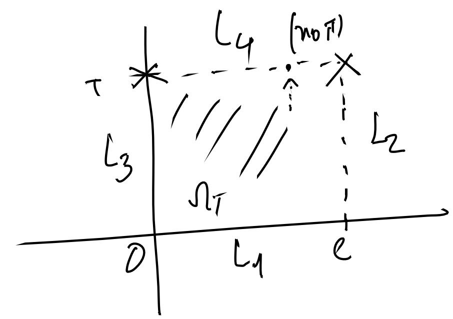
MSO 203B - PDE (Lecture 16 : Maximum Principle for Heat Eqn)

$\Omega_T = \Omega \times (0,T)$, $(\Omega = I \subset \mathbb{R})$ $I = (0,\ell)$

Let $u \in C^{2,1}(\Omega_T)$ be a soln to the eqn $u_t - u_{xx} = 0$. Then the $\underline{\text{max/min}}$ of u is attained on the parabolic boundary $(\partial\Omega_T := L_1 \cup L_2 \cup L_3)$.

$\downarrow$

$v_t - u_{xx} = F(x,t)$.



Proof:- $v(x,t) = u(x,t) + \varepsilon x^2$.

$v_t = u_t \;/\; v_{xx} = u_{xx} + 2\varepsilon$

$v_t - v_{xx} = \underbrace{u_t - u_{xx}}\, - 2\varepsilon = -2\varepsilon < 0$.

Let $v(x_0,t_0) = \max\limits_{\Omega_T} v$

$\therefore v_t(x_0,t_0) - v_{xx}(x_0,t_0) \geq 0$.

$$u_t(x_0,t_0) - u_{xx}(x_0,t_0)$$
$$\overset{\|}{0} + \overset{\|}{\geq 0} \quad \underline{\underline{\geq 0}}$$

$\Rightarrow \max\limits_{\bar{\Omega}_T} v = \max\limits_{\partial\Omega_T \cup L_4} v$

Let us assume $\exists\, (x_0,T)$ s.t. $v(x_0,T) = \max\limits_{\bar{\Omega}_T} v$.

$v_t(x_0,T) = \lim\limits_{h \to 0^+} \dfrac{v(x_0,T) - v(x_0,T-h)}{h} \geq 0$ & $v_{xx}(x_0,T) \leq 0$ $\Rightarrow$ $v_t(x_0,T) - v_{xx}(x_0,T) \geq 0$.

$$\max_{\partial\Omega_T} u \le \max_{\bar\Omega_T} u \le \max_{\bar\Omega_T} v \le \max_{\partial\Omega_T} v \le \max_{\partial\Omega_T} u + \varepsilon l^2 .$$

$v = u + \varepsilon x^2$.
$\le u + \varepsilon l^2$

$\varepsilon \to 0$

$$\boxed{\max_{\partial\Omega_T} u = \max_{\bar\Omega_T} u .}$$

Comparison Principle :-

$$\left.\begin{aligned} &u_t - u_{xx} = 0 . \\ &u(0,t) = u(l,t) = 0 , \\ &u(x,0) = g(x) \end{aligned}\right\} \text{ — } (1)$$

$$\left.\begin{aligned} &v_t - v_{xx} = 0 \\ &v(0,t) = v(l,t) = 0 . \\ &v(x,0) = h(x) \end{aligned}\right\} \text{ — } (2)$$

Assumption :- $g \le h$. on $(0,l)$

Conclusion :- $u \le v$. in Ω_T .

Proof :- Let $\varphi = u - v$, ; RTP: $\varphi \le 0$.

$\varphi_t - \varphi_{xx} = 0$.

$\varphi(0,t) = \varphi(l,t) = 0$ & $\varphi(x,0) = g(x) - h(x) \le 0$.

$\Rightarrow$ max φ in Ω_T is zero. (Maximum Principle)

$\Rightarrow$ $\varphi \le 0$ in Ω_T

$\Rightarrow$ $u \le v$ in Ω_T

Uniqueness Problem :-

$$\left.\begin{aligned} u_t - u_{xx} &= f(x,t) \\ u(0,t) &= u(l,t) = 0 \\ u(x,0) &= g(x). \end{aligned}\right\} \longrightarrow (1).$$

Let u and v be two soln of (1)

RTP :- $u \equiv v$ in Ω_T.

Define, $\varphi = u - v$. Then φ satisfies

$$\varphi_t - \varphi_{xx} = 0$$
$$\varphi(0,t) = \varphi(l,t) = 0$$
$$\varphi(x,0) = 0$$

$\therefore \max\limits_{\Omega_T} \varphi = 0$ and $\min\limits_{\Omega_T} \varphi = 0 \Rightarrow \varphi \equiv 0, \Rightarrow u = v$ in Ω_T.

Stability of Heat Eqn :-

$$u_t - u_{xx} = 0 \quad \text{in } (0,l)\times(0,T)$$
$$u(0,t) = u(l,t) = 0$$
$$u(x,0) = u_0(x)$$

$$v_t - v_{xx} = 0$$
$$v(0,t) = v(l,t) = 0$$
$$v(x,0) = v_0(x).$$

RTP:- $\max_{\bar{\Omega}_T} |u(x,t) - v(x,t)| \leq \max_{x\in(0,l)} |u_0(x) - v_0(x)|.$

Define $\varphi(x,t) = u(x,t) - v(x,t).$

Then, $\varphi_t - \varphi_{xx} = 0.$

$$\varphi(0,t) = \varphi(l,t) = 0.$$
$$\varphi(x,0) = u_0 - v_0.$$

From Maximum Principle :- $\max_{\bar{\Omega}_T} |\varphi(x,t)| \leq \max_{\partial\Omega_T} |u_0 - v_0|.$

$$\Rightarrow \max_{\bar{\Omega}_T} |u(x,t) - v(x,t)| \leq \max_{\partial\Omega_T} |u_0(x) - v_0(x)|.$$