MSO 230B - Partial Differential Equation
Assignment 3

October 11, 2018

Tutorial Problem

1. Consider the first order linear PDE in two variable given by

$$a(x,y)u_x + b(x,y)u_y = c(x,y) \text{ in } \mathbf{R}^2 \tag{1}$$

We know that a curve $\mathbf{C} = (x(s), y(s), z(s))$ is called an integral curve for the vector field $(a(x,y), b(x,y), c(x,y))$ if one has the following

$$x'(s) = a(x(s), y(s)) \quad y'(s) = b(x(s), y(s)) \quad z'(s) = c(x(s), y(s))$$

The above set of equations are known as characterestic equations and the projection of the characteristic curves onto the xy-plane are called the projected characteristic curves.

(a) Find the integral curve to the transport equation $u_t + 5u_x = 0$.

(b) Using the integral curves find solution to the equation.

(c) Verify the solution using chain rule.

(d) Describe the projected characteristic curve for this equation.

2. Let us consider the following problem:

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= f(u) \text{ in } \Omega \\ u &= g \text{ on } \Gamma \end{aligned} \tag{2}$$

(a) Choose $a(x,y) = -y$, $b(x,y) = x$ and $f(x) = x$ and show that if g is assumed to be smooth (e.g, C^∞ function) then u is also smooth provided $\Omega = \{x > 0, y > 0\}$ and $\Gamma = \{x > 0, y = 0\}$.

(b) Choose $f(x) = x^2$, $\Omega = \{y > 0\}$ and Γ to be the boundary of Ω to demonstate that the solution of a semilinear first order PDE given by (2) need not be defined for all Ω provided $a(x,y) = b(x,y) = 1$.

3. We say Γ is noncharacteristic for the Cauchy problem

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= f(x,y,u) \text{ in } \mathbf{R}^2 \\ u &= g \text{ on } \Gamma \end{aligned} \tag{3}$$

if Γ is nowhere tangent to the projected characteristic $(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r)))$. The equation (3) admits a unique solution **NEAR A NEIGHBOURHOOD OF** Γ provided Γ is a noncharacteristic.

(a) Show that the problem (3) admits a unique solution in a neighbourhood of $\Gamma = (1, y)$ if $a(x, y) = 1$, $b(x, y) = x$, $f(x, y, u) = u$ and u restricted to Γ is any smooth function.

(b) For which real number a can you find an unique solution to the problem $u_x + u_y = 0$; $u(x, ax) = h(x)$ for any smooth h.

4. Let us look at a little more complicated situation of Burger's equation given by

$$\begin{aligned} u_t + uu_x &= 0 \\ u(x,0) &= \exp(-x^2) \end{aligned} \tag{4}$$

(a) Firstly solve the equation to see that the solution so obtained is in implicit form.

(b) Draw the projected characteristic curves to show that there exists a singularity. Physically this is the precursor of Shock Waves and gives rise to a discipline of mathematics called "Conservation Law".

Do You Know

The Burger's equation is used to model shock phenomena frequently arising in computational fluid dynamics and mathematically can be obtained from the exotic Navier Stokes equation by dropping the pressure term. Some of you may be wondering what is a Navier Stokes equation. Well for starters it models the motion of viscous fluids. No one fully understands it and the solution in 3 dimensions is an open problem with a million dollar bounty on it. You can always look it up in wikipedia in case you are interested.

Practise Problem

1. Solve $u_x^2 + u_y^2 = 1$ using the seperation of variable $u(x, y) = f(x) + g(y)$.
2. Solve the problem $u_x + 2u_y = 0$ $u(0, y) = \exp(-2y)$ using seperation of variable.
3. Solve: $u(x + y)u_x + u(x - y)u_y = x^2 + y^2$.
4. Find the general solution to the problem $(y - z)u_x + (z - x)u_y + (x - y)u_z = 0$.
5. Reduce the problem $u_x - u_y = u$ to its canonical form and solve it.