Introduction to Latent Variable Models

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Probabilistic Machine Learning (CS772A)

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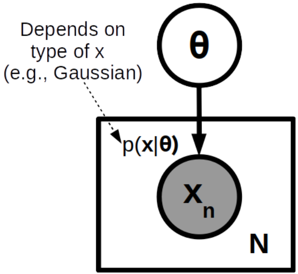
A Simple Generative Model

- All observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ generated from a distribution $p(\boldsymbol{x}|\theta)$



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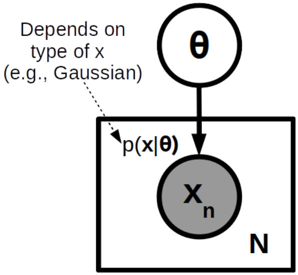

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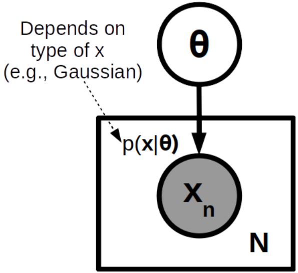
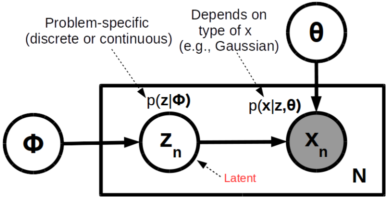


- Unknowns: Parameters θ of the assumed data distribution $p(\boldsymbol{x}|\theta)$
- Many ways to estimate the parameters (MLE, MAP, or Bayesian inference)

Generative Model with Latent Variables

- Assume each observation $\boldsymbol{x}_n$ to be associated with a latent variable $\boldsymbol{z}_n$



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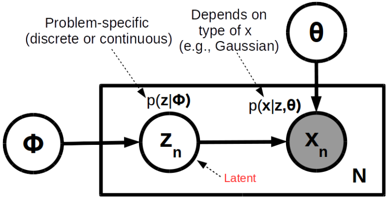

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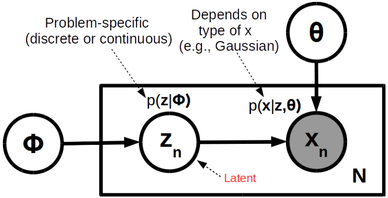

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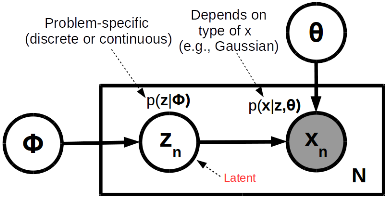

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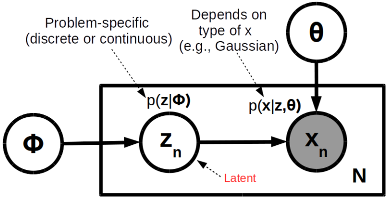

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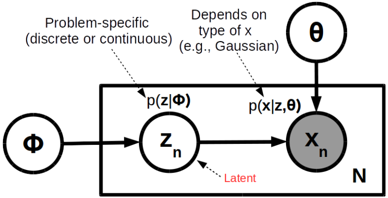

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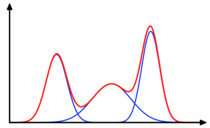

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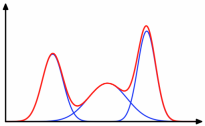

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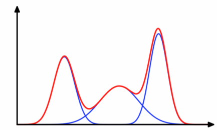

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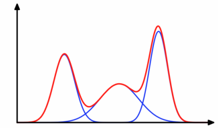

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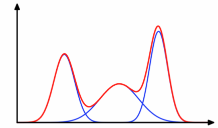

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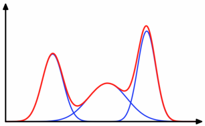

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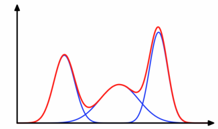

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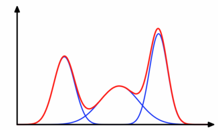

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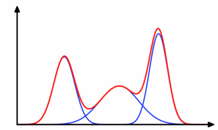

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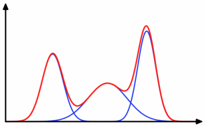

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- If each $p(\boldsymbol{x}|\theta_k)$ is a Gaussian $\Rightarrow$ Gaussian Mixture Model (used for probabilistic or "soft" clustering)

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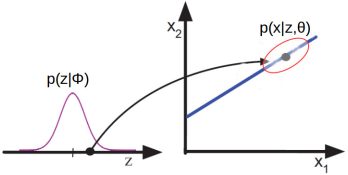

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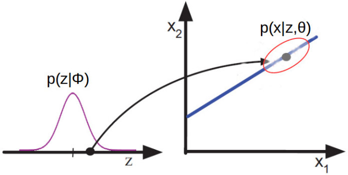

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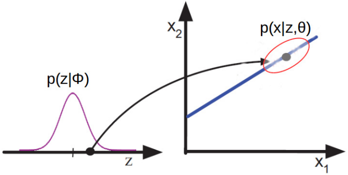

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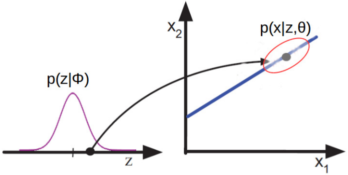

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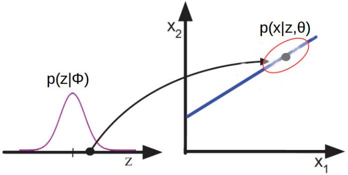

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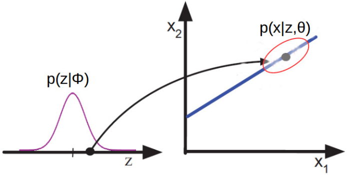

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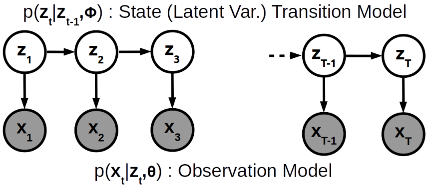
- If $p(\boldsymbol{z}|\phi)$ and $p(\boldsymbol{x}|\boldsymbol{z}, \theta)$ are Gaussians and $\boldsymbol{z}$ to $\boldsymbol{x}$ map linear $\Rightarrow$ factor analysis or probabilistic PCA

Other Examples..

- State-space latent variable models (e.g., Hidden Markov Models, Kalman Filters)



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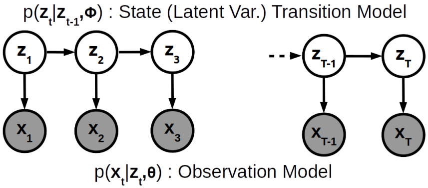

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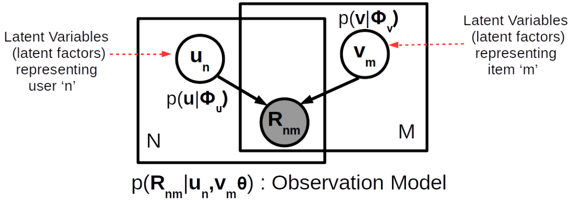

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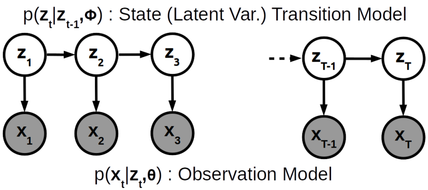

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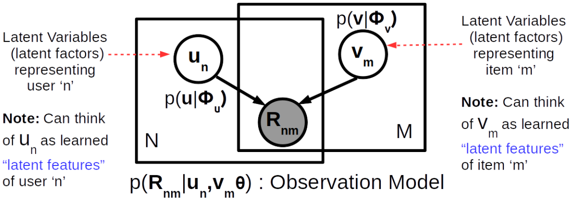
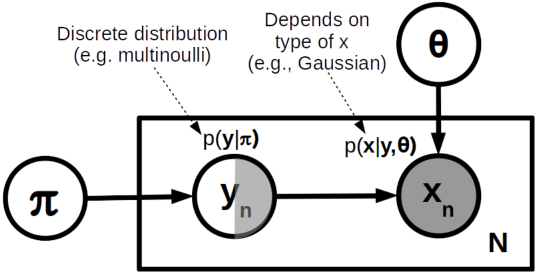


Other Examples..

- Semi-supervised generative classification: Some training inputs can be unlabeled
  - These "missing" labels can be treated as latent variables and inferred



Latent Variable Models for Clustering and Density Estimation

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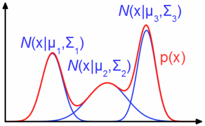
- A generative model for data clustering. Assume data generated from a mixture of K Gaussians



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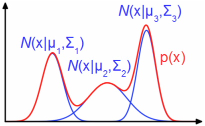

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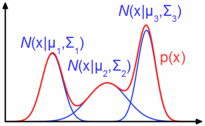

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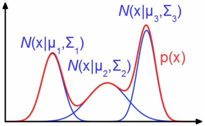

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- π_k is the prior probability than $\boldsymbol{x}_n$ comes from the k-th Gaussian (i.e., cluster id $\boldsymbol{z}_n = k$)

$$p(\boldsymbol{z}_n = k|\pi) = \pi_k$$

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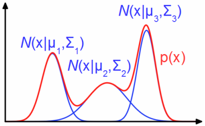

- Let $\pi = (\pi_1, \pi_2, \dots, \pi_K)$ with $\sum_{k=1}^{K} \pi_k = 1$ denote the "mixing weights" of the K Gaussians
- Intuitively $\pi_k \in (0,1)$ is the fraction of data "contributed" by the k-th Gaussian
- π_k is the prior probability than $\boldsymbol{x}_n$ comes from the k-th Gaussian (i.e., cluster id $\boldsymbol{z}_n = k$)

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- Same as a multinoulli prior on $\boldsymbol{z}$, i.e., $p(\boldsymbol{z}_n|\pi) = \prod_{k=1}^{K} \pi_k^{z_{nk}}$

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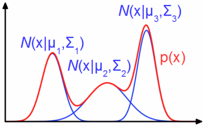

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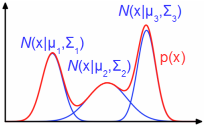

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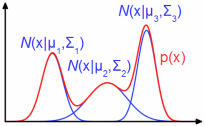

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- Goal: Learn the GMM parameters $\Theta = (\pi, \{\mu_k, \Sigma_k\}_{k=1}^{K})$ (and cluster assignments $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N\}$)

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  - Expectation Maximization (EM) helps solve such problems in a clean and efficient way

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- More details in the next class..