Module 15
Moment Generating Function

- X: a discrete or A.C. r.v. with p.m.f./p.d.f $f_X(\cdot)$;
- Define

$$A_X = \left\{ t \in \mathbb{R} : E\left(e^{tX}\right) < \infty \right\}.$$

- Clearly $0 \in A_x$, and thus $A_x \neq \phi$.

Definition 1: The moment generating function (m.g.f.) of r.v. X is defined by

$$M_X(t) = E\left(e^{tX}\right), \quad t \in A_X.$$

Remark 1:

(a)

$$M_X(0) = 1 \text{ and } M_X(t) > 0, \quad \forall\ t \in A_X;$$

(b) Let $Y = cX + d$, for some real constants $c \neq 0$ and $d \in \mathbb{R}$. Then

$$\begin{aligned} M_Y(t) &= E\left(e^{tY}\right) \\ &= E\left(e^{t(cX+d)}\right) \\ &= e^{td} E\left(e^{ctX}\right) \\ &= e^{td} M_X(ct), \quad t \in A_Y = \left\{\frac{x}{c} : x \in A_X\right\}. \end{aligned}$$

(c) The name moment generating function to the transform $M_X(t),\ t \in A_X$, is attributed to the fact that $M_X(t)$ can be used to generate moments $\left(\mu_r' = E\left(X^r\right),\ r = 1, 2, \ldots\right)$ of r.v. X.

Result 1:

Suppose that, for some $h > 0$, the m.g.f. $M_X(t)$ is finite for every $t \in (-h, h)$. Then

(a) $M_X(t)$ is differentiable any number of times in $(-h, h)$;

(b) for each r $\in \{1, 2, \ldots\}$, $\mu'_r = E(X^r)$ is finite and

$$\mu'_r = M_X^{(r)}(0) = \left[\frac{d}{dt^r} M_X(t)\right]_{t=0};$$

(c) for $t \in (-h, h)$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r.$$

Proof:

(Outline of the proof for A.C. case). Fix $r \in \{1, 2, \ldots\}$.

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
\frac{d^r M_X(t)}{dt^r} &= \frac{d^r}{dt^r} \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{d^r}{dt^r} e^{tx} \right) f_X(x)dx \\
&= \int_{-\infty}^{\infty} x^r e^{tx} f_X(x)dx \\
\left[\frac{d^r M_X(t)}{dt^r} \right]_{t=0} &= \int_{-\infty}^{\infty} x^r f_X(x)dx = \mu_r',
\end{aligned}
$$

where passing of derivative $\frac{d^r}{dt^r}$ under the integral sign can be justified through advanced mathematical arguments.

Also,

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f_X(x)dx \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f_X(x)dx \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r, \;\; r = 1, 2, \ldots,
\end{aligned}
$$

where interchange of integral and summation sign can be justified through advanced mathematical arguments.

Corollary 1:

Under the notation and assumptions of above result, let $\psi_X(t) = \ln M_X(t),\ t \in (-h, h)$. Then

$$\begin{aligned} \mu_1' &= E(X) = \psi_X^{(1)}(0); \\ \text{and } \mu_2 &= \mathrm{Var}(X) = \psi_X^{(2)}(0). \end{aligned}$$

Proof. We have, for $t \in (-h, h)$,

$$\begin{aligned} \psi_X^{(1)}(t) &= \frac{M_X^{(1)}(t)}{M_X(t)} \\ \psi_X^{(1)}(0) &= \frac{M_X^{(1)}(0)}{M_X(0)} = \mu_1' \\ \psi_X^{(2)}(t) &= \frac{M_X(t)M_X^{(2)}(t) - (M_X^{(1)}(t))^2}{(M_X(t))^2} \\ \psi_X^{(2)}(0) &= M_X^{(2)}(0) - (M_X^{(1)}(0))^2 \end{aligned}$$

$$= \mu_2' - (\mu_1')^2 = \mathrm{Var}(X).$$

Remark 2:

(a) The function $\psi_X(t),\ t \in (-h, h)$, is called the cumulant generating function of X;

(b) For $r = 1, 2, \ldots$

$$\mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in Maclaurin's series expansion of } M_X(t).$$

Example 1:

Let X be a r.v. with p.m.f.

$$f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is a fixed constant. Then $S_X = \{0, 1, 2, \ldots\}$ and

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} \\ &= e^{\lambda(e^t - 1)}, \end{aligned}$$

is finite for every $t \in \mathbb{R}$. Thus μ_r' is finite for every $r \in \{1, 2, \ldots\}$. We have, for $t \in \mathbb{R}$,

$$\psi_X(t) = \ln M_X(t) = \lambda(e^t - 1), \quad t \in \mathbb{R}$$

$$\psi_X^{(1)}(0) = \psi_X^{(2)}(0) = \lambda$$

$$\Rightarrow \quad E(X) = \mathrm{Var}(X) = \lambda.$$

Alt.

$$\begin{aligned} M_X(t) &= 1 + \lambda(e^t - 1) + \frac{\lambda^2(e^t-1)^2}{2!} + \frac{\lambda^3(e^t-1)^3}{3!} + \cdots \\ &= 1 + \lambda S + \frac{\lambda^2 S^2}{2!} + \frac{\lambda^3 S^3}{3!} + \cdots, \end{aligned}$$

where

$$S = e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots.$$

Thus

$$\begin{aligned}
M_X(t) &= 1 + \lambda t + (\lambda + \lambda^2)\frac{t^2}{2!} + \left(\frac{\lambda}{3!} + \frac{2\lambda^2}{(2!)^2} + \frac{\lambda^3}{3!}\right)\frac{t^3}{3!} + \cdots \\
E(X) &= \text{Coefficient of } t \text{ in } M_X(t) = \lambda \\
E(X^2) &= \text{Coefficient of } \frac{t^2}{2!} \text{ in } M_X(t) = \lambda + \lambda^2 \\
E(X^3) &= \text{Coefficient of } \frac{t^3}{3!} \text{ in } M_X(t) = \lambda^3 + 3\lambda^2 + \lambda \\
\text{Var}(X) &= E(X^2) - (E(X))^2 = \lambda.
\end{aligned}$$

Example 2:

Let X be a r.v. with p.d.f.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is a constant. Here $S_X = [0, \infty)$. Also

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx \\ &= \left(1 - \frac{t}{\lambda}\right)^{-1} \end{aligned}$$

is finite for every $t < \lambda$. Thus moments of all orders exist.

$$
\begin{aligned}
M_X(t) &= 1+\frac{t}{\lambda}+\frac{t^2}{\lambda^2}+\cdots+\frac{t^r}{\lambda^r}+\cdots, -\lambda<t<\lambda \\
\mu_r' &= \text{coefficient of } \frac{t^r}{r!} \text{ in expansion of } M_X(t) \\
&= \frac{r!}{\lambda^r},\ r=1,2,\ldots
\end{aligned}
$$

Alt.

$$
\begin{aligned}
M_X^{(1)}(t) &= \frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2} \\
M_X^{(2)}(t) &= \frac{2}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-3} \\
M_X^{(r)}(t) &= \frac{r!}{\lambda^r}\left(1-\frac{t}{\lambda}\right)^{-(r+1)},\ -\lambda<t<\lambda \\
\mu_r' &= M_X^{(r)}(0)=\frac{r!}{\lambda^r},\ r=1,2,\ldots.
\end{aligned}
$$

Example 3: Let X be a r.v. with p.d.f.

$$f_X(x) = \frac{1}{\pi}.\frac{1}{1+x^2}, \; -\infty < x < \infty.$$

For $t > 0$

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx}\frac{1}{\pi}.\frac{1}{1+x^2}dx \\ &\geq \frac{1}{\pi}\int_0^{\infty}\frac{e^{tx}}{1+x^2}dx \\ &\geq \frac{1}{\pi}\int_0^{\infty}\frac{tx}{1+x^2}dx \quad (e^y \geq y, \; \forall y > 0) \\ &= \infty \\ \Rightarrow M_X(t) &= \infty, \; \forall t > 0, \end{aligned}$$

i.e., $M_X(t)$ is not finite on any interval $(-h, h)$, $h > 0$. Note that here $E(X)$ does not exists.

Take Home Problems

1. Let X be a r.v. with p.m.f.

$$f_X(x) = \begin{cases} \frac{c_p}{x^p}, & \text{if } x \in \{1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases},$$

where $p > 1$ is a constant and c_p is the normalizing constant. Show that the m.g.f. of X is not finite on any interval $(-h, h)$, $h > 0$. Do the moments exist?

2. Let X be a r.v. with m.g.f.

$$M_X(t) = \frac{1}{2} + \frac{e^{\frac{t}{2}}}{6} + \frac{e^{-\frac{t}{3}}}{3}, \ t \in \mathbb{R}.$$

Find $P(|X| > 0)$ and the p.m.f. of $Y = X^2$.

Abstract of Next Module

- Two r.v.s X and Y are said to have the same distribution (written as $X \stackrel{d}{=} Y$) if they have the same d.f.;
- $X \stackrel{d}{=} Y$ does not imply that $X = Y$;
- We will study properties of r.v.s having the same distribution.

Thank you for your patience