Module 20

Properties of Distribution Function of a Random Vector

- $\underline{X}$: a given p-dimensional r.v. defined on a probability space $(\Omega, \mathcal{P}(\Omega), P)$ with d.f. $F_{\underline{X}}(\cdot)$;

$$\begin{aligned} F_{\underline{X}}(\underline{x}) &= P(\{\underline{X} \leq \underline{x}\}) \\ &= P_{\underline{X}}((-\infty, \underline{x}]), \quad \underline{x} \in \mathbb{R}^p, \end{aligned}$$

  where $P_{\underline{X}}(\cdot)$ is the probability function induced by $\underline{X}$;

- For any p-dimensional rectangle $(\underline{a}, \underline{b}]$ $(\underline{a} < \underline{b})$, we know that

$$P(\{\underline{X} \in (\underline{a}, \underline{b}]\}) = \sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a}, \underline{b}])} F_{\underline{X}}(\underline{z}).$$

Result 1:

Let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional r.v. with d.f. $F_{\underline{X}}(\cdot)$. Then

(a) $\lim_{\substack{x_i \to \infty \\ i=1,\ldots,p}} F_{\underline{X}}(x_1, \ldots, x_p) = 1$; (iterated limit)

(b) For each fixed $i \in \{1, \ldots, p\}$, $\lim_{x_i \to -\infty} F_{\underline{X}}(x_1, \ldots, x_p) = 0$;

(c) $F_{\underline{X}}(x_1, \ldots, x_p)$ is right continuous in each argument, keeping other arguments fixed;

(d) For each rectangle $(\underline{a}, \underline{b}] \subseteq \mathbb{R}^p$ $(\underline{a}, \underline{b} \in \mathbb{R}^p,\ \underline{a} < \underline{b})$

$$\sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a},\underline{b}])} F_{\underline{X}}(\underline{z}) \geq 0.$$

Proof:

For $p = 2$

(a)

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} F_{X_1,X_2}(m,n) &= \lim_{m\to\infty} \left[\lim_{n\to\infty} F_{X_1,X_2}(m,n) \right] \\
&= \lim_{m\to\infty} F_{X_1}(m) \\
&= 1.
\end{aligned}$$

(b) For $x_2 \in \mathbb{R}$,

$$\begin{aligned}
\lim_{x_1 \to -\infty} F_{X_1,X_2}(x_1, x_2) &= \lim_{m \to \infty} F_{X_1,X_2}(-m, x_2) \\
&= \lim_{m \to \infty} P\left(\{X_1 \le -m, X_2 \le x_2\}\right) \\
&= P\left(\bigcap_{m=1}^{\infty} \{X_1 \le -m, X_2 \le x_2\}\right) \\
&= P(\phi) \\
&= 0.
\end{aligned}$$

Similarly, for $x_1 \in \mathbb{R}$,

$$\lim_{x_2 \to -\infty} F_{X_1,X_2}(x_1, x_2) = 0.$$

(c) For $(x_1, x_2) \in \mathbb{R}^2$

$$\begin{aligned}\lim_{n\to\infty} F_{X_1,X_2}(x_1 + \frac{1}{n}, x_2) &= \lim_{n\to\infty} P\left(\left\{X_1 \le x_1 + \frac{1}{n}, X_2 \le x_2\right\}\right) \\ &= P\left(\bigcap_{n=1}^{\infty}\left\{X_1 \le x_1 + \frac{1}{n}, X_2 \le x_2\right\}\right) \\ &= P\left(\{X_1 \le x_1, X_2 \le x_2\}\right) \\ &= F_{X_1,X_2}(x_1, x_2),\end{aligned}$$

implying that $F_{X_1,X_2}(\cdot)$ is right continuous in first argument. Similarly $F_{X_1,X_2}(\cdot)$ is right continuous in second argument.

(d) Let $-\infty < a_1 < b_1 < \infty$ and $-\infty < a_2 < b_2 < \infty$. Then

$$\begin{aligned}\sum_{k=0}^{2}(-1)^k \sum_{\underline{z}\in\Delta_{k,2}((\underline{a},\underline{b}])} F_{\underline{X}}(\underline{z}) &= F_{\underline{X}}(b_1,b_2) - F_{\underline{X}}(a_1,b_2) \\ &\quad - F_{\underline{X}}(b_1,a_2) + F_{\underline{X}}(a_1,a_2) \\ &= P\left(\{a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2\}\right) \\ &\geq 0.\end{aligned}$$

Remark 1:

(a) Let $F_{\underline{X}}(\cdot)$ be the d.f. of a p-dimensional r.v. $\underline{X}$. For $h > 0$ and $(a_1, a_2, \ldots, a_p) \in \mathbb{R}^p$

$$\begin{aligned} F_{\underline{X}}(a_1 + h, a_2, \ldots, a_p) - F_{\underline{X}}(a_1, a_2, \ldots, a_p) \\ &= P(\{X_1 \leq a_1 + h, X_i \leq a_i, i = 2, \ldots, p\}) \\ &\quad - P(\{X_1 \leq a_1, X_i \leq a_i, i = 2, \ldots, p\}) \\ &= P(\{a_1 < X_1 \leq a_1 + h, X_i \leq a_i, i = 2, \ldots, p\}) \\ &\geq 0. \end{aligned}$$

It follows that the d.f. of a random vector is non-decreasing in each argument when other arguments are kept fixed.

(b) For $p = 1$, condition (d) of above theorem is equivalent to

$$\begin{aligned} P(\{a < X \leq b\}) &\geq 0, \ \forall \ -\infty < a < b < \infty \\ \Leftrightarrow \quad F_X(b) &\geq F_X(a) \quad, \forall \ -\infty < a < b < \infty, \end{aligned}$$

i.e., $F_X(\cdot)$ is non-decreasing.

Result 2:

If a function $G : \mathbb{R} \to \mathbb{R}$ satisfies properties mentioned in (a)-(d) of above result then there exists a probability space $(\Omega, \mathcal{P}(\Omega), P)$ and a r.v. $\underline{X} = (X_1, \ldots, X_p)$ on Ω such that $G(\cdot)$ is the d.f. of $\underline{X}$.

Remark 3: Clearly the d.f.

$$\begin{aligned} F_{\underline{X}}(\underline{x}) &= P\left(\{\underline{X} \leq \underline{x}\}\right) \\ &= P_{\underline{X}}\left((-\underline{\infty}, \underline{x}]\right), \ \underline{x} \in \mathbb{R}^p \end{aligned}$$

is determined by the induced probability function $P_{\underline{X}}(\cdot)$. Conversely, it can be shown that, d.f. determines the induced probability function $P_{\underline{X}}(\cdot)$ uniquely. This suggests that to study the induced probability function $P_{\underline{X}}(\cdot)$ it suffices to study the d.f. $F_{\underline{X}}(\cdot)$.

Take Home Problem

Let $F : \mathbb{R} \to \mathbb{R}$ be given by

$$F(x, y) = \begin{cases} 1, & \text{if } x + 2y \geq 1 \\ 0, & \text{if } x + 2y < 1 \end{cases}.$$

Does $F(\cdot)$ defines a d.f. of some random vector?

Abstract of Next Module

We will introduce the notion of independence of random variables.

Thank you for your patience