MSO 203B - Partial Differential Equation
Assignment 5

October 29, 2018

Tutorial Problem

1. The one operator which is commonly encountered in problems of Electrical, Magnetism and Electrodynamics is the Laplace operator given by $L(u) := \Delta u$ where $\Delta u = u_{xx} + u_{yy}$. Any function $u \in C^2$ belonging to the kernel of such an operator is said to be Harmonic. Let $X = \{u \in C^2(\Omega) : \Delta u = 0\}$ be the Kernel of L.

   (a) Show that X is a vector space.

   (b) Show that the operator is rotationally symmetric e.g the equation $\Delta u = 0$ is invariant under rotation about any arbitrary point in $\mathbf{R}^2$.

   (c) Once you know the equation is rotationally symmetric we can look for solution $u(x, y) = \phi(r)$ where $r^2 = x^2 + y^2$ with $(x, y) \in \mathbf{R}^2$. Question is can you find such a solution? This solutions are radial solutions and are called Fundamental Solution of Laplacian in 2-Dimension.

2. The Maximum Principle for the Laplace equation

$$\Delta u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

   says that the maximum or the minimum of any solution can always be found on the boundary of the domain provided Ω is open and bounded. This is the most fundamental result in all of PDE. Proper application of the maximum principle can yield existence, uniqueness and stability of the solution among other results. But for now prove that the above problem admits a unique solution $u \in C^2(\bar{\Omega})$ without using Maximum Principle.

3. A PDE is called well-posed if the following holds:-

   - There exists atleast one solution to the problem.
   - The solution so obtained is unique.
   - For a small change in initial data the solution changes a little. To put it mathematically one needs to show that if u_1 and u_2 are two solution to an equation $Lu = g$ subject to the boundary condition $u = u_a$ and $u = u_b$ respectively then

$$\max_x |u_1(x) - u_2(x)| \leq C \max_x |u_a(x) - u_b(x)|$$

Show that the Laplace equation is not well-posed in an arbitrary domain.

Practise Problem

1. Solve $u_{xx} + u_{yy} = 1$ using the seperation of variable.
2. Show that the problem
$$\Delta u = 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega$$
does not admit a solution $u \in C^2(\bar{\Omega})$ such that there exists $x_0 \in \Omega$ with $u(x_0) = 1$
3. Show that the equation
$$\Delta u = g \text{ in } \Omega; \quad u = f \text{ on } \partial\Omega$$
admit a unique solution provided f and g are smooth.
4. Let $u : \Omega \to \mathbf{R}$ be harmonic such that $f : u(\Omega) \to \mathbf{R}$ is convex. Show that $f \cdot u$ is subharmonic e.g. $\Delta f(u(x)) \geq 0$