Module 6

CONDITIONAL PROBABILITY,

THEOREM OF TOTAL PROBABILITY

&

BAYES’ THEOREM

Conditional Probability

Example 1: A box contains 4 distinct red balls and 3 distinct black balls. Two balls are drawn at random and without replacement from this box. Find the probability that both balls are black.

Solution : Define events:

- A: ball in first draw is black;
- B: ball in second draw is black.
- Then

$$\begin{aligned}\text{Required probability} &= P(A \cap B) \\ &= \frac{\binom{3}{2}}{\binom{7}{2}} = \frac{1}{7}.\end{aligned}$$

- Alternately,

$$P(A) = \frac{3}{7};$$

P(ball in second draw is black given that ball in first draw is black)

$$= \frac{2}{6} = \frac{1}{3};$$

- i.e.,

$$P(B|A) = \frac{1}{3}$$

- $$P\left(A \cap B\right) = P\left(A\right) P\left(B|A\right) = \frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$$

- $$P\left(B|A\right) = \frac{P\left(A \cap B\right)}{P\left(A\right)}, \quad \text{provided } P\left(A\right) > 0.$$

Relative Frequency Interpretation of Conditional Probability Expression

- Repeat the experiment N (a large number) times;

- As per the relative frequency interpretation of probability

$$P(B|A) \approx \frac{f_N(A \cap B)}{f_N(A)} = \frac{\frac{f_N(A \cap B)}{N}}{\frac{f_N(A)}{N}} = \frac{P(A \cap B)}{P(A)}.$$

Definition 1: Let A and B be two events. The conditional probability of B given A is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)},$$

provided $P(A) > 0$.

Result 1: Let $A \in \mathcal{P}(\Omega)$ be a fixed event such that $P(A) > 0$. Then $P(\cdot|A)$ is a probability function for sample space Ω. It is also a probability function for sample space A.

Proof:

- For $E \in \mathcal{P}(\Omega)$

$$P(E|A) = \frac{P(A \cap E)}{P(A)} \geq 0;$$

- Let $\{E_i : i \in S\}$ be a countable collection of disjoint subsets of Ω. Then $E_i \cap A$, $i \in S$, are also disjoint (being subsets of disjoint events $E_i's$). Thus

$$
\begin{aligned}
P\left(\bigcup_{i\in S} E_i \middle| A\right) &= \frac{P\left(\left(\bigcup_{i\in S} E_i\right)\cap A\right)}{P\left(A\right)} \\
&= \frac{P\left(\bigcup_{i\in S}\left(E_i\cap A\right)\right)}{P\left(A\right)} \\
&= \frac{\sum_{i\in S} P\left(E_i\cap A\right)}{P\left(A\right)} \\
&= \sum_{i\in S}\frac{P\left(E_i\cap A\right)}{P\left(A\right)} \\
&= \sum_{i\in S} P\left(E_i \middle| A\right).
\end{aligned}
$$

- Also

$$
P\left(\Omega|A\right) = \frac{P\left(A\cap\Omega\right)}{P\left(A\right)} = \frac{P\left(A\right)}{P\left(A\right)} = 1.
$$

- Similarly $P(\cdot|A)$ is a probability function for sample space A.

Remark 1:

- We have

$$P\left(E_1 \cap E_2\right) = P\left(E_1\right) P\left(E_2 \middle| E_1\right),$$

provided $P\left(E_1\right) > 0$;

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$$\begin{aligned} P\left(E_1 \cap E_2 \cap E_3\right) &= P\left(\left(E_1 \cap E_2\right) \cap E_3\right) \\ &= P\left(E_1 \cap E_2\right) P(E_3 \middle| E_1 \cap E_2) \\ &= P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2), \end{aligned}$$

provided $P\left(E_1 \cap E_2\right) > 0$ (which also guarantees that $P\left(E_1\right) > 0$).

Example 2: 5 cards are drawn at random and without replacement from a deck of 52 cards. Define events

B: all drawn cards are spade;

A: among drawn cards, at least four are spade;

C: among drawn cards, there is a king of spade.

Then

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{P(B)}{P(A)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\binom{13}{5} \Big/ \binom{52}{5}}{\left[\binom{13}{4}\binom{39}{1} + \binom{13}{5}\right] \Big/ \binom{52}{5}}; \\
P\left(C|A\right) &= \frac{P\left(A \cap C\right)}{P\left(A\right)} \\
&= \frac{\left[\binom{12}{3}\binom{39}{1} + \binom{12}{4}\right] \Big/ \binom{52}{5}}{\left[\binom{13}{4}\binom{39}{1} + \binom{13}{5}\right] \Big/ \binom{52}{5}}.
\end{aligned}
$$

Result 2 (Theorem of Total Probability): Let $\{E_i : i \in S\}$ be a countable collection of mutually exclusive and exhaustive events ($E_i \cap E_j = \phi$, $\forall\ i \neq j$ and $P(\bigcup_{i\in S} E_i) = 1$). Then, for any other event E,

$$P(E) = \sum_{i\in S} P(E \cap E_i) = \sum_{i\in S} P(E|E_i)P(E_i).$$

Proof: Note that E_is are disjoint and $E \cap E_i \subseteq E_i \implies E \cap E_i$s are disjoint. Since $P\left(\bigcup_{i\in S} E_i\right) = 1$,

$$\begin{aligned}
P(E) &= P\left(E \cap \left(\bigcup_{i\in S} E_i\right)\right) \\
&= P\left(\bigcup_{i\in S} (E \cap E_i)\right) \\
&= \sum_{i\in S} P(E \cap E_i) \\
&= \sum_{i\in S} P(E|E_i)P(E_i).
\end{aligned}$$

Result 3 (Bayes' Theorem):. Let $\{E_i : i \in S\}$ be a collection of mutually exclusive and exhaustive events such that $P(E_i) > 0$, $\forall\, i \in S$. Suppose that E is any other event. Then, for any fixed $j \in S$,

$$P(E_j|E) = \frac{P(E|E_j)\,P(E_j)}{\sum_{i\in S} P(E|E_i)\,P(E_i)}.$$

Proof: Let $j \in S$ be fixed. Then

$$P(E_j|E) = \frac{P(E \cap E_j)}{P(E)} = \frac{P(E|E_j)\,P(E_j)}{\sum_{i\in S} P(E|E_i)\,P(E_i)},$$

using theorem of total probability.

Remark 2:

- $\{E_j : j \in S\}$: possible (distinct and exhaustive) causes for occurrence of an event;

- E: any given event;

- $P(E_j|E)$: given that event E has occurred, the probability that it was caused by E_j, $j \in S$;

- In Bayes' theorem the probabilities $P(E_i)$, $i \in S$, are called prior probabilities and conditional probabilities $P(E_i|E)$, $i \in S$, are called posterior probabilities.

Example 3:

- There are two bowls (labeled B_1 and B_2);
- Bowl B_1 contains 3 red and 7 black balls;
- Bowl B_2 contains 8 red and 2 black balls;
- All balls are identical (except colour);
- A die is cast and a bowl is selected as per the following scheme:
  - Bowl B_1 is selected if 5 or 6 spots show up on the die, otherwise, bowl B_2 is selected;
- Then one ball is drawn at random from the selected bowl;

(a) Find the probability that the drawn ball is red;

(b) Given that the drawn ball is red, find the conditional probability that it came from bowl B_1.

Solution: Define events:

- E_i: bowl B_i is selected, $i = 1, 2$;
- R: red bowl is drawn.

Clearly $\{E_1, E_2\}$ are mutually exclusive and exhaustive.

(a)

$$\begin{aligned} P(R) &= P(E_1 \cap R) + P(E_2 \cap R) \\ &= P(R|E_1) P(E_1) + P(R|E_2) P(E_2) \\ &= \frac{3}{10} \times \frac{2}{6} + \frac{8}{10} \times \frac{4}{6} = \frac{19}{30}; \end{aligned}$$

(b)

$$\begin{aligned} P(E_1|R) &= \frac{P(R|E_1) P(E_1)}{P(R)} \\ &= \frac{P(R|E_1) P(E_1)}{P(R|E_1) P(E_1) + P(R|E_2) P(E_2)} \\ &= \frac{\frac{3}{10} \times \frac{2}{6}}{\frac{19}{30}} = \frac{3}{19}. \end{aligned}$$

Take Home Problems:

Let A, B and D be events with $P(A) > 0$. Show that

(a)

$$P(B|A) = P(B|A \cap D)\,P(D|A) + P(B|A \cap D^c)\,P(D^c|A);$$

(b)

$$P(B^c \cap D^c|A) = 1 - P(B|A) - P(D|A) + P(B \cap D|A).$$

Abstract of Next Module

We will introduce the concept of independent events and study their properties.

Thank you for your patience