# Module 4

## PROPERTIES OF PROBABILITY FUNCTION

- $(\Omega, \mathcal{P}(\Omega), P)$: a given probability space;
- For an event $E \in \mathcal{P}(\Omega)$, let $E^c = \Omega - E$ denote its complementary event (set of outcomes not favorable to $E$).

**Result 1**: For any event $E$

$$0 \leq P(E) \leq 1 \text{ and } P(E^c) = 1 - P(E).$$

**Proof:** Since $\Omega = E \cup E^c$ and, $E$ and $E^c$ are disjoint

$$\begin{aligned} P(\Omega) &= P(E \cup E^c) \\ 1 &= P(E) + P(E^c) \qquad \text{(Axiom 2 and Axiom 3)} \\ &\geq P(E) \geq 0. \qquad \text{(Axiom 1)} \end{aligned}$$

Thus $0 \leq P(E) \leq 1$. Since $0 \leq P(E) \leq 1 < \infty$, we also have $P(E^c) = 1 - P(E)$).

**Result 2**: Let $E$ and $F$ be events such that $E \subseteq F$. Then

$$P(E) \leq P(F) \quad \text{(monotonicity of probability function)}$$

and

$$P(F - E) = P(F) - P(E).$$

Proof: Let $E$, $F \in \mathcal{P}(\Omega)$ and let $E \subseteq F$. Since $F = E \cup (F - E)$ and, $E$ and $F - E$ are disjoint, we have

$$\begin{aligned} P(F) &= P(E \cup (F - E)) \\ &= P(E) + P(F - E) \qquad \text{(Axiom 2)} \\ &\geq P(E). \qquad \text{(Axiom 1)} \end{aligned}$$

Since $0 \leq P(E) \leq 1 < \infty$, we have

$$P(F - E) = P(F) - P(E).$$

**Result 3**: Let $E_1$ and $E_2$ be two events (not necessarily disjoint). Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

**Proof**: Since

$$E_1 \cup E_2 = E_1 \cup (E_2 - E_1),$$

and, $E_1$ and $E_2 - E_1$ are disjoint, we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2 - E_1).$$

Also, $E_2 - E_1 = E_2 - (E_1 \cap E_2)$ and $E_1 \cap E_2 \subseteq E_2$. Therefore,

$$\begin{aligned} P(E_2 - E_1) &= P(E_2) - P(E_1 \cap E_2) \qquad \text{(Result 2)} \\ \Longrightarrow \quad P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cup E_2). \end{aligned}$$

**Result 4 (Inclusion Exclusion Formula):** For events $E_1, E_2, \ldots, E_n$ $(n \geq 2)$, let

$$\begin{aligned}
p_{1,n} &= \sum_{i=1}^{n} P\left(E_i\right) \\
p_{2,n} &= \sum\sum_{1 \leq i < j \leq n} P\left(E_i \cap E_j\right) \\
&\vdots \\
p_{r,n} &= \sum\sum\cdots\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}), \quad r = 2, \ldots, n \\
p_{n,n} &= P(E_1 \cap E_2 \cap \cdots \cap E_n).
\end{aligned}$$

Then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = p_{1,n} - p_{2,n} + p_{3,n} + \cdots + (-1)^{n-1} p_{n,n}. \tag{1}$$

Proof (By mathematical induction)
The result is clearly true for $n = 2$ (Result 3). Assume that equation (1) holds for any collection of $m$ events. Then

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{m+1} E_i\right) &= P\left(\left(\bigcup_{i=1}^{m} E_i\right) \cup E_{m+1}\right) \\
&= P\left(\bigcup_{i=1}^{m} E_i\right) + P\left(E_{m+1}\right) - P\left(\left(\bigcup_{i=1}^{m} E_i\right) \cap E_{m+1}\right) \\
&= P\left(\bigcup_{i=1}^{m} E_i\right) + P\left(E_{m+1}\right) - P\left(\bigcup_{i=1}^{m} \left(E_i \cap E_{m+1}\right)\right) \\
&= p_{1,m} - p_{2,m} + p_{3,m} \cdots + (-1)^{r-1} p_{r,m} + \ldots + (-1)^{m-1} p_{m,m} + P\left(E_{m+1}\right) \\
&\quad - \left(t_{1,m} - t_{2,m} + t_{3,m} \cdots + (-1)^{r-2} t_{r-1,m} + \cdots (-1)^{m-1} t_m\right),
\end{aligned}
$$

where, for $r = 1, \ldots, m$,

$$
\begin{aligned}
p_{r,m} &= \sum\sum_{1 \le i_1 < i_2 < \cdots < i_r \le m} \cdots \sum P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) \\
t_{r,m} &= \sum\sum_{1 \le i_1 < i_2 < \cdots < i_r \le m} \cdots \sum P((E_{i_1} \cap E_{m+1}) \cap (E_{i_2} \cap E_{m+1}) \cap \cdots \cap (E_{i_r} \cap E_{m+1})) \\
&= \sum\sum_{1 \le i_1 < i_2 < \cdots < i_r \le m} \cdots \sum P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r} \cap E_{m+1}).
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
p_{1,m} + P\left(E_{m+1}\right) &= p_{1,m+1} \\
p_{2,m} + t_{1,m} &= p_{2,m+1} \\
p_{r,m} + t_{r-1,m} &= p_{r,m+1}, \quad r = 2, 3, ..., m \\
t_{m,m} &= p_{m+1,m+1}.
\end{aligned}
$$

Thus

$$
P\left(\bigcup_{i=1}^{m+1} E_i\right) = p_{1,m+1} - p_{2,m+1} + p_{3,m+1} + \cdots + (-1)^m p_{m+1,m+1}.
$$

**Result 5**: For events $E_1$, $E_2 \ldots$, $E_n$

(i)

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i); \qquad \text{(Boole's inequality)}$$

(ii)

$$P\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} P(E_i) - (n-1). \qquad \text{(Bonferroni's inquality)}$$

**Proof:** (i) We have

$$\bigcup_{i=1}^{n} E_i = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap E_1^c \cap E_2^c) \cup \cdots \cup \left(E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c\right),$$

where $E_1, (E_2 \cap E_1^c), (E_3 \cap E_1^c \cap E_2^c), \ldots, (E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c)$ are disjoint events. Also $E_r \cap E_1^c \cap E_2^c \cap \cdots \cap E_{r-1}^c \subseteq E_r,\ r = 2, \ldots, n$.
Thus

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} E_i\right) &= P(E_1) + P(E_2 \cap E_1^c) + (E_3 \cap E_1^c \cap E_2^c) + \cdots + P\left(E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c\right) \\ &\leq P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n). \end{aligned}$$

(ii)

$$\begin{aligned}
P\left(\bigcap_{i=1}^{n} E_i\right) &= P\left(\left(\bigcup_{i=1}^{n} E_i^c\right)^c\right) \\
&= 1 - P\left(\bigcup_{i=1}^{n} E_i^c\right) \\
&\geq 1 - \sum_{i=1}^{n} P\left(E_i^c\right) \qquad \text{(using Boole's inequality)} \\
&= 1 - \sum_{i=1}^{n} \left(1 - P\left(E_i\right)\right) \\
&= 1 - n + \sum_{i=1}^{n} P\left(E_i\right) \\
&= \sum_{i=1}^{n} P\left(E_i\right) - (n-1).
\end{aligned}$$

**Remark 1:** : Under the notation of Result 4, it can be shown that

$$\begin{aligned}
p_{1,n} - p_{2,n} &\leq P\left(\bigcup_{i=1}^{n} E_i\right) \leq p_{1,n}; \\
p_{1,n} - p_{2,n} + p_{3,n} - p_{4,n} &\leq P\left(\bigcup_{i=1}^{n} E_i\right) \leq p_{1,n} - p_{2,n} + p_{3,n}; \\
&\vdots \\
\sum_{i=1}^{2r} (-1)^{i-1} p_{i,n} &\leq P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{2r-1} (-1)^{i-1} p_{i,n}, \quad r = 1, 2, \ldots.
\end{aligned}$$

**Take Home Problems:**

(1) Let $E$ be an event such that $P(E) = 1$. Show that

$$P(F) = P(E \cap F), \qquad \forall\ F \in \mathcal{P}(\Omega).$$

(2) Let $E$ be an event such that $P(E) = 0$. Show that

$$P(F) = P(E \cup F) \qquad \forall\ F \in \mathcal{P}(\Omega).$$

(3) If $P(E_i) = 1,\ i = 1, \ldots, n$, show that

$$P\left(\bigcap_{i=1}^{n} E_i\right) = 1.$$

(4) If $P(E_i) = 0,\ i = 1, \ldots, n$, show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) = 0.$$

# Abstract of Module 5

We will discuss:

- Continuity of probability function.
- Equally likely probability models.

# Thank you for your patience