Module 14
Expectation of a Random Variable

- X: a given r.v. with d.f. $F_X(\cdot)$ and p.m.f./p.d.f. $f_X(\cdot)$.

Definition 1:

(a) Let X be a discrete r.v. with support S_X and p.m.f. $f_X(\cdot)$. We say that the expected value of X (denoted by $E(X)$) exists and equals

$$E(X) = \sum_{x \in S_X} x \, f_X(x),$$

provided $\sum_{x \in S_X} |x| f_X(x) < \infty$.

(b) Let X be an A.C. r.v. with p.d.f. $f_X(\cdot)$. We say that the expected value of X (denoted by $E(X)$) exists and equals

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) dx,$$

provided $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$.

Result 1:

Let X be a discrete or A.C. r.v. Then

$$E(X) = \int_0^{\infty} P(\{X > y\})dy - \int_{-\infty}^{0} P(\{X < y\})dy,$$

provided the expectation exists.

Proof: For A.C. case (the proof for discrete case follows similarly).

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x\ f_X(x)dx = \int_{-\infty}^{0} x\ f_X(x)dx + \int_0^{\infty} x\ f_X(x)dx \\
&= -\int_{-\infty}^{0}\int_{x}^{0} f_X(x)dydx + \int_0^{\infty}\int_0^{x} f_X(x)dydx \\
&= -\int_{-\infty}^{0}\int_{-\infty}^{y} f_X(x)dxdy + \int_0^{\infty}\int_y^{\infty} f_X(x)dxdy \\
&= -\int_{-\infty}^{0} P(\{X < y\})dy + \int_{\infty}^{0} P(\{X > y\})dy.
\end{aligned}$$

Corollary 1:

(a) Let X be a discrete or an A.C. r.v. with $P(\{X \geq 0\}) = 1$. Then

$$E(X) = \int_0^{\infty} P(\{X > y\}) = \int_0^{\infty} (1 - F_X(y))dy.$$

(b) If X is discrete with $P(\{X \in \{0, 1, 2, ...\}\}) = 1$, then

$$E(X) = \sum_{k=1}^{\infty} P(\{X \geq k\}).$$

Result 2:

Let X be a discrete or an A.C. r.v. with p.m.f./p.d.f. $f_X(\cdot)$ and support S_X. Let $Y = g(X)$, for some function $g : \mathbb{R} \to \mathbb{R}$. Then

$$E(Y) = E(g(X)) = \begin{cases} \sum_{x \in S_X} g(x) f_X(x), & \text{if } X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx, & \text{if } X \text{ is A.C.} \end{cases} .$$

Proof: (For discrete case.) Let $f_Y(\cdot)$ be the p.m.f. of Y and let $S_Y = g(S_X) = \{g(x) : x \in S_X\}$. Then, clearly, S_Y is the support of Y and

$$\begin{aligned} E(Y) &= \sum_{y \in S_Y} y \; f_Y(y) \\ &= \sum_{y \in S_Y} y \; P(\{h(X) = y\}) \\ &= \sum_{y \in S_Y} y \sum_{\substack{x \in S_X \\ h(x) = y}} f_X(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{y \in S_Y} \sum_{\substack{x \in S_X \\ h(x)=y}} y \; f_X(x) \\
&= \sum_{y \in S_Y} \sum_{\substack{x \in S_X \\ h(x)=y}} h(x) f_X(x) \\
&= \sum_{x \in S_X} h(x) f_X(x).
\end{aligned}
$$

Example 1:

Let X be a r.v. with p.m.f.

$$f_X(x) = \begin{cases} \frac{c_p}{x^p}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases},$$

where $p > 1$ is a given constant. Then $S_X = \{1, 2, \ldots\}$ and

$$\begin{aligned} E(|X|^r) &= \sum_{x \in S_X} |x|^r f_X(x) \\ &= c_p \sum_{x=1}^{\infty} \frac{1}{x^{p-r}} \end{aligned}$$

is finite iff $r < p - 1$. Thus $E(X^r)$ exists iff $r < p - 1$.

In particular E(X) exists iff $p > 2$. For $p > 2$

$$\begin{aligned} E(X) &= \sum_{x \in S_X} x \, f_X(x) \\ &= c_p \sum_{x=1}^{\infty} \frac{1}{x^{p-1}}. \end{aligned}$$

Example 2:

Let X be a r.v. with p.d.f.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \; -\infty < x < \infty.$$

Then

$$\begin{aligned} E(|X|) &= \int_{-\infty}^{\infty} |x| f_X(x) dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{1+x^2} dx \\ &= \frac{2}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2} dx \\ &= \infty. \end{aligned}$$

Thus $E(X)$ does not exists.

Example 3:

Let X be a r.v. with p.d.f.

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned} E(X^3) &= \int_{-\infty}^{\infty} x^3 f_X(x) dx \\ &= 2\int_0^1 x^4 dx \\ &= \frac{2}{5}. \end{aligned}$$

Result 3:

Let X be a discrete or an A.C. r.v. and let $g_i : \mathbb{R} \to \mathbb{R},\ i = 1, ..., k$, be such that $E(g_i(X))$ exists.

(a) If $P(\{g_1(X) \le g_2(X)\}) = 1$ then $E(g_1(X)) \le E(g_2(X))$. In particular if $P(\{a \le X \le b\}) = 1$, for some real constants a and b, then $a \le E(X) \le b$.

(b) If $P(\{X \ge 0\}) = 1$ and $E(X) = 0$ then $P(\{X = 0\}) = 1$.

(c) If $E(X)$ exists then $|E(X)| \le E(|X|)$.

(d) For real constants $c_1, \ldots, c_k$

$$E\left(\sum_{i=1}^{k} c_i g_i(X)\right) = \sum_{i=1}^{k} c_i E\left(g_i(X)\right).$$

Proof :

(a) (For discrete case.) Let $A = \{x \in \mathbb{R} : g_1(x) \leq g_2(x)\}$. Then, $f_X(x) = 0,\ \forall\ x \in A^c$ $(P(\{g_1(X) \leq g_2(X)\}) = 1)$.

$$\begin{aligned}
E(g_1(X)) &= \sum_{x \in S_X} g_1(x) f_X(x) \\
&= \sum_{x \in S_X \cap A} g_1(x) f_X(x) + \sum_{x \in S_X \cap A^c} g_1(x) f_X(x) \\
&= \sum_{x \in S_X \cap A} g_1(x) f_X(x) \\
&\leq \sum_{x \in S_X \cap A} g_2(x) f_X(x) \\
&= \sum_{x \in S_X \cap A} g_2(x) f_X(x) + \sum_{x \in S_X \cap A^c} g_2(x) f_X(x) \\
&= \sum_{x \in S_X} g_2(x) f_X(x) = E(g_2(X)).
\end{aligned}$$

(b) (Discrete case.) Since $P(\{X \geq 0\}) = 1$, we have $S_X \subseteq [0, \infty)$. Thus, for $n = 1, 2, \ldots$,

$$\begin{aligned}
E(X) &= \sum_{x \in S_X} x\, f_X(x) \\
&= \sum_{x \in S_X \cap [0, \frac{1}{n})} x\, f_X(x) + \sum_{x \in S_X \cap [\frac{1}{n}, \infty)} x\, f_X(x) \\
&\geq \sum_{x \in S_X \cap [\frac{1}{n}, \infty)} x\, f_X(x) \\
&\geq \frac{1}{n} \sum_{x \in S_X \cap [\frac{1}{n}, \infty)} f_X(x) \\
&= \frac{1}{n} P\left(\left\{X \geq \frac{1}{n}\right\}\right) \\
\Rightarrow P\left(\left\{X \geq \frac{1}{n}\right\}\right) &\leq nE(X) = 0, \ \forall\ n = 1, 2, \ldots
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow\ P\left(\left\{X \geq \frac{1}{n}\right\}\right) &= 0,\ \forall\ n = 1, 2, \ldots \\
\Rightarrow\ \lim_{n\to\infty} P\left(\left\{X \geq \frac{1}{n}\right\}\right) &= 0 \\
\Rightarrow\ P\left(\bigcup_{n=1}^{\infty}\left\{X \geq \frac{1}{n}\right\}\right) &= 0 \\
\Rightarrow\ P(\{X > 0\}) &= 0 \\
\Rightarrow\ P(\{X = 0\}) &= 1.
\end{aligned}
$$

(c) Follow from (a) on using the fact that

$$-|X| \leq X \leq |X|.$$

(d) (Discrete case.) For simplicity of notations we prove the result for $k = 2$.

$$\begin{aligned} E(c_1 g_1(X) + c_2 g_2(X)) &= \sum_{x \in S_X} (c_1 g_1(x) + c_2 g_2(x)) f_X(x) \\ &= c_1 \sum_{x \in S_X} g_1(x) f_X(x) + \\ &\quad\ c_2 \sum_{x \in S_X} g_2(x) f_X(x) \\ &= c_1 E(g_1(X)) + c_2 E(g_2(X)). \end{aligned}$$

Remark 1: If $E(X)$ exists then using (c) above it follows that $|E(X)| < \infty$ (i.e., $E(X)$ is finite).

Some special Expectations

- X a r.v.;
- $g : \mathbb{R} \to \mathbb{R}$: a given function;
- Then $Y = g(X)$ is a r.v. and $E(g(X)) =$ expected value of $g(X)$.

Some special expectations are:

(i) $\mu_1' = \mu = E(X) =$ mean of (distribution of) X;

(ii) For $r \in \{1, 2, ...\}$, $\mu_r' = E(X^r) = r$-th moment of X about origin;

(iii) For $r \in \{1, 2, ...\}$, $E(|X|^r) = r$-th absolute moment of X about origin;

(iv) For $r \in \{1, 2, ...\}$, $\mu_r = E((X - \mu)^r) = r$-th moment of X about its mean (or r-th central moment);

(v) $\mu_2 = \sigma^2 = E((X - \mu)^2) =$ Variance of X (written as Var(x)).

Remark 1:

(a)

$$\begin{aligned}\text{Var}(X) &= E((X-\mu)^2)\\ &= E(X^2 - 2\mu X + \mu^2)\\ &= E(X^2) - 2\mu E(X) + \mu^2\\ &= E(X^2) - \mu^2\\ &= E(X^2) - (E(X))^2.\end{aligned}$$

(b) Since $\text{Var(X)} = E((X - E(X))^2) \geq 0$, we have

$$E(X^2) \geq (E(X))^2,$$

for any r.v. X.

(c) $\text{Var}(X) = 0 \Rightarrow P(\{X = E(X)\}) = 1$.

Take Home Problems

1. Let X be a r.v. with p.d.f.

$$f_X(x) = \begin{cases} c\,(x+1), & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant.
(a) Find the value of c;
(b) Find the mean and variance of X.

2. . Let X be a r.v. with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases}.$$

Show that $E(X)$ does not exists.

Abstract of Next Module

In next module we will introduce a transform

$$M_X(t) = E\left(e^{tX}\right),\ t \in \mathbb{R},$$

that can be used to generate moments $(\mu_r' = E(X^r),\ r = 1, 2, \ldots)$ of a r.v. X. This transform is called the moment generating function (m.g.f.) of X. We will study various properties of m.g.f.

Thank you for your patience