ESc201 : Introduction to Electronics

Sinusoidal Steady state Analysis

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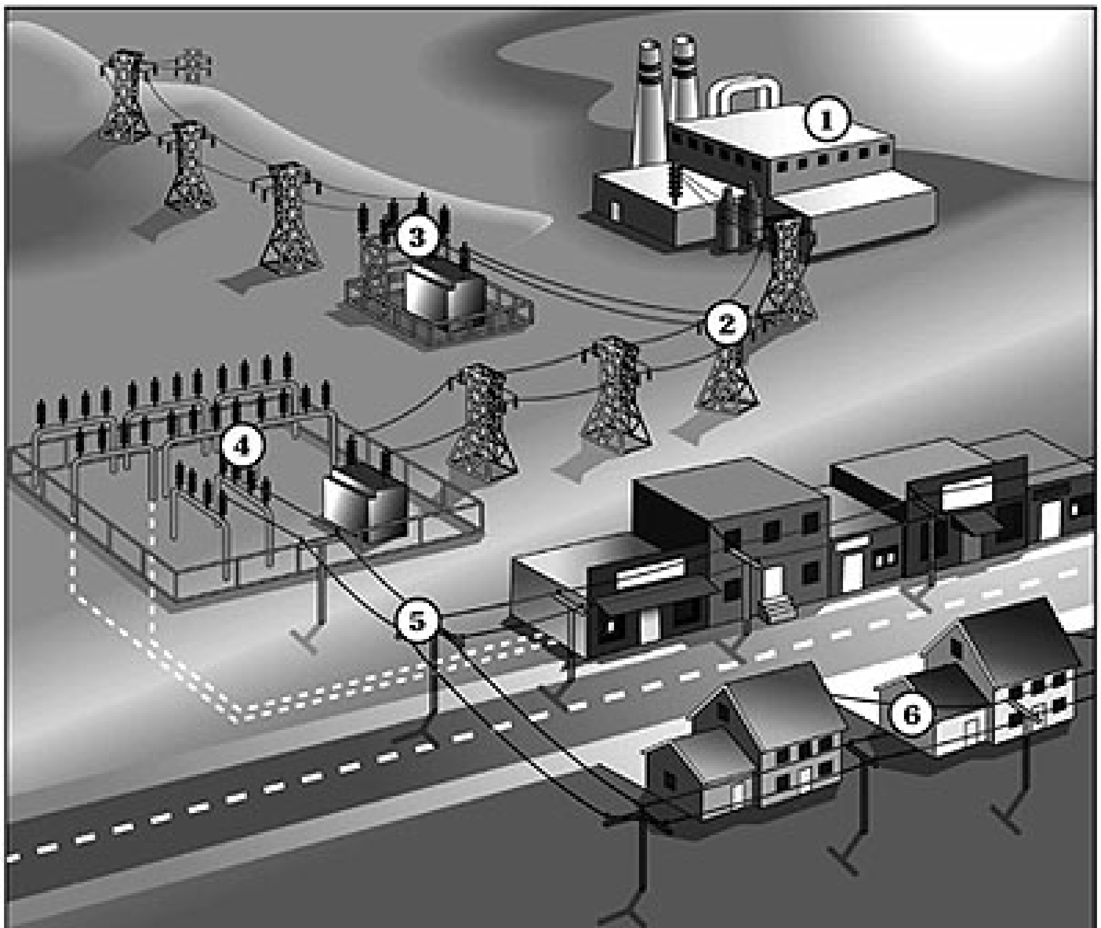
Importance of Sinusoidal Sources

- Appear in many practical applications
  - Electric power is distributed by sinusoidal currents and voltages
  - Sinusoidal signals are used widely in radio communications

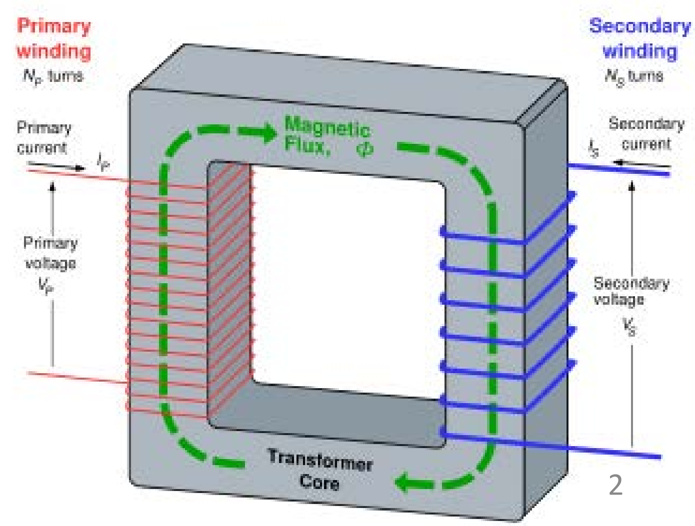


$$Loss = i^2 R_{wire}$$

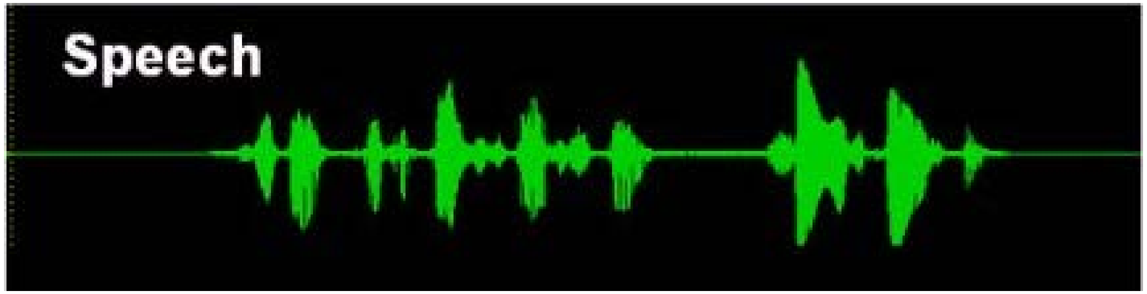
$$p = v \times i$$

$$2.2KW = 2.2KV \times 1A$$

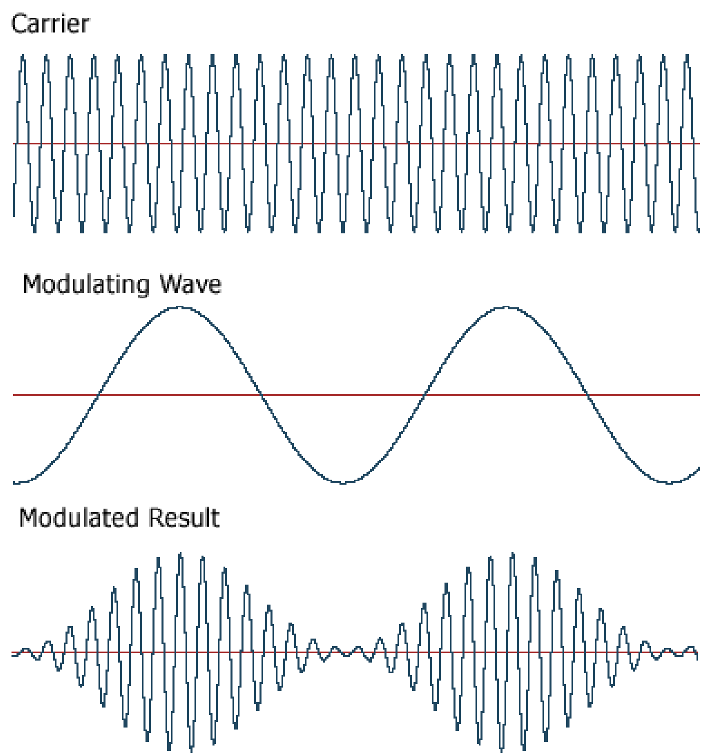
$$2.2KW = 220V \times 10A$$



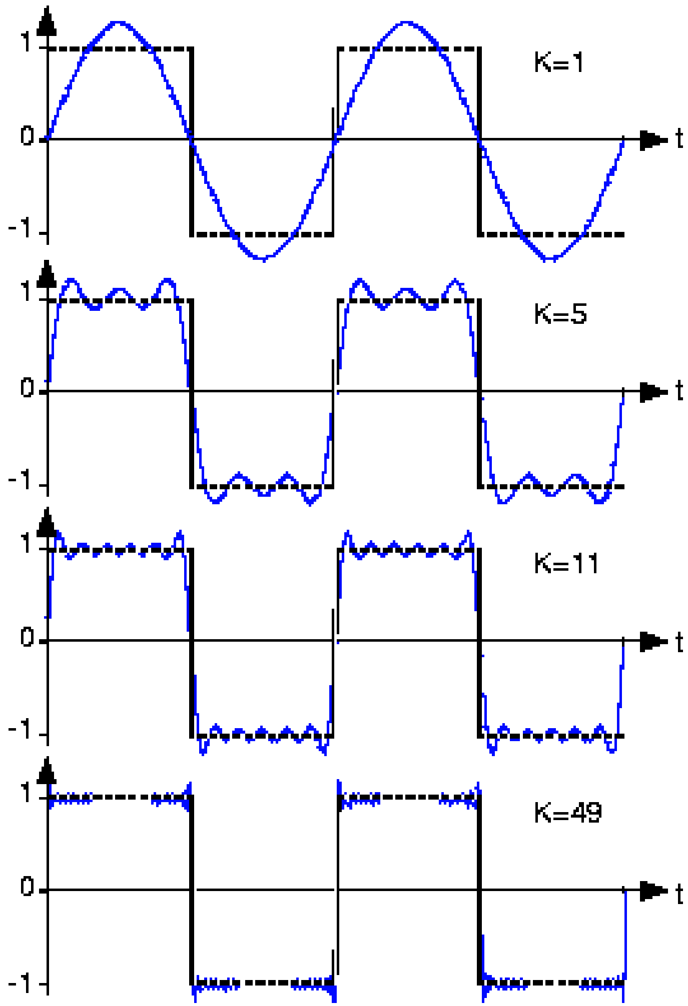
Communication



20 Hz -20KHz



- Any signal can be represented by a sum of sinusoidal components (Fourier Analysis)



$$f(t)=\frac{4}{\pi}\sum_{1,3,5}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi t}{T}\right)$$

- Sinusoids have good mathematical properties
  - Derivative is a sinusoid
  - Integral is a sinusoid

$$\frac{d(Sin\,x)}{dx} = Cos\,x = Sin(90 - x)$$

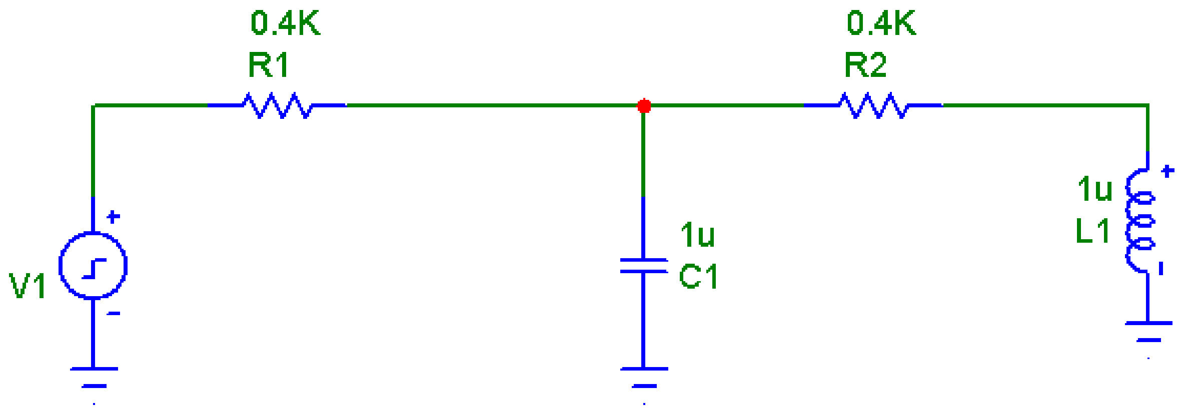
$$i_c = C\frac{dv_c}{dt}$$

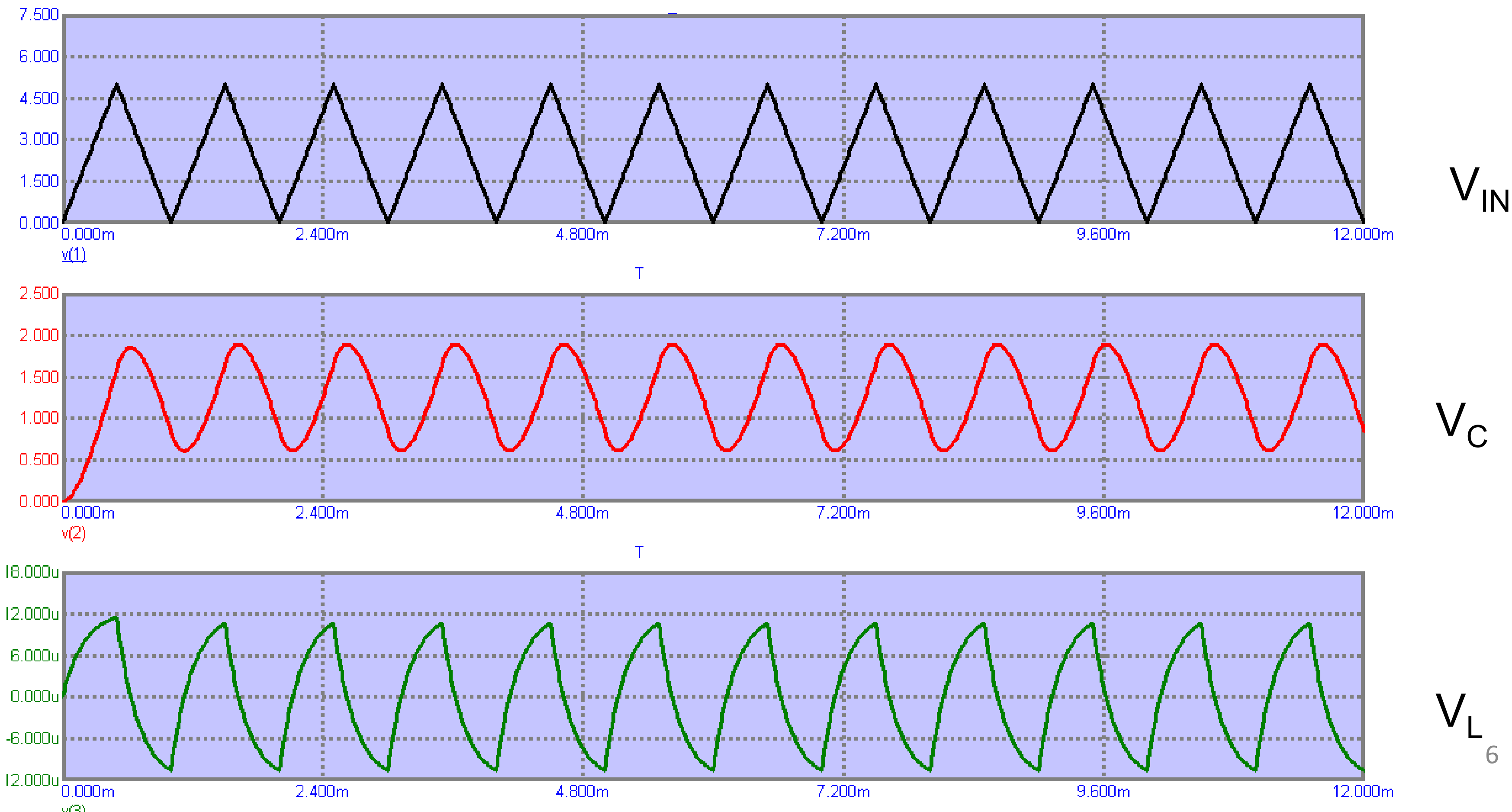
$$\int Sin\,x\,dx = -Cos\,x = Sin(x - 90)$$

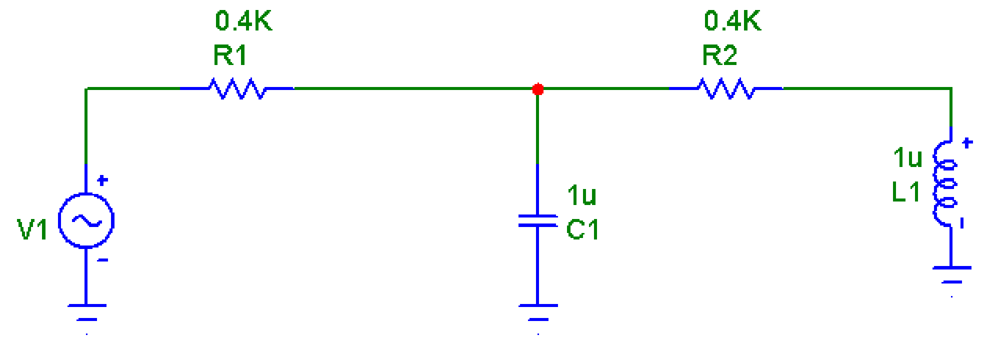
$$v = L\frac{di}{dt}$$

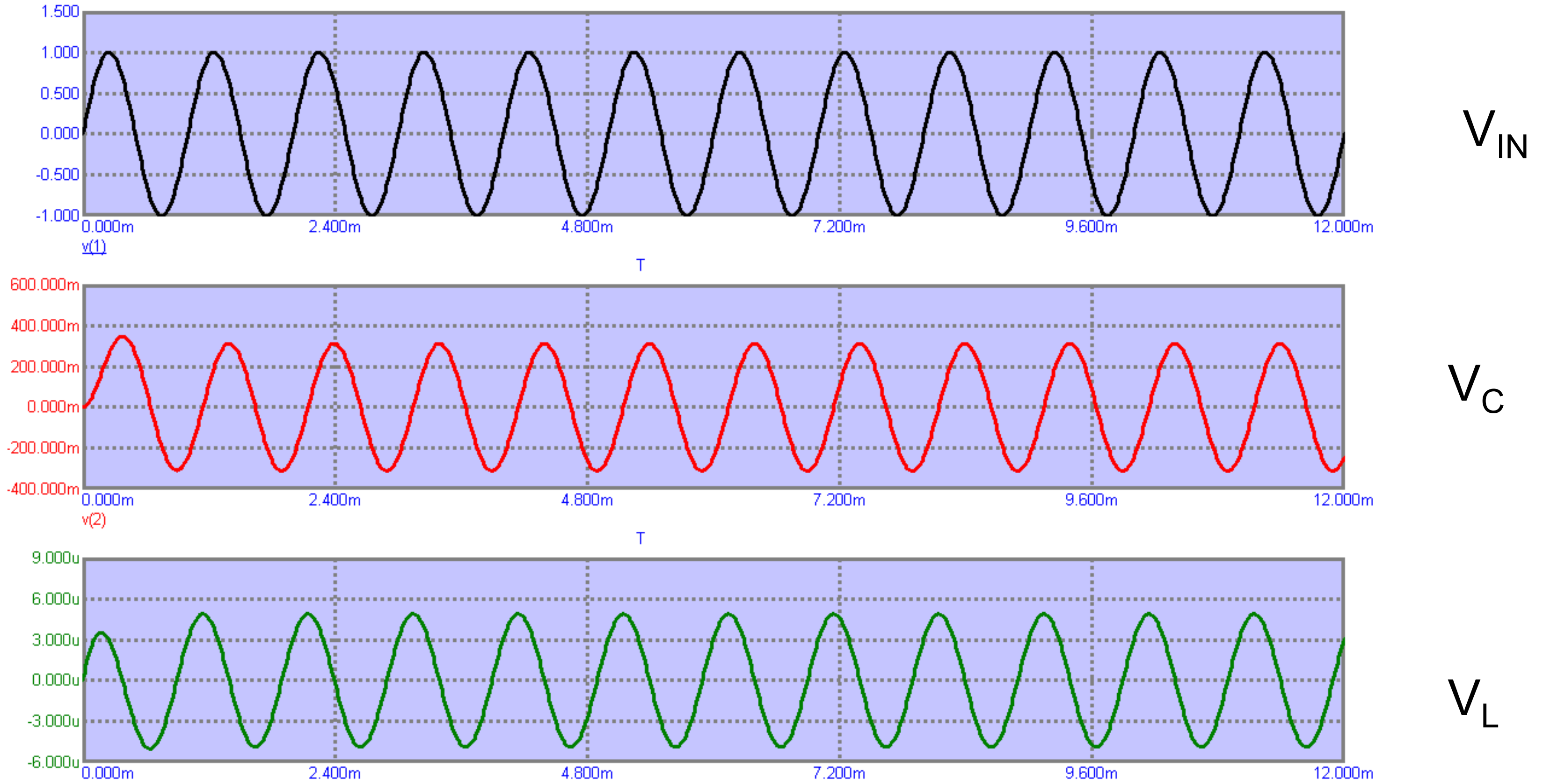
So as a sinusoidal signal goes through a circuit, it remains a sinusoid

This makes analysis easier


0.4K
R1
0.4K
R2
V1
1u
C1
1u
L1

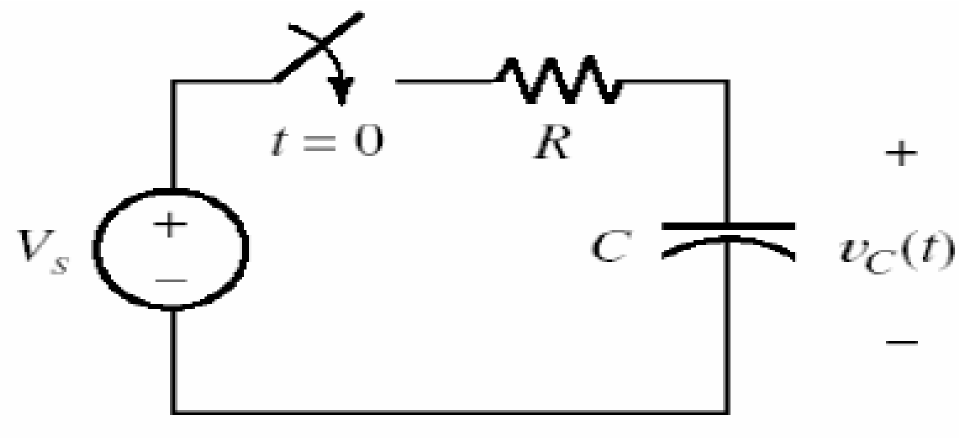

7.500
6.000
4.500
3.000
1.500
0.000
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
v(1)
V_{IN}
2.500
2.000
1.500
1.000
0.500
0.000
v(2)
V_C
18.000u
12.000u
6.000u
0.000u
-6.000u
12.000u
V_L



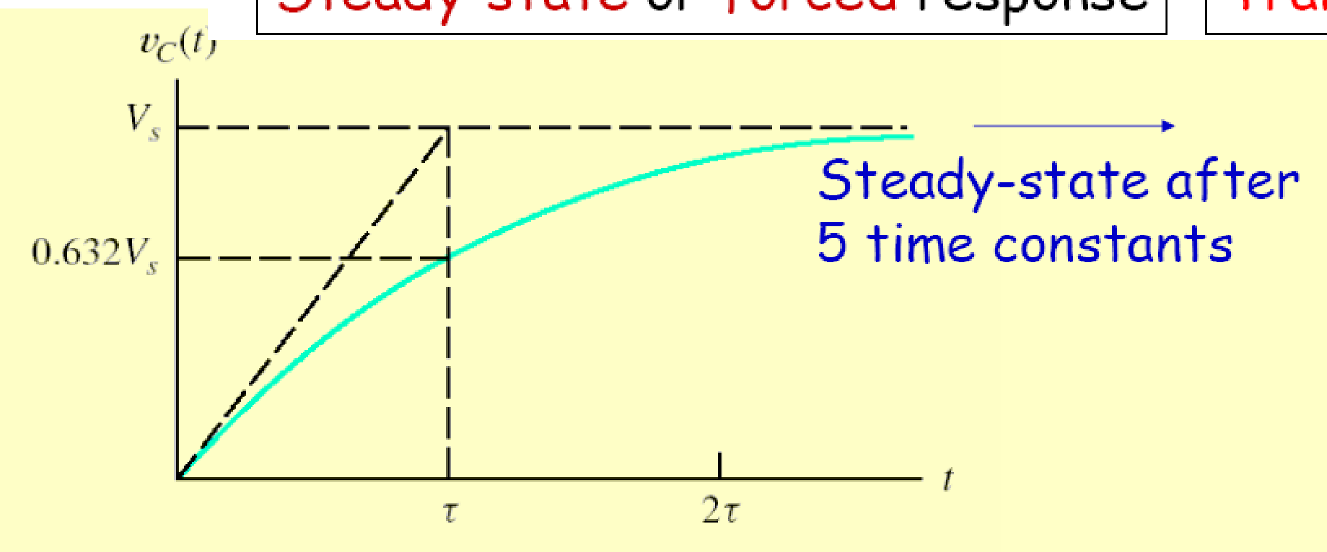


Voltage everywhere in the circuit is sinusoidal

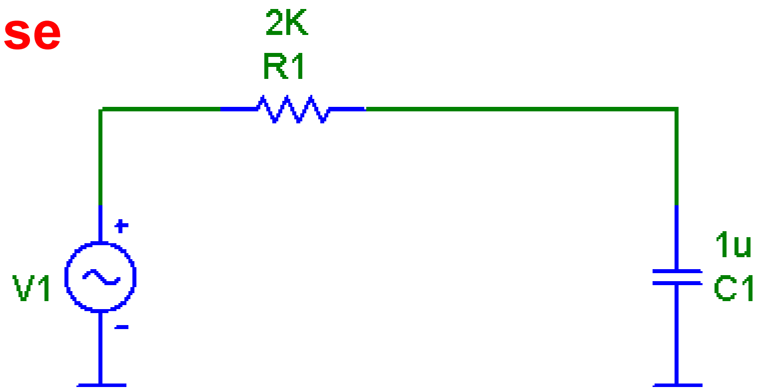
Transient and Forced Response

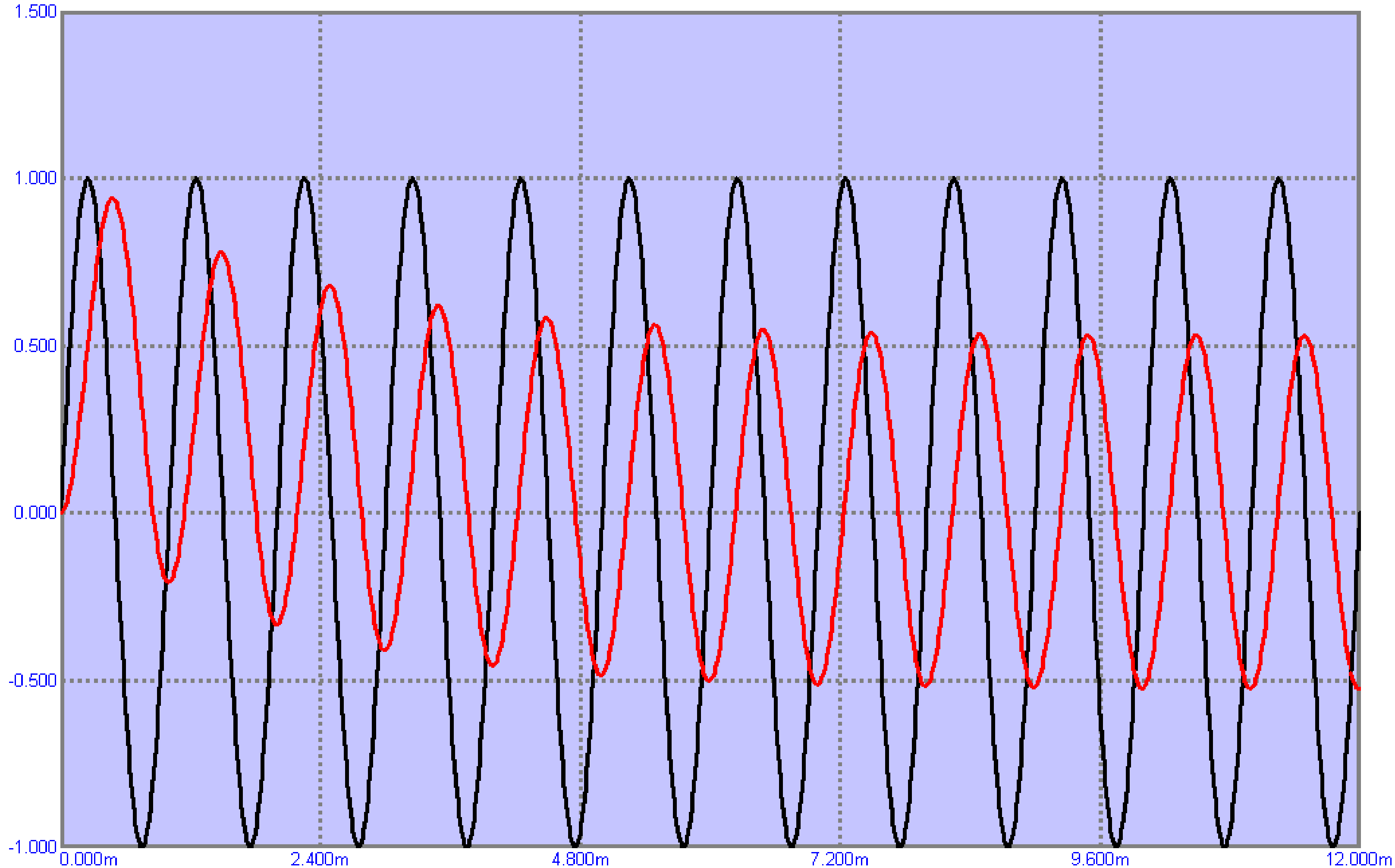


$$v_C(t) = \underbrace{V_s}_{\text{Steady-state or forced response}} - \underbrace{V_s e^{-t/\tau}}_{\text{Transient response}}$$



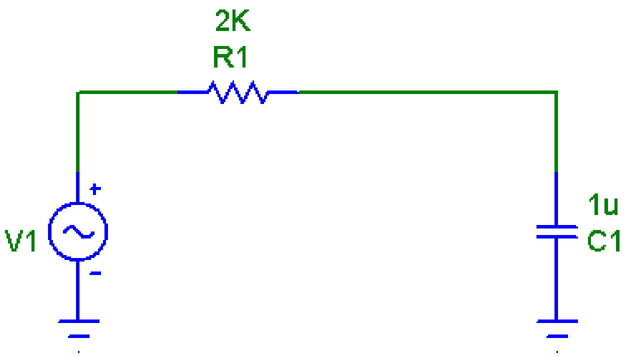
Transient and Forced Response

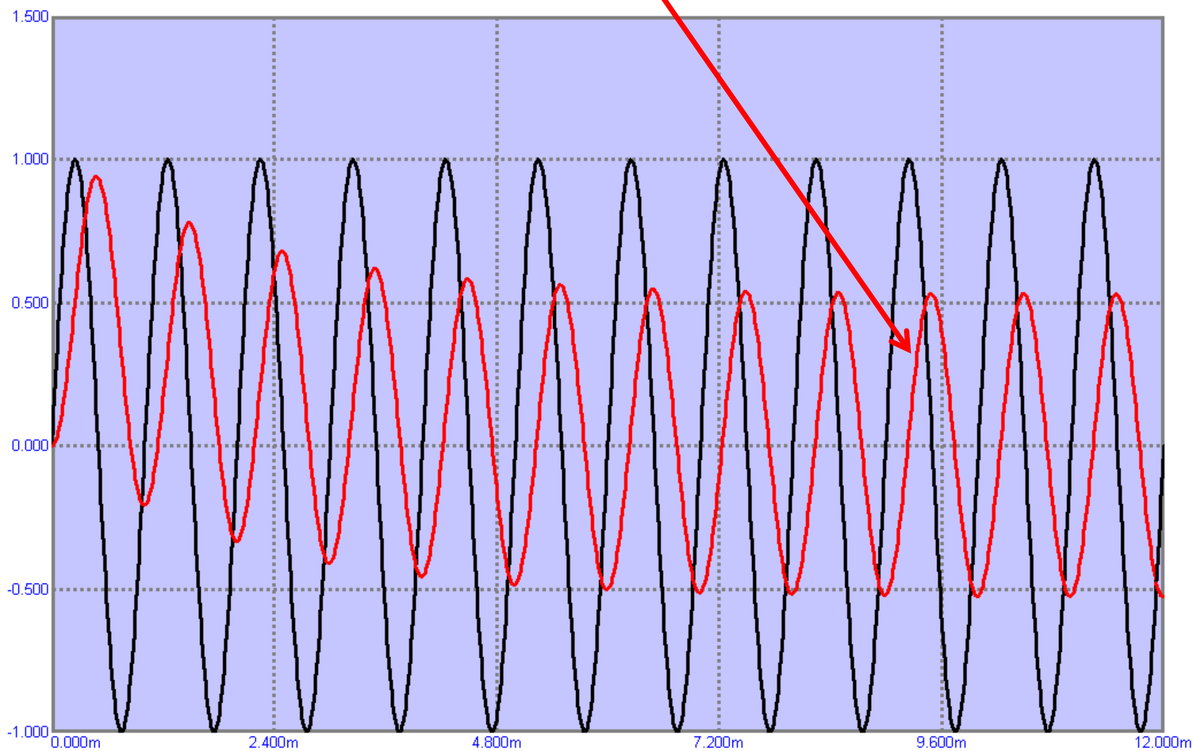




Sinusoidal Steady-State

- Whenever the forced input to the circuit is sinusoidal the response will be sinusoidal
- If the input persists, the response will persist and we call it steady-state response

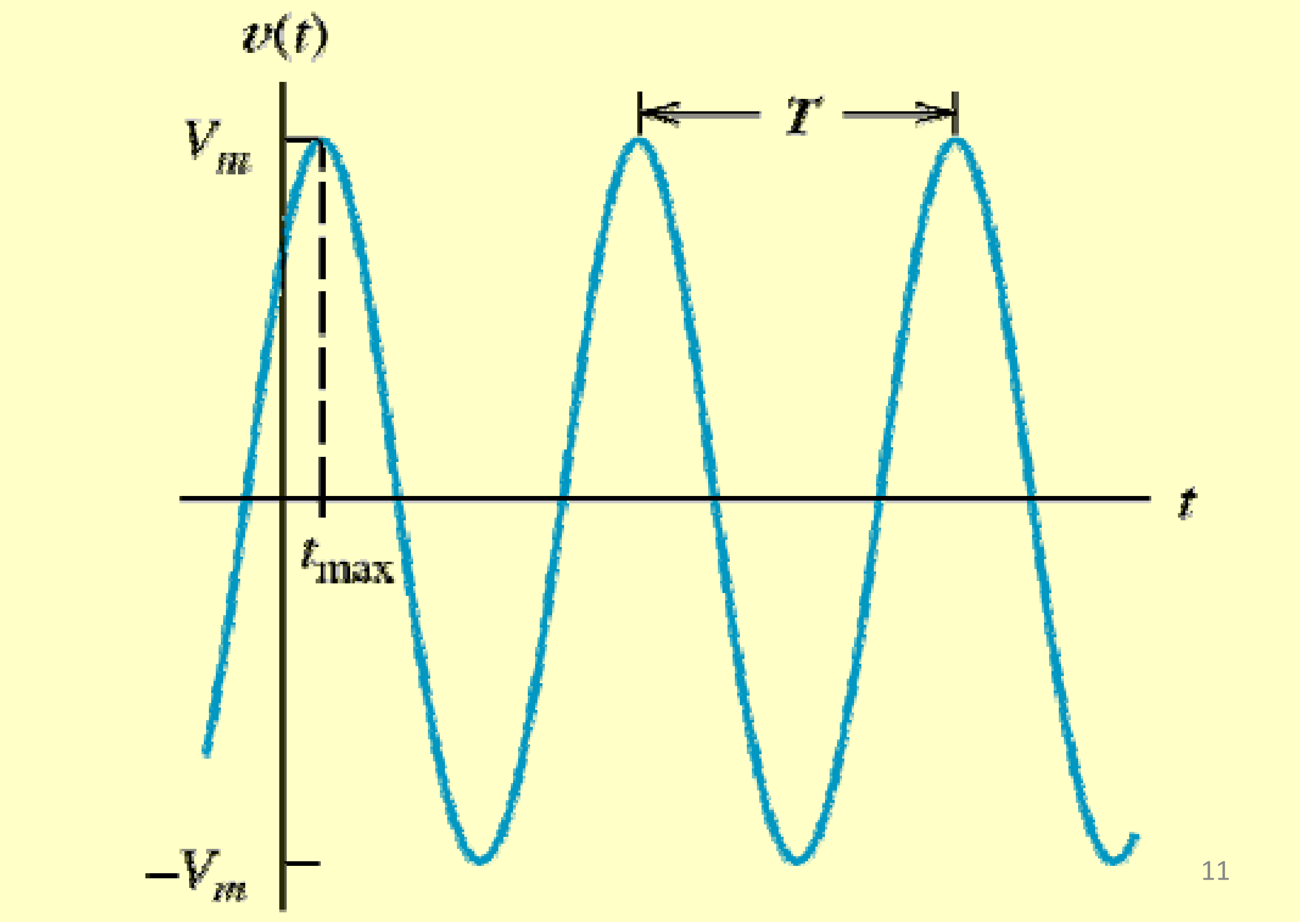




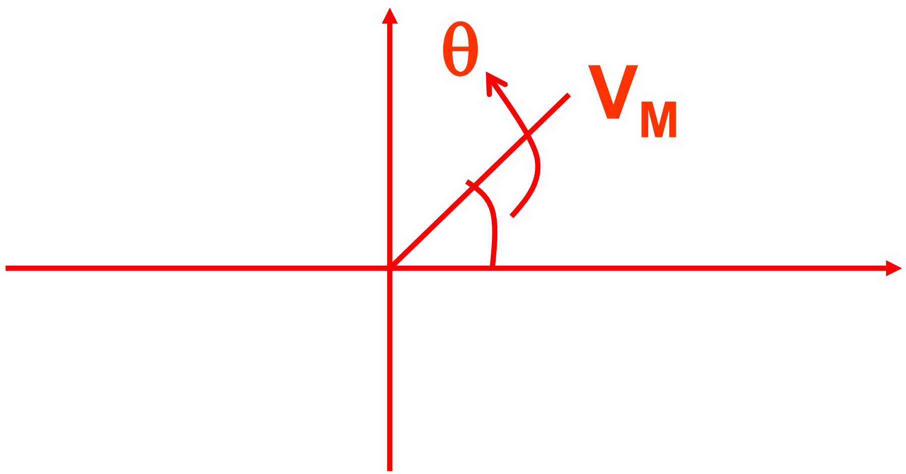
Sinusoidal Currents and Voltages

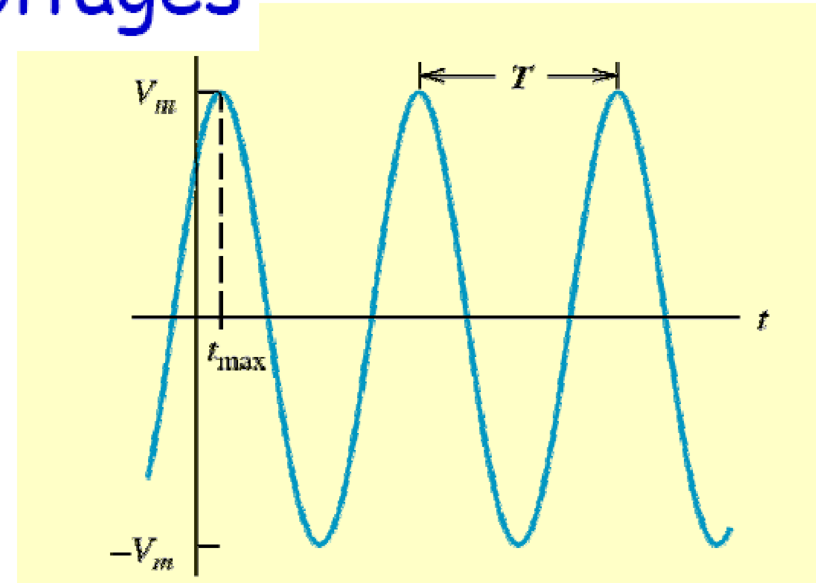
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the peak value



Sinusoidal Currents and Voltages





$$v(t) = V_m \cos(\omega t + \theta)$$

ω is the angular frequency in radians per second

T is the period , where $f = \dfrac{1}{T}$ is the frequency

$\omega = \dfrac{2\pi}{T}$ $\quad \omega = 2\pi f$ $\quad \theta$ is the phase angle

Example-1

$$5\sin(4\pi t - 60^o)$$

What is the amplitude, phase, angular frequency, time period, frequency?

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\sin(z) = \cos(z - 90^\circ)$$

$$v(t) = 5 \cos(4\pi t - 60^o - 90^o)$$

Amplitude = 5 ; Phase = -150^o

Phase in radians:

360^o = 2 π

$$\theta = \frac{-150}{360} \times 2\pi = -2.618 \quad \text{radians}$$

$$\omega = 4\pi \quad r/s$$

$$\omega = \frac{2\pi}{T} = 4\pi \Rightarrow T = 0.5s$$

$$f = \frac{1}{T} = 2Hz$$

Example-2 Find the phase difference between the two currents

$$i_1 = 4\sin(377t + 25^o)$$

$$i_2 = -5\cos(377t - 40^o)$$

$$x(t) = x_m \cos(\omega t + \theta)$$

$$i_1 = 4\cos(377t + 25^o - 90^o)$$

$$\theta_1 = -65^o$$

$$i_2 = 5\cos(377t - 40^o + 180^o)$$

$$\theta_2 = 140^o$$

$$\theta_1 - \theta_2 = -205^o$$

Which signal leads and by how much?

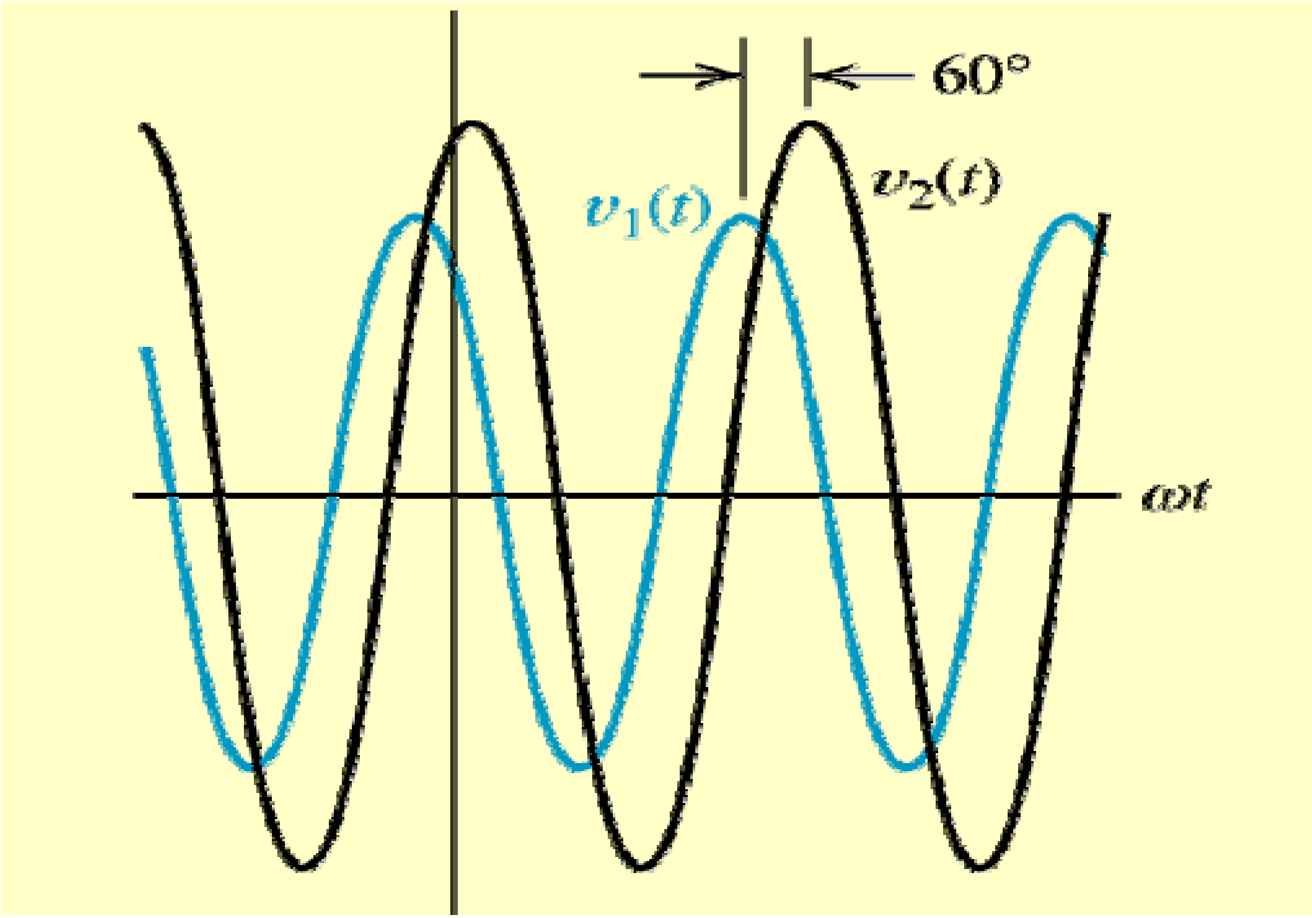
$$\sin(\omega t \pm 180^o) = -\sin \omega t$$

$$\cos(\omega t \pm 180^o) = -\cos \omega t$$

$$\sin(\omega t \pm 90^o) = \pm\cos \omega t$$

$$\cos(\omega t \pm 90^o) = \mp\sin \omega t$$

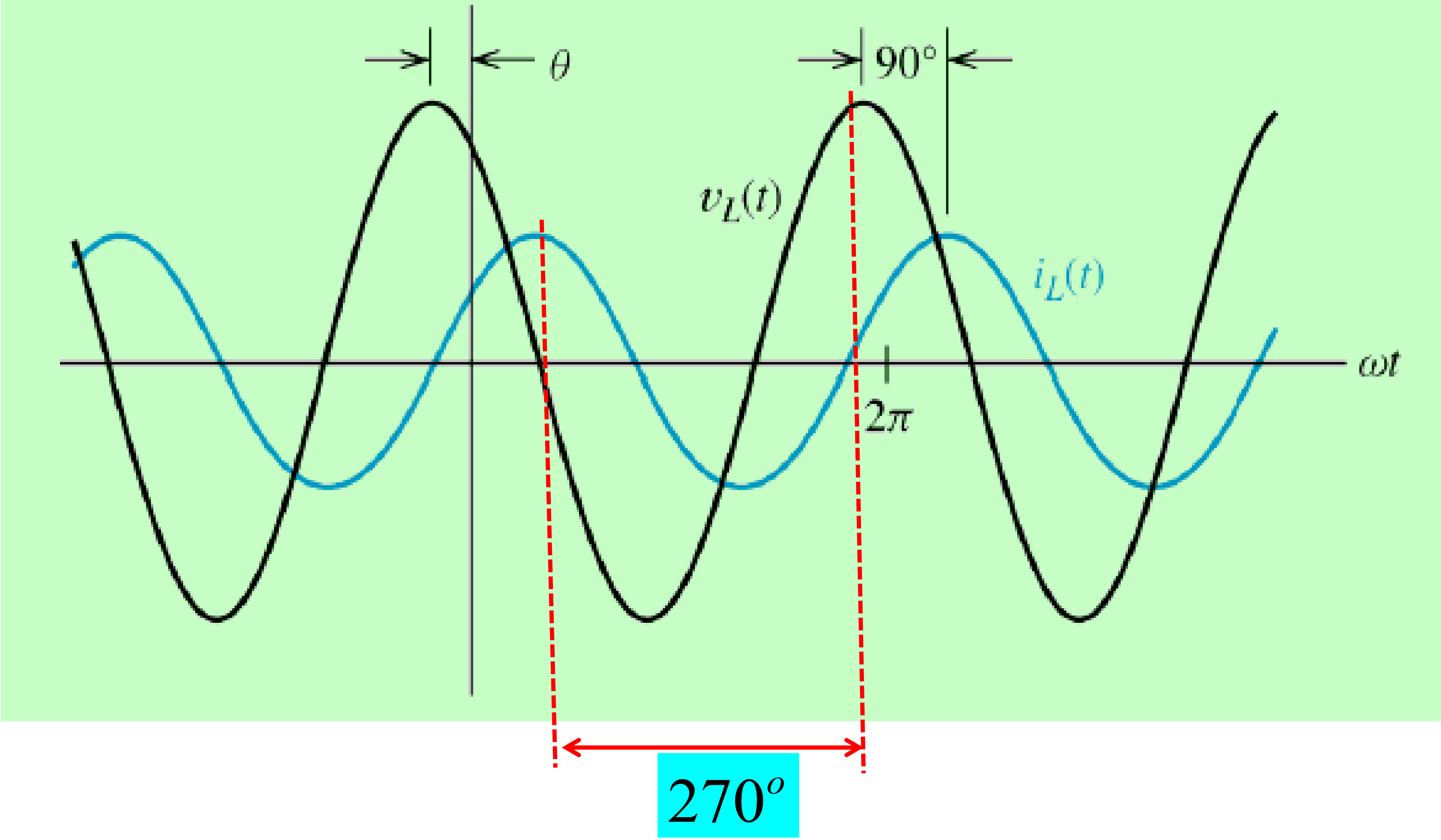
Phase Relationships



$$v_2(t) = v_{2m} \ \cos(\omega t)$$

$$v_1(t) = v_{1m} \ \cos(\omega t + 60^o)$$

The peaks of $v_1(t)$ occur 60° before the peaks of $v_2(t)$. In other words, $v_1(t)$ leads $v_2(t)$ by 60°.



Voltage leads current by 90° or lags current by 270° ?

Phase difference is usually considered between -180 to 180°

Add or subtract 360° to bring the phase between -180 to 180°

$$i_1 = 4\cos(377t - 65^o)$$

$$i_2 = 5\cos(377t + 140^o)$$

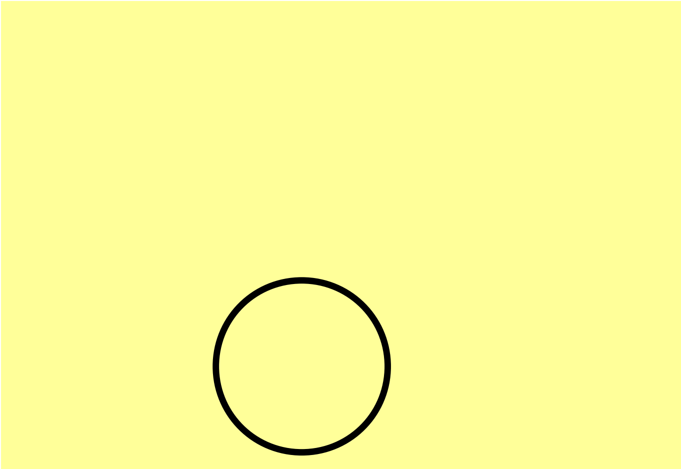
Does i_2 lead i_1 ?

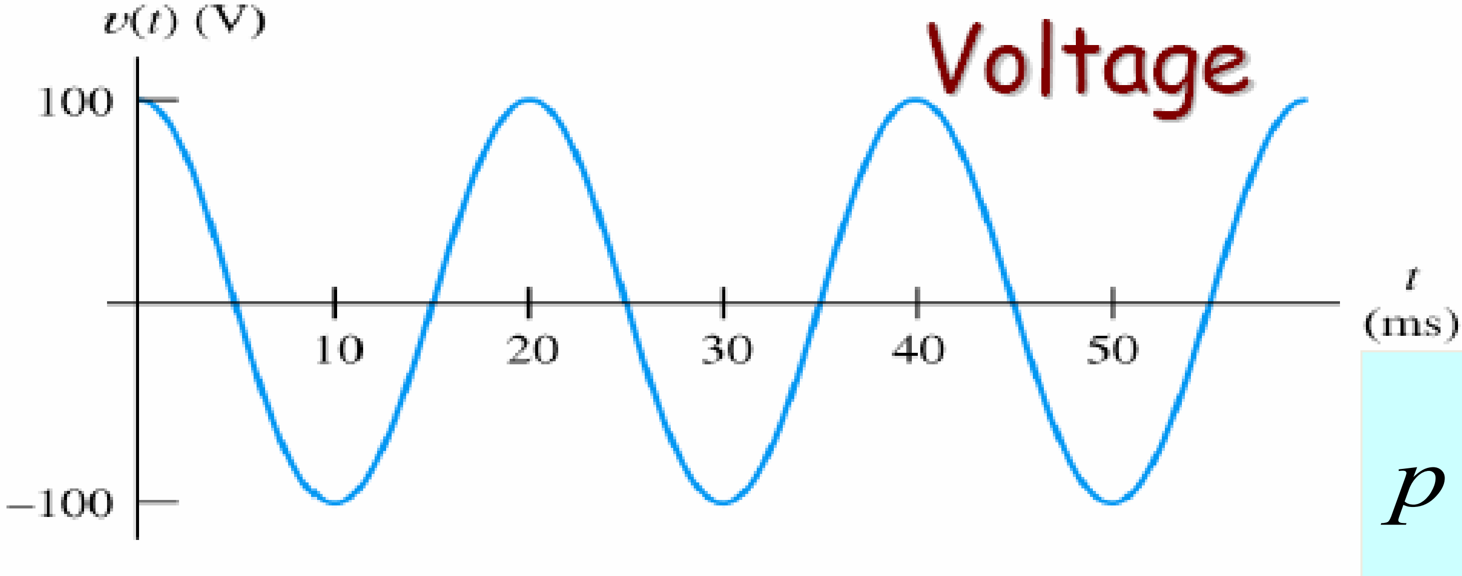
$$\theta_1 - \theta_2 = -205^o$$

$$\theta_1 - \theta_2 = -205^o + 360^o = 155^o$$

i_1 leads i_2 by 155^o

Power dissipation with sinusoidal Voltage

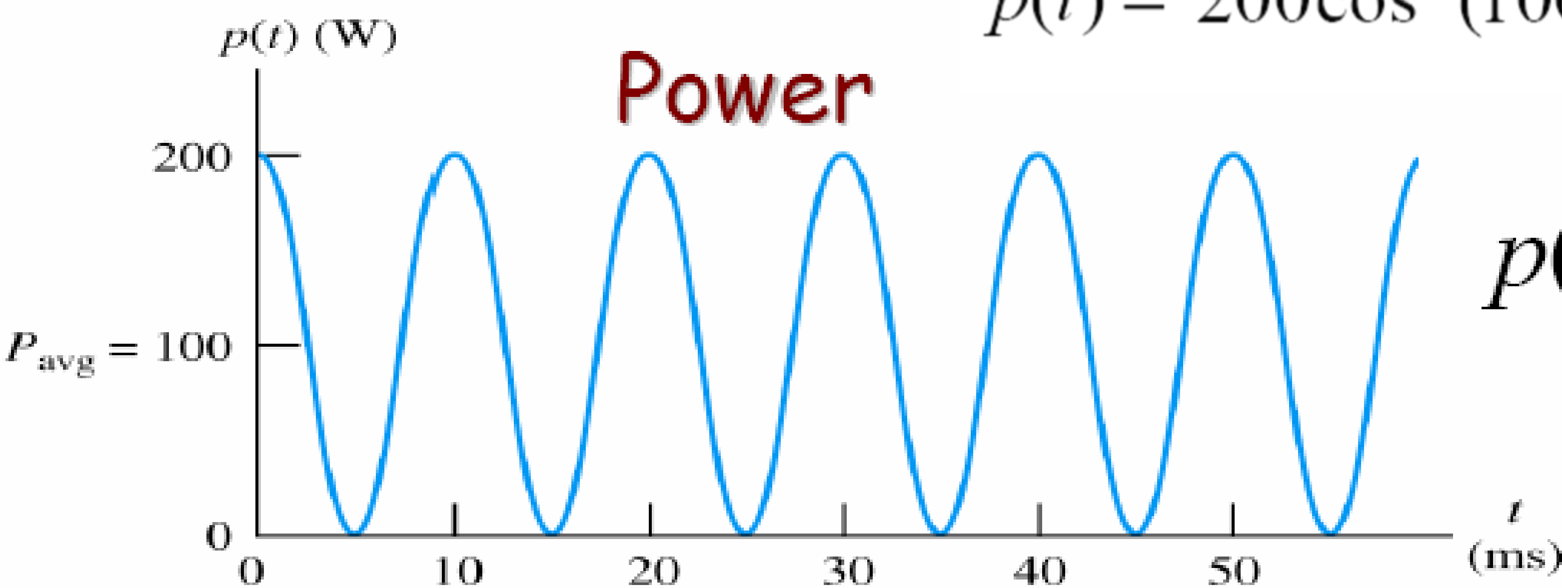




(a)

$$p = \frac{v(t)^2}{R}$$

$$p(t) = 200\cos^2(100\pi t)\ \mathrm{W}$$



$$p(t)$$

Average

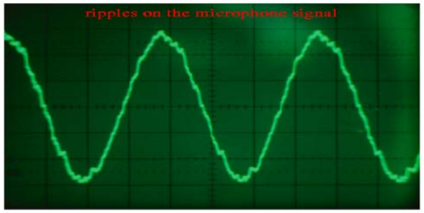
X: $x_1, x_2, x_3, \quad\quad ..x_N$

$$x_{avg} = \frac{1}{N}\sum x_i$$

If X is continuous, its average over a time t_1

$$x_{avg} = \frac{1}{t_1}\int_0^{t_1} x(t)\,dt$$

For periodic signals



$$x_{avg} = \frac{1}{T}\int_0^{T} x(t)\,dt$$

Average Power

$$p_{avg} = \frac{1}{T}\int_0^T \frac{v(t)^2}{R}\,dt$$

We would like to express it like the dc power dissipated in a resistor

$$p_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}\right]^2}{R}$$

$$p = \frac{V^2}{R}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$$

$$p_{avg} = \frac{V_{rms}^2}{R}$$

This is true for any periodic waveform

RMS Value of a Sinusoid

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$$

$$v(t) = V_m \cos(\omega t + \theta)$$

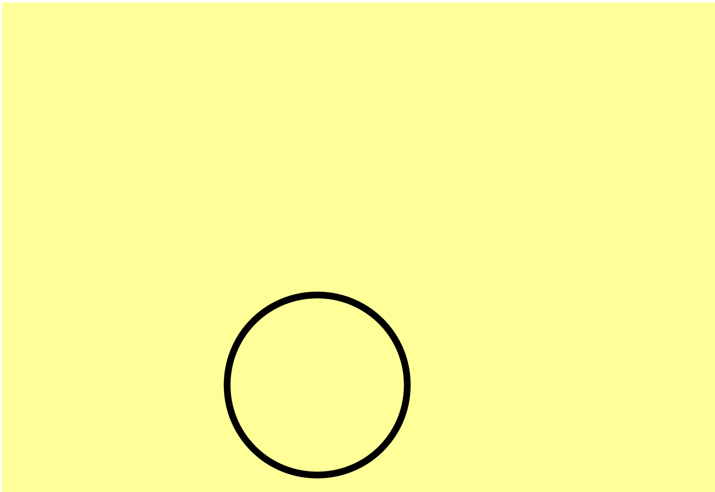
$$\int_0^T \cos^2(\omega t + \theta)dt = \int_0^T \frac{1 - \cos(2\omega t + 2\theta)}{2}dt$$

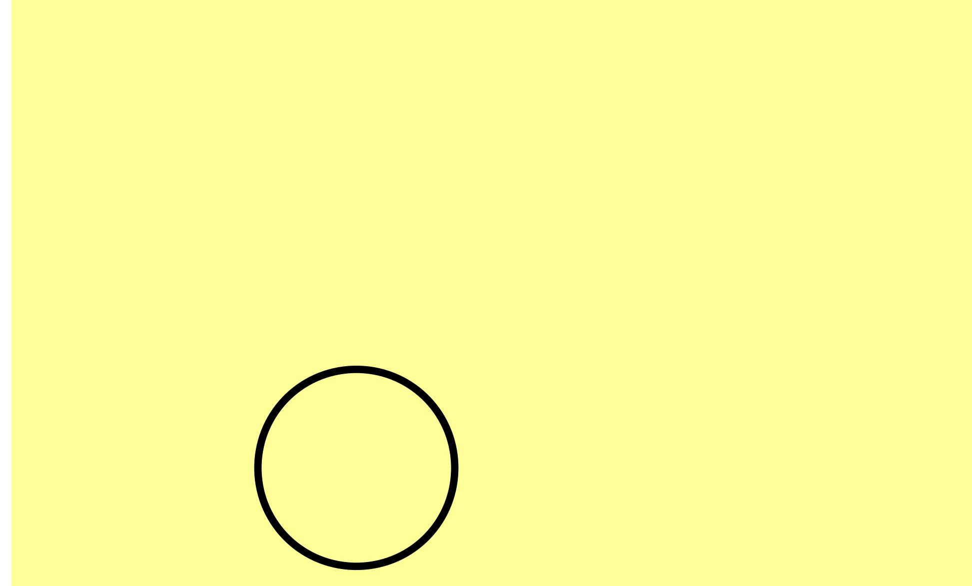
$$= 0.5T - \frac{1}{4\omega}\sin(2\omega t + 2\theta)\Big|_0^T = 0.5T$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

The RMS value for a sinusoid is the peak value divided by the square root of 2

Power dissipation with sinusoidal Voltage



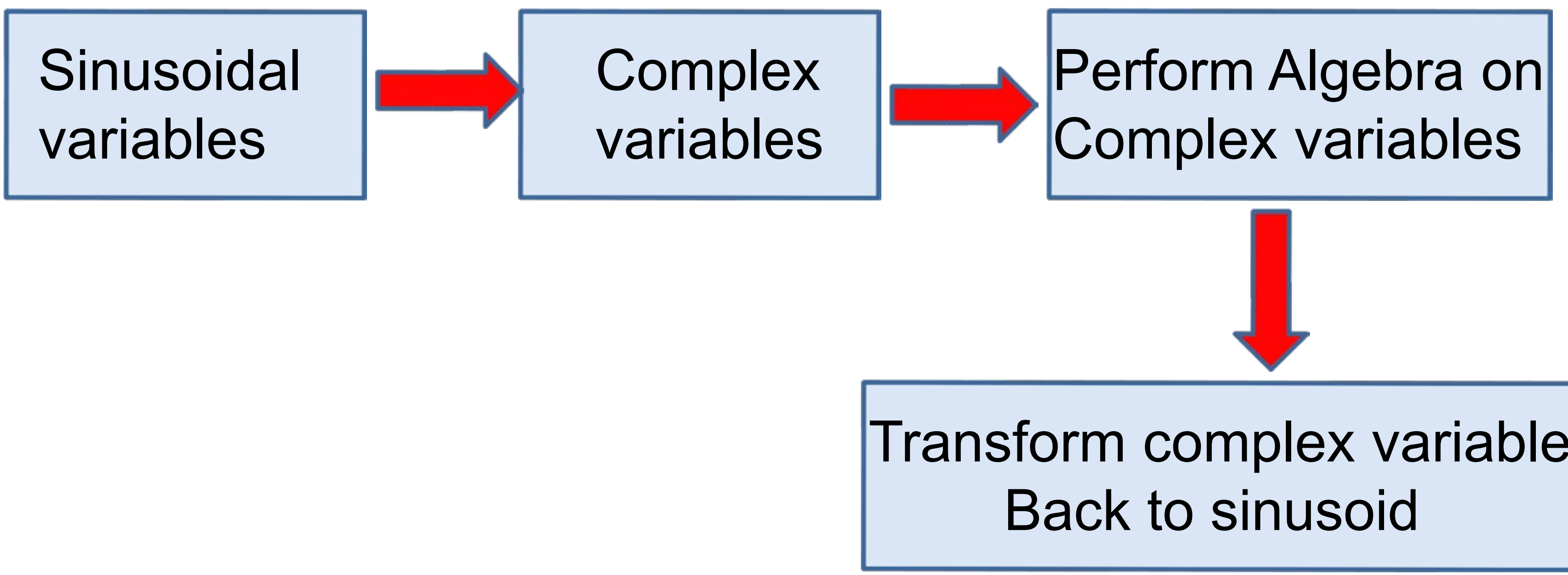


Performing algebra on sinusoids by representing them as complex numbers

$$v_1(t) = 20\cos(\omega t - 45^\circ) \qquad v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$v_1(t) + v_2(t) = ?$$

Strategy



$$20\cos(\omega t - 45^\circ) \longrightarrow \mathbf{V}_1 = 20\angle -45^\circ$$

$$14.14 - j14.14$$

$$10\sin(\omega t + 60^\circ) \longrightarrow \mathbf{V}_2 = 10\angle -30^\circ$$

$$8.660 - j5$$

$$\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}$$

$$v_s(t) = 29.97\cos(\omega t - 39.7^\circ)$$

Complex Numbers

$$z = x + jy$$

Real part

$\sqrt{-1}$

Imaginary part

$$z_1 = 5 + j5 \qquad z_2 = 3 - j4$$

$$z_1 + z_2 = (5 + j5) + (3 - j4) = 8 + j1$$
$$z_1 - z_2 = (5 + j5) - (3 - j4) = 2 + j9$$

Complex conjugate of z is:

$$z^* = x - jy$$

Complex Numbers

$$z_1 = 5 + j5 \qquad z_2 = 3 - j4$$

$$\begin{aligned} z_1 z_2 &= (5 + j5)(3 - j4) \\ &= 15 - j20 + j15 - j^2 20 \\ &= 15 - j20 + j15 + 20 \\ &= 35 - j5 \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{5 + j5}{3 - j4} \times \frac{z_2^*}{z_2^*}$$

$$= \frac{5 + j5}{3 - j4} \times \frac{3 + j4}{3 + j4}$$

$$= \frac{15 + j20 + j15 + j^2 20}{9 + j12 - j12 - j^2 16}$$

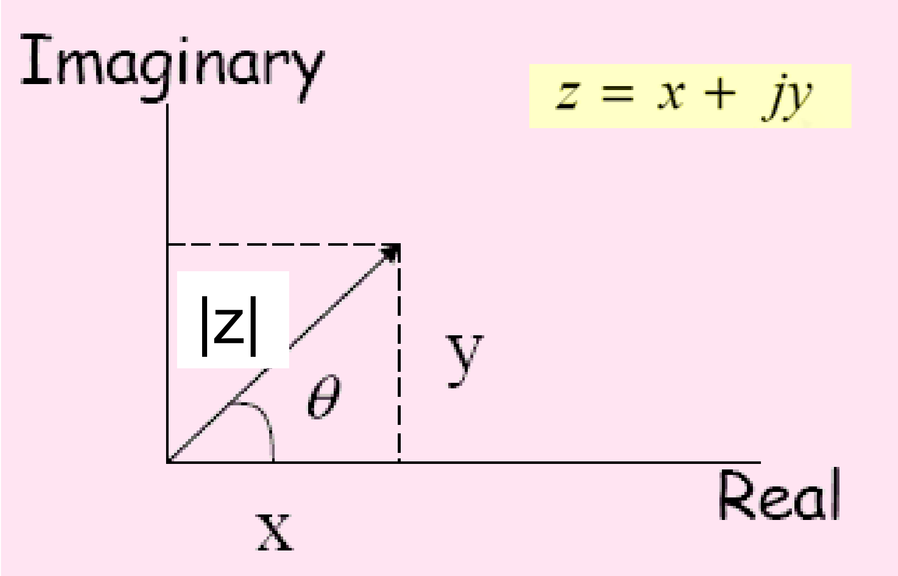
$$= \frac{15 + j20 + j15 - 20}{9 + j12 - j12 + 16}$$

$$= \frac{-5 + j35}{25}$$

$$= -\frac{5}{25} + j\frac{35}{25}$$

$$= 0.2 + j1.4$$

A complex number can be represented as a point in the complex Plane



Represent the complex number by the length of the arrow and the angle between the arrow and the positive real axis

$$|z| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}$$

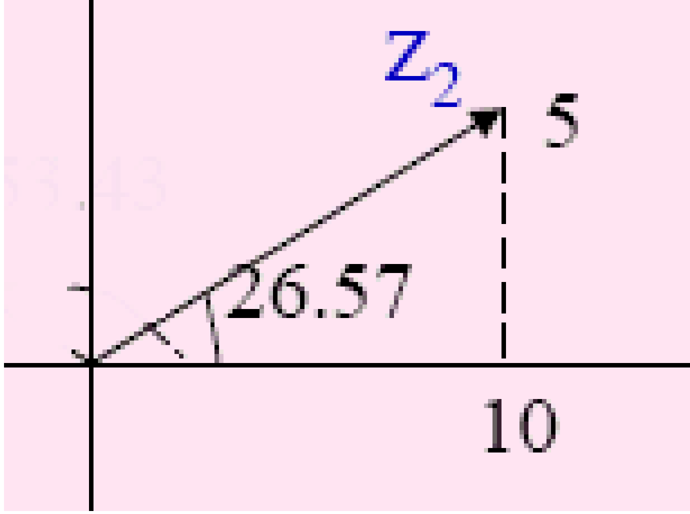
Polar form: $z = |z|\angle\theta$

Rectangular Polar form

$$z_2 = 10 + j5$$

$$z_2 = \sqrt{(10)^2 + (5)^2}\angle\tan^{-1}\left(\frac{5}{10}\right)$$

$$= 11.18\angle 26.57^\circ$$

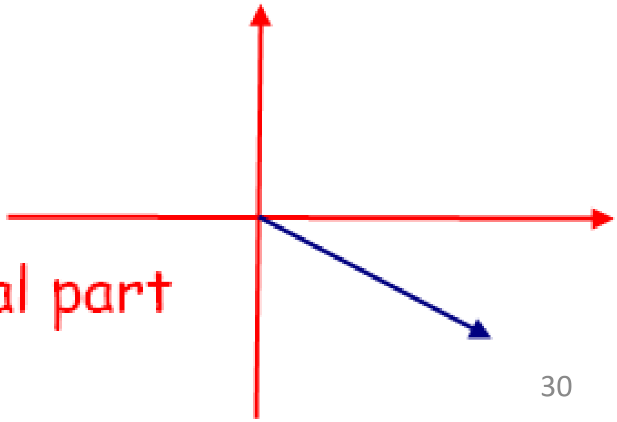


$$z_3 = -10 + j5$$

$$z_3 = \sqrt{(10)^2 + (5)^2}\angle\tan^{-1}\left(\frac{5}{-10}\right)$$

$$= 11.18\angle -26.57^\circ$$

Wrong angle since real part is negative;



Rectangular Polar form:

$$z_3 = -10 + j5$$

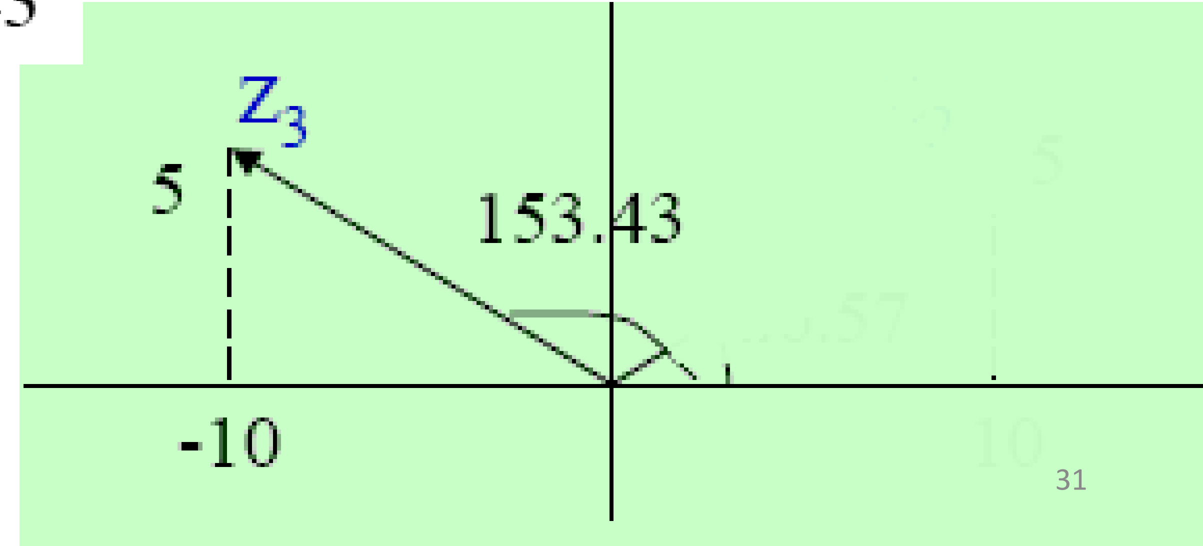
$$z_3 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{-10}\right)$$

the true angle is:

$$\theta = \tan^{-1}(y/x) \pm 180^\circ$$

$$= -26.57 + 180 = 153.43^\circ$$

Be careful while determining the phase angle



$$z = x + jy \iff z = |z|\angle\theta$$

$$? \Leftarrow z_1 = 5\angle 30^\circ$$

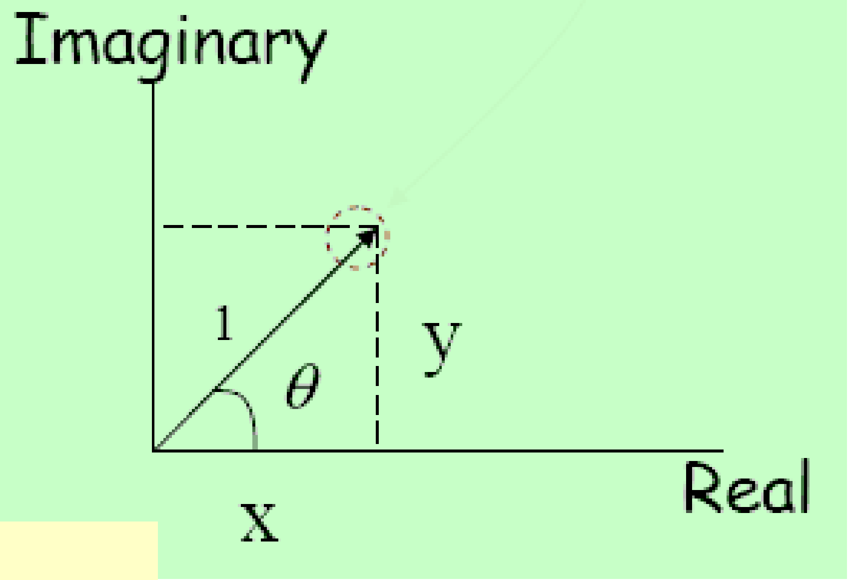
Euler's Identities

$$e^{j\theta} = 1\angle\theta = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$e^{-j\theta} = 1\angle -\theta = \cos\theta - j\sin\theta$$

$$A\angle\theta = A\cos\theta + Aj\sin\theta$$



$$z_1 = 5\angle 30^\circ$$

$$z_1 = 5\cos(30^\circ) + j5\sin(30^\circ)$$

$$= 4.33 + j2.5 = x + jy$$

Forms of a Complex Number

$$z_2 = 10 + j5$$ Rectangular form

$$z_2 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{10}\right)$$

$$= 11.18 \angle 26.57^\circ$$ Polar form

$$= 11.18 e^{j26.57^\circ}$$ Exponential form

Important Note:

$$1\angle 90^\circ = \cos 90 + j \sin 90 = j$$

Arithmetic Operations in Polar and Complex Form

To **add** or **subtract** two complex numbers, convert them first into rectangular form and then perform the operation

- To multiply two complex numbers in polar form

$$z_1\, z_2 = |z_1| \angle \theta_1 \times |z_2| \angle \theta_2$$
$$= |z_1|\,|z_2| \angle (\theta_1 + \theta_2)$$

$$|z_1| \angle \theta_1 \quad = |z_1|\, e^{j\theta_1}$$

$$z_1\, z_2 = |z_1|\, e^{j\theta_1} \times |z_2|\, e^{j\theta_2}$$
$$= |z_1|\,|z_2|\, e^{j(\theta_1+\theta_2)}$$

- To divide two complex numbers in polar form

$$\frac{z_1}{z_2} = \frac{|z_1|\angle\theta_1}{|z_2|\angle\theta_2}$$

$$= \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

$$|z_1|\angle\theta_1 = |z_1| e^{j\theta_1}$$

$$\frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}}$$

$$= \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)}$$

$$j150 \times 0.707\angle -15^\circ = 106.1\angle 75^\circ$$

Important Note:

$$1\angle 90^\circ = \cos 90 + j \sin 90 = j$$

$$v(t) = V_m \cos(\omega t + \theta)$$

$$v(t) = \mathrm{Re}(V_m \times e^{j(\omega t + \theta)})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$v(t) = \mathrm{Re}(V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta))$$

$$v(t) = V_m \cos(\omega t + \theta) \Rightarrow \mathrm{Re}(V_m \angle \omega t + \theta)$$

$$v(t) = V_m \cos(\omega t + \theta) \Leftrightarrow V_m \angle \theta$$

Phasor

$$v(t) = 3\ \cos(\omega t + 45) \Rightarrow 3\ \angle 45$$

$$\Rightarrow 3\ \cos(45) + j3\sin(45)$$

$$v(t) = 5\ \cos(\omega t - 60) \Leftarrow 5\ \angle -60$$

$$v_1(t) = 20\cos\left(\omega t - 45^\circ\right) \qquad v_2(t) = 10\sin\left(\omega t + 60^\circ\right)$$

$$v_1(t) + v_2(t) = ?$$

$$v_1(t) = 20\cos(\omega t - 45^\circ) \qquad v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$20\cos(\omega t - 45^\circ) \to 20\angle -45^\circ$$

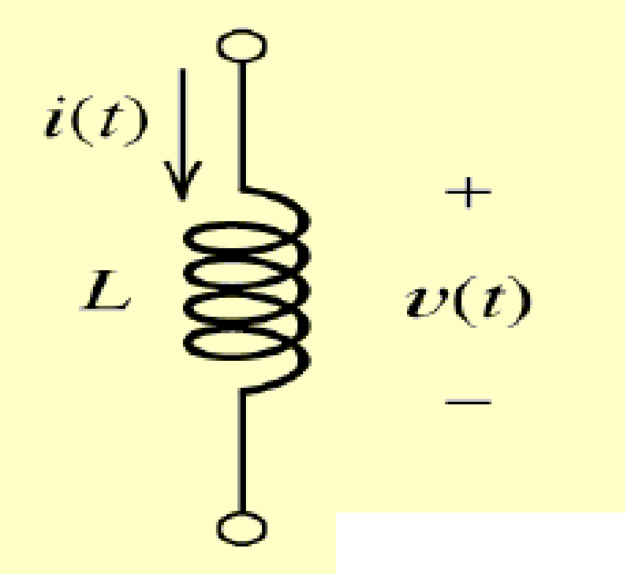
$$10\sin(\omega t + 60^\circ) = 10\cos(\omega t + 60^\circ - 90^\circ) \to 10\angle -30^\circ$$

$$\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}$$

$$v_s(t) = 29.97\cos(\omega t - 39.7^\circ)$$

Complex Impedances

For the purpose of sinusoidal steady state analysis, inductors and capacitors can be represented as Complex Impedances

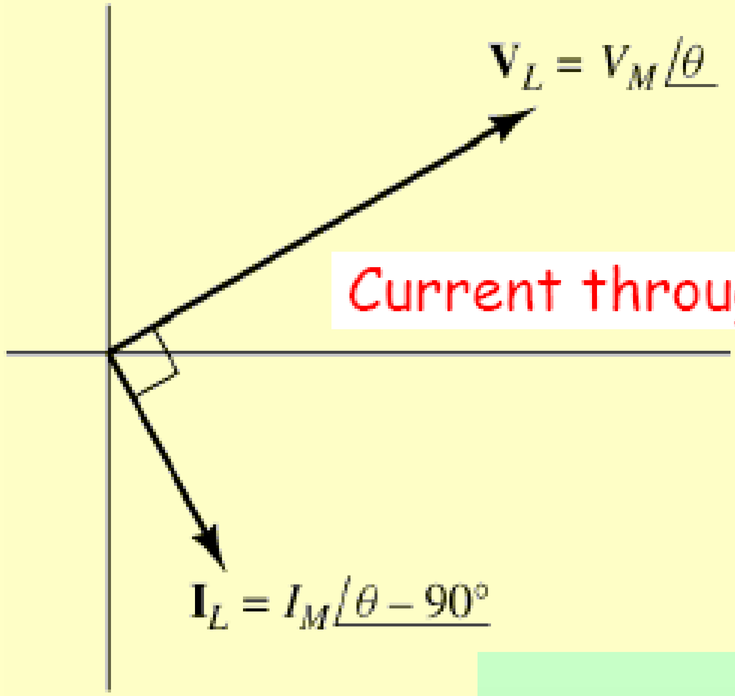


$$i_L = I_m \sin(\omega t + \theta) \longrightarrow \mathbf{I_L} = I_m \angle \theta - 90^\circ$$

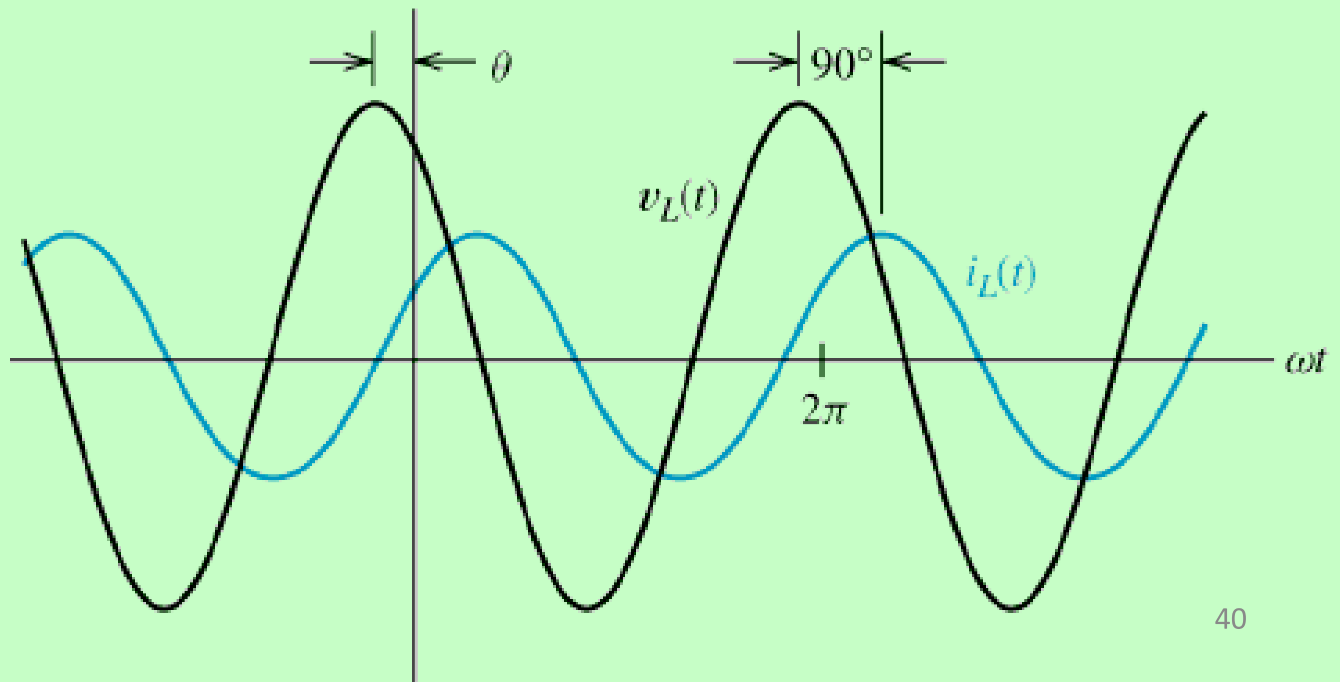
$$v_L = L\frac{di_L}{dt} \longrightarrow \mathbf{V_L} = \omega L I_m \angle \theta$$

$$= \omega L I_m \cos(\omega t + \theta)$$

Current through the inductor lags the voltage by 90°



Current through the inductor lags the voltage by 90°



$$I_L = I_M \angle \theta - 90$$

$$V_L = \omega L I_M \angle \theta$$

$$V_L = \omega L I_M \angle \theta - 90 + 90$$

$$V_L = I_M \angle \theta - 90 \times \omega L \angle 90$$

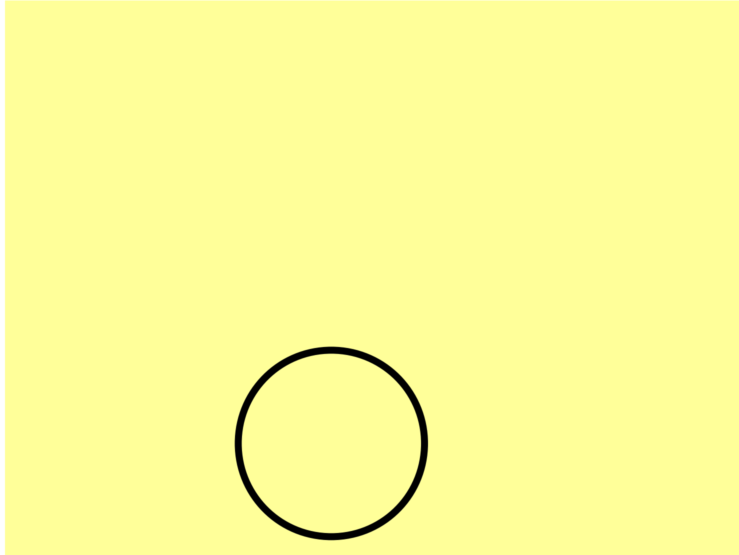
$$V_L = I_L \times \omega L \angle 90$$

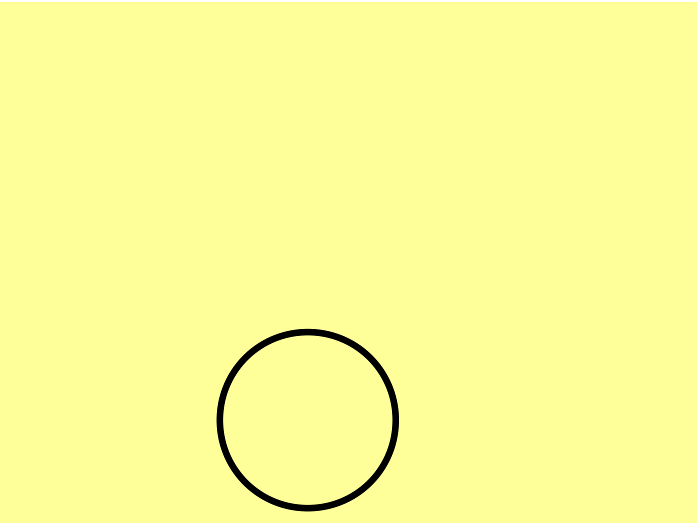
$$V_L = I_L \times j\omega L$$

$$V_L = I_L \times Z_L$$

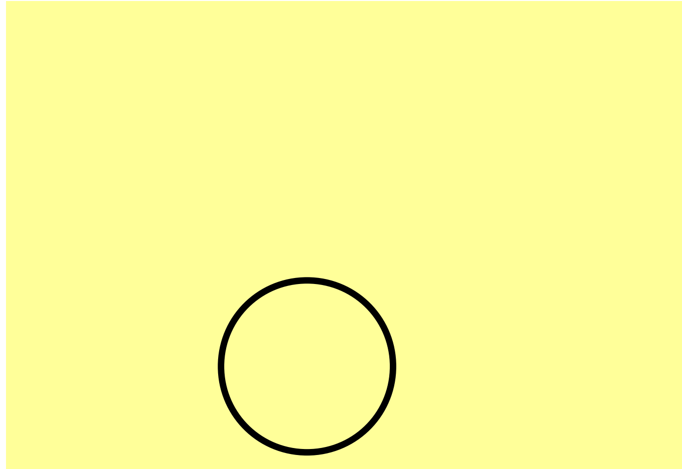
$$Z_L = j\omega L$$

This is like ohms law relationship between phasor voltage and current





Resistor



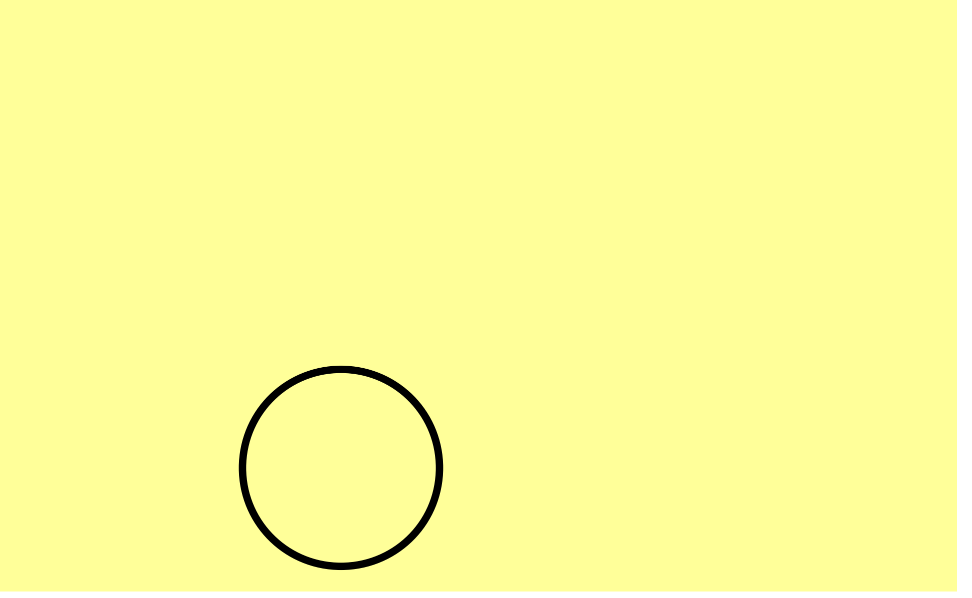
$$v(t) = V_M \cos(\omega t + \theta)$$

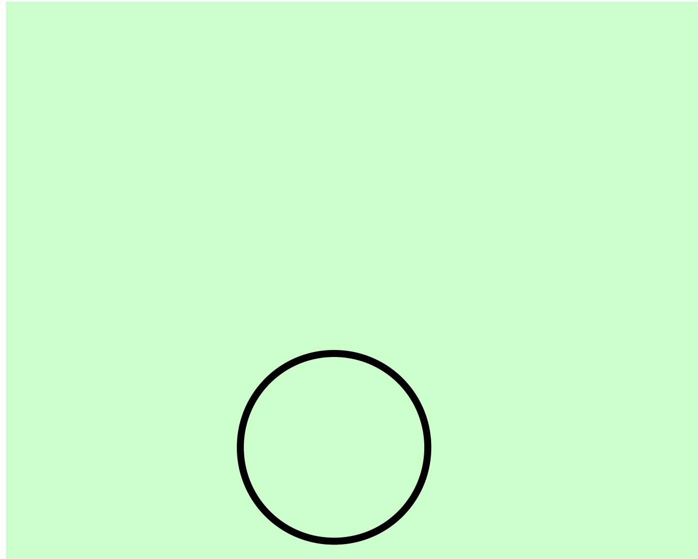
$$i(t) = \frac{V_M}{R} \cos(\omega t + \theta)$$

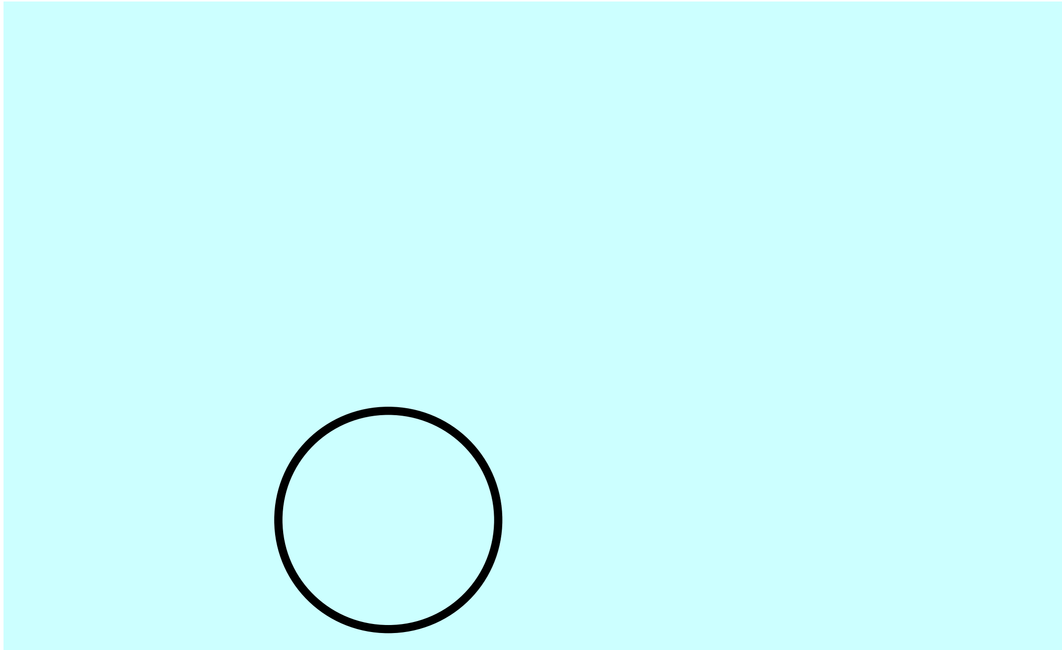
$$V_R = V_M \angle \theta$$

$$I_R = \frac{V_M}{R} \angle \theta$$

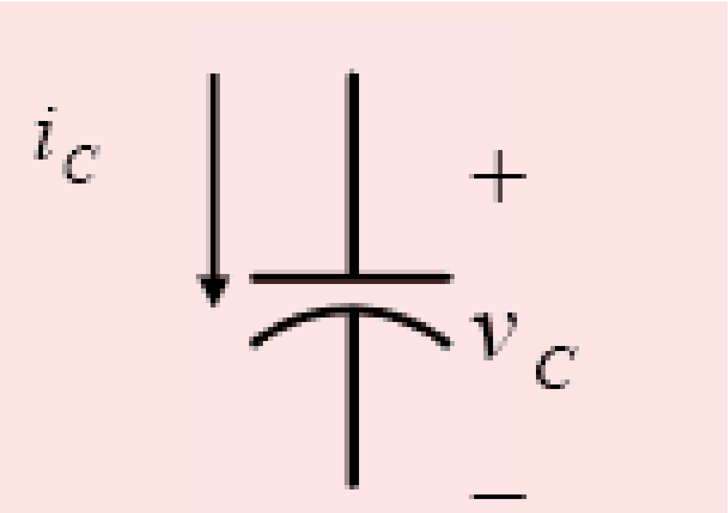
$$I_R = \frac{V_R}{R}$$







Capacitor



$$v(t) = V_M \cos(\omega t + \theta)$$

$$i_c = C \frac{dv_c}{dt}$$

$$i(t) = -\omega C V_M \sin(\omega t + \theta)$$

$$i(t) = \omega C V_M \cos(\omega t + \theta + 90^o)$$

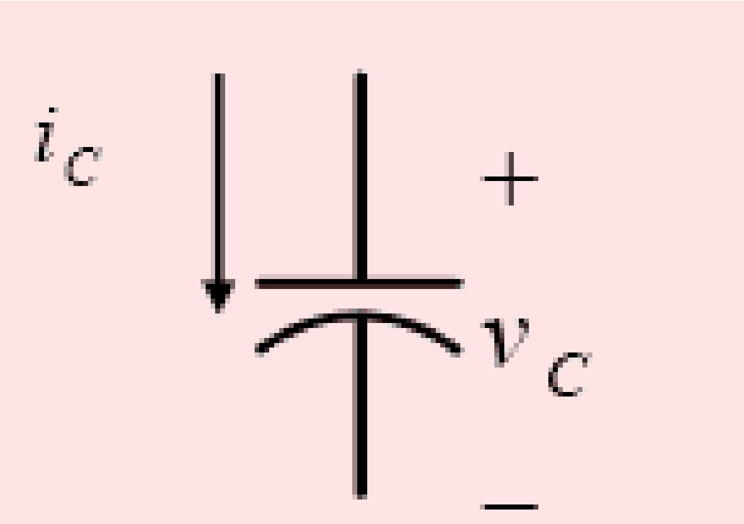
$$V_C = V_M \angle \theta$$

$$I_C = \omega C V_M \angle \theta + 90$$

In a capacitor, current leads voltage by 90^0

Capacitor

$$V_C = V_M \angle \theta$$



$$I_C = \omega C V_M \angle \theta + 90$$

$$I_C = \omega C \angle 90 \times V_M \angle \theta$$

$$I_C = j\omega C V_C$$

$$V_C = I_C \times Z_C$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency)

2. Express components by their complex impedances:

   Replace inductances by their complex impedances

$$Z_L = j\omega L$$

   Replace capacitances by their complex impedances

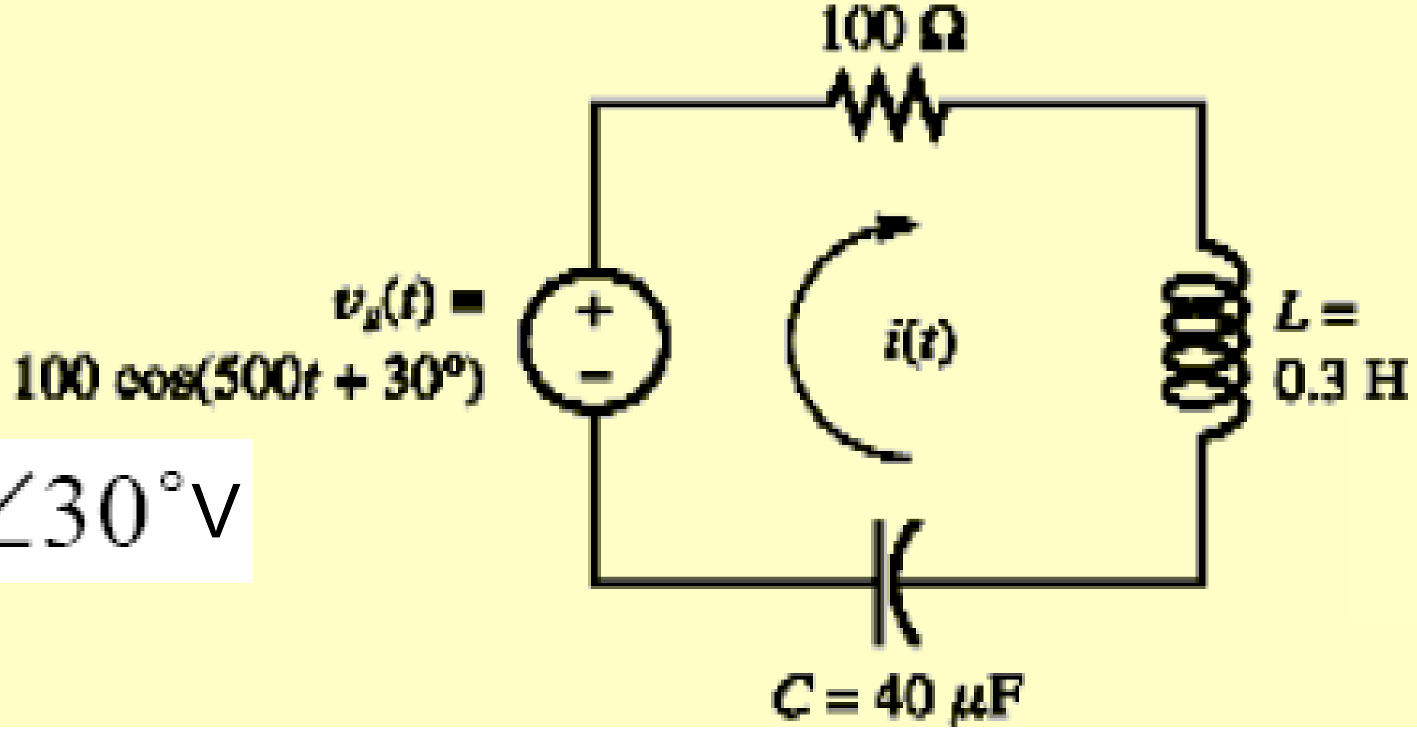
$$Z_C = 1/(j\omega C)$$

   Resistances have impedances equal to their resistances

$$Z_R = R$$

3 Analyze the circuit using any of the techniques studied earlier performing the calculations with complex arithmetic

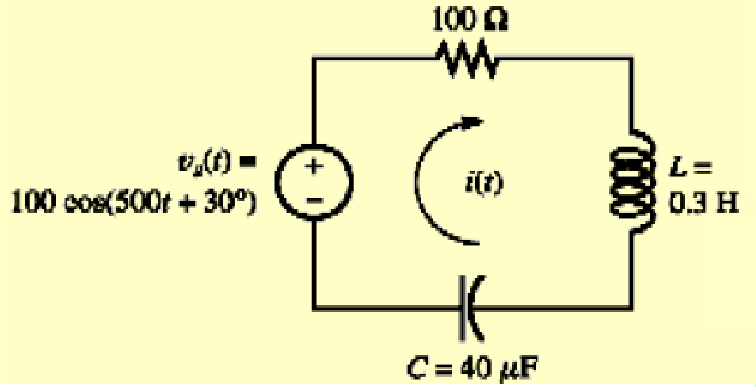
Example-6

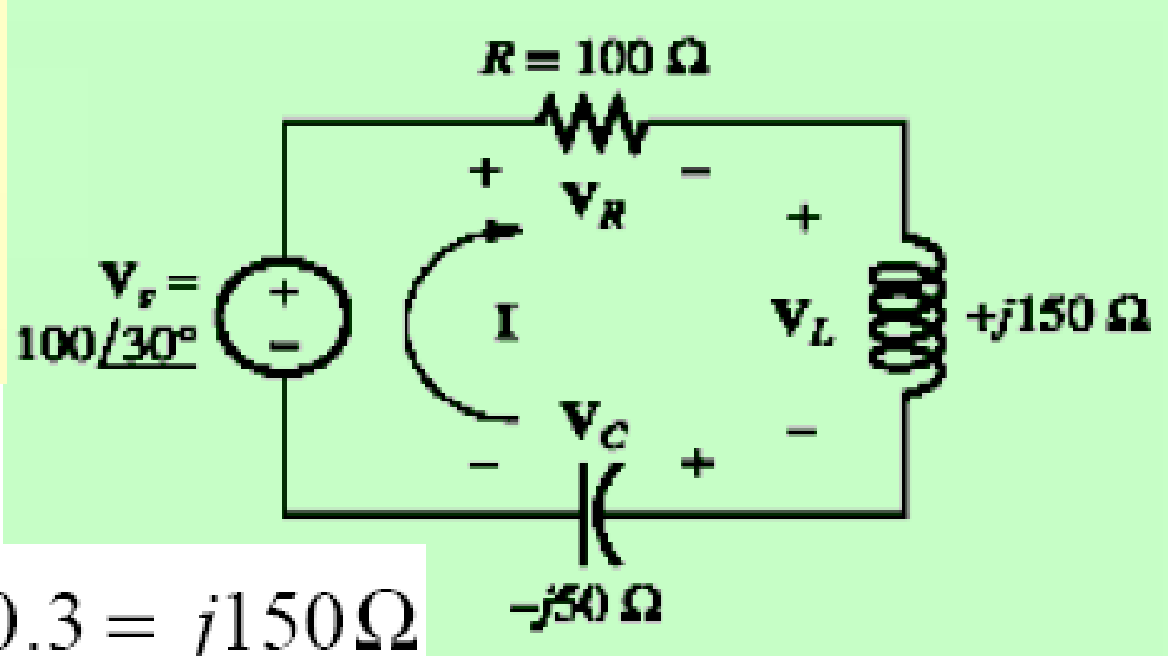


$$\mathbf{V}_s = 100\angle 30^\circ \text{V}$$

$$Z_L = j\omega L = j500\times 0.3 = j150\,\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{500\times 40\times 10^{-6}} = -j50\,\Omega$$





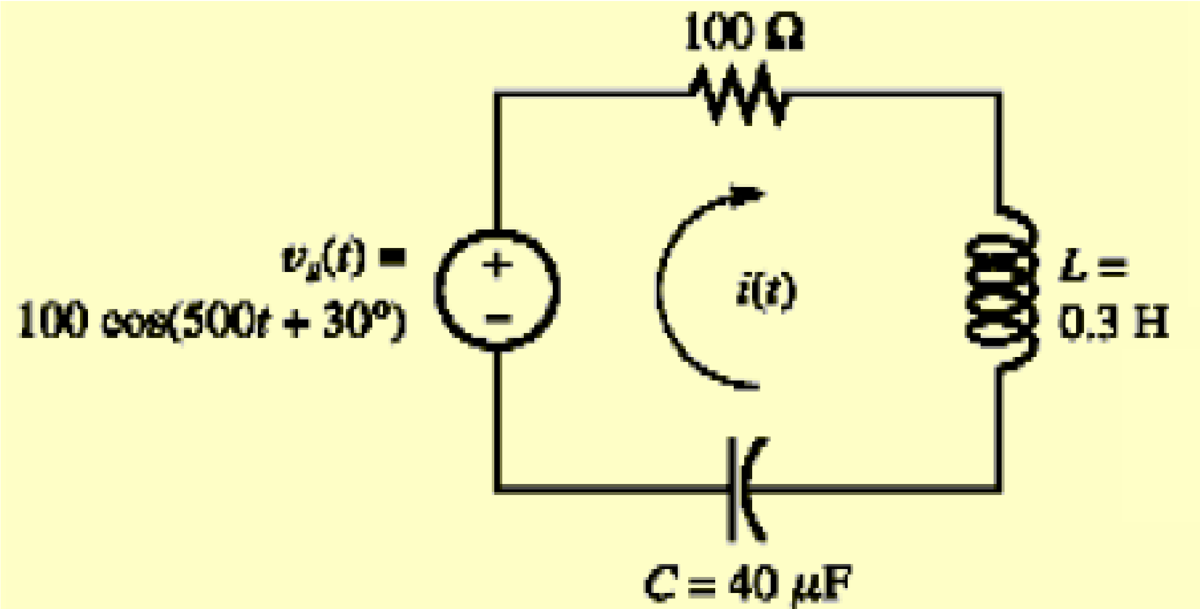
$$Z_L = j\omega L = j500 \times 0.3 = j150\,\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{500 \times 40 \times 10^{-6}} = -j50\,\Omega$$

$$Z_{eq} = 100 + j150 - j50 = 100 + j100 = 141.4\angle 45^\circ\ \Omega$$

$$I = \frac{\mathbf{V}_s}{Z_{eq}} = \frac{100\angle 30^\circ}{141.4\angle 45^\circ} = 0.707\angle -15^\circ \text{ A}$$

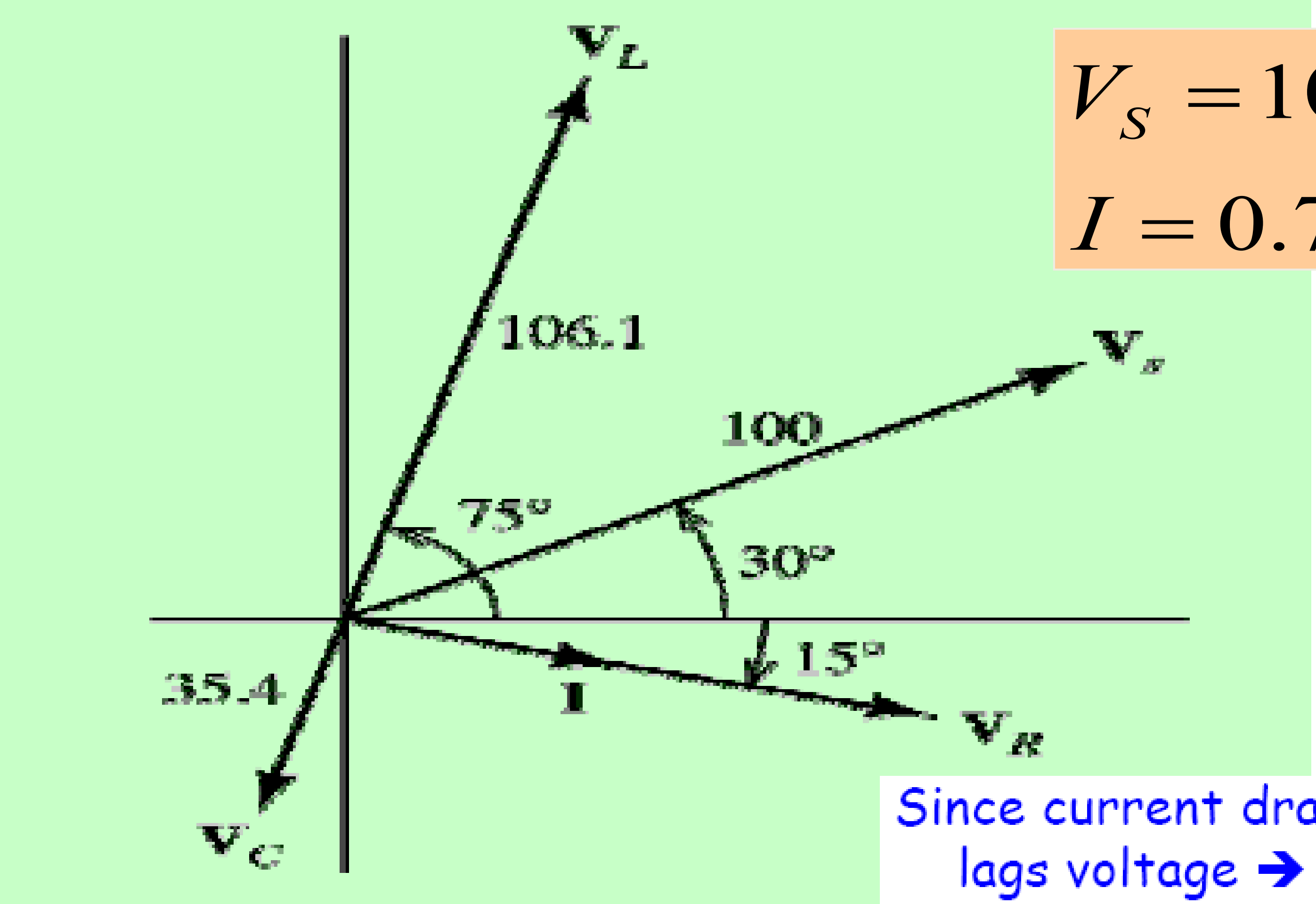
$$i(t) = 0.707\cos(500t - 15^\circ) \text{ A}$$



$$\mathbf{V_R} = R\mathbf{I} = 100\times 0.707\angle -15^\circ = 70.7\angle -15^\circ \text{ V}$$

$$\mathbf{V_L} = j\omega L\mathbf{I} = j150\times 0.707\angle -15^\circ = 106.1\angle 75^\circ \text{ V}$$

$$\mathbf{V_C} = -j\frac{1}{\omega C}\mathbf{I} = -j50\times 0.707\angle -15^\circ = 35.4\angle -105^\circ \text{ V}$$



$$V_S = 100\angle 30^o$$

$$I = 0.707\angle -15^o$$

Since current drawn from the supply lags voltage → inductive circuit