Exponential Family and Generalized Linear Models

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Probabilistic Machine Learning (CS772A)

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Exponential Family (Pitman, Darmois, Koopman, Late 1930s)

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- $h(\boldsymbol{x})$: A constant (doesn't depend on θ)

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- $A(\theta) = \frac{\mu^2}{2\sigma^2} + \log\sigma = \frac{-\theta_1^2}{4\theta_2} - \frac{1}{2}\log(-2\theta_2) - \frac{1}{2}\log(2\pi)$

A General Trick

A general trick to represent any distribution (assuming it is exp-family dist.) in exp-family form

- Write the given distribution as $\exp(\log p())$ and simplify, e.g., for the Binomial

$$\begin{aligned}
\exp\left(\log \text{Binomial}(x|N,p)\right) &= \exp\left(\log \binom{N}{x} p^x (1-p)^{N-x}\right) \\
&= \exp\left(\log \binom{N}{x} + x\log p + (N-x)\log(1-p)\right) \\
&= \binom{N}{x} \exp\left(x \log \frac{p}{1-p} - N\log(1-p)\right)
\end{aligned}$$

- Now compare the resulting expression with the exponential family form

$$p(\boldsymbol{x}|\theta) = h(\boldsymbol{x})\exp(\theta^\top \phi(\boldsymbol{x}) - A(\theta))$$

.. to identify the natural parameters, sufficient statistics, log-partition function, etc.

Other Examples

- Many other distribution belong to the exponential family
  - Bernoulli
  - Beta
  - Gamma
  - Multinoulli/Multinomial
  - Dirichlet
  - Multivariate Gaussian
  - .. and many more (`https://en.wikipedia.org/wiki/Exponential_family`)

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  - Mixture distributions (e.g., mixture of Gaussians)
- If the support of the distribution depends on its parameters, then it is not an exp. family dist.

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- Derivatives of $A(\theta)$ can be used to generate the cumulants of the sufficient statistics $\phi(\boldsymbol{x})$
- Exercise: Assume θ to be a scalar (thus $\phi(\boldsymbol{x})$ is also scalar).

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- Exercise: For Binomial, using its expression of $A(\theta)$, derive the first and second cumulants of $\phi(x)$

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- Suppose we have data $\mathcal{D} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ drawn i.i.d. from an exponential family distribution

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  - Very useful when doing probabilistic/Bayesian inference with large-scale data sets. Also useful in online parameter estimation problems.

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- This is basically matching the expected moments of the distribution with empirical moments ("empirical" here means what we compute using the observed data)

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- Given N observations $x_1, \ldots, x_N$ from a univariate Gaussian $N(x|\mu, \sigma^2)$, doing moment-matching

$$\mathbb{E}[\phi(x)] = \frac{1}{N}\sum_{i=1}^{N}\phi(x_i)$$

- The "true", i.e., expected moments: $\mathbb{E}[\phi(x)] = \mathbb{E}\begin{bmatrix} x \\ x^2 \end{bmatrix}$. Therefore

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- For a univariate Gaussian, note that $\mathbb{E}[x] = \mu$ and $\mathbb{E}[x^2] = \text{var}[x] + \mathbb{E}[x]^2 = \sigma^2 + \mu^2$
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Bayesian Inference for Exponential Family Distributions

- We saw that the total likelihood given N i.i.d. observations $\mathcal{D}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$

$$p(\mathcal{D}|\theta) \propto \exp\left[\theta^\top \phi(\mathcal{D}) - NA(\theta)\right] \qquad \text{where} \quad \phi(\mathcal{D}) = \sum_{i=1}^{N} \phi(\boldsymbol{x}_i)$$

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- Note: Prior's log-partition function $A_c(\nu_0, \boldsymbol{\tau}_0)$ updates to posterior's: $A_c(\nu_0 + N, \boldsymbol{\tau}_0 + \phi(\mathcal{D}))$

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- Note that the posterior hyperparam $\bar{\tau}_0'$ is a convex combination of the average suff-stats $\bar{\tau}_0$ of the ν_0 pseudo-observations and the average suff-stats $\bar{\phi}$ of the N actual observations

Posterior Predictive Distribution

- Assume some past (training) data $\mathcal{D} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ generated from an exp. family distribution

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- Nice Property: If the likelihood is an exponential family distribution, prior is conjugate (and thus is the posterior), the posterior predictive always has a closed form expression (shown next)

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- Recall the form of the likelihood $p(\mathcal{D}|\theta)$ for exp. family dist.

$$p(\mathcal{D}|\theta) = \left[\prod_{i=1}^{N} h(\boldsymbol{x}_i)\right] \exp\left[\theta^\top \phi(\mathcal{D}) - NA(\theta)\right]$$

- The conjugate prior was

$$p(\theta|\nu_0, \boldsymbol{\tau}_0) = h(\theta) \exp\left[\theta^\top \tau_0 - \nu_0 A(\theta) - A_c(\nu_0, \boldsymbol{\tau}_0)\right]$$

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$$p(\mathcal{D}'|\theta) = \left[\prod_{i=1}^{N'} h(\tilde{\boldsymbol{x}}_i)\right] \exp\left[\theta^\top \phi(\mathcal{D}') - N'A(\theta)\right] \qquad \text{where} \quad \phi(\mathcal{D}') = \sum_{i=1}^{N'} \phi(\tilde{\boldsymbol{x}}_i)$$

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- Very useful result. Also holds for $N = 0$
  - In the $N = 0$ case, $p(\mathcal{D}') = \int p(\mathcal{D}'|\theta)p(\theta)d\theta$ is simply the marginal likelihood of $\mathcal{D}'$

Exponential Family and GLM

Generalized Linear Models (GLM)

- (Probabilistic) Linear regression: when response y is real-valued

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.. where η is a scalar-valued natural parameter (depends on $\boldsymbol{x}$) and sufficient statistics $\phi(y) = y$

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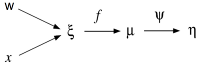

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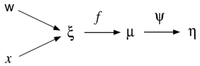

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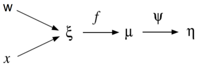

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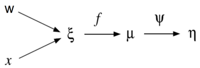

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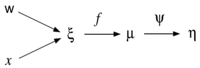

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- Note: Some GLM can be represented as $p(y|\eta,\phi) = h(y,\phi)\exp(\frac{\eta y - A(\eta)}{\phi})$ where ϕ is a dispersion parameter (Gaussian/gamma GLMs use this representation)

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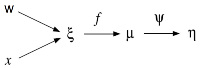

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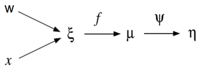

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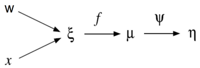

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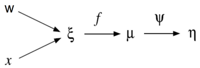

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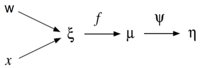

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- Interesting class project idea: Design simple inference algorithms for non-conjugate GLM

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- Also used in designing GLMs
- We will see several use cases when we discuss approximate inference algoritms (e.g., Gibbs sampling, and especially variational inference)