Module 25

CONDITIONAL EXPECTATIONS

- $\underline{X} = (\underline{Y}, \underline{Z})$: a p-dimensional r.v. with joint p.m.f./p.d.f. $f_{\underline{Y},\underline{Z}}(\cdot)$ and support $S_{\underline{X}}$, where
- $\underline{Y} = (Y_1, \ldots, Y_{p_1})$: a p_1-dimensional r.v. with p.m.f./p.d.f. $f_{\underline{Y}}(\cdot)$ and support $S_{\underline{Y}}$,
- $\underline{Z} = (Z_1, \ldots, Z_{p_2})$: a p_2-dimensional r.v. with p.m.f./p.d.f. $f_{\underline{Z}}(\cdot)$ and support $S_{\underline{Z}}$,

and $p = p_1 + p_2$.

- For given $\underline{z} \in S_{\underline{Z}}$ (with $f_{\underline{Z}}(\underline{z}) > 0$) the conditional p.m.f/p.d.f. of $\underline{Y}$ given $\underline{Z} = \underline{z}$ is

$$\begin{aligned} f_{\underline{Y}|\underline{Z}}(\underline{y}|\underline{z}) &= \frac{f_{\underline{Y},\underline{Z}}(\underline{y},\underline{z})}{f_{\underline{Z}}(\underline{z})} \\ &= k(\underline{z}) f_{\underline{Y},\underline{Z}}(\underline{y},\underline{z}), \;\; \underline{y} \in \mathbb{R}^p, \end{aligned}$$

  where, for fixed $\underline{z} \in S_{\underline{Z}}$ (with $f_{\underline{Z}}(\underline{z}) > 0$), $k(\underline{z}) = [f_{\underline{Z}}(\underline{z})]^{-1}$ is a normalizing constant. Thus the conditional p.m.f./p.d.f. is proportional to the joint p.d.f.

Definition 1:

- Let $\psi : \mathbb{R}^{p_1} \to \mathbb{R}$ be a given function and let $\underline{z} \in S_{\underline{Z}}$ (with $f_{\underline{Z}}(\underline{z}) > 0$) be given.
  (a) The conditional expectation of $\psi(\underline{Y})$ given that $\underline{Z} = \underline{z}$ is defined by
  $$E(\psi(\underline{Y})|\underline{Z} = \underline{z}) = \int\limits_{\mathbb{R}^{p_1}} \psi(\underline{y}) f_{\underline{Y}|\underline{Z}}(\underline{y}|\underline{z}) d\underline{y},$$
  provided the expectation is finite.
  (b) The conditional variance of $\psi(\underline{Y})$ given that $\underline{Z} = \underline{z}$ is defined by
  $$\begin{aligned} \mathrm{Var}(\psi(\underline{Y})|\underline{Z} = \underline{z}) &= E\left(\left(\psi(\underline{Y}) - E\left(\psi(\underline{Y})|\underline{Z} = \underline{z}\right)\right)^2 |\underline{Z} = \underline{z}\right) \\ &= E\left(\psi^2(\underline{Y})|\underline{Z} = \underline{z}\right) - \left(E\left(\psi(\underline{Y})|\underline{Z} = \underline{z}\right)\right)^2. \end{aligned}$$
- Note that $E(\psi(\underline{Y})|\underline{Z} = \underline{z})$ is a function of $\underline{z} \in S_{\underline{Z}}$.

Notation:

- $$E(\psi(\underline{Y})|\underline{Z}) = \psi^*(\underline{Z}),$$

  where

  $$\psi^*(\underline{z}) = E(\psi(\underline{Y})|\underline{Z} = \underline{z}).$$

- Similarly we define $\mathrm{Var}(\psi(\underline{Y})|\underline{Z})$, $\mathrm{Cov}(Y_1, Y_2|\underline{Z})$ and $\rho(Y_1, Y_2|\underline{Z})$.

Result 1 :

Under the above notations

(a) $E(\psi(\underline{Y})) = E\big(E(\psi(\underline{Y})|\underline{Z})\big)$.

(b) $\mathrm{Var}(\psi(\underline{Y})) = \mathrm{Var}\big(E(\psi(\underline{Y})|\underline{Z})\big) + E\big(\mathrm{Var}(\psi(\underline{Y})|\underline{Z})\big)$.

Proof : For A.C. case.

(a) Note that

$$E\big(E(\psi(\underline{Y})|\underline{Z})\big) = E(\psi^*(\underline{Z})),$$

where, for $\underline{z} \in S_{\underline{Z}}$,

$$\begin{aligned} \psi^*(\underline{z}) &= E(\psi(\underline{Y})|\underline{Z} = \underline{z}) \\ &= \int\limits_{\mathbb{R}^{p_1}} \psi(\underline{y}) f_{\underline{Y}|\underline{Z}}(\underline{y}|\underline{z}) d\underline{y}. \end{aligned}$$

Thus

$$
\begin{aligned}
E\left(E(\psi(\underline{Y})|\underline{Z})\right) &= E(\psi^*(\underline{Z})) \\
&= \int\limits_{\mathbb{R}^{p_2}} \psi^*(\underline{z}) f_{\underline{Z}}(\underline{z}) d\underline{z} \\
&= \int\limits_{\mathbb{R}^{p_2}} \left\{ \int\limits_{\mathbb{R}^{p_1}} \psi(\underline{y}) f_{\underline{Y}|\underline{Z}}(\underline{y}|\underline{z}) d\underline{y} \right\} f_{\underline{Z}}(\underline{z}) d\underline{z} \\
&= \int\limits_{\mathbb{R}^{p}} \psi(\underline{y}) f_{\underline{Y}|\underline{Z}}(\underline{y}|\underline{z}) f_{\underline{Z}}(\underline{z}) d\underline{y} d\underline{z} \\
&= \int\limits_{\mathbb{R}^{p}} \psi(\underline{y}) f_{\underline{Y},\underline{Z}}(\underline{y},\underline{z}) d\underline{y} \\
&= E(\psi(\underline{Y})).
\end{aligned}
$$

(b) Let $\psi^*(\underline{Z}) = E\big(\psi(\underline{Y})|\underline{Z}\big)$. Then by (a)

$$\begin{aligned}
\mathrm{Var}(\psi(\underline{Y})) &= E\Big(\big(\psi(\underline{Y}) - E(\psi(\underline{Y}))\big)^2\Big) \\
&= E\Bigg(E\Big(\big(\psi(\underline{Y}) - E(\psi(\underline{Y}))\big)^2|\underline{Z}\Big)\Bigg) \\
&= E(\psi_1(\underline{Z}),) \qquad (1)
\end{aligned}$$

where

$$\begin{aligned}
\psi_1(\underline{Z}) &= E\Bigg(\Big(\psi(\underline{Y}) - E\big(\psi(\underline{Y})\big)\Big)^2\Big|\underline{Z}\Bigg) \\
&= E\Bigg(\Big(\big\{\psi(\underline{Y}) - E\big(\psi(\underline{Y})|\underline{Z}\big)\big\} \\
&\qquad + \big\{E\big(\psi(\underline{Y})|\underline{Z}\big) - E\big(\psi(\underline{Y})\big)\big\}\Big)^2\Big|\underline{Z}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= E\left(\left(\psi(\underline{Y}) - E\left(\psi(\underline{Y})|\underline{Z}\right)\right)^2\Big|\underline{Z}\right) + \left(E\left(\psi(\underline{Y})|\underline{Z}\right) - E(\psi(\underline{Y}))\right)^2 \\
&\quad +2\left(E\left(\psi(\underline{Y})|\underline{Z}\right) - E(\psi(\underline{Y}))\right)E\left(\left(\psi(\underline{Y}) - E\left(\psi(\underline{Y})|\underline{Z}\right)\right)\Big|\underline{Z}\right) \\
&= \mathrm{Var}(\psi(\underline{Y})|\underline{Z}) + \left(E\left(\psi(\underline{Y})|\underline{Z}\right) - E\left(E(\psi(\underline{Y})|\underline{Z})\right)\right)^2 + 0.
\end{aligned}
$$

Thus from (1)

$$
\begin{aligned}
\mathrm{Var}(\psi(\underline{Y})) &= E\left(\mathrm{Var}(\psi(\underline{Y})|\underline{Z})\right) + E\left(\left(E\left(\psi(\underline{Y})|\underline{Z}\right) - E\left(E(\psi(\underline{Y})|\underline{Z})\right)\right)^2\right) \\
&= E\left(\mathrm{Var}(\psi(\underline{Y})|\underline{Z})\right) + \mathrm{Var}\left(E\left(\psi(\underline{Y})|\underline{Z}\right)\right).
\end{aligned}
$$

Remark 1:

If $\underline{Y}$ and $\underline{Z}$ are independent then

$$E(\psi(\underline{Y})|\underline{Z}) = E(\psi(\underline{Y}))$$

and

$$\mathrm{Var}(\psi(\underline{Y})|\underline{Z}) = \mathrm{Var}(\psi(\underline{Y})).$$

Example 1:

Let $\underline{X} = (X_1, X_2, X_3)'$ be A.C. r.v.with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{1}{x_1 x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y_1 = 2X_1 - X_2 + 3X_3$, $Y_2 = X_1 - 2X_2 + X_3$ and $Y = X_1 X_2 X_3$.

(a) Find $\rho(Y_1, Y_2)$.

(b) For a fixed $x_2 \in (0, 1)$ find $E(Y|X_2 = x_2)$, $\mathrm{Var}(Y|X_2 = x_2)$ and $\mathrm{Cov}(X_1, X_3|X_2 = x_2)$.

Solution :

(a)

$$
\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= \mathrm{Cov}(2X_1 - X_2 + 3X_3, X_1 - 2X_2 + X_3) \\
&= 2\mathrm{Var}(X_1) - 5\mathrm{Cov}(X_1, X_2) + 5\mathrm{Cov}(X_1, X_3) \\
&\qquad +2\mathrm{Var}(X_2) - 7\mathrm{Cov}(X_2, X_3) + 3\mathrm{Var}(X_3) \\
\mathrm{Var}(Y_1) &= 4\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 9\mathrm{Var}(X_3) - 4\mathrm{Cov}(X_1, X_2) \\
&\qquad +12\mathrm{Cov}(X_1, X_3) - 6\mathrm{Cov}(X_2, X_3) \\
\mathrm{Var}(Y_2) &= \mathrm{Var}(X_1) + 4\mathrm{Var}(X_2) + \mathrm{Var}(X_3) - 4\mathrm{Cov}(X_1, X_2) \\
&\qquad +2\mathrm{Cov}(X_1, X_3) - 4\mathrm{Cov}(X_2, X_3) \\
\rho(Y_1, Y_2) &= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}}
\end{aligned}
$$

For any function $\psi(\cdot) : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} E(\psi(X_1, X_2, X_3)) &= \int\limits_{\mathbb{R}^3} \psi(x_1, x_2, x_3) f_{\underline{X}}(x_1, x_2, x_3) d\underline{x} \\ &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{\psi(x_1, x_2, x_3)}{x_1 x_2} dx_3 dx_2 dx_1 \end{aligned}$$

Thus,

$$\begin{aligned} E(X_1) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_2} dx_3 dx_2 dx_1 = \frac{1}{2}; \\ E(X_1^2) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_1}{x_2} dx_3 dx_2 dx_1 = \frac{1}{3}; \end{aligned}$$

$$
\begin{aligned}
E(X_2) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_1} dx_3 dx_2 dx_1 = \frac{1}{4}; \\
E(X_2^2) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_2}{x_1} dx_3 dx_2 dx_1 = \frac{1}{9}; \\
E(X_3) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_3}{x_1 x_2} dx_3 dx_2 dx_1 = \frac{1}{8}; \\
E(X_3^2) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{x_3^2}{x_1 x_2} dx_3 dx_2 dx_1 = \frac{1}{27}; \\
E(X_1 X_2) &= \int_0^1 \int_0^{x_1} \int_0^{x_2} dx_3 dx_2 dx_1 = \frac{1}{6};
\end{aligned}
$$

$$
\begin{aligned}
E(X_1X_3) &= \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3}{x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{12};\\
E(X_2X_3) &= \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3}{x_1}\,dx_3\,dx_2\,dx_1 = \frac{1}{18};\\
\mathrm{Var}(X_1) &= E(X_1^2) - (E(X_1))^2 = \frac{1}{12};\\
\mathrm{Var}(X_2) &= E(X_2^2) - (E(X_2))^2 = \frac{7}{144};\\
\mathrm{Var}(X_3) &= E(X_3^2) - (E(X_3))^2 = \frac{37}{1728};\\
\mathrm{Cov}(X_1, X_2) &= E(X_1X_2) - E(X_1)E(X_2) = \frac{1}{24};
\end{aligned}
$$

$$\begin{aligned}
\text{Cov}(X_1, X_3) &= E(X_1X_3) - E(X_1)E(X_3) = \frac{1}{48};\\
\text{Cov}(X_2, X_3) &= E(X_2X_3) - E(X_2)E(X_3) = \frac{7}{228}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{Cov}(Y_1, Y_2) &= \frac{1}{6} - \frac{5}{24} + \frac{5}{48} + \frac{7}{72} - \frac{49}{288} + \frac{37}{1728}\\
&= \frac{31}{576};\\
\text{Var}(Y_1) &= 4\text{Var}(X_1) + \text{Var}(X_2) + 9\text{Var}(X_3) - 4\text{Cov}(X_1, X_2)\\
&\qquad +12\text{Cov}(X_1, X_3) - 6\text{Cov}(X_2, X_3)\\
&= \frac{1}{3} + \frac{7}{144} + \frac{37}{192} - \frac{1}{6} + \frac{1}{4} - \frac{7}{48}\\
&= \frac{295}{576};
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}(Y_2) &= \mathrm{Var}(X_1) + 4\mathrm{Var}(X_2) + \mathrm{Var}(X_3) - 4\mathrm{Cov}(X_1, X_2) \\
&\qquad + 2\mathrm{Cov}(X_1, X_3) - 4\mathrm{Cov}(X_2, X_3) \\
&= \frac{1}{12} + \frac{7}{36} + \frac{37}{1728} - \frac{1}{6} + \frac{1}{24} - \frac{7}{72} \\
&= \frac{133}{1728}; \\
\rho(Y_1, Y_2) &= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}} \\
&= 0.2710.
\end{aligned}
$$

(b) Clearly, given $X_2 = x_2$, X_1 and X_3 are independent with p.d.f.s

$$f_{X_1|X_2}(x_1|x_2) = \begin{cases} \frac{c_1(x_2)}{x_1}, & \text{if } x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

$$= \begin{cases} \frac{-1}{(\ln x_2)x_1}, & \text{if } x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$f_{X_3|X_2}(x_3|x_2) = \begin{cases} c_2(x_2), & \text{if } 0 < x_3 < x_2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{x_2}, & \text{if } 0 < x_3 < x_2 \\ 0, & \text{otherwise} \end{cases}.$$

Also, for $0 < x_2 < 1$,

$$\begin{aligned} f_{x_2}(x_2) &= \int_{x_2}^{1} \int_{0}^{x_2} \frac{1}{x_1 x_2} dx_3 dx_1 \\ &= -\ln x_2; \end{aligned}$$

$$f_{x_2}(x_2) = \begin{cases} -\ln x_2, & \text{if } 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned} E(Y|X_2 = x_2) &= E(X_1 X_2 X_3|X_2 = x_2) \\ &= x_2 E(X_1 X_3|X_2 = x_2) \\ &= x_2 E(X_1|X_2 = x_2) E(X_3|X_2 = x_2); \end{aligned}$$

$$\begin{aligned} E(X_1|X_2 = x_2) &= \int_{-\infty}^{\infty} x_1 f_{X_1|X_2}(x_1|x_2) dx_1 \\ &= \int_{x_2}^{1} \frac{-1}{\ln x_2} dx_1 \\ &= \frac{x_2 - 1}{\ln x_2}; \end{aligned}$$

$$
\begin{aligned}
E(X_3|X_2=x_2) &= \int\limits_{-\infty}^{\infty} x_3 f_{X_3|X_2}(x_3|x_2)dx_3 \\
&= \int\limits_{0}^{x_2} \frac{x_3}{x_2} dx_3 \\
&= \frac{x_2}{2};
\end{aligned}
$$

$$
E(Y|X_2=x_2) = \frac{x_2^2(x_2-1)}{2\ln x_2};
$$

$$
\begin{aligned}
E(Y^2|X_2=x_2) &= E(X_1^2X_2^2X_3^2|X_2=x_2) \\
&= x_2^2 E(X_1^2|X_2=x_2)E(X_3^2|X_2=x_2);
\end{aligned}
$$

$$
\begin{aligned}
E(X_1^2|X_2 = x_2) &= \int_{x_2}^{1} \frac{-x_1}{\ln x_2} dx_1 \\
&= -\frac{1 - x_2^2}{2 \ln x_2}; \\
E(X_3^2|X_2 = x_2) &= \int_{0}^{x_2} \frac{x_3^2}{x_2} dx_3 \\
&= \frac{x_2^2}{3}.
\end{aligned}
$$

$$\mathrm{Var}(X_1|X_2 = x_2) = E(X_1^2|X_2 = x_2) - (E(X_1|X_2 = x_2))^2.$$

Since, given $X_2 = x_2$, X_1 and X_3 are independent we have $\mathrm{Cov}(X_1, X_3|X_2 = x_2) = 0$.

Take Home Problem

Let $\underline{X} = (X_1, X_2, X_3)'$ be a discrete r.v. with joint p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{x_1 x_2 x_3}{72}, & \text{if } x_1 = 1, 2, \ x_2 = 1, 2, 3, \ x_3 = 1, 3 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y_1 = 2X_1 - X_2 + 3X_3$, $Y_2 = X_1 - 2X_2 + X_3$ and $Y = X_1 X_2 X_3$.

(a) Find $\rho(Y_1, Y_2)$;

(b) For fixed $x_2 \in \{1, 2, 3\}$ find $E(Y|X_2 = x_2)$ and $\mathrm{Var}(Y|X_2 = x_2)$.

Abstract of Next Module

We will introduce the joint m.g.f. of a random vector and study its properties.

Thank you for your patience