MSO 203B - Lecture 19 (Wave Eqn - II)

Duhamel's Principle:-

$$\left.\begin{aligned} y'(t) &= Ay(t) + f(t) \\ y(0) &= y_0 \end{aligned}\right\} \text{ — } (1)$$

$$\therefore\ e^{-At}y'(t) - Ae^{-At}y(t) = f(t)e^{-At}.$$

$$\Rightarrow \left[y(t)e^{-At}\right]' = f(t)e^{-At}.$$

$$\Rightarrow y(t)e^{-At} - y(0)e^{-A\cdot 0} = \int_0^t f(\tau)e^{-A\tau}\,d\tau$$

$$\Rightarrow y(t)e^{-At} = y_0 + \int_0^t f(\tau)e^{-A\tau}\,d\tau.$$

$$\Rightarrow y(t) = e^{At}y_0 + e^{At}\int_0^t f(\tau)e^{-A\tau}\,d\tau.$$

Define, $S(t) = e^{At}$ (Solution Operator) [Semigroup Theory].

$$\int_0^t$$

$$\boxed{L(y) = f}$$

$$\boxed{L(y) = 0} \qquad \boxed{Ly = f}$$

$$\psi \qquad\qquad \phi$$

$$L(\phi + \psi) = L\phi + L\psi = 0 + f = f$$

$e^{At}y_0 \qquad \psi e^{At}$

$$\boxed{y(t) = S(t)y_0 + \int_0^t S(t-\tau) f(\tau)\, d\tau.}$$ ← Semigroup formulation of an ODE

$$\left.\begin{aligned} u_{tt} - u_{xx} &= f(x,t) \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}\right\} \text{ — (I)}.$$

$$S(t)\begin{bmatrix} g \\ h \end{bmatrix} =$$

$$u^H(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\, ds$$ is the soln of the homogeneous problem. //

u^I is a soln of (I)

$u^I = u^H + u^P$. (u^P is a particular soln of (I))

Choose $u_t = v$. — (1)

$v_t = u_{xx} + f(x,t)$ — (2)

Let us define $U(x,t) = \begin{bmatrix} u \\ v \end{bmatrix}$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ \partial^2/\partial x^2 & 0 \end{bmatrix}}_{B}\begin{bmatrix} u \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ f \end{bmatrix}}_{C} \Rightarrow \boxed{U_t = BU + C \;;\; U(x,0) = \begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}.} \text{ — (II)}$$

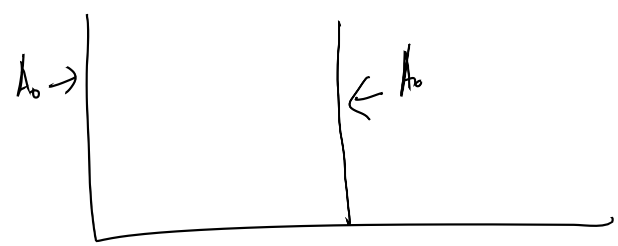
$S(t)y_0 = u^H.$

$$S(t)\begin{bmatrix} g \\ h \end{bmatrix} = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi;$$

$$S(t-\tau)f(\tau) = S(t-\tau)\begin{pmatrix} 0 \\ f \end{pmatrix}.$$

$$= \frac{1}{2}\int_{x-t+\tau}^{x+t-\tau} f(x,\xi)\,d\xi$$

$$u^I(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(x,\xi)\,d\xi\,d\tau$$



Uniqueness :-

$u_{tt} - u_{xx} = H(x,t) \quad ; \; x \in (0,l), \; t \geq 0.$

$u(x,0) = f(x)$

$u_t(x,0) = g(x).$

$u(0,t) = u(l,t) = A_0$

$$v_{tt} - v_{xx} = 0$$
$$v(0,t) = v(l,t) = 0$$
$$v(x,0) = v_t(x,0) = 0$$
$\Big\}$ — IV $\quad v = u_1 - u_2$

RTP :- $v = 0$.

$$E(t) = \frac{1}{2}\int_0^l (v_t^2 + v_x^2)\,dx$$

$$\Rightarrow E'(t) = \frac{1}{2}\cdot 2\int_0^l (v_t v_{tt} + v_x v_{xt})\,dx = \int_0^l (v_t v_{tt} + v_x v_{xt})\,dx$$

$$= \int_0^l (v_t v_{tt} - v_{xx} v_t)\,dx + v_x v_t \Big|_0^l$$

$$= \int_0^l \left[(v_{tt} - v_{xx})\, v_t\right] dx + v_x v_t \Big|_0^l$$

$$= v_x(l,t)\, v_t(l,t) - v_x(0,t)\, v_t(0,t)$$

$$= 0$$

$\Rightarrow E(t)$ is constant

$$E(0) = \frac{1}{2}\int_0^l \left[v_t^2(x,0) + v_x^2(x,0)\right] dx$$

$$= 0$$

$$\Rightarrow E(t) = 0$$

$$v_t^2 = 0$$

$$v_x^2 = 0$$

$$\Rightarrow \nabla v = 0 \left[(v_t, v_x) = \nabla v\right]$$

$$\Rightarrow v \equiv 0$$