MSO 203B - Partial Differential Equation

Assignment 6

November 8, 2018

Tutorial Problem

1. We know that the D'Alembert's Formula
$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$
solves the wave equation
$$u_{tt} - c^2u_{xx} = 0;\ x \in \mathbf{R},\ t > 0$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x),\ x \in \mathbf{R}$$
Note that the solution at any given point (x_0, t_0) only depends on the value of f at the points $x_0 \pm ct_0$ and what g does between these two points. The interval $[x_0 - ct_0, x_0 + ct_0]$ is called the domain of dependence of the variable (x, t). The area $t > t_0$ bounded by the two characteristics $x \pm ct = constant$ passing through (x_0, t_0) is called the range of influence. The solution at any point (x, t) is completely determined by Cauchy data on the domain of dependence. This region is called the region of determinancy of the solution. With this information answer the following:-
   (a) Solve the homogeneous wave equation provided $f(x) = \sin x$ and $g(x) = \cos x$.
   (b) Find the domain of dependence, range of influence and region of determinancy for the problem.
   (c) Show that the solution makes sense even if f and g are not continuous everywhere.

2. A general problem in linear hyperbolic equations investigates the solution of the following problem
$$u_{xy} = Au_x + Bu_y + Cu + D$$
where A, B, C, D are continuous function of (x, y) with the condition that $u(x, y) = f(x)$ on a characteristic and $u(x, y)$ on a monotonic increasing curve $y = y(x)$. Can you use the ideas explored in the class to find the solution for the following:-
$$u_{tt} - c^2u_{xx} = 0;\ x \in \mathbf{R},\ t > 0$$
$$u(x,t) = f(x) \text{ on } \{x + ct = 0\}$$
$$u(x,t) = g(x) \text{ on } \{x - ct = 0\}$$

with $f(0) = g(0)$.

3. From the Elliptic and Parabolic theory we saw that the Maximum Princple roughly says that the extremum to any solution will always be obtained on the elliptic or the parabolic boundary. This can be used in a number of ways to explore the number of solution to a given equation, answer stability of the equation and also provides information on the solutions in cases where no explicit solution(s) are available. Now the question is why don't we look for maximum principle for wave equation. The answer is it is not possible to obtain such a result for wave equation. Can you explore this situation and provide an example to illustrate the point.

4. Separation of Variable is a very useful tool of obtaining solution as long as we are in a nice rectangular region bounded or not. This works by reducing the PDE to Sturm-Liouville problems and then solving the problem using Fourier series. Use this to solve for the following:

$$\begin{gathered} u_t - ku_{xx} = 0;\ x \in (0, l),\ t > 0 \\ u(0, t) = 0,\ u(l, t) = 1,\ t > 0 \\ u(x, 0) = f(x),\ 0 < x < l \end{gathered}$$

5. Solving Nonhomogeneous problems for Heat equation can be easier when the source term is only function of x. Using a suitable transformation solve the problem

$$\begin{gathered} u_t - ku_{xx} = 2;\ x \in (0, l),\ t > 0 \\ u(0, t) = 0,\ u(l, t) = 0,\ t > 0 \\ u(x, 0) = f(x),\ 0 < x < l \end{gathered}$$

6. Let us conclude our last assignment with what I consider the most fundamental class of result in PDE aka the Maximum Principle. Define $A := \{(x, t) : 0 < x < \pi, 0 < t \leq T\}$ and let u solves

$$\begin{gathered} u_t - u_{xx} = 0 \text{ in } A \\ u(0, t) = 0,\ u(\pi, t) = 0,\ 0 \leq t \leq T \\ u(x, 0) = \sin^2 x,\ 0 < x < \pi \end{gathered}$$

   Using maximum principle to show that $0 \leq u(x, t) \leq e^{-t} \sin x$ in A.

Practise Problem

1. Solve the following:-

$$\begin{gathered} u_{tt} - c^2 u_{xx} = 0;\ x \in \mathbf{R},\ t > 0 \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x),\ x \in \mathbf{R} \\ u_x(0, t) = 0,\ t \geq 0 \end{gathered}$$

2. Solve:-

$$
\begin{gathered}
u_{tt} - u_{xx} = \cos x;\ x \in (0, l),\ t > 0 \\
u(x, 0) = x(1 - x),\ u_t(x, 0) = 0:\ 0 \leq x \leq 1 \\
u(0, t) = u(1, t) = 0, t \geq 0
\end{gathered}
$$

3. Show that the compatibility condition for the Neumann problem

$$
\begin{gathered}
\Delta u = f \text{ in } \Omega \\
\tfrac{\partial u}{\partial \eta} = g \text{ on } \partial\Omega
\end{gathered}
$$

4. Solve using SOV

$$
\begin{gathered}
u_t - u_{xx} = 0;\ x \in (0, l),\ t > 0 \\
u(0, t) = 0,\ u(l, t) = 0,\ t > 0 \\
u(x, 0) = e^x,\ 0 < x < l
\end{gathered}
$$

5. Solve:-

$$
\begin{gathered}
u_t - k u_{xx} = 0;\ x \in (0, l),\ t > 0 \\
u(0, t) = 0,\ t > 0 \\
u(x, 0) = f(x),\ 0 < x < l \\
3u(l, t) + u'(l, t) = 0,\ t > 0
\end{gathered}
$$