MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-VII

A. Illustrative Discussion Problems

1. Let
$$F(x,y)=\begin{cases}1, & \text{if } x+2y\geq 1\\ 0, & \text{if } x+2y<1\end{cases}.$$
Does $F(\cdot,\cdot)$ define a d.f.?

2. Let
$$F(x,y)=\begin{cases}0, & \text{if } x<0 \text{ or } x+y<1 \text{ or } y<0\\ 1, & \text{otherwise}\end{cases}.$$
Does $F(\cdot)$ define a d.f.?

3. Let $F(x,y)$ be the d.f. of some two-dimensional r.v. (X,Y), and let $F_1(x)$ and $F_2(y)$, respectively, be the marginal d.f.s of X and Y. Define $U(x,y)=\min\{F_1(x),F_2(y)\}$ and $L(x,y)=\max\{F_1(x)+F_2(y)-1,0\}$. Prove that:

   (i) $L(x,y)$ and $U(x,y)$ are each d.f.s and that their marginal d.f.s are the same as those of $F(x,y)$;

   (ii) $L(x,y)\leq F(x,y)\leq U(x,y)$.
   (**Note:** Let the r.v. X have distribution function $F_1(x)$ and let $Y=g(X)$ have distribution function $F_2(y)$, where $g(\cdot)$ is some function. If $g(\cdot)$ is increasing (decreasing), then $F_{X,Y}(x,y)=U(x,y)$ $(F_{X,Y}(x,y)=L(x,y))$.

4. Let $\underline{X}=(X_1,X_2)$ be a bivariate random vector with distribution function
$$F(x_1,x_2)=\begin{cases}0, & \text{if } x_1<0 \text{ or } x_2<0\\ \frac{x_1x_2}{8}, & \text{if } 0\leq x_1<1, 0\leq x_2<2 \text{ or } 1\leq x_1<2, 0\leq x_2<1\\ \frac{x_1}{4}, & \text{if } 0\leq x_1<1, x_2\geq 2\\ \frac{4+x_1x_2}{8}, & \text{if } 1\leq x_1<2, 1\leq x_2<2\\ \frac{2+x_1}{4}, & \text{if } 1\leq x_1<2, x_2\geq 2\\ \frac{x_2}{4}, & \text{if } x_1\geq 2, 0\leq x_2<1\\ \frac{2+x_2}{4}, & \text{if } x_1\geq 2, 1\leq x_2<2\\ 1, & \text{if } x_1\geq 2, x_2\geq 2\end{cases}.$$
Find $P(\{(X_1,X_2)=(0,0)\})$ and $P(\{(X_1,X_2)=(1,1)\})$. Is (X_1,X_2) discrete?

5. For the bivariate p.m.f. (called negative binomial distribution)

$$f_{X,Y}(x,y) = P(X = x, Y = y) = \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x\theta_2^y(1-\theta_1-\theta_2)^k, \ x, y = 0, 1, 2, \ldots,$$

where $k \geq 1$ is an integer, $0 < \theta_i < 1$, $i = 1, 2$ and $\theta_1 + \theta_2 < 1$, find the marginal p.m.f.s of X and Y.

6. Three balls are randomly placed in three empty boxes B_1, B_2 and B_3. Let N denote the total number boxes which are occupied and let X_i denote the number of balls in the box B_i, $i = 1, 2, 3$.

   (a) Find the joint p.m.f. of (N, X_1);

   (b) Find the joint p.m.f. of (X_1, X_2);

   (c) Find the marginal distributions of N and X_2;

   (d) Find the marginal p.m.f. of X_1 from the joint p.m.f. of (X_1, X_2).

7. Let $\underline{X} = (X_1, X_2, X_3)$ be a discrete type random vector with p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} cx_1x_2x_3, & \text{if } (x_1, x_2, x_3) \in \{1, 2\} \times \{1, 2, 3\} \times \{1, 3\} \\ 0, & \text{otherwise} \end{cases},$$

   where c is a real constant.
   (a) Find the value of c;
   (b) Find the marginal p.m.f.s. of X_1; of X_2; of X_3; (c) Find the marginal p.m.f. of $\underline{Y} = (X_1, X_3)$; (d) Find $P(X_1 = X_2 = X_3)$.

B. Practice Problems from the Text Book

Chapter 2: Multivariate Distributions, Problem Nos.: 1.8, 1.9.

MSO 201a: Probability and Statistics
2016-2017 - II Semester
Assignment VII (Solutions)

Problem No. 1 Suppose that $F(x, y)$ is a d.f. of r.v. (X, Y). Then the marginal d.f. of X is

$$F_X(x) = \lim_{y \to \infty} F(x, y) = 1, \quad \forall x \in \mathbb{R}.$$

Since $F_X(x)$ is not a d.f. ($F_X(-\infty) \neq 0$) in $\mathbb{R}$, $F(x, y)$ is not a d.f. in $\mathbb{R}^2$.

Alt. For rectangle $(\frac{1}{4}, 1] \times (\frac{1}{4}, 1]$

$F(1, 1) - F(\frac{1}{4}, 1) - F(1, \frac{1}{4}) + F(\frac{1}{4}, \frac{1}{4}) = 1 - 1 - 1 + 0 = -1 < 0$

$\Rightarrow F(x, y)$ is not a d.f.

Problem No. 2 For rectangle $(\frac{1}{4}, 1] \times (\frac{1}{4}, 1]$

$F(1, 1) - F(\frac{1}{4}, 1) - F(1, \frac{1}{4}) + F(\frac{1}{4}, \frac{1}{4}) = 1 - 1 - 1 + 0 = -1 < 0$

$\Rightarrow F(x, y)$ is not a d.f.

Problem No. 3 (i) $U(x, y) = \dfrac{F_1(x) + F_2(y) - |F_1(x) - F_2(y)|}{2}$, $x, y \in \mathbb{R}$

$$L(x, y) = \frac{F_1(x) + F_2(y) - 1 + |F_1(x) + F_2(y) - 1|}{2}.$$

Clearly

$$\lim_{x \uparrow \infty,\, y \uparrow \infty} U(x, y) = \lim_{x \uparrow \infty,\, y \uparrow \infty} L(x, y) = 1,$$

$\lim_{x \downarrow -\infty} U(x, y) = \lim_{x \downarrow -\infty} L(x, y) = 0 \;\forall y \in \mathbb{R}$ and $\lim_{y \downarrow -\infty} U(x, y) = \lim_{y \downarrow -\infty} L(x, y) = 0 \;\forall x \in \mathbb{R}$

Also

$\lim_{h \downarrow 0} U(x+h, y) = \lim_{h \downarrow 0} U(x, y+h) = U(x, y) \quad \forall x, y \in \mathbb{R}$

and $\lim_{h \downarrow 0} L(x+h, y) = \lim_{h \downarrow 0} L(x, y+h) = L(x, y) \quad \forall x, y \in \mathbb{R}$.

For rectangle $(a_1, b_1] \times (a_2, b_2]$, $a_1 < b_1$, $a_2 < b_2$,

$U(b_1, b_2) - U(b_1, a_2) - U(a_1, b_2) + U(a_1, a_2)$

$$= \frac{|F_1(b_1) - F_2(a_2)| + |F_1(a_1) - F_2(b_2)| - |F_1(b_1) - F_2(b_2)| - |F_1(a_1) - F_2(a_2)|}{2}$$

≥ 0 (Since, for $-\infty < x_1 \leq x_2 < \infty$, $-\infty < y_1 \leq y_2 < \infty$, $|x_1 - y_1| + |x_2 - y_2| \leq |x_1 - y_2| + |x_2 - y_1|$ and $F_1(a_1) \leq F_1(b_1)$, $F_2(a_2) \leq F_2(b_2)$)

$L(b_1, b_2) - L(b_1, a_2) - L(a_1, b_2) + L(a_1, a_2)$

$$= \frac{|F_1(b_1) + F_2(b_2) - 1| + |F_1(a_1) + F_2(a_2) - 1| - |F_1(b_1) + F_2(a_2) - 1| - |F_1(a_1) + F_2(b_2) - 1|}{2}$$

≥ 0 (since, for $-\infty < x_1 \leq x_2 < \infty$, $-\infty < y_1 \leq y_2 \leq \infty$, $|x_1 - y_1| + |x_2 - y_2| \leq |x_1 - y_2| + |x_2 - y_1|$ and $F_1(a_1) \leq F_1(b_1)$, $1 - F_2(b_2) \leq 1 - F_2(a_2)$)

Marginal d.f.s of $U(\cdot,\cdot)$ are

$U_1(x) = \lim_{y \to \infty} U(x, y) = F_1(x)$ $\forall x \in \mathbb{R}$, and $U_2(y) = \lim_{x \to \infty} U(x, y) = F_2(y)$ $\forall y \in \mathbb{R}$

and marginal d.f.s of $L(\cdot,\cdot)$ are

$L_1(x) = \lim_{y \to \infty} L(x, y) = F_1(x)$ $\forall x \in \mathbb{R}$ and $L_2(y) = \lim_{x \to \infty} U(x, y) = F_2(y)$, $\forall y \in \mathbb{R}$.

(ii) Since, for any probability function $P(\cdot)$ and events A and B,

$$\max\{P(A) + P(B) - 1, 0\} \leq P(A \cap B) \leq \min\{P(A), P(B)\},$$

we have

$$L(x, y) \leq F(x, y) \leq U(x, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

Problem No. 4 $P(\{(X_1, X_2) = (0, 0)\}) = P(X_1 \leq 0, X_2 \leq 0) - P(X_1 < 0, X_2 \leq 0) - P(X_1 \leq 0, X_2 < 0) + P(X_1 < 0, X_2 < 0)$

$$= F(0, 0) - F(0-, 0) - F(0, 0-) + F(0-, 0-)$$
$$= 0 - 0 - 0 + 0$$
$$= 0$$

$$P(\{(X_1, X_2) = (1, 1)\}) = F(1, 1) - F(1-, 1) - F(1, 1-) + F(1-, 1-)$$
$$= \frac{5}{8} - \frac{1}{8} - \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

We have

$$F_{X_1}(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 < 0 \\ \frac{x_1}{4}, & \text{if } 0 \leq x_1 < 1 \\ \frac{2 + x_1}{4}, & \text{if } 1 \leq x_1 < 2 \\ 1, & \text{if } x_2 \geq 2 \end{cases}$$

is discontinuous only at $x_1 = 1$, with $P(X_1 = 1) = \frac{1}{2} < 1$. This implies that X_1 is not discrete which in turn implies that (X_1, X_2) is not discrete. (Since marginal distributions of discrete r.v. are discrete)

Problem No. 5 We know that, for positive integer m and $|x|<1$,

$$(1-x)^{-m} = 1 + \binom{m}{1}x + \binom{m+1}{2}x^2 + \binom{m+2}{3}x^3 + \dots + \binom{m+k-1}{k}x^k + \dots$$

Clearly X and Y have the same support $S_X = S_Y = \{0, 1, 2, \dots\}$.

For $x \in S_X$

$$f_X(x) = P(X=x) = \sum_{y=0}^{\infty} P(X=x, Y=y) = \frac{(x+y+k-1)!\;\theta_1^x(1-\theta_1-\theta_2)^k}{x!\,(k-1)!} \sum_{y=0}^{\infty} \frac{(x+y+k-1)!}{y!}\theta_2^y$$

$$= \binom{x+k-1}{k-1}\theta_1^x(1-\theta_1-\theta_2)^k \sum_{y=0}^{\infty}\binom{x+y+k-1}{y}\theta_2^y$$

$$= \binom{x+k-1}{k-1}\theta_1^x(1-\theta_1-\theta_2)^k(1-\theta_2)^{-(x+k)} = \binom{x+k-1}{k-1}\left(1-\frac{\theta_1}{1-\theta_2}\right)^k\left(\frac{\theta_1}{1-\theta_2}\right)^x$$

Thus

$$f_X(x) = \begin{cases} \binom{x+k-1}{k-1}\left(1-\frac{\theta_1}{1-\theta_2}\right)^k\left(\frac{\theta_1}{1-\theta_2}\right)^x, & \text{if } x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

By symmetry

$$f_Y(y) = \begin{cases} \binom{y+k-1}{k-1}\left(1-\frac{\theta_2}{1-\theta_1}\right)^k\left(\frac{\theta_2}{1-\theta_1}\right)^y, & \text{if } y = 0, 1, 2, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Problem No. 6 Possible allocations of 3 balls to 3 boxes are

Boxes

No. of Balls

B_1	B_2	B_3
3	0	0
2	1	0
2	0	1
1	2	0
1	1	1
1	0	2
0	3	0
0	2	1
0	1	2
0	0	3

$\rightarrow$ Each with probability $\frac{1}{10}$.

Corresponding values of (N, X_1, X_2, X_3)

N	X_1	X_2	X_3
1	3	0	0
2	2	1	0
2	2	0	1
2	1	2	0
3	1	1	1
2	1	0	2
1	0	3	0
2	0	2	1
2	0	1	2
1	0	0	3

→ Each with probability $\frac{1}{10}$

(a) & (c): Joint p.m.f. of (N, X_1) is:

$N \backslash X_1$	0	1	2	3	Total
1	$\frac{2}{10}$	0	0	$\frac{1}{10}$	$\frac{3}{10}$
2	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	0	$\frac{6}{10}$
3	0	$\frac{1}{10}$	0	0	$\frac{1}{10}$
Total	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	

→ Marginal p.m.f. of N

Marginal p.m.f. of X_1

(b), (c) & (d): Joint p.m.f. of (X_1, X_2) is:

$X_1 \backslash X_2$	0	1	2	3	Total
0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{4}{10}$
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	0	$\frac{3}{10}$
2	$\frac{1}{10}$	$\frac{1}{10}$	0	0	$\frac{2}{10}$
3	$\frac{1}{10}$	0	0	0	$\frac{1}{10}$
Total	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	

→ Marginal p.m.f of X_1

Marginal p.m.f. of X_2

Problem No. 7 (a) $S_{\underline{X}} = \{(x_1, x_2, x_3) : x_1 = 1, 2,\ x_2 = 1, 2, 3,\ x_3 = 1, 3\}$.

$$\sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(x_1, x_2, x_3) = 1 \Rightarrow \sum_{x_1=1,2} \sum_{x_2=1,2,3} \sum_{x_3=1,3} c\, x_1 x_2 x_3 = 1$$

$$\Rightarrow c \left(\sum_{x_1=1,2} x_1\right)\left(\sum_{x_2=1,2,3} x_2\right)\left(\sum_{x_3=1,3} x_3\right) = 1 \Rightarrow c(3 \times 6 \times 4) = 1$$

$$\Rightarrow c = \frac{1}{72}$$

(b) We have $S_{X_1} = \{1, 2\}$, $S_{X_2} = \{1, 2, 3\}$, $S_{X_3} = \{1, 3\}$.

For $x_1 \in S_{X_1}$

$$f_{X_1}(x_1) = P(X_1 = x_1) = \sum_{x_2=1,2,3} \sum_{x_3=1,3} P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

$$= \frac{1}{72} \sum_{x_2=1,2,3} \sum_{x_3=1,3} x_1 x_2 x_3 = \frac{x_1}{72}\left(\sum_{x_2=1,2,3} x_2\right)\left(\sum_{x_3=1,3} x_3\right)$$

$$= \frac{x_1}{72} \times 6 \times 4 = \frac{x_1}{3}.$$

Thus

$$f_{X_1}(x_1) = \begin{cases} \frac{x_1}{3}, & \text{if } x_1 = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly

$$f_{X_2}(x_2) = \begin{cases} \frac{x_2}{6}, & \text{if } x_2 = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}, \quad f_{X_3}(x_3) = \begin{cases} \frac{x_3}{4}, & \text{if } x_3 = 1, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(c) Clearly $S_{\underline{Y}} = \{(y_1, y_2) : y_1 = 1, 2,\ y_3 = 1, 3\}$.

For $\underline{y} \in S_{\underline{Y}}$

$$f_{\underline{Y}}(\underline{y}) = P(X_1 = y_1, X_3 = y_2) = \sum_{x=1,2,3} P(X_1 = y_1, X_2 = x, X_3 = y_2)$$

$$= \sum_{x=1,2,3} \frac{y_1 x y_2}{72} = \frac{y_1 y_2}{12}$$

Thus

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} \frac{y_1 y_2}{12}, & \text{if } y_1 = 1, 2,\ y_2 = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(d) $$P(X_1 = X_2 = X_3) = \sum_{\substack{\underline{x} \in S_{\underline{X}} \\ x_1 = x_2 = x_3}} \frac{x_1 x_2 x_3}{72} = \frac{1}{72}.$$

$\boxed{5/5}$