Module 19

Joint Distribution of Random Variables

- $(\Omega, \mathcal{P}(\Omega), P)$: a given probability space;
- In many situations we may be interested in two or more numerical characteristics of the sample space simultaneously;

Example 1:

$\mathcal{E}$: Casting a red and a white die;

$$\Omega = \{(i,j): \ i,j \in \{1,2,\ldots,6\}\},$$

where, in $(i,j) \in \Omega$, i denotes the number of spots on upper face of red die and j denotes the number of spots on upper face of white die.

Define r.v.s $X_1 : \Omega \to \mathbb{R}$ and $X_2 : \Omega \to \mathbb{R}$ as:

$$\begin{aligned} X_1((i,j)) &= i+j = \text{ sum of numbers on upper faces of two dice;} \\ X_2((i,j)) &= |i-j| = \text{ absolute difference of numbers on upper faces} \\ &\qquad \text{of two dice.} \end{aligned}$$

- We may be interested in studying X_1 and X_2 simultaneously, i.e., we may be interested in studying the function

$$\underline{X} : \Omega \to \mathbb{R}^2,$$

where $\underline{X} = (X_1, X_2)', \ \underline{X}((i,j)) = (X_1(i,j), X_2(i,j))', \ (i,j) \in \Omega.$

Definition 1: A function

$$\underline{X} = (X_1, \ldots, X_p)' : \Omega \to \mathbb{R}^p$$

is called a p-dimensional random vector (r.v.); here

$$\mathbb{R}^p = \{\underline{x} = (x_1, \ldots, x_p) : \ -\infty < x_i < \infty, i = 1, \ldots, p\}$$

denotes the p-dimensional Euclidean space.

- For A $\subseteq \mathbb{R}^p$ (i.e., $A \in \mathcal{P}(\mathbb{R}^p)$), define

$$\underline{X}^{-1}(A) = \{\omega \in \Omega : \underline{X}(\omega) \in A\}.$$

- Further define the set function $P_{\underline{X}} : \mathcal{P}(\mathbb{R}^p) \rightarrow [0, 1]$ as

$$\begin{aligned} P_{\underline{X}}(A) &= P\left(\underline{X}^{-1}(A)\right) \\ &= P\left(\{\omega \in \Omega : \underline{X}(\omega) \in A\}\right), \; A \in \mathcal{P}(\mathbb{R}^p). \end{aligned}$$

Result 1:

$\left(\mathbb{R}^p, \mathcal{P}\left(\mathbb{R}^p\right), P_{\underline{X}}\right)$ is a probability space, i.e., $P_{\underline{X}}(\cdot)$ is a probability function on $\mathcal{P}(\mathbb{R}^p)$.

Proof: Same as for $p = 1$

Definition 2 : The probability space $(\mathbb{R}^p, \mathcal{P}(\mathbb{R}^p), P_{\underline{X}})$ is called the probability space induced by r.v. $\underline{X}$ (or simply the induced probability space).

- $(\Omega, \mathcal{P}(\Omega), P) \xrightarrow{\underline{X}} \left(\mathbb{R}^p, \mathcal{P}\left(\mathbb{R}^p\right), P_{\underline{X}}\right)$

Notations:

- For $A \in \mathcal{P}(\mathbb{R}^p)$, $\{\underline{X} \in A\} \equiv \{\omega \in \Omega : \underline{X}(\omega) \in A\}$;

- For $\underline{a} = (a_1, \ldots, a_p)$, $\underline{b} = (b_1, \ldots, b_p) \in \mathbb{R}^p$

$$\begin{aligned} (\underline{a}, \underline{b}] &= (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_p, b_p] \\ &= \{\underline{x} = (x_1, \ldots, X_p) : a_i < x_i \leq b_i, i = 1, \ldots, p\} \end{aligned}$$

  denotes the p-dimensional semi-closed rectangle with vertices

$$\{\underline{z} = (z_1, \ldots, z_p) \in \mathbb{R}^p : z_i \in \{a_i, b_i\}, i = 1, \ldots, p\},$$

  (in all 2^p vertices).

$$
\begin{aligned}
\Delta_p &\equiv \Delta_p((\underline{a}, \underline{b}]) \\
&= \left\{\underline{z} = (z_1, \ldots, z_p) \in \mathbb{R}^p : \; z_i \in \{a_i, b_i\}, i = 1, \ldots, p\right\} \\
&= \text{set of all vertices of p-dimensional rectangle } (\underline{a}, \underline{b}] \\
&= \bigcup_{k=0}^{p} \Delta_{k,p}((\underline{a}, \underline{b}]),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{k,p} &\equiv \Delta_{k,p}((\underline{a}, \underline{b}]) \\
&= \{\underline{z} = (\underline{z}_1, \ldots, \underline{z}_p) \in \Delta_p : k \text{ of } z_i\text{s are } a_j\text{s}\}, \; k = 0, 1, \ldots, p.
\end{aligned}
$$

Example 2:

- $p = 2$

$$\begin{aligned}
\Delta_2 &= \{(b_1, b_2), (a_1, b_2), (b_1, a_2), (a_1, a_2)\} \\
\Delta_{0,2} &= \{(b_1, b_2)\} \\
\Delta_{1,2} &= \{(a_1, b_2), (b_1, a_2)\} \\
\Delta_{2,2} &= \{(a_1, a_2)\}
\end{aligned}$$

Notation :

- For $\underline{a} = (a_1, \ldots, a_p),\ \underline{b} = (b_1, \ldots, b_p) \in \mathbb{R}^p$

$$
\begin{aligned}
(-\underline{\infty}, \underline{a}] &= \{\underline{z} \in \mathbb{R}^p :\ z_i \leq a_i,\ i = 1, \ldots, p\} \\
(\underline{a}, \underline{\infty}) &= \{\underline{z} \in \mathbb{R}^p :\ z_i > a_i,\ i = 1, \ldots, p\} \\
(\underline{a}, \underline{b}) &= \{\underline{z} \in \mathbb{R}^p :\ a_i < z_i < b_i,\ i = 1, \ldots, p\}
\end{aligned}
$$

  $[a, b],\ [\underline{a}, \underline{b}),\ (-\underline{\infty}, \underline{a})$ and $[\underline{a}, \underline{\infty})$ are defined similarly.

- $\underline{a} \ \leq \ \underline{b}$ means $a_i \leq \ b_i,\ i = 1, \ldots, p$.

- $\underline{a} \ < \ \underline{b}$ means $a_i \leq \ b_i,\ i = 1, \ldots, p$, with $a_j < b_j$, for some $j \in \{1, \ldots, p\}$.

Definition 3:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be a p-dimensional r.v.

(a) The joint distribution function (d.f.) of $\underline{X}$ is the function $F_{\underline{X}} : \mathbb{R}^p \to \mathbb{R}$ defined by

$$\begin{aligned} F_{\underline{X}}(\underline{x}) &= P_{\underline{X}}((-\underline{\infty}, \underline{x}]) \\ &= P(\underline{X}^{-1}((-\underline{\infty}, \underline{x}])) \\ &= P(\{\omega \in \Omega : \underline{X}(\omega) \leq \underline{x}\}) \\ &= P(\{\omega \in \Omega : X_i(\omega) \leq x_i, i = 1, \ldots, p\}), \ \underline{x} \in \mathbb{R}^p. \end{aligned}$$

(b) The joint d.f. of any subset of r.v.s $\{X_1, \ldots, X_p\}$ is called a marginal d.f. of $F_{\underline{X}}(\cdot)$.

Example 3:

Let $\underline{X} = (X_1, X_2, X_3, X_4)'$ be a r.v. with d.f. $F_{\underline{X}}(\cdot)$. Then

$$\begin{aligned} F_{X_1}(.) &\quad : \quad \text{d.f. of } X_1; \\ F_{x_2,x_4}(.) &\quad : \quad \text{joint d.f. of } (X_2, X_4)' \\ F_{x_1,x_3,x_4}(\cdot) &\quad : \quad \text{joint d.f. of } (X_1, X_3, X_4)' \end{aligned}$$

are marginal d.f.s of $\underline{X} = (X_1, X_2, X_3, X_4)'$.

Result 2:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be a r.v. with joint d.f. $F_{X_1,\ldots,X_p}(\cdot)$. Then the marginal d.f. of $(X_1, \ldots, X_{p-1})$ is

$$F_{X_1,\ldots,X_{p-1}}(x_1, \ldots, x_{p-1}) = \lim_{t\to\infty} F_{X_1,\ldots,X_{p-1},X_p}(x_1, \ldots, x_{p-1}, t),$$
$$(x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1}.$$

Proof: Let $(x_1, \ldots, x_{p-1}) \in \mathbb{R}^{p-1}$ be fixed. Then

$$\begin{aligned}
F_{X_1,\ldots,X_{p-1}}(x_1, \ldots, x_{p-1}) &= P\left(\{X_1 \le x_1, \ldots, X_{p-1} \le x_{p-1}\}\right) \\
&= P\left(\{X_1 \le x_1, \ldots, X_{p-1} \le x_{p-1}, X_p < \infty\}\right) \\
&= P\left(\bigcup_{t=1}^{\infty} \{X_1 \le x_1, \ldots, X_{p-1} \le x_{p-1}, X_p \le t\}\right) \\
&= \lim_{t\to\infty} P\left(\{X_1 \le x_1, \ldots, X_{p-1} \le x_{p-1}, X_p \le t\}\right) \\
&= \lim_{t\to\infty} F_{X_1,\ldots,X_{p-1},X_p}(x_1, \ldots, x_{p-1}, t).
\end{aligned}$$

Remark 1:

The above result suggests that to get a marginal d.f. take (in limit) the arguments of unwanted variables in the joint d.f. to ∞.

Result 3: Let $\underline{X} = (X_1, \ldots, X_p)'$ be a p-dimensional r.v. with joint d.f. $F_{\underline{X}}(\cdot)$. Then, for any p-dimensional rectangle $(\underline{a}, \underline{b}]$ $(\underline{a}, \underline{b} \in \mathbb{R}^p, \underline{a} < \underline{b})$,

$$\begin{aligned} P\left(\{\underline{X} \in (\underline{a}, \underline{b}]\}\right) &= P\left(\{a_i < X_i \leq b_i, i = 1, \ldots, p\}\right) \\ &= \sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a}, \underline{b}])} F_{\underline{X}}(\underline{z}). \end{aligned}$$

Proof: For $p = 2$

$$\begin{aligned} P\left(\{a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2\}\right) &= P\left(\{X_1 \leq b_1, a_2 < X_2 \leq b_2\}\right) \\ &\quad - P\left(\{X_1 \leq a_1, a_2 < X_2 \leq b_2\}\right) \\ P\left(\{X_1 \leq b_1, a_2 < X_2 \leq b_2\}\right) &= P\left(\{X_1 \leq b_1, X_2 \leq b_2,\}\right) \\ &\quad - P\left(\{X_1 \leq b_1, X_2 \leq a_2,\}\right) \end{aligned}$$

$$\begin{aligned} &= F_{X_1,X_2}(b_1, b_2) - F_{X_1,X_2}(b_1, a_2) \\ P(\{X_1 \le a_1, a_2 < X_2 \le b_2\}) &= P(\{X_1 \le a_1, X_2 \le b_2, \}) \\ &\quad -P(\{X_1 \le a_1, X_2 \le a_2, \}) \\ &= F_{X_1,X_2}(a_1, b_2) - F_{X_1,X_2}(a_1, a_2). \end{aligned}$$

Therefore

$$\begin{aligned} P(\{\underline{X} \in (\underline{a}, \underline{b}]\}) &= F_{X_1,X_2}(b_1, b_2) - [F_{X_1,X_2}(a_1, b_2) + F_{X_1,X_2}(b_1, a_2)] \\ &\quad +F_{X_1,X_2}(a_1, a_2) \\ &= \sum_{k=0}^{2} (-1)^k \sum_{z \in \Delta_{k,2}((\underline{a},\underline{b}])} F_{X_1,X_2}(z_1, z_2). \end{aligned}$$

Thus the result holds for $p = 2$. Suppose, for any probability function, result holds for some $p = m$.

Let

$\underline{Y} = (X_1, \ldots, X_m, X_{m+1}) = (\underline{X}, X_{m+1})$, $\underline{a_0} = (a_1, \ldots, a_m)$, $\underline{b_0} = (b_1, \ldots, b_m)$, $\underline{a} = (a_1, \ldots, a_m, a_{m+1})$ and $\underline{b} = (b_1, \ldots, b_m, b_{m+1})$. Then

$$
\begin{aligned}
P\left(\{\underline{Y} \in (\underline{a}, \underline{b}]\}\right) &= P\left(\left\{\underline{X} \in (\underline{a_0}, \underline{b_0}],\ X_{m+1} \in (a_{m+1}, b_{m+1}]\right\}\right) \\
&= P\left(\left\{\underline{X} \in \left(\underline{a_0}, \underline{b_0}\right]\right\} \mid \{X_{m+1} \in (a_{m+1}, b_{m+1}]\}\right) \times \\
&\qquad P\left(\{X_{m+1} \in (a_{m+1}, b_{m+1}]\}\right).
\end{aligned}
$$

Using the result for $p = m$ (for conditional probability function) we have

$$
\begin{aligned}
& P\left(\left\{\underline{X} \in \left(\underline{a_0}, \underline{b_0}\right]\right\} \mid \{X_{m+1} \in (a_{m+1}, b_{m+1}]\}\right) \\
&= \sum_{k=0}^{m} (-1)^k \sum_{\underline{z} \in \Delta_{k,m}((\underline{a_0}, \underline{b_0}])} P\left(\{\underline{X} \leq \underline{z}\} \mid \{X_{m+1} \in (a_{m+1}, b_{m+1}]\}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
& P\left(\{\underline{Y} \in (\underline{a}, \underline{b}]\}\right) \\
&= \sum_{k=0}^{m} (-1)^k \sum_{\underline{z} \in \Delta_{k,m}((\underline{a_0}, \underline{b_0}])} P(\{\underline{X} \leq \underline{z},\quad a_{m+1} < X_{m+1} \leq b_{m+1}\})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{m}(-1)^k \sum_{\underline{z}\in\Delta_{k,m}((\underline{a_0},\underline{b_0}])} [P(\{\underline{X} \le \underline{z},\ X_{m+1} \le b_{m+1}\}) \\
&\qquad\qquad\qquad\qquad\qquad -P(\underline{X} \le \underline{z},\ X_{m+1} \le a_{m+1})] \\
&= \sum_{k=0}^{m+1}(-1)^k \sum_{\underline{z}\in\Delta_{k,m+1}((\underline{a},\underline{b}])} P(\{X_i \le z_i,\ i = 1,\dots,m+1\}) \\
&= \sum_{k=0}^{m+1}(-1)^k \sum_{\underline{z}\in\Delta_{k,m+1}((\underline{a},\underline{b}])} (-1)^k F_{\underline{X}}(\underline{z}).
\end{aligned}
$$

Take Home Problems

1. Let $\underline{X} = (X_1, X_2)$ be a r.v. having the joint d.f.

$$F_{x_1,x_2}{}^{(x,y)} = \begin{cases} xy^2, & \text{if } 0 \leq x < 1, 0 \leq y < 1 \\ x, & \text{if } 0 \leq x < 1, y \geq 1 \\ y^2, & \text{if } x \geq 1, 0 \leq y < 1 \\ 1, & \text{if } x \geq 1, y \geq 1 \\ 0, & \text{otherwise} \end{cases} .$$

Find the marginal d.f.s of X_1 and X_2.

2. Consider the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$G(x,y) = \begin{cases} x, & \text{if } 0 \leq x < 1, y \geq 1 \\ y^2, & \text{if } x \geq 1, 0 \leq y < 1 \\ 1, & \text{if } x \geq 1, y \geq 1 \\ 0, & \text{otherwise} \end{cases}.$$

Verify whether or not

$$\sum_{k=0}^{2} (-1)^k \sum_{\underline{z} \in \Delta_{k,2}(\underline{a},\underline{b}]} F_{\underline{X}}(z_1, z_2) \geq 0,$$

holds for all rectangles $(\underline{a}, \underline{b}]$, where $a_1 \in [0,1), a_2 \in [0,1), b_1 \in [1,\infty)$ and $b_2 \in [1,\infty)$.

Abstract of Next Module

- We will derive various properties of joint distribution function

Thank you for your patience