Lecture 11 : Canonical Form - II (MSO 203B).

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{--- (1)}$$

$\exists\ (x,y) \mapsto (\xi(x,y), \eta(x,y))$ s.t assuming $w(\xi,\eta) := u(x,y)$ we have. (For 2-D)

$$\tilde{A} w_{\xi\xi} + \tilde{B} w_{\xi\eta} + \tilde{C} w_{\eta\eta} + \tilde{D} w_{\xi} + \tilde{E} w_{\eta} + \tilde{F} w = \tilde{G} \quad \text{--- (2)}$$

where $\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$\tilde{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$.

$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

$\tilde{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$.

$\tilde{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$.

$\tilde{F} = F$ & $\tilde{G} = G$.

$A = -1,\ B = 0,\ C = 1.$

$2(-1)(-1) + 2\cdot 1\cdot 1 = 4$

(1) is parabolic if $B^2 - 4AC = 0$

(1) is hyperbolic if $B^2 - 4AC > 0$.

(1) is elliptic if $B^2 - 4AC < 0$.

$$\boxed{\tilde{B}^2 - 4\tilde{A}\tilde{C} = J^2(B^2 - 4AC)}$$

Canonical Form of an Hyperbolic PDE :-

Choose $\xi$ & $\eta$ s.t $\tilde{A} = \tilde{C} = 0$ ie

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$
$$A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0.$$

$$\tilde{B}^2 - 4\tilde{A}\tilde{C} > 0$$

$$\boxed{w_{\xi\eta} + l[w_\xi, w_\eta, w, \xi, \eta] = 0} \leftarrow \text{Canonical Form.}$$

$A\theta_x^2 + B\theta_x\theta_y + C\theta_y^2 = 0$ — (*)

$\xi$ & $\eta$ are roots of (*)

$\Rightarrow A\left(\frac{\theta_x}{\theta_y}\right)^2 + B\frac{\theta_x}{\theta_y} + C = 0$

$\Rightarrow \frac{\theta_x}{\theta_y} = \frac{-B \pm \sqrt{B^2-4AC}}{2A} = p, \tilde{p}$

$\therefore$ $\theta_x - p\theta_y = 0$ & $\theta_x - \tilde{p}\theta_y = 0$

$\Downarrow$

$\theta$ is constant along the char. curves

$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A}$.

$\Downarrow$

$\phi_1(x,y) = c_1$ & $\phi_2(x,y) = c_2$.

Choose $\xi(x,y) = \phi_1(x,y)$

$\eta(x,y) = \phi_2(x,y)$.

Ex 1:- $u_{tt} - u_{xx} = 0$ $A = -1,\ B = 0,\ C = 1$ ....

To calculate $\xi$ & $\eta$ we look for

Ch. curves $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm 1.$

Family of Char. curves are given by $y + x = c_1$ and $y - x = c_2$

$\xi(x,y) = x + y$.

$\eta(x,y) = y - x$.

$$\boxed{\begin{array}{ll} \xi_x = 1, & \xi_y = 1 \\ \eta_x = -1, & \eta_y = 1 \end{array}}$$

Canonical Form:- $4w_{\xi\eta} = 0 \Rightarrow w_{\xi\eta} = 0$

Soln:- $w(\xi,\eta) = F(\xi) + G(\eta)$. $(F, G \in C^1)$

Original Soln $u(x,y) := w(\xi,\eta) = F(\xi) + G(\eta) = F(x+y) + G(y-x)$

Parabolic Eqn :-

$\tilde{B}^2 - 4\tilde{A}\tilde{C} = 0 \Rightarrow \tilde{B}^2 = 4\tilde{A}\tilde{C}$.

Assume, $\tilde{A} = 0 \Rightarrow \tilde{B} = 0$ $(\tilde{C} \neq 0)$.

$\therefore \boxed{w_{\eta\eta} + \ell[w_\xi, w_\eta, w, \xi, \eta] = 0}$ ← Canonical Form.

$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$.

$\Rightarrow (\sqrt{A}\xi_x + \sqrt{C}\xi_y)^2 = 0$

$\Rightarrow \sqrt{A}\xi_x + \sqrt{C}\xi_y = 0$

$\xi$ = constant along the char curves $\frac{dy}{dx} = \frac{B}{2A}$. $\Rightarrow \phi_1(x,y) = C_1$. (Only One family (real)).

Choose, $\xi(x,y) = \phi_1(x,y)$

and $\eta(x,y)$ s.t. $J(\xi,\eta) \neq 0$.

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

$B^2 = 4AC$

$$\boxed{\begin{array}{l} x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0 \\ w_{\eta\eta} = 0 \quad \boxed{\eta = y,\ \xi = \frac{y}{x}} \end{array}}$$