ASSIGNMENT - 4

Given

$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + fu = g$ —— (1)

is the general form.

Comparing with (1) the given equation we have

$a = x^2$	$e = y$
$b = xy$	$f = 0$
$c = y^2$	$g = 0$
$d = x$	

$\therefore \Delta = b^2 - ac = x^2y^2 - x^2y^2 = 0.$

Given a change of variable $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ the transformed equation is given by

$A w_{\xi\xi} + 2B w_{\xi\eta} + C w_{\eta\eta} + D w_\xi + E w_\eta + F w = G$ —— (i)

where

$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$ —— (ii)

$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\eta_y\xi_y$

$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$ —— (iii)

$D = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y$

$E = a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y$

The canonical form of parabolic eqn is

$w_{\xi\xi} + \bar{D} w_\xi + \bar{E} w_\eta + \bar{F} w = \bar{G}$

$\because b^2 - ac = 0$ hence $A = \frac{1}{a}[a\xi_x + b\xi_y]^2 = 0$ $\boxed{OR}$ $C = \frac{1}{a}[a\eta_x + b\eta_y]^2 = 0$

Hence, we only have one family of curve given by

$a X_x + b X_y = 0$ —— (2) where $X = \xi$ or η but not both.

~~...~~

Let, $a\eta_x + b\eta_y = 0$

$\Rightarrow x^2\eta_x + (-xy)\eta_y = 0$

$\Rightarrow x\eta_x - y\eta_y = 0 \quad [\because x > 0]$ — (3)

(3) is our well known form of 1^{st} order eqn.

Hence, $\frac{dy}{dx} = -\frac{y}{x} \Rightarrow xy = \text{constant}$

Hence $\eta(x,y) = f(xy)$ $(f \in C^1)$.

For simplicity assume, $f(z) = z$. Hence $\eta(x,y)$ can be chosen as $\eta(x,y) = xy$.

To find $\xi(x,y)$ we assume,

$$J(\xi,\eta) = \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix} = \begin{vmatrix} \xi_x & y \\ \xi_y & x \end{vmatrix} = x\xi_x - y\xi_y \neq 0$$

Choosing $\xi(x,y) = x$ we have, $J(\xi,\eta) \neq 0$.

$\therefore u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + y w_\eta$

$u_y = w_\xi \xi_y + w_\eta \eta_y = x w_\eta$

$u_{xx} = [w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x] + y[w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x]$

$= w_{\xi\xi} + 2y w_{\xi\eta} + y^2 w_{\eta\eta}$

$u_{xy} = [w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y] + w_\eta + y[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y]$

$= w_\eta + x w_{\xi\eta} + xy w_{\eta\eta}$

$u_{yy} = x[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y] = x^2 w_{\eta\eta}$

putting in the values, $x^2 w_{\xi\xi} + x w_\xi = 0 \Rightarrow w_{\xi\xi} + \frac{1}{\xi} w_\xi = 0$.

Hence it remains parabolic in the new co-ordinate.

2. $u_{xx} + 4u_{xy} + u_x = 0$

$a=1,\ b=2,\ c=0,\ d=1,\ e=0,\ f=g=0$

$\therefore b^2 - ac = 4 - 1.0 = 4 > 0$

Hence the eqn is hyperbolic.

The char eqn is given by

$\xi_x + 4\xi_y = 0$ and $\eta_x = 0$. (Putting $A = C = 0$ in (i) & (iii))

Solving we obtain

$\xi(x,y) = 4x - y$ and $\eta(x,y) = y$. (By choosing the simplest such soln).

$\therefore u_x = w_\xi \xi_x + w_\eta \eta_x = 4w_\xi$

$u_{xx} = 4[w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x] = 16w_{\xi\xi}$

$u_{xy} = 4[w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y] = -4w_{\xi\xi} + 4w_{\xi\eta}$

Putting the values in the original eqn we obtain

$w_{\xi\eta} + \frac{1}{4}w_\xi = 0 \leftarrow$ CANONICAL FORM

Assume, $w_\xi = \phi$

$\therefore \phi_\eta = -\frac{\phi}{4} \Rightarrow \phi(\xi,\eta) = \tilde{g}(\xi)\, e^{-\eta/4}$ for some $\tilde{g} \in C^1$.

So, $w(\xi,\eta) = g(\xi)\, e^{-\eta/4}$ for some $g \in C^1$.

is a particular solution.

$\therefore u(x,y) := w(\xi,\eta) = g(4x-y)\, e^{-y/4}$; $g \in C^1$ is our required soln.

Scanned by CamScanner

3. $xu_{xx} + 2x^2 u_{xy} = u_x - 1. \quad (x>0)$.

$a = x,\ b = x^2,\ c = 0,\ d = -1,\ e = 0,\ f = 0,\ g = 1.$

$\therefore \Delta := b^2 - ac = (x^2)^2 - x \cdot 0 = x^4 > 0$

So, the eqn is hyperbolic provided $x>0$.

The Char eqn ~~is~~ are

$x\xi_x^2 + 2x^2 \xi_x \xi_y = 0.$

$\Rightarrow \xi_x(\xi_x + 2x\xi_y) = 0$

Either, $\xi_x = 0$ or $\xi_x + 2x\xi_y = 0.$

$x\eta_x^2 + 2x^2 \eta_x \eta_y = 0$

$\Rightarrow \eta_x(\eta_x + 2x\eta_y) = 0$

Either, $\eta_x = 0$ or

$\eta_x + 2x\eta_y = 0.$

Choose, $\xi(x,y) = y$. & $\eta(x,y) = -y + x^2$.

(One can choose the other way) ~~with~~ ~~the~~

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$\quad = w_\xi \cdot 0 + w_\eta \cdot (+2x) = +2x w_\eta.$

$u_y = w_\xi \xi_y + w_\eta \eta_y$

$\quad = w_\xi \cdot 1 + w_\eta(-1) = w_\xi + w_\eta(-1) = w_\xi - w_\eta$

$u_{xx} = +2[xw_\eta]_x = +2\left[w_\eta + x(w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x)\right]$

$\quad = +2\left[w_\eta + 2x^2 w_{\eta\eta}\right]$

$\quad = 4x^2 w_{\eta\eta} + 2w_\eta.$

$u_{xy} = \cancel{w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x + w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x} = +2x\left[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y\right]$

$\quad = +2x w_{\eta\xi} - 2x w_{\eta\eta}.$

$\cancel{= -2x[w_{\xi\eta} + w_{\eta\eta}]}$

Putting the values in the eqn we get

Scanned by CamScanner

$$4x^3 \cancel{w_{\eta\eta}} + \cancel{2x w_\eta} + 4x^3 w_{\xi\eta} - \cancel{4x^3 w_{\eta\eta}} = +\cancel{2x w_\eta} - 1.$$

$$\Rightarrow 4x^3 w_{\xi\eta} = 1.$$

$$\Rightarrow w_{\xi\eta} = \frac{1}{4(\xi+\eta)^{3/2}} \leftarrow \text{CANONICAL FORM.}$$

Note that the eqn is linear.

Hence, let $u = u_g + u_p$ (u_g = general soln to homogeneous problem, u_p = particular soln to inhomogeneous problem).

Note, $u_p(x,y) = x$ is a soln of the eqn,

Now u_g is the soln of $x u_{xx} + 2x^2 u_{xy} - u_x = 0$.

Using the earlier change of variable we have

$$w_{\xi\eta} = 0 \Rightarrow w(\xi,\eta) = f(\xi) + g(\eta)$$

$$\therefore u_g(x,y) = f(x^2-y) + g(y) \text{ for } f,g \in C^2.$$

$$\therefore u = x + f(x^2-y) + g(y) \text{ for any } f,g \in C^2.$$

4. $u_{xx} + x^2 u_{yy} = 0$.

$a = 1, b = 0, c = x^2, d = e = f = g = 0$.

$\therefore \Delta = b^2 - ac = -x^2 < 0$ for all $x \neq 0$.

The char eqn are given by

$$\phi_x \pm i x \phi_y = 0. \text{ where } \phi = \xi + i\eta.$$

$\therefore$ We choose, $\xi = y$ and $\eta = \frac{x^2}{2}$ [This is obtained by solving for ϕ and then choosing ξ & η].

$$\therefore\ u_{xx} = w_{\xi\xi}\,\xi_x^2 + 2w_{\xi\eta}\,\eta_x\xi_x + w_{\eta\eta}\,\eta_x^2 + w_\xi\,\xi_{xx} + w_\eta\,\eta_{xx}$$

$$= x^2 w_{\eta\eta} + w_\eta$$

and, $u_{yy} = w_{\xi\xi}\,\xi_y^2 + 2w_{\xi\eta}\,\xi_y\eta_y + w_{\eta\eta}\,\eta_y^2 + w_\xi\,\xi_{yy} + w_\eta\,\eta_{yy}$

$$= w_{\xi\xi}$$

Substituting this we obtain,

$$w_{\xi\xi} + w_{\eta\eta} + \frac{1}{2\eta} w_\eta = 0.$$

FAMILY OF CHAR CURVES

ξ = constant $\Rightarrow$ Family of st lines.

& η = constant $\Rightarrow$ Family of parabola.