Module 10

DISCRETE RANDOM VARIABLES

- X: a given random variable defined on some probability space $(\Omega, \mathcal{P}(\Omega), P)$;
- $F_X(\cdot)$: d.f. of X;
- D_X: the set of discontinuity points of $F_X(\cdot)$ (a countable set);

Definition 1: The r.v. X is said to be discrete if there exists a countable set S_X such that

(i)

$$P(\{X = x\}) > 0, \ \forall \, x \in S_X;$$

(ii)

$$P(\{X \in S_X\}) = \sum_{x \in S_X} P(\{X = x\}) = 1.$$

The Set S_X is called the support of random variable X.

Remark 1:

(a) For a discrete r.v. X, $S_X = D_X$; (Exercise)

(b) A r.v. X is discrete iff

$$\begin{aligned} & P(X \in D_X) && = 1 \\ \Leftrightarrow \quad & \textstyle\sum_{x \in D_X} P\left(\{X = x\}\right) && = 1 \\ \Leftrightarrow \quad & \textstyle\sum_{x \in D_X} \left(F_X\left(x\right) - F_X\left(x-\right)\right) && = 1 \\ \Leftrightarrow \quad & \text{sum of sizes of jumps} && = 1. \end{aligned}$$

(c) Let X be a discrete r.v., $D_X = \{x_1, x_2, \ldots\}$ and

$$\begin{aligned} p_i &= P(\{X = x_i\}) \\ &= F_X(x_i) - F_X(x_i -), \ i = 1, 2, \ldots. \end{aligned}$$

For simplicity let $x_1 < x_2 < x_3 < \cdots$. Then

$$\begin{aligned} F_X(x) &= P(\{X \le x\}) \\ &= P(\{X \le x, X \in D_X\}) \\ &= \sum_{\substack{t \le x \\ t \in D_X}} P(\{X = t\}). \end{aligned}$$

Clearly

$$F_X(x) = \begin{cases} 0, & \text{if } x < x_1 \\ p_1, & \text{if } x_1 \le x < x_2 \\ p_1 + p_2, & \text{if } x_2 \le x < x_3 \\ \vdots & \\ \sum_{j=1}^{i} p_j, & \text{if } x_i \le x < x_{i+1}, \ i = 1, 2, \ldots \\ \vdots & \end{cases}$$

Thus the d.f. of a discrete r.v. is a step function with jump sizes $p_1, p_2, \ldots$.

Example 1: Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{3}, & \text{if } 0 \le x < 1 \\ \frac{2}{3}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

- Clearly $F_X(\cdot)$ is right continuous, non-decreasing, $F_X(-\infty) = 0$ and $F_X(\infty) = 1$;
- $D_X = \{0, 1, 2\}$;

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$$
\begin{aligned}
P(\{X \in D_X\}) &= P(\{X=0\}) + P(\{X=1\}) \\
&\quad + P(\{X=2\}) \\
&= (F_X(0) - F_X(0-)) \\
&\quad + (F_X(1) - F_X(1-)) \\
&\quad + (F_X(2) - F_X(2-)) \\
&= \left(\frac{1}{3} - 0\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(1 - \frac{2}{3}\right) \\
&= 1.
\end{aligned}
$$

- Thus X is a discrete r.v.

Example 2: Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}.$$

- Clearly $F_X(\cdot)$ is continuous, non-decreasing, $F_X(-\infty) = 0$, and $F_X(\infty) = 1$;
- $D_X = \phi$ (the empty set);
- Thus X is not a discrete r.v..

Example 3: Let X be a r.v. with d.f.

$$F_X(x) \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \leq x < 1 \\ \frac{1}{2}, & \text{if } 1 \leq x < 2 \\ \frac{2}{3}, & \text{if } 2 \leq x < 3 \\ 1, & \text{if } x \geq 3 \end{cases} .$$

- Clearly $F_X(\cdot)$ is right continuous, non-decreasing, $F_X(-\infty) = 0$, $F_X(\infty) = 1$ and

$$D_X = \{1, 2, 3\}.$$

- $$\begin{aligned} P(\{X \in D_X\}) &= P(\{X = 1\}) + P(\{X = 2\}) \\ &\quad + P(\{X = 3\}) \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(1 - \frac{2}{3}\right) \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \neq 1 \end{aligned}$$

- Thus X is not a discrete r.v..

Definition 2: Let X be a discrete r.v. with d.f. $F_X(\cdot)$ and support S_X (so that $S_X = D_X$, $P(\{X = x\}) > 0$, $\forall\, x \in S_X$ and $\sum_{x\in S_X} P((\{X = x\}) = 1$. Define $f_X : \mathbb{R} \to \mathbb{R}$ as

$$f_X(x) = \begin{cases} P(\{X = x\}), & \text{if } x \in S_X \\ \\ 0, & \text{otherwise} \end{cases}.$$

The function $f_X(\cdot)$ is called the probability mass function (p.m.f) of r.v. X.

Remark 2:

(a) Let $f_X(\cdot)$ be the p.m.f of discrete r.v. X having support S_X. Then $f_X(x) \geq 0,\ \forall\, x \in \mathbb{R}$, $f_X(x) > 0,\ \forall\, x \in S_X$ and $\sum_{x \in S_X} f_X(x) = 1$. Conversely, it can be shown that any function $g(\cdot)$ satisfying the above properties (i.e., $g(x) \geq 0,\ \forall\, x \in \mathbb{R}$, $g(x) > 0,\ \forall\, x \in S$ and $\sum_{x \in S} g(x) = 1$, for some countable set S) is a p.m.f of some r.v. Y.

(b) For a discrete r.v. with support S_X and p.m.f $f_X(\cdot)$

$$
\begin{aligned}
F_X(x) &= P(\{X \leq x\}) \\
&= P(\{X \leq x, X \in S_X\}) \\
&= \sum_{\substack{t \leq x \\ t \in S_X}} f_X(t)
\end{aligned}
$$

and

$$f_X(x) = P(\{X = x\})$$

$$= \begin{cases} F_X(x) - F_X(x-), & \text{if } x \in S_X \\ 0, & \text{otherwise} \end{cases},$$

i.e., p.m.f. $f_X(\cdot)$ determines d.f. $F_X(\cdot)$ and conversely. Thus the probability function $P_X(\cdot)$ induced by r.v. X can be studied through p.m.f $f_X(\cdot)$.

(c) The p.m.f of a discrete r.v. is unique.

Example 4:

- $\mathcal{E}$: a fair die is tossed repeatedly and independently;
- X: No. of tosses required to get 6 for the first time;
- $S_X = \{1, 2, 3, \ldots\}$;
- For $x \in S_X$, the event $\{X = x\}$ occurs iff first $x - 1$ trials do not result in 6 and x^{th} trial results in a 6. Thus, for $x \in S_X$,

$$P(\{X = x\}) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}.$$

- The p.m.f of X is

$$f_X(x) = \begin{cases} \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

- The d.f. of X is

$$\begin{aligned} F_X(x) &= P(\{X \leq x\}) \\ &= \begin{cases} 0, & \text{if } x < 1 \\ \sum_{j=1}^{[x]} \left(\frac{5}{6}\right)^{j-1} \frac{1}{6}, & \text{if } x \geq 1 \end{cases}. \\ &= \begin{cases} 0, & \text{if } x < 1 \\ 1 - \left(\frac{5}{6}\right)^{[x]}, & \text{if } x \geq 1 \end{cases}. \end{aligned}$$

Example 5: Let X be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{1}{6}, & \text{if } 2 \leq x < 3 \\ \frac{2}{3}, & \text{if } 3 \leq x < 4 \\ \frac{5}{6}, & \text{if } 4 \leq x < 5 \\ 1, & \text{if } x \geq 5 \end{cases}.$$

$$
\begin{aligned}
D_X &= \{2,3,4,5\} \\
P(\{X=2\}) &= F_X(2) - F_X(2-) = \frac{1}{6}; \\
P(\{X=3\}) &= F_X(3) - F_X(3-) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}; \\
P(\{X=4\}) &= \frac{5}{6} - \frac{2}{3} = \frac{1}{6}; \\
P(\{X=5\}) &= 1 - \frac{5}{6} = \frac{1}{6}; \\
P(\{X \in D_X\}) &= P(\{X=2\}) + P(\{X=3\}) \\
&\quad + P(\{X=4\}) + P(\{X=5\}) \\
&= 1.
\end{aligned}
$$

Thus X is a discrete r.v. with p.m.f

$$
f_X(x) = \begin{cases} \frac{1}{6}, & \text{if } x \in \{2,4,5\} \\ \frac{1}{2}, & \text{if } x = 3 \\ 0, & \text{otherwise} \end{cases}.
$$

Take Home Problems

1. For any discrete r.v. X, show that $S_X = D_X$.

2. A fair coin is independently tossed 5 times. Let X denote the number of heads observed in 5 tosses. Find the p.m.f of X.

3. For any discrete r.v. X, show that

$$\begin{aligned} S_X &= \{x \in \mathbb{R} : F_X(x+\epsilon) - F_X(x-\epsilon) > 0, \\ &\qquad \forall\, \epsilon > 0\} \\ &= \{x \in \mathbb{R} : P(\{x-\epsilon < X \leq x+\epsilon\}) > 0, \\ &\qquad \forall\, \epsilon > 0\}. \end{aligned}$$

Abstract of Next Module

We defined discrete r.v.s through a property of d.f $F_X(\cdot)$. In next module we will introduce:

(a) Continuous r.v s ($F_X(\cdot)$ is continuous everywhere);

(b) Absolutely continuous r.v.s ($F_X(\cdot)$ is a definite integral of some non-negative function).

Thank you for your patience