Q1: (i) The Harmonic functions in Ω forms a vector space.

Soln:- $u \in C^2(\Omega)$ is said to be harmonic if $\Delta u = 0$ in Ω.

To check if it forms a vector space we consider

$u_1, u_2 \in C^2(\Omega)$ s.t $\Delta u_1 = \Delta u_2 = 0$.

Hence, $\Delta(u_1+u_2) := \Delta u_1 + \Delta u_2 = 0$ & $u_1+u_2 \in C^2(\Omega)$

Thus, u_1+u_2 is harmonic.

Again for $c \in \mathbb{R}$ and $\Delta u = 0$ we have

$\Delta(cu) := c\Delta(u) = 0$.

So, cu is harmonic

$\therefore$ Harmonic Functions in Ω forms a vector space

(ii) Rotational Symmetry of the Laplace operator.

ie, the eqn $\Delta u = 0$ is invariant under rotation about an arbitrary center 'y' in $\mathbb{R}^2$

ie if $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$ then define ~~$w(x',y')$~~ $w(x',y') = u(x,y)$.

$\therefore$ $\boxed{u_{xx} + u_{yy} = w_{x'x'} + w_{y'y'}}$ — RTP

~~$u_x = (u_x)(x')_x + (u_y)(y')_x$~~

To show (RTP)

$u_x = w_{x'}(x')_x + w_{y'}(y')_x$

$= \cos\theta\, w_{x'} + \sin\theta\, w_{y'}$

$\Rightarrow u_{xx} = \cos\theta\,[w_{x'x'}(x')_x + w_{x'y'}(y')_x] + \sin\theta\,[w_{y'x'}(x')_x + w_{y'y'}(y')_x]$

$= \cos^2\theta\, w_{x'x'} + \sin\theta\cos\theta\, w_{x'y'} + \sin\theta\cos\theta\, w_{x'y'} + \sin^2\theta\, w_{y'y'}$ — (1)

$u_y = w_{x'}(x')_y + w_{y'}(y')_y$

~~$= \cos\theta\, w_{x'} + \sin\theta\, w_{y'}$~~

$= -\sin\theta\, w_{x'} + \cos\theta\, w_{y'}$.

$u_{yy} = -\sin\theta\,[w_{x'x'}(x')_y + w_{x'y'}(y')_y] + \cos\theta\,[w_{y'x'}(x')_y + w_{y'y'}(y')_y]$

$= \sin^2\theta\, w_{x'x'} - \sin\theta\cos\theta\, w_{x'y'} - \sin\theta\cos\theta\, w_{x'y'} + \cos^2\theta\, w_{y'y'}$ — (2)

Adding (1) and (2) we get (RTP).

(iii) $u(x,y) = \varphi(r)$; $r = \sqrt{x^2+y^2}$.

$\therefore$ $u_x(x,y) = \varphi'(r) r_x = \varphi'(r)\frac{x}{r}$.

& $u_y(x,y) = \varphi'(r) r_y = \varphi'(r)\frac{y}{r}$.

& $u_{xx}(x,y) = \varphi'(r)\frac{1}{r} + \frac{x^2}{r^2}\varphi''(r)$.

& $u_{yy}(x,y) = \varphi'(r)\cdot 0 + \frac{y}{r}\cdot\varphi''(r)\cdot\frac{y}{r}$.

$= \frac{y^2}{r^2}\varphi''(r)$.

So, $u_{xx} + u_{yy} = \frac{\varphi'(r)}{r} + \frac{\varphi''(r)}{r^2}\left[x^2+y^2\right]$

$= \varphi''(r) + \frac{\varphi'(r)}{r}$.

Now if $\Delta u = 0$

Then, $\varphi''(r) + \frac{\varphi'(r)}{r} = 0$

Taking, $\varphi'(r) := \psi(r)$ (~~Definition~~) (Define)

$\psi'(r) = -\frac{\psi(r)}{r}$.

$\Rightarrow \psi(r) = \frac{K}{r}$; K is a constant

Again, $\varphi'(r) = \psi(r) = K/r \Rightarrow \varphi(r) = K\ln|r| + K_1$ (K_1 = constant)

$\therefore u(x,y) = K\ln(\sqrt{x^2+y^2}) + K_1$

(2) The problem $\Delta u = 0$ in Ω
$u = 0$ on $\partial\Omega$ $\Bigg\} \longrightarrow (I)$

has a unique soln given by $u = 0$.

Let u_1 and u_2 be two soln of (I)

Then, $v := u_1 - u_2$ (Define).

From (I) and using linearity of laplacian,

$$\left.\begin{array}{l}\Delta v = 0 \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega\end{array}\right\} \longrightarrow (II)$$

Hence, multiplying v with $\Delta v = 0$ and integrating by parts we get,

$$\int_\Omega v\Delta v = 0 \quad (\text{Since, } v \in C^2(\bar{\Omega}) \text{ all of this is valid})$$

$$\Rightarrow -\int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v\frac{\partial v}{\partial \eta}\,ds = 0 \quad (\eta = \text{unit outward normal to } \Omega)$$

$$\Rightarrow -\int_\Omega |\nabla v|^2 = 0 \quad (\because v = 0 \text{ on } \partial\Omega)$$

$$\Rightarrow \nabla v = 0 \; \cancel{(c = \text{constant})} \text{ in } \Omega.$$

~~in Ω~~

$\therefore v =$ constant in Ω.

But, $v = 0$ on $\partial\Omega$ and v is continuous. Hence

$v = 0$ in Ω.

③ Laplacian Equation is ill-posed

Consider, $\Delta u = 0$ in $\{(x,y) \in \mathbb{R}^2 : x > 0\}$

and, $u(0,y) = 0$

$$\frac{\partial u}{\partial x}(0,y) = e^{-\sqrt{n}}\, n \sin ny.$$

It is easy to see that

$$u_n(x,y) = e^{-\sqrt{n}}\, e^{nx} \sin ny$$

satisfies the eqn.

However we can make $\frac{\partial u}{\partial x}(0,y) = n e^{-\sqrt{n}} \sin ny$ as small as we like if n is sufficiently large, but the soln

$$u_n(x,y) = e^{-\sqrt{n}}\, e^{nx} \sin ny$$

and some (x,y) may be very large for large enough 'n' and some (x,y).