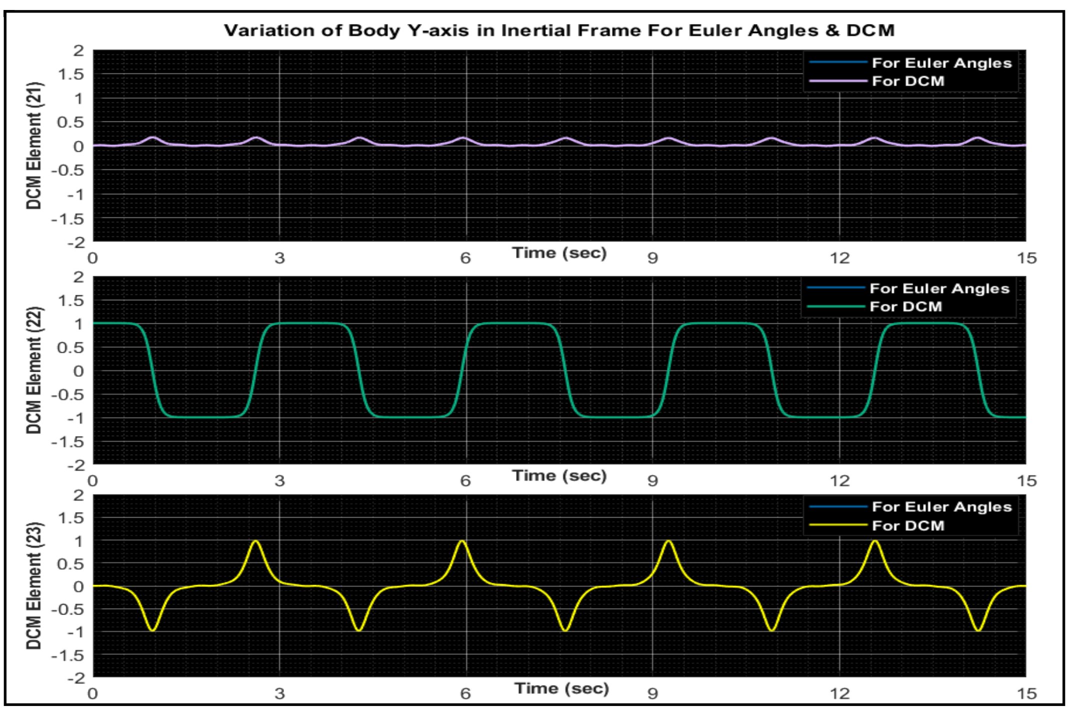
To solve these 12 nonlinear 1st Order ODEs with the help of the ODE45 Solver, we assumed the **initial conditions** for the nominal orientation of the CubeSat body frame to be as follows:

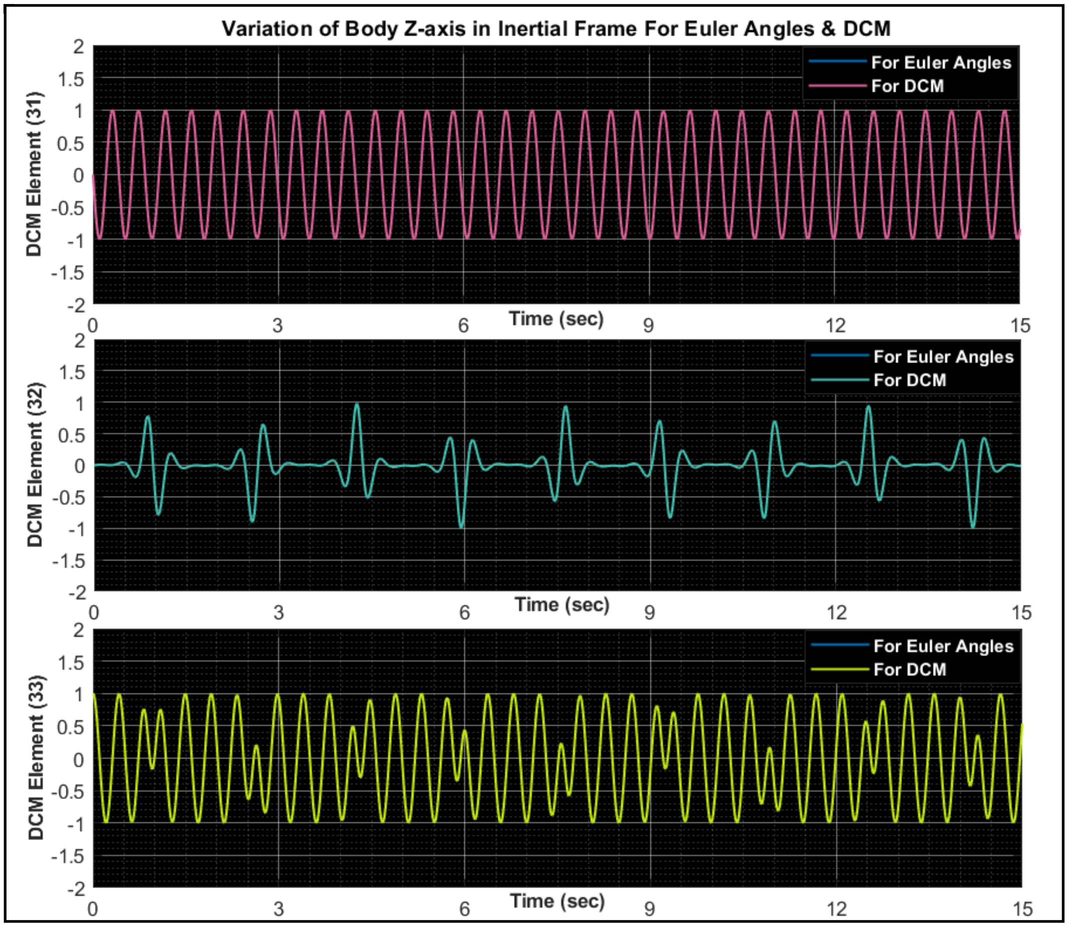
$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{Bmatrix} \omega_1(0) \\ \omega_2(0) \\ \omega_3(0) \end{Bmatrix}_B = \begin{Bmatrix} 0.1 \\ 15 \\ 0.1 \end{Bmatrix} \text{rad/s}$$

Validating the Simulation Results for the DCM by comparing it with the DCM of Euler Angles:

Solving Poisson's Kinematic equations and the rotational EoMs with the help of the ODE45 Solver directly gives us the elements of the DCM, which, when plotted in pairs with the DCM elements of Euler Angles, coincide with one another.







Animating the 3D Rotational Motion of the CubeSat:

To animate the rotational motion of the CubeSat, we will use the Direction Cosine Matrix, which will be computed at each timestep to update the orientation of the CubeSat geometry in a 3D plot. Besides, we'll also animate the rotation of the satellite fins, keeping them fixed relative to the satellite body so they rotate together.