Latent Variable Models for Sequential/Time-Series Data

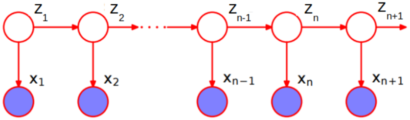
Piyush Rai

Probabilistic Machine Learning (CS772A)

Nov 7, 2017

Latent Variable Models for Sequential Data

- Task: Given a sequence of observations, infer the latent state of each observation



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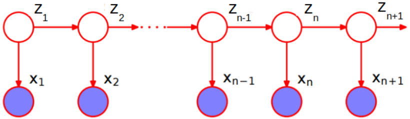
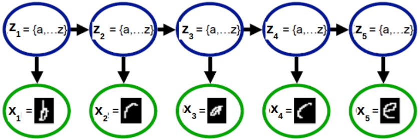


- An example: Recognizing a sequence of handwritten characters



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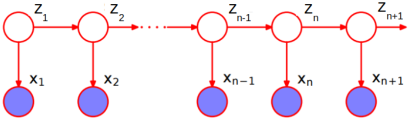

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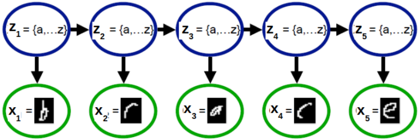

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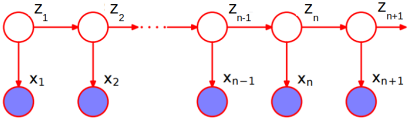

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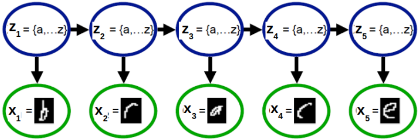

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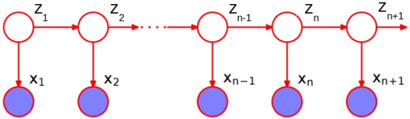

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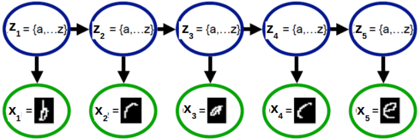

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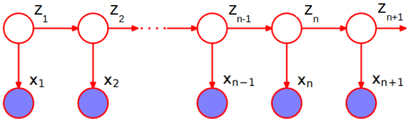
Latent Variable Models for Sequential Data

- For both cases (discrete/continuous $\boldsymbol{z}_n$), the generic model can be written as follows

$$\begin{aligned} \boldsymbol{z}_n|\boldsymbol{z}_{n-1} &\sim p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) && \text{(first-order dependence b/w } \boldsymbol{z}_n\text{'s)} \\ \boldsymbol{x}_n|\boldsymbol{z}_n &\sim p(\boldsymbol{x}_n|\boldsymbol{z}_n) && \text{(i.i.d. draws of } \boldsymbol{x}_n \text{ given } \boldsymbol{z}_n\text{)} \end{aligned}$$



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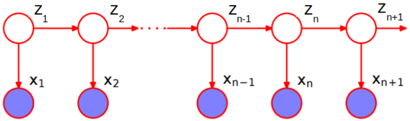

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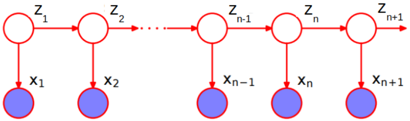

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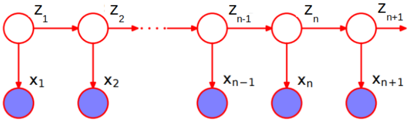

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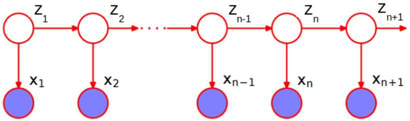

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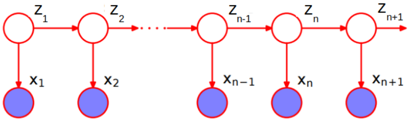

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- If states $\boldsymbol{z}_n$ are discrete, we get a Hidden Markov Model (HMM)
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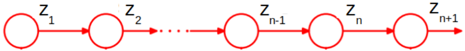
- In both cases, observations $\boldsymbol{x}_n$ can be anything (discrete/real)

State-Transition Model



- For discrete states case (HMM), $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ will be a discrete distribution, e.g.,

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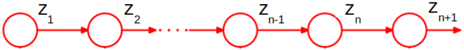

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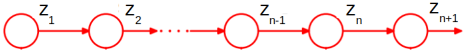

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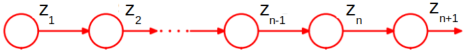

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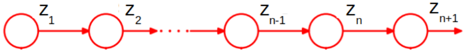

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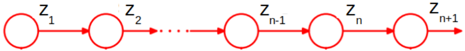

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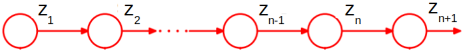

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- For both HMM and SSM, there is also an initial state distribution $p(\boldsymbol{z}_1)$, e.g.,

$$\begin{aligned} p(\boldsymbol{z}_1) &= \text{multinoulli}(\boldsymbol{\pi}_0) && \text{(for HMM)} \\ p(\boldsymbol{z}_1) &= \mathcal{N}(\mathbf{0}, \mathbf{I_K}) && \text{(for SSM)} \end{aligned}$$

Observation/Emission Model

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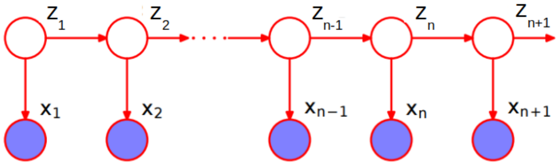

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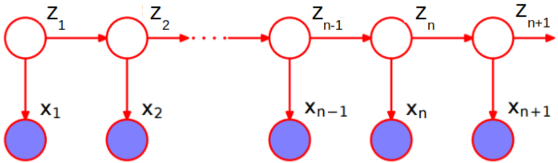

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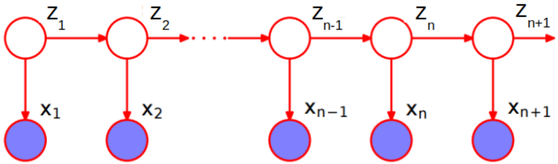

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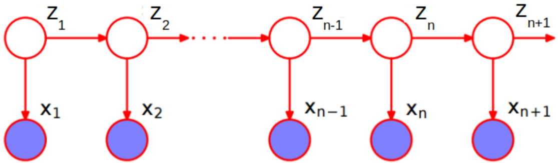

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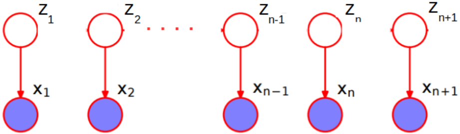
A Special Case

- What if we have i.i.d. latent states, i.e.,. $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) = p(\boldsymbol{z}_n)$?



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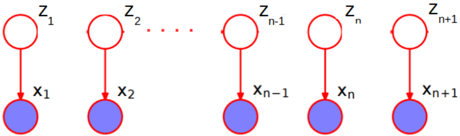

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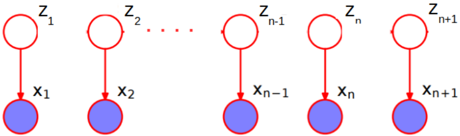

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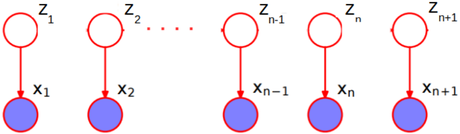

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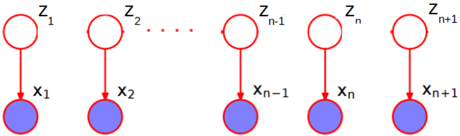

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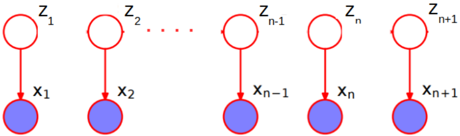

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- SSM becomes PPCA/GPLVM/DLGM

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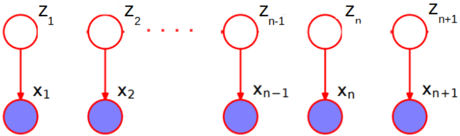

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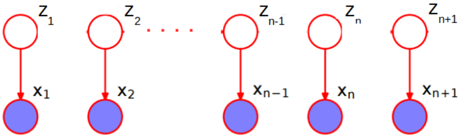

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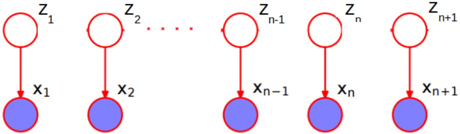

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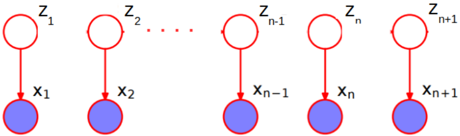

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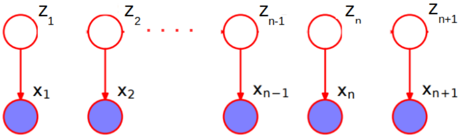

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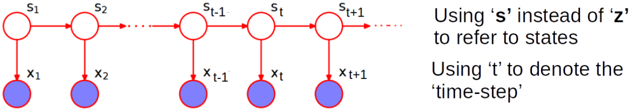

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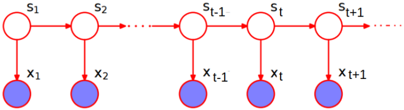

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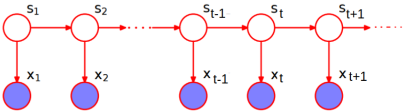

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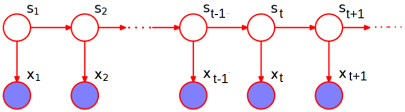

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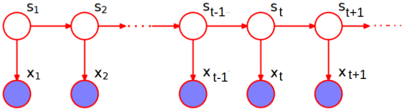

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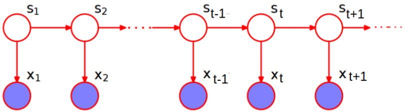

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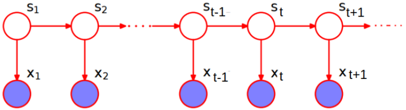

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- Note: If $g_t, h_t, \mathbf{Q}_t, \mathbf{R}_t$ are independent of t then the model is called stationary

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- This is a Linear Gaussian SSM; also called Linear Dynamical System (LDS)
- Note: $\mathbf{A}_t$, $\mathbf{B}_t, \mathbf{Q}_t, \mathbf{R}_t$ may be known (fixed) or may be required to be learned

Linear Gaussian SSM (LDS): An Example

- Consider the linear Gaussian SSM: $\boldsymbol{s}_t|\boldsymbol{s}_{t-1} = \mathbf{A}_t\boldsymbol{s}_{t-1} + \epsilon_t$ and $\boldsymbol{x}_t|\boldsymbol{s}_t = \mathbf{B}_t\boldsymbol{s}_t + \delta_t$

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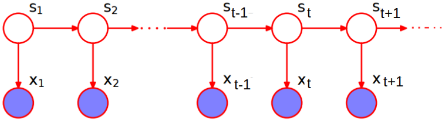

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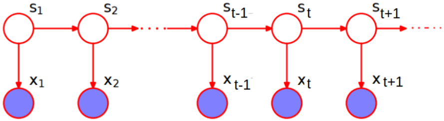

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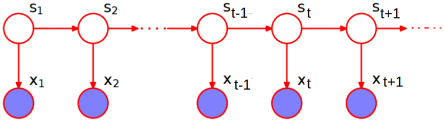

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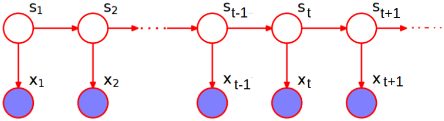

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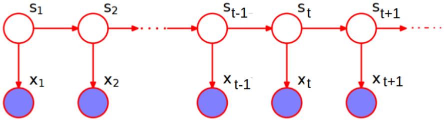

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Typical Inference Tasks in Gaussian SSM

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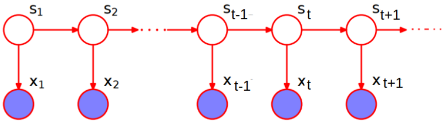

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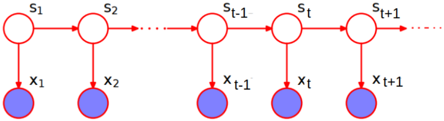

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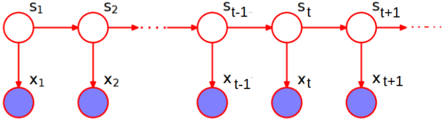

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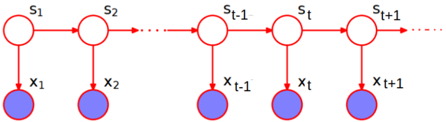
- Today, we'll mainly focus on the filtering problem (solved using the Kalman Filtering algorithm)

Kalman Filtering



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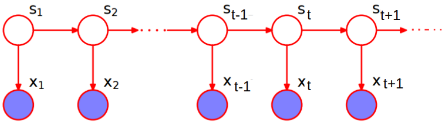

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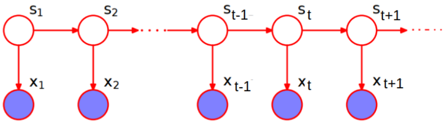

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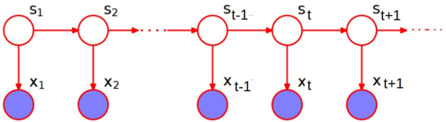

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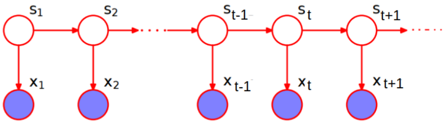

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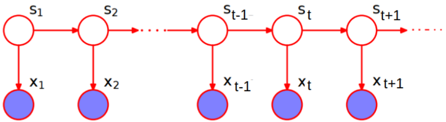

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- The "prior" above is: $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1}) = \int p(\boldsymbol{s}_t|\boldsymbol{s}_{t-1})p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})d\boldsymbol{s}_{t-1}$

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  - Reason: Starting with $p(\boldsymbol{s}_0) = \mathcal{N}(\boldsymbol{s}_0|\mathbf{0}, \mathbf{I}_K)$, the posterior over $\boldsymbol{s}_t$ will be Gaussian at each step t
- Also, using Gaussian's properties, we know that

$$\int \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{s}_{t-1}, \mathbf{Q})\mathcal{N}(\boldsymbol{s}_{t-1}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) d\boldsymbol{s}_{t-1} = \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{\mu}, \mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

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- Thus we get closed form expressions for the parameters $(\boldsymbol{\Sigma}',\boldsymbol{\mu}')$ of $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_t)$ in terms of the parameters $(\boldsymbol{\Sigma},\boldsymbol{\mu})$ of $p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{t-1})$

Kalman Filtering: Predicting Future Observations

- We saw how to compute $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ which was a Gaussian $\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')$

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p(\boldsymbol{x}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) &= \int p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1}) p(\boldsymbol{s}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) d\boldsymbol{s}_{t+1} \\
&= \int \underbrace{p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1})}_{\mathcal{N}(\boldsymbol{x}_{t+1}|\mathbf{B}\boldsymbol{s}_{t+1}, \mathbf{R})} \int \underbrace{p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t)}_{\mathcal{N}(\boldsymbol{s}_{t+1}|\mathbf{A}\boldsymbol{s}_t, \mathbf{Q})} \underbrace{p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)}_{\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')} d\boldsymbol{s}_t d\boldsymbol{s}_{t+1}
\end{aligned}
$$

Kalman Filtering: Predicting Future Observations

- We saw how to compute $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ which was a Gaussian $\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')$
- Often we are also interested in predicting the future observations

$$\begin{aligned} p(\boldsymbol{x}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) &= \int p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1}) p(\boldsymbol{s}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) d\boldsymbol{s}_{t+1} \\ &= \int \underbrace{p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1})}_{\mathcal{N}(\boldsymbol{x}_{t+1}|\mathbf{B}\boldsymbol{s}_{t+1}, \mathbf{R})} \int \underbrace{p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t)}_{\mathcal{N}(\boldsymbol{s}_{t+1}|\mathbf{A}\boldsymbol{s}_t, \mathbf{Q})} \underbrace{p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)}_{\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')} d\boldsymbol{s}_t d\boldsymbol{s}_{t+1} \end{aligned}$$

- This requires two integrals but the final result is again a Gaussian (expression not shown here)

Kalman Filtering: Some Notes

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- We can use iterative methods to estimate these parameters
    - Basically, we can alternate between inferring the states and inferring the parameters
- This can be done using approximate inference methods such as EM, MCMC, or VB

An Application: Online Dynamic Linear Regression

- Consider a dynamic linear model for regression
- The underlying ("true") weight vector $\boldsymbol{w}$ is not static (fixed) but can change with each observation

$$\begin{aligned}
\boldsymbol{w}_t &= \boldsymbol{w}_{t-1} + \eta_t \quad \text{where} \quad \eta_t \sim \mathcal{N}(0, \mathbf{I}) \\
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- Can model it as a linear dynamical system (Kalman Filtering) problem with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_T$ as the states and $y_1, \ldots, y_T$ as observations

$$\begin{aligned} p(\boldsymbol{w}_t | \boldsymbol{w}_{t-1}) &= \mathcal{N}(\boldsymbol{w}_t | \mathbf{I}\boldsymbol{w}_{t-1}, \mathbf{I}) \qquad \text{(state-transition model)} \\ p(y_t | \boldsymbol{x}_t, \boldsymbol{w}_t) &= \mathcal{N}(y_t | \boldsymbol{x}_t^\top \boldsymbol{w}_t, \sigma^2) \qquad \text{(observation model)} \end{aligned}$$

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$$

- Can now apply Kalman filtering to solve for $\boldsymbol{w}_t$ and y_t at future time-steps (note: this problem is also called the "recursive least squares" problem)

Other Extensions of SSM/LDS

- Nonlinear dynamical systems: Assume state-transition and observation models to be nonlinear

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$$

  - It's a hybrid LDS – the "state" consists of two latent variables $c_t, \boldsymbol{z}_t$ (discrete and continuous)

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- The general principle (time-evolving latent variables) can be applied in a wide range of probabilistic models to enable them handle dynamic/time-evolving data
  - E.g., in LDA, we can make the topic assignments of adjacent words follow a Markov relationship (results in an HMM-LDA type model)

Backup Slides: Kalman Smoothing

Kalman Smoothing in SSMs

Goal: Infer $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T)$ given <u>all the observations</u> (both past and future)

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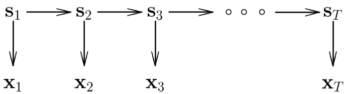
Note that each state variable $\boldsymbol{s}_t$ separates the graph into three independent parts



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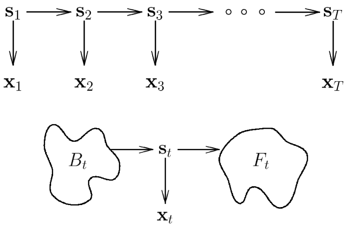

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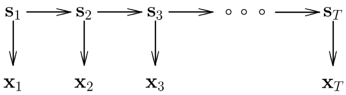

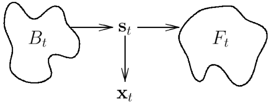

$$
\begin{aligned}
B_t &= \{\mathbf{x}_1..\mathbf{x}_{t-1}, \mathbf{s}_1..\mathbf{s}_{t-1}\} \\
F_t &= \{\mathbf{x}_{t+1}..\mathbf{x}_T, \mathbf{s}_{t+1}..\mathbf{s}_T\}
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$$
\begin{tikzcd}
\mathbf{s}_1 \arrow[r] \arrow[d] & \mathbf{s}_2 \arrow[r] \arrow[d] & \mathbf{s}_3 \arrow[r] \arrow[d] & \circ \circ \circ \arrow[r] & \mathbf{s}_T \arrow[d] \\
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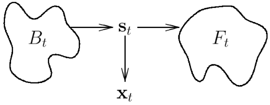

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Kalman Smoothing in SSMs

- Goal: marginal probability $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ of each state (i.e., smoothing)
- Let's look at the joint probability first:

$$\begin{aligned}
p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) &= \int_{\mathbf{s}_1..\mathbf{s}_{t-1}}\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(B_t,\mathbf{s}_t,\mathbf{x}_t,F_t) \\
&= \left(\int_{\mathbf{s}_1..\mathbf{s}_{t-1}} p(B_t,\mathbf{s}_t)\right) p(\mathbf{x}_t|\mathbf{s}_t) \left(\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(F_t|\mathbf{s}_t)\right) \\
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$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) &= p(B_t^x, \mathbf{s}_t) p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x, \mathbf{x}_t, \mathbf{s}_t) \\
\beta_t(\mathbf{s}_t) &= p(F_t^x|\mathbf{s}_t)
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$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) &= p(B_t^x, \mathbf{s}_t) p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x, \mathbf{x}_t, \mathbf{s}_t) \\
\beta_t(\mathbf{s}_t) &= p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

$$
\boxed{p(\mathbf{s}_t, \mathbf{x}_1..\mathbf{x}_T) = \alpha_t(\mathbf{s}_t)\beta_t(\mathbf{s}_t)}
$$

Kalman Smoothing in SSMs

- Goal: marginal probability $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ of each state (i.e., smoothing)
- Let's look at the joint probability first:

$$
\begin{aligned}
p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) &= \int_{\mathbf{s}_1..\mathbf{s}_{t-1}}\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(B_t,\mathbf{s}_t,\mathbf{x}_t,F_t) \\
&= \left(\int_{\mathbf{s}_1..\mathbf{s}_{t-1}} p(B_t,\mathbf{s}_t)\right) p(\mathbf{x}_t|\mathbf{s}_t)\left(\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(F_t|\mathbf{s}_t)\right) \\
&= p(B_t^x,\mathbf{s}_t)p(\mathbf{x}_t|\mathbf{s}_t)p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

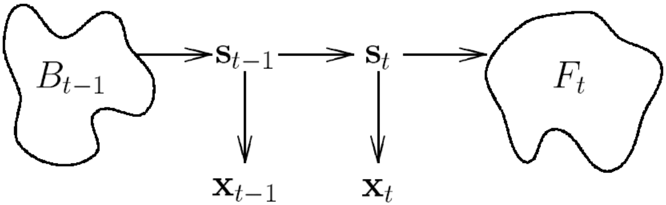
$$
\begin{aligned}
B_t^x &= \{\mathbf{x}_1..\mathbf{x}_{t-1}\} \\
F_t^x &= \{\mathbf{x}_{t+1}..\mathbf{x}_T\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) &= p(B_t^x,\mathbf{s}_t)p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x,\mathbf{x}_t,\mathbf{s}_t) \\
\beta_t(\mathbf{s}_t) &= p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

$$
\boxed{p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) = \alpha_t(\mathbf{s}_t)\beta_t(\mathbf{s}_t)}
$$

- From the joint, we can compute $p(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T) = \sum_{\boldsymbol{s}_t} p(\boldsymbol{s}_t,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$, and $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ using Bayes rule

Estimation via Forward-Backward Recursion



Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}
$$

Forward recursion for α

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}
$$

Forward recursion for α

$$
\begin{aligned}
\beta_{t-1}(\mathbf{s}_{t-1}) = p(F_{t-1}^x|\mathbf{s}_{t-1}) &= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}, \mathbf{x}_t, F_t^x|\mathbf{s}_{t-1}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})p(F_t^x|\mathbf{s}_t = \mathbf{z}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})\beta_t(\mathbf{z})
\end{aligned}
$$

Backward recursion for β

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}$$

Forward recursion for α

$$\begin{aligned}
\beta_{t-1}(\mathbf{s}_{t-1}) = p(F_{t-1}^x|\mathbf{s}_{t-1}) &= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}, \mathbf{x}_t, F_t^x|\mathbf{s}_{t-1}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})p(F_t^x|\mathbf{s}_t = \mathbf{z}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})\beta_t(\mathbf{z})
\end{aligned}$$

Backward recursion for β

Initialize as $\alpha_1(\boldsymbol{s}_1) = p(\boldsymbol{s}_1)p(\boldsymbol{x}_1|\boldsymbol{s}_1)$ and $\beta_T(\boldsymbol{s}_T) = 1$