Locally (Conditionally) Conjugate Models

Piyush Rai

Probabilistic Machine Learning (CS772A)

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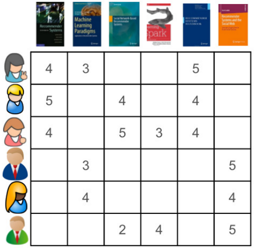
  - Based on the idea of local or conditional conjugacy (yes, conjugacy helps even in complex problems!)
  - Leads to "update one parameter keeping others fixed" style updates (similar in spirit to EM)
  - Local conjugacy makes each individual update have a simple form!

An Example Problem: Matrix Completion



4	3			5	
5		4		4	
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	3				5
	4				4
		2	4		5

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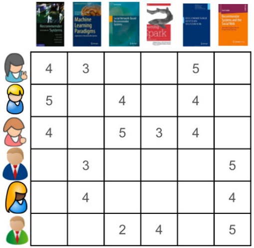

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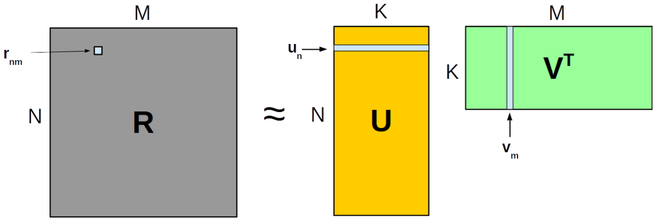
  - Note: "users" and "items" could mean other things too (depends on the data)
- $(i, j) \in \Omega$ denotes an observed user-item pair. Ω is the set of all such pairs
- Only a small number of user-item ratings observed, i.e., $|\Omega| \ll NM$
- We would like to predict the unobserved values $r_{ij} \notin \mathcal{D}$

Matrix Completion via Matrix Factorization

- Suppose the "ratings" matrix **R** can be approximately factorized as a product of two matrices



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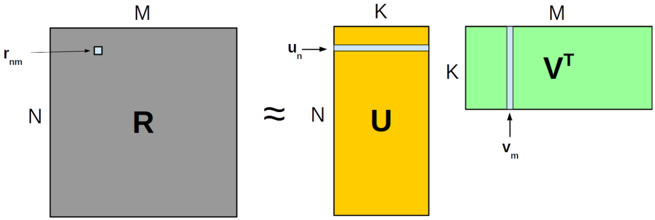

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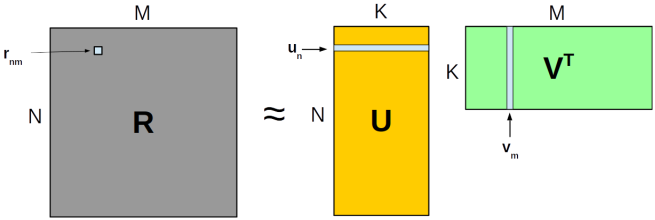

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$$r_{nm} \approx \boldsymbol{u}_n^\top \boldsymbol{v}_m = \sum_{k=1}^{K} u_{nk} v_{mk}$$

- Note: Assume $K \ll N, M \Rightarrow \mathbf{R}$ is approximately a low-rank matrix

- Given $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$, any missing element r_{nm} in $\mathbf{R}$ can be predicted as $r_{nm} \approx \boldsymbol{u}_n^\top \boldsymbol{v}_m$

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- Can assume **R** to be a low-rank matrix plus some noise **E**

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- $\mathbf{E} = \{\epsilon_{ij}\}$ consists of the "noise" in **R** (not captured by the low-rank assumption)
- We can write each element of matrix **R** as

$$r_{ij} = \boldsymbol{u}_i^\top \boldsymbol{v}_j + \epsilon_{ij} \qquad (i = 1, \dots, N, \quad j = 1, \dots, M)$$

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- The goal is to infer latent factors $\mathbf{U} = \{\boldsymbol{u}_i\}_{i=1}^N$ and $\mathbf{V} = \{\boldsymbol{v}_j\}_{j=1}^M$, given observed ratings from $\mathbf{R}$

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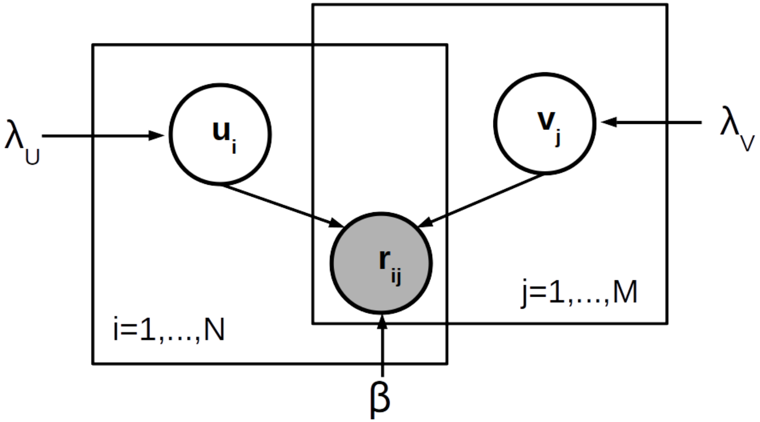
- For simplicity, we will assume the hyperparams $\beta, \lambda_u, \lambda_v$ to be fixed and not to be learned

Matrix Completion via Matrix Factorization



The Posterior

- Our target posterior distribution for this model will be

$$p(\mathbf{U}, \mathbf{V}|\mathbf{R}) = \frac{p(\mathbf{R}|\mathbf{U}, \mathbf{V})p(\mathbf{U}, \mathbf{V})}{\int \int p(\mathbf{R}|\mathbf{U}, \mathbf{V})p(\mathbf{U}, \mathbf{V})d\mathbf{U}d\mathbf{V}}$$

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- $\mathbf{U}_{-i}$ denotes all of $\mathbf{U}$ except $\boldsymbol{u}_i$. $\mathbf{V}, \mathbf{U}_{-i}$ is the set of all other unknowns (fixed to a “current” value)

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- The denominator (and hence the posterior) is intractable! **Question: Why?**
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- **Can we compute these CPs easily?** Yes, if the model has "local conjugacy"

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  - This will be more clear from the example of the Bayesian matrix factorization model

Representation of Posterior

- With the conditional posterior based approximation, the target posterior

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- One way to get the overall representation of the posterior can be can be using sampling based inference algorithms like Gibbs sampling or MCMC, or Variational Inference (more on this later)

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