Tutorial 3

1. $u_y + 5u_x = 0$ — (1)

Let $c(s) = (x(s), y(s), z(s))$ be the integral curve associated with the eqn (1)

Hence the characteristic eqns are given by

$x'(s) = 5$; $y'(s) = 1$; $z'(s) = 0$

Solving we get,

$$(*)\begin{cases} x(s) = 5s + c_1 \\ y(s) = s + c_2 \\ z(s) = c_3 \end{cases}$$ c_1, c_2, c_3 are constants.

a) Hence one of the integral curves can be written as

$c(s) = \{(5s, s, 1) : s \in \mathbb{R}\}$

b) From (*) we have, $x = 5y + c_4 \Rightarrow x - 5y = c_4$ (c_4 = constant).

and $z = c_3$.

Thus we have z is constant along the characteristics $x - 5y = c$.

Define, $u(x,y) := z(x(s), y(s)) = f(x - 5y)$ for an arbitrary $f \in C^1$.

c) $u(x,y) = f(x-5y)$ for $f \in C^1$.

$\therefore u_x = f'$ and $u_y = -5f'$ (By chain Rule).

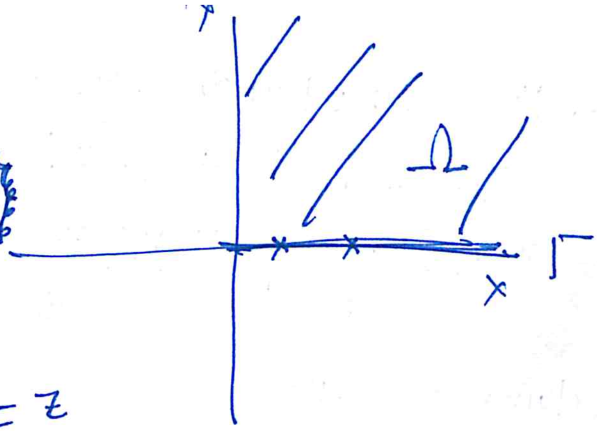
So, $u_y + 5u_x = 0$

d) The projected characteristic curves are characteristic curves projected on the plane $z = 0$.

Hence the projected char. curves in this case are the st. line's

$x - 5y = c$. where c is an arbitrary constant.

2. (a) $-yu_x + xu_y = u$ in $\{x>0, y>0\} = \Omega$
$u = g$ on $\Gamma = \{x>0; y=0\}$



Let $c(r,s) = \{(x'(r,s), y'(r,s), z'(r,s)) : r \geq 0, s>0\}$

be the ~~char~~ integral curve.

Hence the char eqn are given by

$x'(r,s) = -y$ | $y'(r,s) = x$ | $z'(r,s) = z$
$x(r,0) = r$. — (I) | $y(r,0) = 0$ — (II) | $z(r,0) = g(r)$. — (III)

From (I) and (II) we get.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow x^2+y^2 = c$$

Now, $x^2(r,0) + y^2(r,0) = c$
$\Rightarrow r^2 + 0 = c \Rightarrow c = r^2$.

Hence, the projected char. curves are $x^2+y^2=r^2$.

From (III), $z(r,s) = f(r)e^s$.
$g(r) = z(r,0) = f(r)e^0 \Rightarrow z(r,s) = g(r)e^s$.

$\therefore$ the char curves are given by $c(s) = \left\{(r\cos s, r\sin s, g(r)e^s : 0 \leq s \leq \frac{\pi}{2}, r \geq 0\right\}$

Define, $u(x,y) := z(r,s) = g(r)e^s$.
$$= g\left(\sqrt{x^2+y^2}\right) e^{\tan^{-1}\left(\frac{y}{x}\right)}.$$

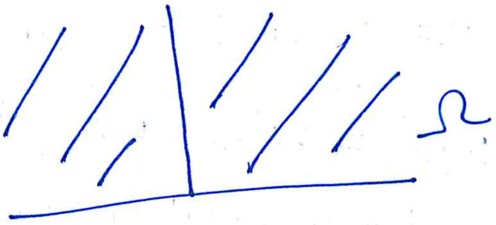
Clearly if g is C^∞ so is u ($\because$ It is the composition of a smooth fn with a smooth fn multiplied with a smooth fn).

(b) $u_x + u_y = u^2$ in $\Omega = \{y>0\}$ — (1)
$u = g$ on $\Gamma = \{y=0\}$

If $C(r,s) = \{(x(r,s), y(r,s), z(r,s)) : r, s \in \mathbb{R}\}$ is a characteristic curve to the eqn (1).

$\therefore$ The characteristic eqn is

$$x'(r,s) = 1 \quad \Big| \quad y'(r,s) = 1 \quad \Big| \quad z'(r,s) = z^2$$
$$x(r,0) = r \qquad \Big| \quad y(r,0) = 0 \quad \Big| \quad z(r,0) = g(r)$$
$$\text{(I)} \qquad\qquad \text{(II)} \qquad\qquad \text{(III)}$$



Solving we have,

$$x(r,s) = s + r$$
$$y(r,s) = s$$

and, $$z(r,s) = \frac{g(r)}{1 - s\,g(r)}.$$

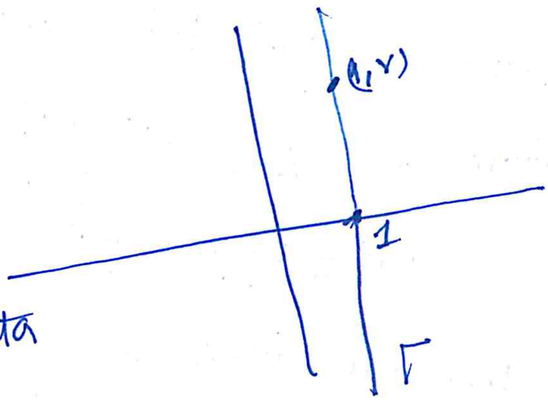
Clearly we can see if

$$u(x,y) = z(r,s) = \frac{g(x-y)}{1 - y\,g(x-y)}$$ is the solution if $1 - y\,g(x-y) \neq 0$.

Hence, the solution maynot be defined everywhere.

3. (a) $u_x + x u_y = ~~x~~u$ in $\mathbb{R}^2$

$u|_\Gamma = g(y)$ where $\Gamma = \{(1,y) : y \in \mathbb{R}\}$



To check for uniqueness of solution.

Let $\ell(r)$ = $\{(1,r,0) : r \in \mathbb{R}\}$ be the ~~char~~ data curve.

We need to check that Γ is nowhere tangent to for $(\gamma_1(r), \gamma_2(r)) = (1, r)$.

$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r)))$ i.e. ~~(1,0)·(1,x)~~

~~Hence~~ that is,

$$\underbrace{(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r)))}_{A} \cdot \underbrace{(-\gamma_2'(r), \gamma_1'(r))}_{B} \neq 0$$

Now, $A \cdot B = (1,1) \cdot (-1, 0) = -1 \neq 0$.

So, Γ is nowhere tangent to the projected Characteristics.

(b) $u_x + u_y = 0$

$u(x, ax) = f(x)$.

$\Gamma = (\gamma_1(r), \gamma_2(r)) = \{(r, ar) : r \in \mathbb{R}\}$ is the projected data curve.

Here, $(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) = (1, 1)$.

Hence for non-characteristic we have

$(1, 1) \cdot (-a, 1) \neq 0$

$\Rightarrow -a+1 \neq 0 \Rightarrow a \neq 1$.

Hence if $a \neq 1$ then we have a unique soln to the problem in a nbd of $\{(r, ar) : r \in \mathbb{R}\}$.

(a) $u_t + uu_x = 0$.

$u(x, 0) = \exp(-x^2)$.

Let $\Gamma(r) = \{(r, 0) : r \in \mathbb{R}\}$ be the initial data curve (or projected data curve).

Γ is non-characteristic since,

$\gamma_1'(r) - \gamma_2'(r)\,\phi(r) \neq 0$.

or, $1 - 0.\phi(r) \neq 0$.

Char Eqn are given by

$x'(r,s) = z$ | $t'(r,s) = 1$ | $z'(r,s) = 0$

$x(r,0) = r$ | $t(r,0) = 0$ | $z(r,0) = e^{-x^2}$.

Solving, $t(r,s) = s$

$z(r,s) = e^{-r^2}$.

$x(r,s) = se^{-r^2} + r$.

$\Rightarrow$ $x = tz + r$.

ie, $r = x - tz$

Define, $u(x,y) := z(r,s) = e^{-(x-tu)^2}$

$\therefore$ The soln is given by $u(x,t) = \exp[-(x-tu)^2]$.

(b) The projected char curves are given by

$x = \phi(r)s + r$ and $t = s$

where $\phi(r) = e^{-r^2}$.

Clearly $\exists\ r_1 < r_2$ s.t $e^{-r_1^2} > e^{-r_2^2}$.

Hence, the characteristic meet at the pt (x_0, y_0) thus creating a multivalued sol^n at that pt. We call such a solution as a singularity.