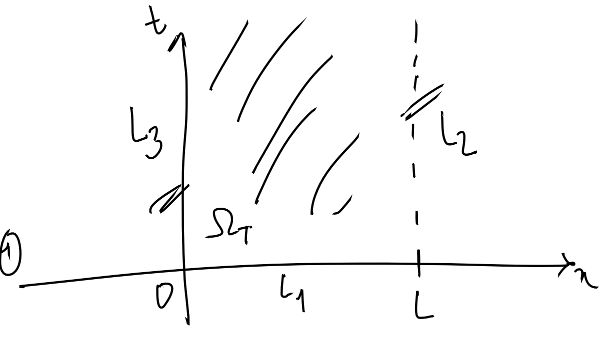
MSO 203B (PDE) Lecture 16 : Heat Eqn

1-D Heat Eqn

$$u_t - u_{xx} = 0 \quad \text{in} \quad \underset{\overset{\Uparrow}{x}}{(0,L)} \times \underset{\overset{\Uparrow}{t}}{(0,\infty)} = \Omega_T$$

$$\left.\begin{aligned} &u(x,0) = \varphi(x), \quad x \in (0,L). \\ &u(0,t) = u(L,t) = 0, \quad t \in (0,\infty) \end{aligned}\right\} - (1)$$



$\boxed{\partial\Omega_T := \text{Bdry of } \Omega_T \text{ is defined as } L_1 \cup L_2 \cup L_3}$

Soln of (1) is a fxn $u \in C^{2,1}(\Omega_T) \cup C(\bar{\Omega}_T)$. s.t u satisfies (1) where $\bar{\Omega}_T = \Omega_T \cup \partial\Omega_T$

$u(x,t) = X(x)\,T(t)$

$\Rightarrow -X''T + XT' = 0$

$\Rightarrow XT' = X''T$

$\Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = \lambda$

$\cancel{X(0)T(t)} = 0$

$X(L) = 0$

$\therefore X'' = \lambda X \;;\; X(0) = X(L) = 0 \Rightarrow \lambda_n = -\left(\frac{n\pi}{L}\right)^2 ; \quad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$

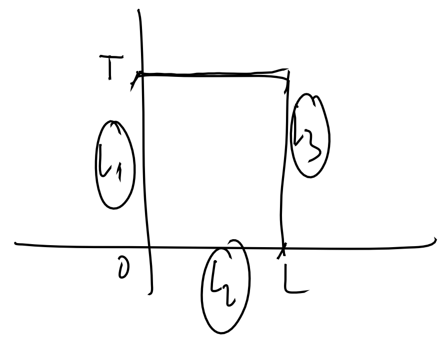
$T' = \lambda T \Rightarrow T_n' = -\left(\frac{n\pi}{L}\right)^2 T_n \Rightarrow T_n(t) = \exp\left(-\frac{n^2\pi^2 t}{L^2}\right).$

2D Heat Eqn

$u_t - (u_{xx} + u_{yy}) = 0$

$u_t - \Delta u = 0$

$\underset{\overset{\Uparrow}{x}}{(0,L)} \times \underset{\overset{\Uparrow}{t}}{(0,T)}$



$$u_n(x,t) = \exp\left(-\frac{n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore\ u(x,t) = \sum_1^\infty C_n u_n(x,t) \quad \left(\text{Superposition}\right).$$

$$= \sum_{n=1}^\infty C_n \exp\left(-\frac{n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right).$$

$$f(x) = u(x,0) = \sum_{n=1}^\infty C_n \exp(0) \sin\left(\frac{n\pi x}{L}\right).$$

$$\Rightarrow\ C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The General Soln is $u(x,t) = \sum_{n=1}^\infty \left(\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right).$

<u>Inhomogeneous ~~Hea~~ Eqn:-</u>

$$u_t - u_{xx} = f(x,t) \quad \text{in} \quad (0,L)\times(0,\infty)$$

$$u(x,0) = g(x) \quad , \quad x \in (0,L)$$

$$u(0,t) = u(L,t) = 0 \quad , \quad t \in (0,\infty)$$

Special Case :- Let $f(x,t) = h(x)$.

$u_t = u_{xx} + h(x)$, $x \in (0,L)$, $t \in (0,\infty)$.

$u(0,t) = u(L,t) = 0$ B.C

$u(x,0) = g(x)$. I.C

Step 1 :- Solve $w_{xx} + h(x) = 0$

[Q: Find w s.t $w_{xx} + h(x) = 0$] (Stabilized Problem)

Ex :- $u_t = u_{xx} + \sin x$, B.C & I.C ; $h(x) = \sin x$.

Find w s.t $w_{xx} + \sin x = 0$.

$\Rightarrow$ $w(x) = \sin x$

Step 2 :- Define $v(x,t) := u(x,t) - w(x)$.

$v_t = u_t - 0$

$v_{xx} = u_{xx} - w_{xx} = u_{xx} + \sin x$

$v_t - v_{xx} = u_t - u_{xx} - \sin x = 0$

$w = \sin x$

$w' = \cos x$

$w'' = -\sin x$

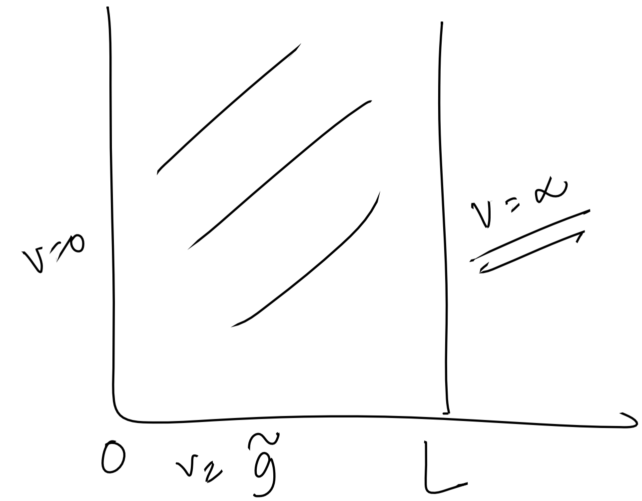
Our transformed eqn is

$$v_t = v_{xx} \quad \text{on } (0,L)\times(0,\infty)$$

$$v(x,0) = u(x,0) - w(x) = g(x) - \sin x \; \left(:= \tilde{g}(x)\right)$$

$$v(0,t) = u(0,t) - w(0) = 0$$

$$v(L,t) = u(L,t) - w(L) = 0 - \sin L \; \left(:= \alpha\right)$$



Define $\theta(x,t) = v(x,t) - \frac{\alpha x}{L}$

$$\theta_t = v_t$$

$$\theta_{xx} = v_{xx} - 0$$

$$\theta_t - \theta_{xx} = 0$$

$$\theta(x,0) = \tilde{g}(x) - \frac{\alpha x}{L} \; \left(:= \tilde{\tilde{g}}(x)\right)$$

$$\theta(0,t) = 0$$

$$\theta(L,t) = v(L,t) - \frac{\alpha \cdot L}{L} = \alpha - \alpha = 0 .$$

Maximum Principle for Heat Eqn :- Let Ω is bounded in $\mathbb{R}$. and 'u' solves $u_t - u_{xx} = 0$. in $(0,L) \times (0,T)$;- $T > 0$.

Then 'u' assumes its max on the parabolic boundary $(\partial\Omega_T := L_1 \cup L_2 \cup L_3)$.