MSO 230B - Partial Differential Equation

Assignment 2

October 4, 2018

1 Tutorial Problem

1. Consider a general 2nd order ODE given by

$$P(x)y'' + Q(x)y' + R(x)y + \lambda y = 0 \tag{1}$$

where P, Q and R are smooth function on $I = [a, b]$ and $\lambda \in \mathbf{C}$.
It is possible to reduce the above equation to a self adjoint form given by

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0 \tag{2}$$

provided p, q, r are smooth functions on I by multiplying equation (1) with a smooth function $\mu(x)$ called the integrating factor (IF) and then exploiting the special structure of equation (2).

- Show that one arrives at the ODE

$$\mu' = \frac{Q - P'}{P}\mu$$

if we follow this route. Solve the ODE to show that the IF so obtains takes the form $\mu = \frac{\exp(\int_x \frac{Q}{P} dx)}{P}$.

- Use this information to reduce the problem

$$y'' + xy' + \lambda y = 0$$

into its self adjoint form by first finding the integrating factor $\mu(x)$.

Remark. *The advantage of this reduction is that if one directly wants to calculate the eigenpairs of a ODE in normal form then the complex case of λ has to be taken into account no matter what the boundary conditions are. On the other hand by reducing the ODE into self adjoint form and provided we have required boundary condition (i.e, Periodic/Regular) we only need to concentrate on the real eigenpairs.*

2. Consider a general second order ODE in self adjoint form i.e,

$$(py')' + qy + \lambda ry = 0 \tag{3}$$

with smooth coefficients such that $p > 0$ and $r > 0$. Assume that there exists an eigenpair (λ_n, ϕ_n) then show that

$$\lambda_n = \frac{-p\phi_n\phi_n'|_a^b + \int_a^b [p\phi_n'^{\,2} - q\phi_n^2]dx}{\int_a^b r\phi_n\, dx}$$

This is called the Raleigh quotient and is used to get an estimate for the eigenvalues when one can't quantitatively derive them.

3. You can actually solve equations by having knowledge about the eigenvalues of the corresponding operator. To understand let us look at the problem

$$\begin{gathered}(xy')' + \tfrac{y}{x} = \tfrac{1}{x},\ x \in [1, e] \\ y(1) = y(e) = 0\end{gathered}$$

- Show that the eigenvalue corresponding to the operator $Ly = (xy')' + \frac{y}{x}$ are $\lambda_n = n^2\pi^2 - 1$ and the eigenfunctions are given as $\phi_n = \sin(n\pi \ln x)$ for $n \in \mathbf{Z}$.
- Transform the eigenvalues to an orthonormal set of function and call it $\{\Phi_n\}$.
- Assume $y(x) = \sum\limits_1^\infty C_n\Phi_n$ be the solution of the equation (4) and solve the equation by finding C_n for $n \in \mathbf{N}$.

4. Let f is 2π periodic function such that its derivative of first and second order are piecewise continuous on $[-\pi, \pi]$. If

$$f(x) = a_0 + \sum_1^\infty [a_n \cos(nx) + b_n \sin(nx)]$$

then

$$f'(x) = \sum_1^\infty [-na_n \sin(nx) + nb_n \cos(nx)]$$

where a_n and b_n are Fourier coefficients corresponding to f.

- Try to prove the above result.
- For the second part start with a Fourier series and instead of differentiating it try integrating and document the relation between the coefficients.

5. Let $\phi_1, \phi_2 ...$ be the normalized eigenfunction of the equation (2) with regular boundary data and suppose f is piecewise continuous on $[0, 1]$.
Then if $c_n := \int_0^1 fr\phi_n\, dx$ then the series $\sum c_n\phi_n$ converges to $\frac{f(x+)+f(x-)}{2}$ at each point on $(0, 1)$.
Can you use it talk about the convergence and also evaluate the sum of the series

$$\sum_1^\infty \frac{(-1)^n}{n^2}$$

2 Practise Problem

1. Apply Parseval's formula to the function $f(x) = x$ in $[-\pi, \pi]$ to find the sum of the series $\sum \frac{1}{n^2}$.

2. We call $\phi(s) = \frac{1}{n^s}$ for s complex to be Riemann zeta function. Using the modulus function on $[-\pi, \pi]$ show the series converges for $s = 2$ and verify it with the last problem.

3. Calculate the Fourier cosine series of the function $f(x) = \cos x$ on $[0, \pi]$.

4. For this problem start by solving the equation $y'' + \lambda y = 0$ subject to the condition $y(0) = y(L) = 0$.
   Now let us recall some properties of a RSLBVP:-

   - There exists a sequence of eigenvalues $\{\lambda_n\}_{n \in \mathbf{N}}$ such that
   $$-\infty < \lambda_1 < \lambda_2 < ...... < \lambda_n < .....$$
   such that $\lim\limits_{n \to \infty} \lambda_n = \infty$
   - Eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal to each other.

   Use this two properties to show that $\int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) = 0$ provided $m \neq n$.