MSO 201a: Probability and Statistics

2016-2017-II Semester
Assignment-VIII

A. Illustrative Discussion Problems

1. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2, x_3) \in A \\ 0, & \text{otherwise} \end{cases},$$

where $A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$.

(a) Are X_1, X_2, X_3 independent?

(b) Are X_1, X_2, X_3 pairwise independent?

(c) Are $X_1 + X_2$ and X_3 independent?

2. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{x_1+2x_2}{18}, & \text{if } x_1 = 1, 2,\ x_2 = 1, 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine the conditional p.m.f. of X_2 given $X_1 = x_1$; $x_1 = 1, 2$.

(b) Find $P(X_1 + X_2 > 2)$, $P(X_1^2 X_2 > 3)$, $P(X_1 = 2|X_2 = 1)$ and $P(X_1 + X_2 \geq 2|X_2 = 1)$.

3. Consider the following joint p.m.f. of r.v. (X, Y):

	$f_{X,Y}(x, y)$			
$y \downarrow \quad x \rightarrow$	1	2	3	4
4	.08	.11	.09	.03
5	.04	.12	.21	.05
6	.09	.06	.08	.04

(a) Find the conditional p.m.f. of X given $Y = 5$.

(b) Find the probabilities $P(X + Y \leq 8)$, $P(X + Y > 7)$, $P(XY \leq 14)$, $P(XY > 18)$, $P(X = 3|Y = 5)$ and $P(Y = 5|X = 3)$.

4. The joint p.d.f. of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} c(1 - x - y), & \text{if } x > 0, y > 0, x + y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the value of the constant c;

(b) Find the marginal p.d.f.s of X and Y;

(c) Find the conditional d.f. of Y given $X = x$, where $x \in (0,1)$ is a fixed constant;

(d) Find $P(\{2X + 3Y < 1\})$.

5. Let (X, Y) be a random vector such that the p.d.f. of X is

$$f_X(x) = \begin{cases} 4x(1-x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0,1)$, the conditional p.d.f. of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2}, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(a) For $y \in (0,1)$, find conditional p.d.f. of X given $Y = y$;

(b) Find $P(\{0 < Y < \frac{1}{3}\})$ and $P(\{\frac{1}{3} < Y < \frac{2}{3}\}|\{X = \frac{1}{2}\})$.

6. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables and that $P(X_i = 0) = 1 - p = 1 - P(X_i = 1)$, $i = 1, \ldots, n$, for some $p \in (0,1)$. Let $X =$ number of $X_1, \ldots, X_n$ that are as large as X_1. Find the p.m.f. of X.

7. Let us choose at random a point from the interval $(0,1)$ and let the r.v. X_1 be equal to the number which corresponds to that point. Then choose a point at random from the interval $(0, X_1)$ and let X_2 be equal to the number which corresponds to this point. Compute $P(X_1 + X_2 \geq 1)$.

8. Let f and g be two p.d.f.s with respective distribution functions F and G. Define $h : \mathbb{R}^2 \to [0, \infty)$ as

$$h(x,y) = [1 + \alpha \{2F(x) - 1\} \{2G(y) - 1\}] f(x)g(y),$$

where $\alpha \in [-1, 1]$.

(a) Show that h is a p.d.f. of some random vector (X, Y);

(b) Show that the marginal p.d.f.s of X and Y are f and g, respectively;

(c) Does there exists a value of $\alpha \in [-1, 1]$ such that X and Y are independent.

B. Practice Problems from the Text Book

Chapter 2: Multivariate Distributions, Problem Nos.: 1.1, 1.5, 1.8, 3.2, 3.6 (a), 3.9

MSO 201a: Probability and Statistics
2016-2017 II Semester
Assignment VIII (Solutions)

$\boxed{\text{Problem No. 1}}$ (a)-(b) Clearly

$P((X_1, X_2) = (0, 0)) = P((X_1, X_2) = (1, 0)) = P((X_1, X_2) = (0, 1)) =$
$P((X_1, X_2) = (1, 1)) = \frac{1}{4}$.

Also $(X_1, X_2) \overset{d}{=} (X_1, X_3) \overset{d}{=} (X_2, X_3)$ (since
$P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$, $i = 1, 2, 3$

Thus X_1 and X_2 are independent ($P((X_1, X_2) = (x_1, x_2))$
$= \frac{1}{4} = P(X_1 = x_1) P(X_2 = x_2)$, $(x_1, x_2) \in \{(1, 0), (0, 1), (0, 0), (1, 1)\}$
$= B$ say and $P((X_1, X_2) = (x_1, x_2)) = 0 = P(X_1 = x_1) P(X_2 = x_2)$ if
$(x_1, x_2) \notin B$).

$P((X_1, X_2, X_3) = (0, 0, 0)) = \frac{1}{4} \neq P(X_1 = 0) P(X_2 = 0) P(X_3 = 0) = \frac{1}{8}$

Thus X_1, X_2 and X_3 are not independent

(c) Let $Y = X_1 + X_2$ so that $S_Y = \{0, 1, 2\}$.

$$f_Y(y) = P(Y = y) = \begin{cases} \frac{1}{4}, & \text{if } y = 0, 2 \\ \frac{1}{2}, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y, X_3}(y, z) = \begin{cases} \frac{1}{4}, & \text{if } (y, z) = (0, 1), (2, 1) \\ \frac{1}{2}, & \text{if } (y, z) = (1, 0) \\ 0, & \text{otherwise} \end{cases}$$

$P(Y = 0, X_3 = 1) = \frac{1}{4} \neq P(Y = 0) P(X_3 = 1) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$\Rightarrow Y = X_1 + X_2$ and X_3 are not independent.

$\boxed{\text{Problem No. 2}}$ (a) $P(X_1 = 1) = \frac{4}{9}$, $P(X_1 = 2) = \frac{5}{9}$, $P(X_2 = 1) = \frac{7}{18}$, $P(X_2 = 2) = \frac{11}{18}$

$$f_{X_2|X_1}(x_2|1) = P(X_2 = x_2 | X_1 = 1) = \frac{P(X_1 = 1, X_2 = x_2)}{P(X_1 = 1)} = \begin{cases} \frac{3}{8}, & \text{if } x_2 = 1 \\ \frac{5}{8}, & \text{if } x_2 = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X_2|X_1}(x_2|2) = P(X_2 = x_2 | X_1 = 2) = \frac{P(X_1 = 2, X_2 = x_2)}{P(X_1 = 2)}$$

$$= \begin{cases} \frac{2}{5}, & \text{if } x_2 = 1 \\ 3/5, & \text{if } x_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(X_1 + X_2 > 2) = P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) + P(X_1 = 2, X_2 = 2)$

$= \frac{5}{18} + \frac{4}{18} + \frac{6}{18} = \frac{5}{6}$

$P(X_1^2 X_2 > 3) = P(X_1 = 2) = \frac{5}{9}$; $P(X_1 = 2 | X_2 = 1) = \frac{P(X_1 = 2, X_2 = 1)}{P(X_2 = 1)} = \frac{4}{7}$

$P(X_1 + X_2 \geq 2 | X_2 = 1) = P(X_1 \geq 1 | X_2 = 1) = 1$.

Problem No. 3 (a)

$$f_{X|Y}(x|5) = P(X=x|Y=5) = \frac{P(X=x, Y=5)}{P(Y=5)} = \frac{P(X=x, Y=5)}{.04+.12+.21+.05}$$

$$= \begin{cases} \frac{.04}{.42} = \frac{2}{21}, & \text{if } x=1 \\ \frac{.12}{.42} = \frac{2}{7}, & \text{if } x=2 \\ \frac{.21}{.42} = \frac{1}{2}, & \text{if } x=3 \\ \frac{.05}{.42} = \frac{5}{42}, & \text{if } x=4 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(X+Y \le 8) = 1 - P(X+Y \ge 9) = 1 - [.08 + .05 + .04] = 0.83$

$P(X+Y > 7) = P(X+Y \ge 8) = P(X+Y=8) + 0.17 = (.06+.21+.03) + 0.17$
$= 0.47$

$P(XY \le 14) = 1 - P(XY \ge 15) = 1 - [.21 + .08 + .03 + .05 + .04] = 0.59$

$P(XY > 18) = P(XY \ge 19) = 0.05 + 0.04 = 0.09$

$$P(X=3|Y=5) = \frac{P(X=3, Y=5)}{P(Y=5)} = \frac{.21}{.04+.12+.21+.05} = \frac{1}{2}$$

$$P(Y=5|X=3) = \frac{P(X=3, Y=5)}{P(X=3)} = \frac{.21}{.09+.21+.08} = \frac{21}{38}.$$

Problem No. 4 (a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1 \int_0^{1-x} c(1-x-y)\,dy\,dx = 1 \Rightarrow c = 6$$

(b) Clearly $S_X = S_Y = [0,1]$. For $x \in S_X = [0,1]$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 6(1-x-y)\,dy = 3(1-x)^2.$$

Thus

$$f_X(x) = \begin{cases} 3(1-x)^2, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}. \quad \text{By symmetry} \quad f_Y(y) = \begin{cases} 3(1-y)^2 & \text{if } 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) For $x \in (0,1)$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2(1-x-y)}{(1-x)^2}, & \text{if } 0 < y < 1-x \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$P(2X+3Y<1) = \underset{2x+3y<1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} f_{X,Y}(x,y)\,dx\,dy = \underset{\substack{x+y<1 \\ 2x+3y<1}}{\int_0^{\infty}\int_0^{\infty}} 6(1-x-y)\,dx\,dy$$

$$= 6 \int_0^{1/2} \left[\int_0^{\frac{1-2x}{3}} (1-x-y)\, dy \right] dx.$$

Problem No. 5 (a) $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} 8xy, & \text{if } 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

For $y \in (0,1)$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = 8y \int_0^y x\, dx = 4y^3$

$$\Rightarrow f_Y(y) = \begin{cases} 4y^3, & \text{if } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

For $y \in (0,1)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{y^2}, & \text{if } 0<x<y \\ 0, & \text{otherwise} \end{cases}$$

Alt. For $y \in (0,1)$, $f_{X|Y}(x|y)$ = a quantity proportional to

$$f_{X,Y}(x,y) = \begin{cases} c(y)x, & \text{if } 0<x<y \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{2x}{y^2}, & \text{if } 0<x<y \\ 0, & \text{otherwise} \end{cases}$$

here, for fixed $y \in (0,1)$, $c(y)$ is proportionality constant.

(b) From (a), $f_Y(y) = \begin{cases} 4y^3, & \text{if } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

$$P(0<Y<\tfrac{1}{3}) = \int_0^{1/3} 4y^3\, dy = \frac{1}{81}$$

As in (a) the conditional p.d.f. of Y given $X=\frac{1}{2}$ is

$$f_{Y|X}(y|\tfrac{1}{2}) = \begin{cases} cy, & \text{if } \frac{1}{2}<y<1 \\ 0, & \text{otherwise} \end{cases} \quad \int_{-\infty}^{\infty} f_{Y|X}(y|\tfrac{1}{2})\, dy = 1 \Rightarrow c = \frac{8}{3}$$

$$P(\tfrac{1}{3}<Y<\tfrac{2}{3} \mid X=\tfrac{1}{2}) = \int_{1/2}^{2/3} \frac{8y}{3}\, dy = \frac{7}{81}.$$

Problem No. 6 X = No. of $(X_1 \ldots X_n)$ which are $\geq X_1$

$P(X=m) = P(\text{exactly } m \text{ of } (X_1, \ldots, X_n) \geq X_1)$

$= P(\text{exactly } m \text{ of } (X_1, \ldots, X_n) \geq 0 \mid X_1=0)(1-p)$

$\quad + P(\text{exactly } m \text{ of } (X_1, \ldots, X_n) \geq 1 \mid X_1=1)\, p$

$= P(\text{exactly } m-1 \text{ of } (X_2, \ldots, X_n) \geq 0 \mid X_1=0)(1-p)$

$\quad + P(\text{exactly } (m-1) \text{ of } (X_2, \ldots, X_n) \geq 1 \mid X_1=1)\, p$

$= P(\text{exactly } m-1 \text{ of } (X_2, \ldots, X_n) \geq 0)(1-p)$

$\quad + P(\text{exactly } (m-1) \text{ of } (X_2, \ldots, X_n) = 1)\, p$ (independence)

$$= \begin{cases} 0 + \binom{n-1}{m-1} p^{m-1} (1-p)^{n-m} p, & \text{if } m = 1, 2, \dots, n-1 \\ 1 \times (1-p) + p^{n-1} \times p, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

Problem No. 7 The p.d.f. of X_1 is $f_{X_1}(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

The conditional p.d.f. of X_2 given $X_1 = x_1$ (where $0 < x_1 < 1$ is fixed) is

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \frac{1}{x_1}, & \text{if } 0 < x_2 < x_1 \\ 0, & \text{otherwise} \end{cases}$$

Thus the joint p.d.f. of (X_1, X_2) is

$$f_{X_1, X_2}(x_1, x_2) = f_{X_2|X_1}(x_2|x_1) f_{X_1}(x_1) = \begin{cases} \frac{1}{x_1}, & \text{if } 0 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X_1 + X_2 \geq 1) = \underset{x_1 + x_2 \geq 1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2 = \underset{\substack{0 < x_2 < x_1 < 1 \\ x_1 + x_2 \geq 1}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} \frac{1}{x_1}\, dx_1\, dx_2$$

$$= \int_{\frac{1}{2}}^{1} \int_{1-x_1}^{x_1} \frac{1}{x_1}\, dx_2\, dx_1 = 1 - \ln 2.$$

Problem No. 8 (a) We have

$$\left.\begin{array}{l} -1 \leq \alpha \leq 1 \\ -1 \leq 2F(x) - 1 \leq 1, \ \forall x \\ -1 \leq 2G(y) - 1 \leq 1, \ \forall y \end{array}\right\} \Rightarrow \begin{array}{l} \alpha[2F(x) - 1][2G(y) - 1] \geq -1 \\ \Rightarrow h(x, y) \geq 0, \quad \forall (x, y) \in \mathbb{R}^2 \end{array}$$

Also

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) g(y)\, dx\, dy + \alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [2F(x) - 1][2G(y) - 1] f(x) g(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} f(x)\, dx\right)\left(\int_{-\infty}^{\infty} g(y)\, dy\right) + \alpha \left(\int_{-\infty}^{\infty} (2F(x) - 1) f(x)\, dx\right)\left(\int_{-\infty}^{\infty} [2G(y) - 1] g(y)\, dy\right)$$

$$= 1 + \frac{\alpha}{4}\left(\int_{-1}^{1} t\, dt\right)\left(\int_{-1}^{1} t\, dt\right) \quad \left(\begin{array}{ll} 2F(x) - 1 = t, & 2F'(x)\, dx = dt \\ & 2f(x)\, dx = dt \\ 2G(y) - 1 = t, & 2G'(y)\, dy = dt \\ & 2g(y)\, dy = dt \end{array}\right)$$

$$= 1$$

$\Rightarrow h$ is a p.d.f.

(b) For $x \in \mathbb{R}$

$$f_X(x) = \int_{-\infty}^{\infty} h(x,y)\,dy = f(x)\int_{-\infty}^{\infty} g(y)\,dy + \alpha\,[2F(x)-1]\int_{-\infty}^{\infty}[2G(y)-1]\,g(y)\,dy$$

$$= f(x) + \frac{\alpha}{2}[2F(x)-1]\int_{-1}^{1} t\,dt = f(x)$$

By symmetry, $f_Y(y) = g(y)$, $y \in \mathbb{R}$.

(c) For $\alpha = 0$

$h(x,y) = f(x)\,f(y)$, $(x,y) \in \mathbb{R}^2$

$\Rightarrow$ X and Y are independent.

Conversely suppose that X and Y are independent. Then

$h(x,y) = f(x)\,g(y)$ $\forall x \in \mathbb{R}, y \in \mathbb{R}$

$\Rightarrow \alpha\,[2F(x)-1]\,[2G(y)-1] = 0$ $\forall x \in \mathbb{R}, y \in \mathbb{R}$

$\Rightarrow \alpha = 0$ (taking $x \to \infty$ and $y \to \infty$)

Thus X and Y are independent $\Leftrightarrow \alpha = 0$.

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5/5