Module 7

INDEPENDENT EVENTS

Definition 1: Events E_1, E_2, ..., E_n are said to be

(a) pairwise indenpendent if

$$P\left(E_i \bigcap E_j\right) = P(E_i)P(E_j), \forall i \neq j;$$

(b) mutually independent if $\forall\ k \in \{2, 3, \ldots, n\}$ and distinct $d_1, d_2, \ldots,\ d_k \in \{1, 2, \ldots, n\}$

$$P\left(E_{d_1} \bigcap E_{d_2} \bigcap \cdots \bigcap E_{d_k}\right) = P\left(E_{d_1}\right)\left(E_{d_2}\right)\cdots\left(E_{d_k}\right)$$

$(\sum_{j=2}^{n} \binom{n}{j} = 2^n - n - 1$ conditions).

- Mutual independence of events $E_1, \ldots, E_n \quad \Rightarrow \quad$ pairwise independence of events $E_1, \ldots, E_n$. Converse may not be true, i.e., in general

  pairwise independence of events $E_1, \ldots, E_n \quad \not\Rightarrow \quad$ mutual independence of events $E_1, \ldots, E_n$; as the following example illustrates.

Example 1:

- Let $\Omega = \{1, 2, 3, 4\}$ and let $P(\cdot)$ be such that

$$P(\{i\}) = \frac{1}{4}, \quad i = 1, 2, 3, 4.$$

- Let $E_1 = \{1, 4\}$, $E_2 = \{2, 4\}$ and $E_3 = \{3, 4\}$. Then

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$$

$$P\left(E_1 \bigcap E_2\right) = P\left(E_1 \bigcap E_3\right) = P\left(E_2 \bigcap E_3\right) = P(\{4\}) = \frac{1}{4}$$

and

$$P\left(E_1 \bigcap E_2 \bigcap E_3\right) = P(\{4\}) = \frac{1}{4}.$$

- Clearly

$$P\left(E_i \bigcap E_j\right) = P(E_i)\,P(E_j) = \frac{1}{4} \quad \forall\, i \neq j,$$

implying that E_1, E_2 and E_3 are pairwise independent.

- However

$$P\left(E_1 \bigcap E_2 \bigcap E_3\right) = \frac{1}{4} \neq \frac{1}{8} = P(E_1)\,P(E_2)\,P(E_3),$$

implying that E_1, E_2 and E_3 are not mutually independent.

Remark 1:

(a) Events in any subcollection of independent events are independent.

(b) Suppose that E_1, E_2, ..., E_n are independent, $k \in \{1, 2, \ldots, n-1\}$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$. Then

$$\begin{aligned} P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k} \bigcap E_n^c\right) &= P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k}\right) \\ &\quad - P\left(E_{i_1} \bigcap E_{i_2} \bigcap \ldots \bigcap E_{i_k} \bigcap E_n\right) \\ &= \prod_{j=1}^{k} P\left(E_{i_j}\right) - \left(\prod_{j=1}^{k} P\left(E_{i_j}\right)\right) P\left(E_n\right) \\ &= \prod_{j=1}^{k} P\left(E_{i_j}\right) \left[1 - P\left(E_n\right)\right] \\ &= \left(\prod_{j=1}^{k} P\left(E_{i_j}\right)\right) P\left(E_n^c\right). \end{aligned}$$

Thus

E_1, E_2, …, E_n are independent $\implies$ $E_{i_1}, \dots, E_{i_k}$, $E^c_{i_{k+1}}, \dots, E^c_{i_n}$ are independent, where $1 \leq k \leq n-1$ and $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$.

Also E_1, E_2, …, E_n are independent $\implies$ $E^c_1, E^c_2, \dots, E^c_n$ are independent.

(c) When we say that two random experiments are performed independently what it means is that associated events are independent

(d) Suppose that $P(E_1) > 0$. Then E_1 and E_2 are independent if, and only if,

$$\begin{aligned} P\left(E_1 \bigcap E_2\right) &= P(E_1)\,P(E_2) \\ \Leftrightarrow \qquad \frac{P(E_1 \bigcap E_2)}{P(E_1)} &= P(E_2) \\ \Leftrightarrow \qquad P(E_2|E_1) &= P(E_2) \end{aligned}$$

$\Leftrightarrow$ Conditional probability of E_2 given E_1 is the same as unconditional probability of E_2.

(e) If E_1, E_2, …, E_n are independent events then

- E_1^c and $E_2 \bigcup E_3^c \bigcup E_4$ are independent;
- $E_1 \bigcup E_2^c$ and E_3^c and $E_4 \bigcap E_5^c$ are independent.

QUIZ

Let E_1, E_2 and E_3 be independent events with $P\left(E_i\right) = \frac{1}{i+1}$, $i = 1, 2, 3$. Find the value of $P\left(E_1 \bigcup E_2^c \bigcup E_3\right)$.

Take Home Problems:

Let $\{E_n\}_{n\geq 1}$ be a sequence of independent events (i.e., event in any finite sub-collection of $\{E_n\}_{n\geq 1}$ are independent).

(a) Show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \geq 1 - e^{-\sum_{i=1}^{n} P(E_i)}, \quad n = 1, 2, \ldots$$

(b) If $\sum_{i=1}^{\infty} P(E_i) = \infty$, show that

$$P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = 0.$$

Abstract of Next Module

- In many situations we may not be directly interested in the sample space Ω. Rather we may be interested in some numerical aspect of Ω, i.e., we may be interested in a function $X : \Omega \to \mathbb{R}$. Such functions are called random variables (r.v.s)

- We will formally define r.v. and study the properties of probability functions induced by them.

Thank you for your patience