Module 32

Limiting Distributions

- $\underline{T} = (T_1, \ldots, T_n)$: a r.v. having joint p.m.f./p.d.f. $f_{\underline{T}}(\cdot)$;

- $h : \mathbb{R}^n \to \mathbb{R}$: a given function;

- Distribution of $X_n = h(\underline{T})$ is desired;

- Very often it is not possible to derive the expression for distribution (i.e., p.m.f. or p.d.f.) of X_n.

Example 1.

Let $T_1, \ldots, T_n$ be a random sample from $\mathrm{B}(a, b)$ distribution (beta distribution), where $-\infty < a < b < \infty$. Let $\bar{T}_n = \frac{1}{n} \sum_{i=1}^{n} T_i$. The form of p.d.f. (or d.f.) of $\bar{T}_n$ is so complicated (it involves multiple integrals which cannot be expressed in closed form) that hardly anybody would be interested in using it. It will be helpful to approximate distribution of $\bar{T}_n$ by a distribution which is mathematically tractable.

Convergence in Distribution and Probability

- $\{X_n\}_{n\geq 1}$: a sequence of r.v.s;
- F_n: d.f. of X_n, $n = 1, 2, \ldots$;
- An approximation to distribution X_n (i.e., F_n) is desired for large values of n (i.e., as $n \to \infty$);
- it may be tempting to approximate $F_n(\cdot)$ by

$$F(x) = \lim_{n\to\infty} F_n(x), \ x \in \mathbb{R}.$$

- **Question:** If F_n's are d.f.s, does

$$F(x) = \lim_{n\to\infty} F_n(x), \ x \in \mathbb{R}$$

define a d.f.?

Answer: No.

Example 2.

(i) Suppose that

$$P\left(X_n = \frac{1}{n}\right) = 1,\ n = 1, 2, \ldots .$$

Then,

$$F_n(x) = P(X_n \leq x) = \begin{cases} 0, & \text{if } x < \frac{1}{n} \\ 1, & \text{if } x \geq \frac{1}{n} \end{cases}.$$

and

$$F(x) = \lim_{n \to \infty} = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$$

is not a d.f. (it is not right continuous at $x = 0$).

However, F can be converted to a d.f.

$$F^*(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

by changing value of F at $x = 0$.

Since $\lim_{n \to \infty} \frac{1}{n} = 0$, a natural approximation of F_n seems to be the d.f. of r.v. X degenerate at 0 (i.e., F^*).

(ii) Let $X_n \sim \mathrm{N}(0, \frac{1}{n})$, $n = 1, 2, \ldots$. Then

$$F_n(x) = P(X_n \leq x) = \Phi(\sqrt{n}x)$$

$$F(x) = \lim_{n\to\infty} F_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

Clearly, F is not a d.f. (it is not right continuous at $x = 0$). However, F can be converted to a d.f.

$$F^*(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

by changing value of F at $x = 0$. Since $E(X_n) = 0$ and $\mathrm{Var}(X_n) = \frac{1}{n} \to 0$, as $n \to \infty$, a natural approximation of F_n seems to be the d.f. of r.v. X degenerate at 0 (i.e., F^*).

(iii) Suppose that

$$P(X_n = 0) = 1 - P(X_n = n) = \frac{1}{n}, n = 1, 2, \ldots .$$

Then,

$$F_n(x) = P(X_n \leq x) = \begin{cases} 0, & \text{if } x < n \\ 1, & \text{if } x \geq n \end{cases}.$$

and

$$F(x) = \lim_{n \to \infty} F_n(x) = 0, \forall x \in \mathbb{R}.$$

Here, $F(x)$ cannot be converted to a d.f. by changing its value at countable number of points.

- The above examples suggest that if a sequence $\{F_n\}_{n\geq 1}$ of d.f.s converges at every point then it may be two restrictive to require that $\{F_n\}_{n\geq 1}$ converges to a d.f. (i.e., to require that $\lim_{n\to\infty} F_n(x) = F(x),\ \forall x \in \mathbb{R}$, for some d.f. F.)

Definition 1:

Let X be a r.v. with d.f. F.

(a) The sequence $\{X_n\}_{n\geq 1}$ is said to converge in distribution to X, as $n \to \infty$ (written as $X_n \xrightarrow{d} X$, as $n \to \infty$), if

$$\lim_{n\to\infty} F_n(x) = F(x), \ \forall \ x \in C_F,$$

where C_F is the set of continuity points of F. The d.f. F (or, the corresponding p.m.f./p.d.f.) is called the limiting distribution of X_n, as $n \to \infty$.

(b) Let $c \in \mathbb{R}$. The sequence $\{X_n\}_{n\geq 1}$ is said to converge in probability to c, as $n \to \infty$ (written as $X_n \xrightarrow{p} c$, as $n \to \infty$), if $X_n \xrightarrow{d} X$, as $n \to \infty$, where X is a r.v. degenerate at c (i.e., $P(X = c) = 1$).

Remark 1:

(i) Since C_F^c is countable,

$$X_n \xrightarrow{d} X, \text{ as n } \to \infty \Rightarrow \lim_{n\to\infty} F_n(x) = F(x),$$

everywhere except possibly at a countable number of points.

(ii) For $c \in \mathbb{R}$,

$$X_n \xrightarrow{p} c, \text{ as n } \to \infty \Leftrightarrow \text{For } x \neq c, \lim_{n\to\infty} F_n(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x > c \end{cases}.$$

(iii) If r.v. X is continuous, then

$$X_n \xrightarrow{d} X \Leftrightarrow \lim_{n\to\infty} F_n(x) = F(x), \ \forall\, x \in \mathbb{R};$$

(iv) For $c \in \mathbb{R}$,

$$X_n \xrightarrow{p} c \Leftrightarrow X_n - c \xrightarrow{p} 0.$$

Example 3

Let

$$P(X_n = 0) = 1 - P(X_n = n) = \frac{1}{n}, \; n = 1, 2, \ldots .$$

Then,

$$F_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{n}, & \text{if } 0 \leq x < \frac{1}{n} \\ 1, & \text{if } x \geq \frac{1}{n}. \end{cases}$$

$$\xrightarrow{n \to \infty} \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0. \end{cases} \quad \text{(not a d.f.)}$$

Clearly, $X_n \xrightarrow{p} 0$.

Example 4

Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with $X_1 \sim \mathrm{U}(0, 1)$, $\theta > 0$. Find limiting distribution of $X_{n:n} = \max\{X_1, \ldots, X_n\}$ and $Y_n = n(\theta - X_{n:n})$.

Solution For $x \in \mathbb{R}$,

$$\begin{aligned} F_{X_{n:n}}(x) &= P(\max\{X_1, \ldots, X_n\} \leq x) \\ &= P(X_i \leq x, i = 1, \ldots, n) \\ &= \prod_{i=1}^{n} P(X_i \leq x) \\ &= \Big[F(x)\Big]^n, \end{aligned}$$

where $F(x)$ is the d.f. of X.

We have

$$F(x) = P(X_1 \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{\theta}, & \text{if } 0 \leq x < \theta \\ 1, & \text{if } x \geq \theta \end{cases}.$$

Thus,

$$F_{X_{n:n}}(x) = \begin{cases} 0, & \text{if } x < 0 \\ \left(\frac{x}{\theta}\right)^n, & \text{if } 0 \leq x < \theta \\ 1, & \text{if } x \geq \theta \end{cases}$$

$$\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } x < \theta \\ 1, & \text{if } x \geq \theta \end{cases}$$

$$\Rightarrow X_{n:n} \xrightarrow{p} \theta.$$

Also,

$$
\begin{aligned}
F_{Y_n}(x) &= P\Big(n(\theta - X_{n:n}) \le x\Big) \\
&= P\Big(X_{n:n} \ge \theta - \frac{x}{n}\Big) \\
&= 1 - F_{X_{n:n}}\Big(\theta - \frac{x}{n}\Big) \\
&= \begin{cases} 0, & \text{if } x \le 0 \\ 1 - \left(1 - \frac{x}{n\theta}\right)^n, & \text{if } 0 < x < n\theta \\ 1, & \text{if } x > n\theta. \end{cases} \\
&\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } x \le 0 \\ 1 - e^{-\frac{x}{\theta}}, & \text{if } x > 0 \end{cases} . (\text{d.f. of Exp}(\theta)) \\
&= G(x), \text{say}
\end{aligned}
$$

Thus, $X_n \xrightarrow{d} X$, where $X \sim \text{Exp}(\theta)$.

Result 1.

Let $\{X_n\}_{n\geq 1}$ be a sequence of r.v.s and let c be a real constant. Then

$$X_n \xrightarrow{p} c, \text{ as n } \to \infty \ \Leftrightarrow \ \forall \ \epsilon > 0, \ \lim_{n\to\infty} P\Big(|X_n - c| \geq \epsilon\Big) = 0,$$

Proof:

Suppose that $X_n \xrightarrow{p} c$, as n $\to \infty$, i.e., $\lim\limits_{n\to\infty} F_n(x) = F(x), \forall \ x \neq c$, where

$$F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}.$$

Fix $\epsilon > 0$. Then

$$\begin{aligned}
\lim_{n\to\infty} P\Big(|X_n - c| \geq \epsilon\Big) &= \lim_{n\to\infty} \Big[P(X_n \leq c - \epsilon) + P(X_n \geq c + \epsilon)\Big] \\
&= \lim_{n\to\infty} \Big[F_n(c - \epsilon) + 1 - F_n((c + \epsilon)-)\Big] \\
&= F(c - \epsilon) + 1 - F((c + \epsilon)-) \\
&= 0 + 1 - 1 \ = \ 0.
\end{aligned}$$

Conversely suppose that

$$
\begin{aligned}
& \lim_{n\to\infty} P\Big(|X_n - c| \geq \epsilon\Big) = 0,\ \forall \epsilon > 0 \\
& \Rightarrow \lim_{n\to\infty} \Big[F_n(c-\epsilon) + 1 - F_n((c+\epsilon)-)\Big] = 0,\ \forall \epsilon > 0 \\
& \Rightarrow \lim_{n\to\infty} F_n(c-\epsilon) = 0,\ \forall \epsilon > 0 \text{ and } \lim_{n\to\infty} F_n((c+\epsilon)-) = 1,\ \forall \epsilon > 0 \\
& \Rightarrow \lim_{n\to\infty} F_n(c-\epsilon) = 0,\ \forall \epsilon > 0 \text{ and } \lim_{n\to\infty} F_n(c+\epsilon) = 1,\ \forall \epsilon > 0 \\
& \qquad\qquad\qquad\qquad (\text{since } F_n(c+\epsilon) \geq F_n(c+\epsilon)-) \\
& \Rightarrow \lim_{n\to\infty} F_n(x) = 0,\ \forall x < c \text{ and } \lim_{n\to\infty} F_n(x) = 1,\ \forall x > c \\
& \Rightarrow \forall x \neq c, \lim_{n\to\infty} F_n(x) = F(x),
\end{aligned}
$$

where

$$
F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}
$$

. Thus, $X_n \xrightarrow{d} X$.

Remark 2.

Above result suggests that if $X_n \xrightarrow{p} c$, as $n \to \infty$, then X_n is stochastically (in probability) very close to c for large values of n. Such an interpretation does not hold for the concept of convergence in distribution. Specifically, if $X_n \xrightarrow{d} X$, as $n \to \infty$ (where X is some non-degenerate r.v.), then it can not be inferred that X_n is getting close to X (as $n \to \infty$) in any sense. All we know in that case is, for large values of n, the distribution of X_n is getting close to that of X.

Result 2.

Let $\{X_n\}_{n\geq 1}$ be a sequence of r.v.s with $E(X_n) = \mu_n \in (-\infty, \infty)$ and $\text{Var}(X_n) = \sigma_n^2 \in (0, \infty)$, $n = 1, 2, \ldots$. Suppose that $\lim_{n\to\infty} \mu_n = \mu \in \mathbb{R}$, and $\lim_{n\to\infty} \sigma_n^2 = 0$. Then $X_n \xrightarrow{p} \mu$, as $n \longrightarrow \infty$.

Proof.

Fix $\epsilon > 0$. Then

$$0 \leq P\Big(|X_n - \mu| \geq \epsilon\Big) \leq \frac{E\big((X_n - \mu)^2\big)}{\epsilon^2}.$$

Also,

$$\begin{aligned} E\big((X_n - \mu)^2\big) &= E\big((X_n - \mu_n + \mu_n - \mu)^2\big) \\ &= E\big((X_n - \mu_n)^2\big) + (\mu_n - \mu)^2 \\ &= \sigma_n^2 + (\mu_n - \mu)^2. \end{aligned}$$

Then,

$$
\begin{aligned}
0 \leq P\Big(|X_n - \mu| \geq \epsilon\Big) &\leq \frac{\sigma_n^2 + (\mu_n - \mu)^2}{\epsilon^2} \\
&\xrightarrow{n\to\infty} 0 \\
\Rightarrow \lim_{n\to\infty} P\Big(|X_n - \mu| \geq \epsilon\Big) &= 0 \\
\Rightarrow X_n \xrightarrow{p} \mu,\ \text{as } n \to \infty.
\end{aligned}
$$

We state the following useful result without providing it proof.

Result 3.

Let $\{X_n\}_{n\geq 1}$ be a sequence of r.v.s with m.g.f.s $\{M_n(\cdot)\}_{n\geq 1}$ and let X be another r.v. with m.g.f. $M(\cdot)$. Suppose there exists an $h > 0$ such that $M_n(\cdot)$, $n = 1, 2, \ldots$ and $M(\cdot)$ are finite on $(-h, h)$ and

$$\lim_{n\to\infty} M_n(t) = M(t), \ \forall t \in (-h, h).$$

Then $X_n \xrightarrow{d} X$.

Lemma 1.

Let $\{c_n\}_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = c$. Then,

$$\lim_{n\to\infty}\left(1+\frac{c_n}{n}\right)^n = e^c.$$

$\Big($ Hint: For any $x \in \mathbb{R}$, $\ln(1+x) = \frac{x}{1+\xi_x}$, for some ξ_x between 0 and x. $\Big)$

Result 4 (Poisson Approximation to Binomial Distribution)

Let $X_n \sim \text{Bin}(n, \theta_n)$, where $\theta_n \in (0,1)$, $n = 1, 2, \ldots$ and $\lim\limits_{n\to\infty} (n\theta_n) = \theta > 0$. Then $X_n \xrightarrow{d} X$, where $X \sim \text{Poisson}(\theta)$, the Poisson distribution with mean θ.

Proof. We know that the m.g.f. of X is

$$M(t) = e^{\theta(e^t - 1)}, \ t \in \mathbb{R}$$

and the m.g.f. of X_n $(n = 1, 2, \ldots)$ is

$$\begin{aligned} M_n(t) &= (1 - \theta_n + \theta_n e^t)^n \\ &= \left(1 + \frac{c_n(t)}{n}\right)^n, t \in \mathbb{R}, \end{aligned}$$

where

$$c_n(t) = n\theta_n(e^t - 1), \ t \in \mathbb{R}$$
$$\stackrel{n\to\infty}{\longrightarrow} \theta(e^t - 1), \ t \in \mathbb{R}.$$

Thus,

$$\lim_{n\to\infty} M_n(t) = e^{\theta(e^t-1)}, \ t \in \mathbb{R}$$

$$\Rightarrow X_n \xrightarrow{d} X.$$

Result 5 (Weak Law of Large Numbers (WLLN) and the Central Limit Theorem (CLT))

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. r.v.s and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i,\ n = 1, 2, \ldots$

(i) **(WLLN)** Suppose that $E(X_1) = \mu$ is finite. Then

$$\bar{X}_n \xrightarrow{p} \mu, \text{as } n \to \infty,$$

(ii) **(CLT)** Suppose that $0 < \mathrm{Var}(X_1) = \sigma^2 < \infty$. Then

$$Z_n \stackrel{\text{def}}{=} \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0, 1), \text{as } n \to \infty.$$

Proof.

(i) For simplicity we will assume that $\text{Var}(X_1) = \sigma^2 < \infty$. Then

$$E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(X_1) = \mu,$$

$$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \frac{\sigma^2}{n} \to \infty, \text{as } n \to \infty.$$

Thus, $\bar{X}_n \xrightarrow{p} \mu$, by Result 2.

(ii) For simplicity assume that the common m.g.f. $M(\cdot)$ of $X_1, X_2, \dots$ is finite in an interval $(-h, h)$, for some $h > 0$. Let $Y_i = \frac{X_i - \mu}{\sigma}$, $i = 1, 2, \dots$, so that $\{Y_n\}_{n \geq 1}$ is a sequence of i.i.d. r.v.s with $E(Y_1) = 0$ and $\text{Var}(Y_1) = 1$. Also,

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu + \sigma\frac{1}{n}\sum_{i=1}^{n} Y_i = \mu + \sigma\bar{Y}_n$$

$$Z_n = \sqrt{n}\bar{Y}_n, n = 1, 2, \dots, \text{where } \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

The common m.g.f. of $Y_1, Y_2, \dots$ is

$$\begin{aligned} M_Y(t) &= E(e^{\frac{t(X_1-\mu)}{\sigma}}) \\ &= e^{-\frac{\mu t}{\sigma}} M_{X_1}(\frac{t}{\sigma}), \ -h\sigma < t < h\sigma. \end{aligned}$$

Then

$$M_Y^{(1)}(0) = E(Y_1) = 0$$
$$M_Y^{(2)}(0) = E(Y_1^2) = 1.$$

Let $\psi_2 : (-h\sigma, h\sigma) \to \mathbb{R}$ be such that

$$M_Y(t) = M_Y(0) + tM_Y^{(1)}(0) + \frac{t^2}{2}\left(M_Y^{(2)}(0) + \psi_2(t)\right), \ -h\sigma < t < h\sigma;$$

i.e., for $t \in (-h\sigma, h\sigma)$

$$\psi_2(t) = \frac{M_Y(t) - M_Y(0) - tM_Y^{(1)}(0)}{t^2/2} - M_Y^{(2)}(0),$$

$$\begin{aligned}
\Rightarrow \lim_{t\to 0} \psi_2(t) &= \lim_{t\to 0} \frac{M_Y^{(1)}(t) - M_Y^{(1)}(0)}{t} - M_Y^{(2)}(0), \ \text{(L' Hospital Rule)} \\
&= M_Y^{(2)}(0) - M_Y^{(2)}(0) \\
&= 0
\end{aligned}$$

The m.g.f. of $Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$ is

$$\begin{aligned}
M_{Z_n}(t) &= E\left(e^{\frac{t}{\sqrt{n}} \sum_{i=1}^{n} Y_i}\right) \\
&= \prod_{i=1}^{n} M_Y\left(\frac{t}{\sqrt{n}}\right) \\
&= \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n \\
&= \left[M_Y(0) + \frac{t}{\sqrt{n}} M_Y^{(1)}(0) + \frac{t^2}{2n}\left(M_Y^{(2)}(0) + \psi_2\left(\frac{t}{\sqrt{n}}\right)\right)\right]^n \\
&= \left[1 + \frac{t^2}{2n}\left(1 + \psi_2\left(\frac{t}{\sqrt{n}}\right)\right)\right]^n, \; t \in (-\sqrt{n}h\sigma, \sqrt{n}h\sigma).
\end{aligned}$$

Clearly,

$$\lim_{n\to\infty} M_{Z_n}(t) = e^{\frac{t^2}{2}} = K(t),\ \forall t \in \mathbb{R}.$$

Note that $K(t), t \in \mathbb{R}$, is the m.g.f. of $Z \sim N(0,1)$. Now the assertion follows from Result 3.

Remark 3:

(i) The WLLN implies that the sample mean based on a random sample from any parent distribution can be made arbitrary close to the population mean (in probability) by choosing sufficiently large sample sizes.

(ii) The CLT states that, irrespective of the nature of the parent distribution, the probability distribution of a normalized version of the sample mean, based on a random sample of large size, is approximately normal.

Justification of Relative Frequency Method of Assigning Probabilities

- $(\Omega, \mathcal{P}(\Omega), P)$: Probability space associated with a random experiment $\mathcal{E}$.
- We are interested in assigning probability, say $P(E)$, to an event $E \in \mathcal{P}(\Omega)$.
- Repeat the experiment $\mathcal{E}$ a large (say N) number of times.
- Define, for $i = 1, \ldots, N$

$$Y_i = \begin{cases} 1, & \text{if ith trial results in E} \\ 0, & \text{otherwise.} \end{cases}$$

- Clearly $Y_1, \ldots, Y_N$ are i.i.d. r.v.s with $\mu = E(Y_1) = P(E)$.
- $f_N(E) =$ No. of times event E occurs in first N trials $= \sum_{i=1}^{N} Y_i$.
- $$r_N(E) = \frac{f_N(E)}{N} = \frac{1}{N}\sum_{i=1}^{N} Y_i = \bar{Y}_N.$$
- By WLLN
$$r_N(E) = \bar{Y}_N \xrightarrow{p} \mu = P(E).$$
- Thus, the WLLN justifies the relative frequency approach to assign probabilities.

Result 6.

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of r.v.s and let X be another r.v.

(i) $g : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ and $X_n \xrightarrow{p} c$, as $n \to \infty$,

$$\Rightarrow g(X_n) \xrightarrow{p} g(c), \text{ as } n \to \infty.$$

(ii) $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(c_1, c_2) \in \mathbb{R}^2$, $X_n \xrightarrow{p} c_1$, and $Y_n \xrightarrow{p} c_2$, as $n \to \infty$,

$$\Rightarrow h(X_n, Y_n) \xrightarrow{p} h(c_1, c_2), \text{as } n \to \infty.$$

(iii) $g : \mathbb{R} \to \mathbb{R}$ is continuous on S_X (support of X) and $X_n \xrightarrow{d} X$, as $n \to \infty$,

$$\Rightarrow g(X_n) \xrightarrow{d} g(X), \text{as } n \to \infty.$$

(iv) $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $D = \{(x, b) : x \in S_X\}$, $X_n \xrightarrow{d} X$, and $Y_n \xrightarrow{p} b$, as $n \to \infty$

$$\Rightarrow h(X_n, Y_n) \xrightarrow{d} h(X, b), \text{as } n \to \infty.$$

Remark 4.

(i) Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. r.v.s with $E(X_1) = \mu$ and $\text{Var}(X_1) = \sigma^2$. The CLT asserts that, as $n \to \infty$,

$$\begin{aligned}
&\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1) \\
\Rightarrow\ &\frac{1}{\sqrt{n}} \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} 0 \times Z = 0 \\
\Rightarrow\ &\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{p} 0 \\
\Rightarrow\ &\bar{X}_n \xrightarrow{p} \mu.
\end{aligned}$$

Thus, under finiteness of variance, the CLT is a stronger result than WLLN.

(ii) For real constants c and d ($d \neq 0$)

$$X_n \xrightarrow{p} c,\ Y_n \xrightarrow{p} d \Rightarrow$$

$$X_n \pm Y_n \xrightarrow{p} c \pm d,\ X_n Y_n \xrightarrow{p} cd,\ \frac{X_n}{Y_n} \xrightarrow{p} \frac{c}{d}.$$

(iii) For real constant c

$$X_n \xrightarrow{d} X,\ Y_n \xrightarrow{p} c \Rightarrow$$

$$X_n \pm Y_n \xrightarrow{d} X \pm c,\ X_n Y_n \xrightarrow{d} cX,\ \frac{Y_n}{X_n} \xrightarrow{d} \frac{X}{c},$$

(provided $c \neq 0$).

Take Home Problems

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. r.v.s with finite mean μ. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, $n = 1, 2, \ldots$ be sequences of sample means and sample variances, respectively. Define $T_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n}$, $n = 1, 2, \ldots$.

(a) If $\sigma^2 = \mathrm{Var}(X_1) \in (0, \infty)$, then show that $S_n^2 \xrightarrow{p} \sigma^2$, $S_n \xrightarrow{p} \sigma$ and $T_n \xrightarrow{d} Z \sim \mathrm{N}(0, 1)$, as $n \to \infty$;

(b) Suppose that the kurtosis $\gamma_1 = \frac{E((X_1-\mu)^4)}{\sigma^4} < \infty$. Then show that $\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} W \; N(0, (\gamma_1 - 1)\sigma^4)$, as $n \to \infty$.

(c) Show that the Student t-distribution with large degrees of freedom (i.e., as degrees of freedom $\nu \to \infty$) can be approximated by a N(0, 1) distribution.

Thank you for your patience