CS203B, Assignment 4

Prof. Manindra Agarwal

August 2016

In this assignment, we further develop the theory of ideals of Dedekind domains. Recall that a Dedekind domain is an integral domain, where all prime ideals are maximal. Let R be a Dedekind domain and F be its field of fractions, i.e.,

$$F = \{\frac{a}{b} \mid a, b \in R, \text{ and } b \neq 0\}.$$

Set $J \subseteq F$ is called a *fractional ideal* of F if (1) J is a commutative group under addition, (2) for every $a \in R$, $a*J \subseteq J$, and (3) there exists an $a \in R$ such that $a*J \subseteq R$. In this definition, set $a*J = \{a*b \mid b \in J\}$.

It is clear that every ideal of R is also a fractional ideal of F.

1. For any $a \in F$, $a \neq 0$, let
$$(a) = \{b * a \mid b \in R\}.$$
Prove that (a) is a fractional ideal. (a) is called a *principle fractional ideal* of F.

2. Prove that J is a fractional ideal of F if and only if there exists an ideal I of R and $a \in R$ such that $J = \frac{1}{a} * I$.

Let $\mathcal{J}$ be the set of all fractional ideals of F. We can define multiplication of fractional ideals in exactly the same way as multiplication of ideals:

$$J_1 * J_2 = \{\sum_{i=1}^{k} a_i * b_i \mid a_i \in J_1, b_i \in J_2\}.$$

Let $\mathcal{J}_0$ be the collection of all principle fractional ideals of F. The sets $\mathcal{J}$ and $\mathcal{J}_0$ carry important information about factorization in the ring R.

3. For a prime ideal I of R, define
$$I^{-1} = \{a \in F \mid a * I \subseteq R\}.$$
Prove that I^{-1} is a fractional ideal of F.

4. Prove that $1 \in I^{-1}$.

5. Prove that $I * I^{-1} = (1)$.

Once we have the inverse of prime ideals in $\mathcal{J}$, we can use unique factorization of ideals of R into prime ideals to obtain inverses of all ideals of R. Let I be any ideal of R and $I = P_1 * P_2 * \cdots * P_k$ where P_i are prime ideals. Then define

$$I^{-1} = P_1^{-1} * P_2^{-1} * \cdots * P_k^{-1}.$$

6. Prove that for any ideal I of R, $I^{-1} \in \mathcal{J}$ and $I * I^{-1} = (1)$.

7. Prove that $\mathcal{J}$ is a group under multiplication.

8. Prove that $\mathcal{J}_0$ is a subgroup of $\mathcal{J}$.

The quotient group $\mathcal{J}/\mathcal{J}_0$ can be shown to be finite. The size of quotient group describes how badly unique factorization fails in the ring R. We only prove a part of this fact.

9. If $\mathcal{J}/\mathcal{J}_0$ has size 1, then prove that (1) every ideal of R is a principle ideal, and (2) elements of R uniquely factorize into irreducible elements.

The size of group $\mathcal{J}/\mathcal{J}_0$ is called the *class number* of the field F. Amongst the rings we considered in the class, $Z[i\sqrt{2}]$ has class number 1 and $Z[i\sqrt{5}]$ has class number 2.