MSO 203B - PDE (Lecture 17 : Wave Equation).

$u_{tt} - u_{xx} = 0$ in $(x,t) \in \mathbb{R} \times (0,\infty)$.

$A = 1,\ B = 0,\ C = 1.$

$B^2 - 4AC > 0$

Char Eqn $\quad \frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{0 \pm \sqrt{0 + 4\cdot 1\cdot 1}}{2\cdot 1} = \pm 1.$

$\Rightarrow t = \pm x + c.$

Choose, $\xi(x,t) = x + t.$

$\eta(x,t) = x - t.$

Define $w(\xi,\eta) := u(x,y)$. The canonical form will look like

$w_{\xi\eta} = 0. \Rightarrow w(\xi,\eta) = F(\xi) + G(\eta)$. where F and G are C^2 fns.

$\therefore u(x,t) = w(\xi,\eta) = F(x+t) + G(x-t).$

$u_{tt} - u_{xx} = 0$

$X(x)T(t)$

$X''T - XT'' = 0$

$\frac{X''}{X} = \frac{T''}{T} = \lambda$

Canonical Form

$(x,y) \mapsto (\xi,\eta)$

$w(\xi,\eta) = u(x,y)$

$w_{\xi\eta} + l[w_\xi, w_\eta, w, \xi, \eta] = 0$

$$\begin{cases} u_{tt} - u_{xx} = 0, & x\in\mathbb{R},\ t>0 \\ u(x,0) = f(x); & x\in\mathbb{R}. \\ u_t(x,0) = g(x); & x\in\mathbb{R}. \end{cases}$$

(f and g are Smooth).

$u(x,t) = F(x+t) + G(x-t)$; F and G are C^2.

$\underline{\underline{\text{I.C}}}$ $\quad f'(x) = u(x,0) = F'(x) + G'(x)$. — ①.

$g(x) = u_t(x,0) = F'(x) - G'(x)$ — ②.

$2F'(x) = f'(x) + g(x)$.

$\Rightarrow F'(x) = \frac{1}{2} f'(x) + \frac{1}{2} g(x)$.

$\Rightarrow F(x) = \frac{1}{2}\int_0^x f'(\xi)\,d\xi + \frac{1}{2}\int_0^x g(\xi)\,d\xi = \frac{1}{2}\left[f(x) - f(0)\right] + \frac{1}{2}\int_0^x g(\xi)\,d\xi$.

and, $G(x) = f(x) - F(x)$

$= \frac{1}{2} f(x) + \frac{1}{2} f(0) - \frac{1}{2}\int_0^x g(\xi)\,d\xi$.

Q: $f(x) = x$

$g(x) = 1$.

$u(x,t) = F(x+t) + G(x-t)$.

$= \frac{1}{2}\left[f(x+t) - f(0)\right] + \frac{1}{2}\int_0^{x+t} g(\xi)\,d\xi$

$+ \frac{1}{2}\left[f(x-t) + f(0)\right] - \frac{1}{2}\int_0^{x-t} g(\xi)\,d\xi$.

$= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi$.

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi.}$$

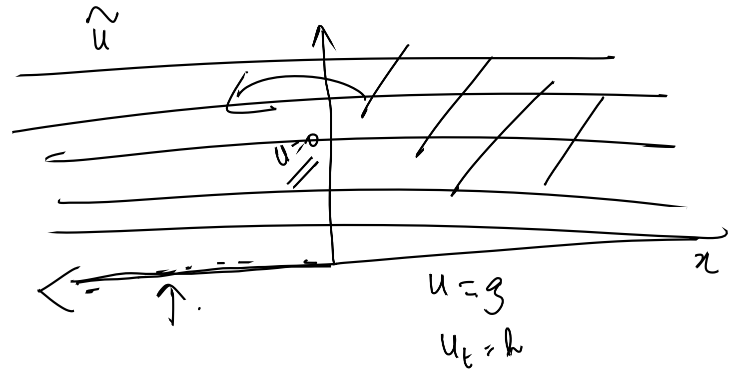
D'Alembert Formulae

$u_{tt} - u_{xx} = 0$ in $\mathbb{R}_+ \times (0,\infty)$

$u = g$; $u_t = h$ on $\mathbb{R}_+ \times \{t=0\}$,

$u = 0$ on $\{x=0\} \times (0,\infty)$.

$g(0) = h(0) = 0$.



$f(x) = -f(-x)$

$x \to 0$

$f(0) = -f(0)$

$f(0) = 0$

$f(x) = f(-x)$

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & ; \; x > 0, t > 0 . \\ -u(-x,t) & ; \; x < 0 , \; t > 0 . \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x), & x > 0 . \\ -g(-x), & x < 0 . \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x), & x > 0 . \\ -h(-x), & x < 0 . \end{cases}$$

$x - t \geq 0$

$x \geq t \geq 0$

So, $\tilde{u}_{tt} - \tilde{u}_{xx} = 0$ on $\mathbb{R} \times (0,\infty)$

$\tilde{u} = \tilde{g}$, $\tilde{u}_t = \tilde{h}$ on $\mathbb{R} \times \{t=0\}$

$u = 0$ on $\{x=0\} \times (0,\infty)$.

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(\xi)\, d\xi .$$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi. & \text{if } \underline{x \geq t \geq 0.} \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{t-x}^{x+t} h(\xi)\,d\xi. & \text{if } \underline{\underline{0 < x \leq t}} \end{cases}$$

$$-\int_{x-t}^{0} h(-\xi)\,d\xi \qquad -\int_{0}^{x-t} h(-\xi)\,d\xi$$