Assignment 1
MSO 203B-Partial Differential Equation

(1) The Gauss Divergence Theorem says that for a continuously differentiable vector field F (any function from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called a vector field in case you are wondering) on a closed bounded region in space with smooth, orientable boundary, one has

$$\int_\Omega \text{div}F \, dx = \int_{\partial\Omega} F.\eta \, dS$$

where η is the outward unit normal vector to $\partial\Omega$. Using this prove that for $u, v \in C^2(\bar{\Omega})$ the following holds:
(a) $\int_\Omega u_{x_i} v \, dx = -\int_\Omega u v_{x_i} dx + \int_{\partial\Omega} uv\eta_i \, dS$
(b) $\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \, dS$
(c) $\int_\Omega \nabla u.\nabla v \, dx = -\int_\Omega u\Delta v dx + \int_\Omega u\frac{\partial v}{\partial \eta} \, dS$

(2) For an unknown function $u : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$, an expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., u(x), x) = 0$$

is called the k-th order PDE where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

Note that we can classify this PDE according to its linearity as semilinear, quasilinear and nonlinear equation depending on the nature of the coefficient of the highest order term which was already discussed in class. Based on the above information classify the following PDE :
(a) $\Delta u = u^2$
(b) $u_{tt} - u_{xx} = \sin x$
(c) $u_t - u_{xx} = \cos(u)$
(d) $|\nabla u| = 1$
(e) $\text{div}(|\nabla u|^{p-2}\nabla u) = 0$. Discuss the case $p = 2$.
also discuss the order of PDE.

(3) Define the differential operator $L(y) := y'' + a(x)y' + b(x)y$ for $a, b \in C(I)$. Using Picard's Theorem (You may be wondering that Picard's theorem is for first order equation, but it can be suitable modified to suit any n-th order equation provided the coefficients are sufficiently smooth) one can show that the problem $L(y) = f$ admits a unique solution $y \in C^2(\mathcal{N}(x_0, y_0, y_1))$ provided $y(x_0) = y_0$ and $y'(x_0) = y_1$ with $f \in C(I)$. Here I is any interval containing x_0 and $\mathcal{N}(x_0, y_0, y_1)$ is a neighbourhood of the point (x_0, y_0, y_1).
On the contrary if we consider the boundary value equation $L(y) = 0$ with the condition that $y(x_0) = y_0$ and $y(x_1) = y_1$ for $x_0 \neq x_1 (\in I)$, then the uniqueness is lost. Give examples such the boundary value problem has no solution or infinitely many solutions.

(4) An inner product on a vector space V over $\mathbb{R}$ is a function $\langle , \rangle : V \times V \to \mathbb{R}$ satisfying the following:
- $\langle x, y\rangle = \langle y, x\rangle$ (Symmetry)
- $\langle ax, y\rangle = a\langle x, y\rangle$ and $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ (Linearity)
- $\langle x, x\rangle \geq 0$ equality holds iff $x = 0$ (Positivity)

for $a \in \mathbb{R}$ and $x, y, z \in V$.
Given $f, g \in L^2([a, b])$ (Please don't worry about $L^2([a, b])$, just think of it as the vector space of all functions such that $\int_a^b |f(x)|^2 dx < \infty$), we call them orthogonal to each other provided $\langle f, g\rangle := \int_I f(x)g(x) \, dx = 0$. One can show using something called Cauchy-Schwatz inequality that the inner product is well-defined.
Keeping this information in mind prove that the function $\cos(\frac{n\pi}{\log 2} \log x)$ is an orthogonal set of functions in $L^2([1, 2])$ using the eigenfunctions of the operator

$$L(y) = x^2 y'' + xy' + 2y; \; x \in [1, 2]$$

subject to the boundary condition that $y'(1) = y'(2) = 0$.

(5) Show that the eigenvalues of the equation

$$y'' + \lambda y = 0$$
$$y(0) = y(\pi); \;\; y'(0) = y'(\pi)$$

are not simple, thus proving that the above Sturm-Liouville problem is not regular because remember that the eigenfunctions of distinct eigenvalues are mutually orthogonal provided you are dealing with a regular Sturm-Liouville problem .

(6) (A juicy piece of full toss in case you are tired of so much reading) Find the eigenvalues and eigenfunctions of the equation

$$y'' + \lambda y = 0; \;\; y(0) = y(\pi) = 0$$

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true - Bertrand Russell