MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-III

A. Illustrative Discussion Problems

1. For each of the following, find the value of constant c so that $f(\cdot)$ is a pmf of some discrete random variable (say X). Also, for each of the following, find $P(\{X > 2\})$, $P(\{X < 4\})$, and $P(\{1 < X < 2\})$:

$$\text{(a) } f(x) = \begin{cases} c(1-p)^x, & \text{if } x \in \{1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}; \quad \text{(b) } f(x) = \begin{cases} \frac{c\lambda^x}{x!}, & \text{if } x \in \{1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases};$$

here $p \in (0, 1)$ and $\lambda > 0$ are fixed constants.

2. In each of the following, find the value of constant c so that $f(\cdot)$ is a pdf of some absolutely continuous random variable (say X). Also, for each of the following, find $P(\{X > 3\})$, $P(\{X \leq 3\})$, and $P(\{3 < X < 4\})$:

$$\text{(a) } f(x) = \begin{cases} cxe^{-x^2}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}; \quad \text{(b) } f(x) = \begin{cases} cxe^{-(x-2)}, & \text{if } x \geq 2 \\ 0, & \text{otherwise} \end{cases}.$$

3. (a) Let X be a discrete random variable with support $S_X = \{0, 1, 2, 3, 4\}, P(\{X = 0\}) = P(\{X = 1\}) = \frac{1}{10}, P(\{X = 2\}) = P(\{X = 3\}) = P(\{X = 4\}) = \frac{4}{15}$. Find the distribution function of X and sketch its graph.

(b) Let the random variable X have the pmf

$$f_X(x) = \begin{cases} \frac{x}{5050}, & \text{if } x \in \{1, 2, \ldots, 100\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that the distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{[x]([x]+1)}{10100}, & \text{if } 1 \leq x < 100 \\ 1, & \text{if } x \geq 100 \end{cases}.$$

Also compute $P(\{3 < X < 50\})$.

4. For each of the following pdfs of some absolutely continuous random variable (say X), find the distribution function and sketch its graph. Also compute $P(\{|X| < 1\})$ and $P(\{X^2 < 9\})$.

$$\text{(a) } f(x) = \begin{cases} \frac{x^2}{18}, & \text{if } -3 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}; \quad \text{(b) } f(x) = \begin{cases} \frac{x+2}{18}, & \text{if } -2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases};$$

$$\text{(c) } f(x) = \begin{cases} \frac{1}{2x^2}, & \text{if } |x| \ge 1 \\ 0, & \text{otherwise} \end{cases}.$$

5. Let the random variable X have the distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \le x < 1 \\ \frac{2}{3}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

Show that X is neither of discrete type nor of absolutely continuous type.

6. (a) Let $F_X(\cdot)$ be the distribution function of a random variable X. Show that $F_X(\cdot)$ can be decomposed as $F_X(x) = \alpha F_d(x) + (1-\alpha)F_c(x),\ x \in \mathbb{R}$, where $\alpha \in [0,1]$, $F_d(\cdot)$ is a distribution function of some discrete random variable and $F_c(\cdot)$ is a distribution function of some continuous random variable. (Prove the assertion only for the case when F_X has finite number of discontinuities).

(b) Let Y be a random variable having the distribution function

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{4}, & \text{if } 0 \le x < 1 \\ \frac{x}{3}, & \text{if } 1 \le x < 2 \\ \frac{3x}{8}, & \text{if } 2 \le x < \frac{5}{2} \\ 1, & \text{if } x \ge \frac{5}{2} \end{cases}.$$

Decompose $H(\cdot)$ as $H(x) = \alpha H_d(x) + (1-\alpha)H_c(x),\ x \in \mathbb{R}$, where $\alpha \in [0,1]$, H_d is a distribution function of some discrete random variable Y_d and H_c is a distribution function of some random continuous variable Y_c.

7. Let X be a random variable with

$$P(\{X=-2\}) = \tfrac{1}{21},\quad P(\{X=-1\}) = \tfrac{2}{21},\quad P(\{X=0\}) = \tfrac{1}{7},$$
$$P(\{X=1\}) = \tfrac{4}{21},\quad P(\{X=2\}) = \tfrac{5}{21},\quad P(\{X=3\}) = \tfrac{2}{7}.$$

Find the p.m.f. and distribution function of $Y = X^2$.

8. Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{3}(\frac{2}{3})^x, & \text{if } \; x \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases};$$

Find the distribution function of of $Y = X/(X+1)$ and hence determine the p.m.f. of Y.

B. Practice Problems from the Text Book

Chapter 1: Probability and Distributions, Problem Nos.: 5.2, 5.9, 6.3, 6.7, 6.9, 7.5, 7.14, 7.18.

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Solutions

Problem 1 (a) $\sum_{x=1}^{\infty} f(x)=1 \Rightarrow c\sum_{x=1}^{\infty}(1-p)^x=1 \Rightarrow c\frac{(1-p)}{p}=1 \Rightarrow c=\frac{p}{1-p}.$

$P(X>2)=P(X\geq 3)=c\sum_{x=3}^{\infty}(1-p)^x=(1-p)^2;$

$P(X<4)=1-P(X\geq 4)=1-c\sum_{x\geq 4}(1-p)^x=1-c\frac{(1-p)^4}{p}=1-(1-p)^3.$

$P(1<X<2)=0.$

(b) $\sum_{x=1}^{\infty} f(x)=1 \Rightarrow c\sum_{x=1}^{\infty}\frac{\lambda^x}{x!}=1 \Rightarrow c(e^{\lambda}-1)=1 \Rightarrow c=\frac{1}{e^{\lambda}-1}$

$P(X>2)=1-P(X\leq 2)=1-[P(X=1)+P(X=2)]=1-c[\lambda+\frac{\lambda^2}{2!}];$

$P(X<4)=\sum_{j=1}^{3}P(X=j)=c[\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}];$

$P(1<X<2)=0.$

Problem 2 (a) $\int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow c\int_0^{\infty}xe^{-x^2}dx=1 \Rightarrow \frac{c}{2}\int_0^{\infty}e^{-t}dt \Rightarrow c=2$

$P(X>3)=\int_3^{\infty}2xe^{-x^2}dx=\int_9^{\infty}e^{-t}dt=e^{-9}$

$P(X\leq 3)=1-P(X>3)=1-e^{-9}$

$P(3<X<4)=\int_3^4 2xe^{-x^2}dt=\int_9^{16}e^{-t}dt=e^{-9}(1-e^{-7}).$

(b) $\int_{-\infty}^{\infty}f(x)dx=1 \Rightarrow c\int_2^{\infty}xe^{-(x-2)}dx=1 \Rightarrow c\int_0^{\infty}(z+2)e^{-z}dz=1$

$\Rightarrow c=\frac{1}{3}.$

$P(X>3)=c\int_3^{\infty}xe^{-(x-2)}dx=c\int_1^{\infty}(z+2)e^{-z}dz=\frac{4c}{e}=\frac{4}{3e};$

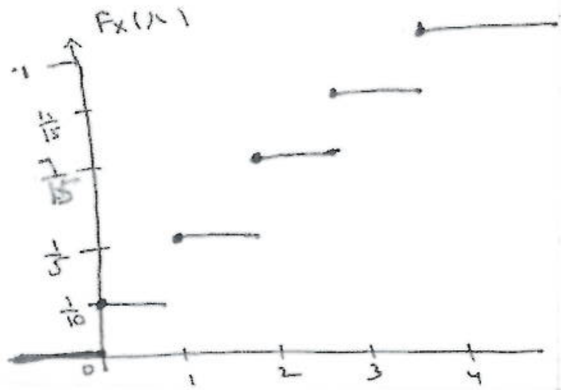
$P(X\leq 3)=1-P(X>3)=1-\frac{4}{3e};$

$P(3<X<4)=c\int_3^4 xe^{-(x-2)}dx=c\int_1^2(z+2)e^{-z}dz=\frac{c(4e-5)}{e^2}$

$=\frac{4e-5}{3e^2}.$

Problem 3 (a)

$$F_X(x)=\begin{cases}0, & \text{if } x<0\\ \frac{1}{10}, & \text{if } 0\leq x<1\\ \frac{1}{5}, & \text{if } 1\leq x<2\\ \frac{7}{15}, & \text{if } 2\leq x<3\\ \frac{11}{15}, & \text{if } 3\leq x<4\\ 1, & \text{if } x\geq 4\end{cases};$$



(b) For $x<1$ $F_X(x) = P(X \le x) = 0$. For $x \ge 1$

$$F_X(x) = P(X \le x) = \sum_{j=1}^{[x]} \frac{j}{5050} = \frac{[x]([x]+1)}{10100}.$$

$$P(3 < X < 50) = F_X(50-) - F_X(3) = \frac{49 \times 50}{10100} - \frac{3 \times 4}{10100} = \frac{1219}{5050}$$

Problem 4 (a)

$$F_X(x) = \begin{cases} 0, & \text{if } x<-3 \\ \frac{1}{18}\int_{-3}^{x} t^2\,dt, & \text{if } -3 \le x < 3 \\ 1, & \text{if } x \ge 3 \end{cases} = \begin{cases} 0, & \text{if } x<-3 \\ \frac{x^3+27}{54}, & \text{if } -3 \le x<3 \\ 1, & \text{if } x \ge 3 \end{cases}$$

$$P(|X|<1) = P(-1<X<1) = F_X(1-) - F_X(-1) = \frac{28}{54} - \frac{26}{54} = \frac{1}{27}$$

$$P(X^2<9) = P(-3<X<3) = F_X(3-) - F_X(-3) = 1-0 = 1$$

(b)

$$F_X(x) = \begin{cases} 0, & \text{if } x<-2 \\ \frac{1}{18}\int_{-2}^{x} (t+2)\,dt, & \text{if } -2 \le x < 4 \\ 1, & \text{if } x \ge 4 \end{cases} = \begin{cases} 0, & \text{if } x<-2 \\ \frac{(x+2)^2}{36}, & \text{if } -2 \le x<4 \\ 1, & \text{if } x \ge 4 \end{cases}$$

$$P(|X|<1) = F_X(1-) - F_X(-1) = \frac{1}{4} - \frac{1}{36} = \frac{2}{9}$$

$$P(X^2<9) = F_X(3-) - F_X(-3) = \frac{25}{36} - 0 = \frac{25}{36}.$$

(c)

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} \int_{-\infty}^{x} \frac{1}{2t^2}\,dt, & \text{if } x<-1 \\ \int_{-\infty}^{-1} \frac{1}{2t^2}\,dt & \text{if } -1 \le x < 1 \\ \int_{-\infty}^{-1} \frac{1}{2t^2}\,dt + \int_{1}^{x} \frac{1}{2t^2}\,dt, & \text{if } x>1 \end{cases} = \begin{cases} -\frac{1}{2x}, & \text{if } x<-1 \\ \frac{1}{2}, & \text{if } -1 \le x<1 \\ 1-\frac{1}{2x}, & \text{if } x \ge 1 \end{cases}$$

$$P(|X|<1) = F_X(1-) - F_X(-1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$P(X^2<9) = F_X(3-) - F_X(-3) = \left(1-\frac{1}{6}\right) - \frac{1}{6} = \frac{2}{3}.$$

Problem 5 $D_X = \{1, 2\} \neq \emptyset \Rightarrow X$ is not continuous $\Rightarrow X$ is not A.C.

$$P(X \in D_X) = P(X=1) + P(X=2) = [F_X(1) - F_X(1-)] + [F_X(2) - F_X(2-)]$$
$$= \left(\frac{2}{3} - \frac{1}{3}\right) + \left(1 - \frac{2}{3}\right) = \frac{2}{3} < 1$$

$\Rightarrow X$ is not of discrete type.

Problem 6 We will prove the result for the case when $D_X = \{a_1, a_2, \ldots, a_n\}$ is finite $(-\infty < a_1 < a_2 < \cdots < a_n < \infty)$. The idea of the proof for the case when D_X is countably infinite is similar but slightly involved.

<u>Case I</u> $D_X = \phi$

The result is trivial with $\alpha = 0$ and $F_c \equiv F_X$.

<u>Case II</u> $D_X = \{a_1, a_2, \ldots, a_n\}$, for some $n \in \mathbb{N}$.

Let $p_i = P(\{X = a_i\}) = F_X(a_i) - F_X(a_i -)$, $i = 1, 2, \ldots, n$.

Then $p_i > 0$, $i = 1, \ldots, n$. Let $\alpha = \sum_{i=1}^{n} p_i$ and define $F_d : \mathbb{R} \to \mathbb{R}$ by

$$F_d(x) = \begin{cases} 0 & \text{if } x < a_1 \\ \dfrac{\sum_{j=1}^{i} p_j}{\alpha}, & \text{if } a_i \le x < a_{i+1},\ i = 1, 2, \ldots, n-1, \\ 1, & \text{if } x \ge a_n \end{cases}$$

Then $F_d \uparrow$, F_d is right continuous, $F_d(-\infty) = 0$ and $F_d(\infty) = 1$. Also $F_d(\cdot)$ is a step function. Thus F_d is d.f. of some discrete r.v.

Define $F_c : \mathbb{R} \to \mathbb{R}$ by

$$F_c(x) = \frac{F_X(x) - \alpha F_d(x)}{1 - \alpha}, \quad x \in \mathbb{R}$$

(here we assumed that $\alpha < 1$. For $\alpha = 1$ the result follows trivially with $F_d \equiv F_X$).

For $A \subseteq \mathbb{R}$, let $S(A) = \{i \in \{1, 2, \ldots, n\} : a_i \in A\}$. Then, for $-\infty < x < y < \infty$

$$F_d(y) - F_d(x) = \sum_{i \in S((-\infty, y])} \frac{p_i}{\alpha} - \sum_{i \in S((-\infty, x])} \frac{p_i}{\alpha} = \sum_{i \in S((x, y])} \frac{p_i}{\alpha}$$

$$F_X(y) - F_X(x) = P(x < X \le y) \ge \sum_{i \in S((x, y])} p_i = \alpha (F_d(y) - F_d(x)),$$

where, for $A \subseteq \mathbb{R}$, $\sum_{i \in S(A)} p_i = 0$, if $S(A) = \phi$. Thus, for $-\infty < x < y < \infty$

$$F_c(y) - F_c(x) = \frac{F_X(y) - F_X(x) - \alpha (F_d(y) - F_d(x))}{1 - \alpha} \ge 0$$

$\Rightarrow F_c \uparrow$

Note that

$$F_X(a_i) - F_X(a_i-) = \alpha(F_d(a_i) - F_d(a_i-)) = p_i, \quad i=1,2,\dots,n$$

and $F_X(x) - F_X(x-) = F_d(x) - F_d(x-) = 0$, if $x \notin \{a_1,\dots,a_n\}$.

It follows that

$$F_c(x) - F_c(x-) = \frac{F_X(x) - F_X(x-) - \alpha(F_d(x) - F_d(x-))}{1-\alpha} = 0, \ \forall x \in \mathbb{R}.$$

$\Rightarrow F_c$ is continuous everywhere

clearly $F_c(\infty) = 1$ and $F_c(-\infty) = 0$ (since $F_X(\infty) = F_d(\infty) = 1$ and $F_X(-\infty) = F_d(-\infty) = 0$).

Thus F_c is d.f. of some continuous r.v

(b) $D_Y = \{1, 2, \frac{5}{2}\}$,

$p_1 = P(Y=1) = H(1) - H(1-) = \frac{1}{12}$; $p_2 = P(Y=2) = H(2) - H(2-) = \frac{1}{12}$

and $p_3 = P(Y = \frac{5}{2}) = H(\frac{5}{2}) - H(\frac{5}{2}-) = \frac{1}{16}$.

$\alpha = p_1 + p_2 + p_3 = \frac{11}{48}$

$P(Y_d = 1) = \frac{p_1}{\alpha} = \frac{4}{11}$, $P(Y_d = 2) = \frac{p_2}{\alpha} = \frac{4}{11}$, $P(Y_d = \frac{5}{2}) = \frac{p_3}{\alpha} = \frac{3}{11}$.

$$H_d(x) = \begin{cases} 0, & x < 1 \\ \frac{4}{11}, & 1 \le x < 2 \\ \frac{8}{11}, & 2 \le x < \frac{5}{2} \\ 1, & x \ge \frac{5}{2} \end{cases}$$

$$H_c(x) = \frac{H(x) - \alpha H_d(x)}{1-\alpha} = \begin{cases} 0, & x < 0 \\ \frac{12x}{37}, & 0 \le x < 1 \\ \frac{4(4x-1)}{37}, & 1 \le x < 2 \\ \frac{2(9x-4)}{37}, & 2 \le x < \frac{5}{2} \\ 1, & x \ge \frac{5}{2} \end{cases}$$

Problem 7 $\quad S_X = \{-2, -1, 0, 1, 2, 3\}$, $S_Y = \{0, 1, 4, 9\}$

$P(Y=0) = P(X=0) = \frac{1}{7}$; $P(Y=1) = P(X \in \{-1, 1\}) = \frac{2}{21} + \frac{4}{21} = \frac{2}{7}$

$P(Y=9) = P(X=3) = \frac{2}{7}$, $P(Y=4) = P(X \in \{-2, 2\}) = \frac{6}{21} = 2/7$

$$f_Y(y) = P(Y=y) = \begin{cases} \frac{1}{7}, & \text{if } y=0 \\ \frac{2}{7}, & \text{if } y \in \{1, 4, 9\} \\ 0, & \text{o.w.} \end{cases}; \quad F_Y(y) = P(Y \le y) = \begin{cases} 0, & \text{if } y<0 \\ \frac{1}{7}, & \text{if } 0 \le y < 1 \\ \frac{3}{7}, & \text{if } 1 \le y < 4 \\ \frac{5}{7}, & \text{if } 4 \le y < 9 \\ 1, & \text{if } y \ge 9 \end{cases}$$

Problem 8

$S_X = \{0, 1, 2, \dots\}$, $Y = \frac{X}{1+X} \uparrow \Rightarrow S_Y = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

Thus

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \sum_{j=0}^{i} \frac{1}{3}\left(\frac{2}{3}\right)^j, & \text{if } \frac{i}{i+1} \le y < \frac{i+1}{i+2},\ i = 0, 1, 2, \dots \end{cases} = \begin{cases} 0, & \text{if } y < 0 \\ 1 - \left(\frac{2}{3}\right)^{i+1}, & \text{if } \frac{i}{i+1} \le y < \frac{i+1}{i+2} \\ & i = 0, 1, 2, \dots \end{cases}$$

$$\begin{aligned} P\left(Y = \tfrac{i}{i+1}\right) &= F_Y\left(\tfrac{i}{i+1}\right) - F_Y\left(\tfrac{i}{i+1}^-\right) \\ &= \left[1 - \left(\tfrac{2}{3}\right)^{i+1}\right] - \left[1 - \left(\tfrac{2}{3}\right)^{i}\right] \\ &= \tfrac{1}{3}\left(\tfrac{2}{3}\right)^{i}, \quad i = 0, 1, 2, \dots \end{aligned}$$

Thus the p.m.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & \text{if } y \in \{0, \frac{1}{2}, \frac{1}{3}, \dots\} \\ 0, & \text{o.w.} \end{cases}$$

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