CS201A/201: Math for CS I/Discrete Mathematics
Quiz-2

Max marks: 60
Time: 60 mins. 3-Nov-2014

1. *Answer all 5 questions. Answer in the separate answer booklet provided.*
2. *Please start each answer on a fresh page. Keep anwers to parts of a question together.*
3. *You can consult* **only your own handwritten notes**. *Other material like photocopies, books, articles, electronic gadgets etc. is* **NOT** *allowed.*

1. Given a query q and a set of web pages W, web search engines give a rating (a positive integer) representing the relevance of the web pages to the query. Let this function be $r : W \times q \rightarrow \mathbb{N}$. We can now define a ranking relation R between the elements of W as follows: If $w_1 R w_2$ then $r(w_1, q) > r(w_2, q)$. So, w_1 is more relevant to query q than w_2.

   Let $W = \{a, b, c, d, e, f\}$ and let the function r be defined by the following table:

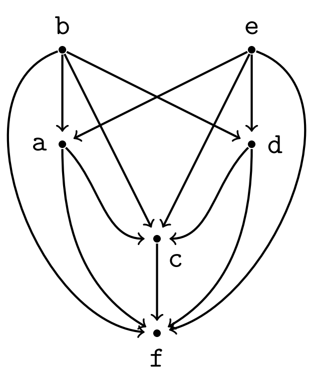
w	a	b	c	d	e	f
$r(w,q)$	7	8	5	7	8	4

   (a) What are the 2-tuples of the relation R?

   Solution:
   $R = \{(b,a),(b,c),(b,d),(b,f),(e,a),(e,c),(e,d),(e,f),(a,c),(a,f),(d,c),(d,f),(c,f)\}$

   (b) Draw the digraph corresponding to the relation R.

   Solution:



   (c) Which of the following properties does R have? Reflexive, symmetric, antisymmetric, transitive.

   Solution:
   R is just the $>$ relation. So, it is not reflexive, not symmetric and not anti-symmetric (note that it is asymmetric that is $xRy \implies \sim yRx$). It is transitive.

[5,3,2=10]

2. (a) Given a set S with 3 elements how many distinct (that is non-isomorphic) posets are possible? Draw a Hasse diagram for each one.

Solution:

There are 5 possible posets whose Hasse diagrams are shown below.

(b) Consider a room with a bulb and 3 doors labelled x, y, z with a switch at each door. We want that whenever any one switch changes state (that is on to off or off to on) the state of the bulb changes as well (that is if the bulb is on it switches off and if it is off it switches on). Using the (1,0)-Boolean algebra (where 1 is on and 0 is off) along with the operations join ($\vee$), meet ($\wedge$) and complement ($\bar{\ }$) derive an expression for the bulb being lighted (that uses x, y, z) and which behaves as required. Explain your approach properly.

Solution:

Let $x = 0$, $y = 0$, $z = 0$ stand for light is off (i.e. 0). The following table defines the transitions.

x	y	z	Bulb state
0	0	0	0

Difference of 1 from 0 0 0.

x	y	z	Bulb state
0	0	1	1
0	1	0	1
1	0	0	1

Difference of 2 from 0 0 0.

x	y	z	Bulb state
1	1	0	0
1	0	1	0
0	1	1	0

Difference of 3 from 0 0 0.

x	y	z	Bulb state
1	1	1	1

We require the expression when the bulb is lighted i.e. value is 1. This gives:

$$(x \wedge y \wedge z) \vee (\bar{x} \wedge \bar{y} \wedge z) \vee (\bar{x} \wedge y \wedge \bar{z}) \vee (x \wedge \bar{y} \wedge \bar{z})$$

Note that the expression can be different depending on what you choose as your initial states for the switches and the corresponding state of the bulb. But the principle remains the same - it will be an expression containing a join of 4 meets.

[(2,3),10=15]

3. (a) Let G be a simple graph with 10 vertices and 28 edges. Will G contain a cycle of length 4? Justify your answer.

Solution:

The answer is yes.

We have $\sum_{v_i \in V} d(v_i) = 2 \times 28 = 56$. With 10 nodes this works out to an average degree of 5.6. This implies there are at least two vertices, say u, v whose degree sum is ≥ 12. Even if u is adjacent to v the degree sum goes down by 2. So, they are connected to 10 other vertices - remember the graph is simple so no multi-edges. Since there are only 8 other vertices excluding u and v there must be an overlap of two vertices, say x and y, to which both u and v are connected giving a 4-cycle u, x, v, y, u.

Note that the problem does not ask you to show an instance of a graph which has a 4-cycle. It is asking whether every simple graph with 10 vertices and 28 edges has a 4-cycle.

(b) A simple graph is regular if all its vertices have the same degree. If a simple regular graph G has 22 edges how many vertices can it have? Show your calculation and reasoning.

Solution:

If there are n vertices all of degree d then $n \times d = 2 \times 22 = 44$. So, n must be a positive integer dividing 44. The (n, d) combinations we get are: $(1, 44), (2, 22), (4, 11), (11, 4), (22, 2), (44, 1)$. The first three combinations are impossible since the graph is simple. $(11, 4)$ is possible (construct C_{11} then connect each node to its second neighbour giving degree 4 for each node); $(22, 2)$ is just C_{22} (giving a degree of 2 for all nodes); $(44, 1)$ is also possible (we get 22 components with each component containing an edge between 2 nodes giving a degree of 1 for all nodes). Note that a simple graph need not be connected.

(c) You are given the graph G in figure 1.

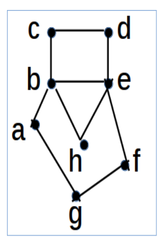


Figure 1: Graph for colourability

i. Is G 2-colourable? Why (justify)?

Solution:

G is not 2-colourable since it contains odd cycles as sub-graphs. An odd cycle requires at least 3 colours.

ii. What is the chromatic number of G? Give a one line justification why your answer is correct.

Solution:

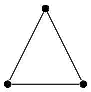
$chv(G) = 3$. We get the following partition of nodes that are not adjacent: $\{g, h\}$, $\{d, b, f\}$ and $\{a, c, e\}$. G is not 2-colourable and the next higher number is 3 so 3 is the minimum number of colours required.

[5,5,(1,2),(1,1)=15]

4. (a) Give one example each (draw graph) of a non-trivial graph (> 2 vertices) that:

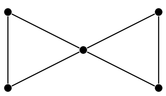
i. Has a Hamiltonian cycle and an Eulerian cycle.

Solution:
K_3 has both an Eulerian and Hamiltonian cycle.



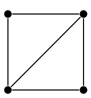
ii. Has an Eulerian cycle but no Hamiltonian cycle.

Solution:
Eulerian cycle present since all degrees even. No Hamiltonian cycle due to bridge vertex connecting two cycles.



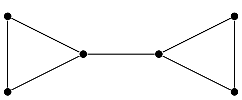
iii. Has a Hamiltonian cycle but no Eulerian cycle.

Solution:
Hamiltonian cycle is obvious. No Eulerian cycle since vertices of odd degree present.



iv. Has neither a Hamiltonian cycle nor an Eulerian cycle.

Solution:
No Eulerian cycle since odd degree vertices present. No Hamiltonian cycle due to bridge edge connecting two cycles.



(b) Does the graph in figure 2 have a Hamiltonian cycle? Why?

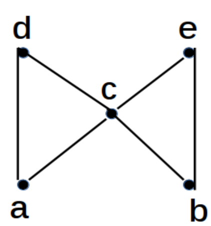


Figure 2: Graph for Hamiltonian circuit

Solution:
No, Since c is a bridge vertex connecting two cycles.

[2x4,2=10]

5. Let $M = \{(b,g),(e,f),(h,i)\}$ be a matching in the graph G in figure 3.

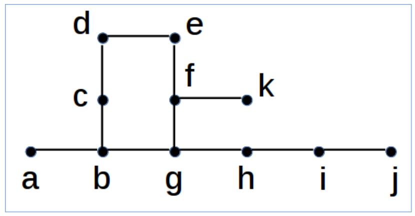


Figure 3: Graph for matching

(a) Find an M-augmenting path in G.

Solution:
An M-augmenting path is (a,b,g,f,e,d). Anothe one is (a,b,g,h,i,j). Note that every alternate edge is part of the matching M and the start and terminal nodes are not in M.

(b) Using part (a) find a match M' such that $|M'| > |M|$.

Solution:
An enhanced matching M' using the first augmented path is $M' = \{(a,b),(g,f),(e,d),(h,i)\}$, $|M'| > |M|$. Similarly, if we choose the other augmenting path the enhanced matching is $M' = \{(a,b),(g,h),(i,j),(e,f)\}$.

(c) Find a maximum matching. Is it unique? Is it perfect?

Solution:
A maximum matching is $\{(a,b),(c,d),(e,f),(g,h),(i,j)\}$. It is not unique. Another matching with the same size is $\{(a,b),(g,h),(i,j),(f,k),(d,e)\}$. Neither is perfect. The first one leaves vertex k uncovered, while the second one leaves vertex c uncovered.

[4,2,(2,1,1)=10]