Module 30

DISTRIBUTIONS BASED ON SAMPLING FROM A NORMAL DISTRIBUTION

- We will introduce two new probability distributions, called the Student t-distribution and the Snedecor F-distribution, which arise as probability distributions of various statistics based on a random sample from normal distribution.

Definition 1

(i) For a given positive integer m, a random variable X is said to have the Student t-distribution with m degrees of freedom (written as $X \sim t_m$) if the p.d.f. of X is given by

$$f_X(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\,\Gamma(\frac{m}{2})} \frac{1}{\left(1+\frac{x^2}{m}\right)^{\frac{m+1}{2}}}, \quad -\infty < x < \infty.$$

(ii) The Student t-distribution with 1 degree of freedom is also called the standard Cauchy distribution.

(iiii) For positive integers n_1 and n_2, a random variable X is said to have the Snedecor F-distribution with (n_1, n_2) degrees of freedom (written as $X \sim F(n_1, n_2)$) if the p.d.f. of X is given by

$$f_X(x) = \frac{\left(\frac{n_1}{n_2}\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{\left(\frac{n_1}{n_2}x\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}}} I_{(0,\infty)}(x).$$

Remark 1

The following observations are obvious:

(i) $X \sim t_m \Rightarrow X \stackrel{d}{=} -X$ (since $f_X(x) = f_X(-x), \ \forall\ x \in \mathbb{R}$), i.e., the distribution of $X \sim t_m$ is symmetric about 0. Moreover the distribution of t_m is unimodal with mode at 0.;

(ii) The p.d.f. of Cauchy distribution is given by

$$f(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}, -\infty < y < \infty.$$

If a random variable X has the Cauchy distribution (i.e., if $X \sim t_1$) then $E(X)$ does not exist;

(iii) $X \sim F_{n_1,n_2}, \Rightarrow Y = \frac{\frac{n_1}{n_2}X}{1+\frac{n_1}{n_2}X} \sim \mathrm{Beta}(\frac{n_1}{2}, \frac{n_2}{2})$, the beta distribution with shape parameter $(\frac{n_1}{2}, \frac{n_2}{2})$.

Result 1:

(i) Let $Z \sim \mathrm{N}(0,1)$ and $Y \sim \chi^2_m$ (where $m \in \{1,2,\ldots\}$) be independent random variables. Then

$$T = \frac{Z}{\sqrt{\frac{Y}{m}}} \sim t_m.$$

(ii) For positive integers n_1 and n_2, let $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ be independent random variables. Then

$$U = \frac{X_1/n_1}{X_2/n_2} \sim F_{n_1,n_2}.$$

(iii) Let m and r be positive integers and let $X \sim t_m$. Then $E(X^r)$ is not finite if $r \in \{m, m+1, \ldots\}$. For $r \in \{1,2,\ldots,m-1\}$ and $m \geq r+1$

$$E(X^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \frac{m^{\frac{r}{2}} r! \Gamma(\frac{m-r}{2})}{2^r (\frac{r}{2})! \Gamma(\frac{m}{2})}, & \text{if } r \text{ is even} \end{cases}.$$

(iv) If $X \sim t_m$, then

$$\begin{aligned} \mu_1^{'} &= E(X) = 0, \text{ for } m \in \{2, 3, \ldots\} \\ \mu_2 &= \mathrm{Var}(X) = \frac{m}{m-2}, \text{ for } m \in \{3, 4, \ldots\} \\ \beta_1 &= \text{coefficient of skewness} = 0, \text{ for } m \in \{4, 5, \ldots\} \\ \text{and } \gamma_1 &= \text{kurtosis} = \frac{3(m-2)}{m-4}, \text{ for } m \in \{5, 6, \ldots\}. \end{aligned}$$

(v) Let n_1, n_2 and r be positive integers and let $X \sim F_{n_1,n_2}$. Then, for $n_2 \in \{1, 2, \ldots, 2r\}$ and $r \geq \frac{n_2}{2}$, $E(X^r)$ is not finite. For $n_2 \in \{2r+1, 2r+2, \ldots\}$

$$E(X^r) = \left(\frac{n_2}{n_1}\right)^r \prod_{i=1}^{r} \left(\frac{n_1 + 2(i-1)}{n_2 - 2i}\right).$$

(vi) If $X \sim F_{n_1,n_2}$ then

$$\begin{aligned}
\mu_1' &= E(X) = \frac{n_2}{n_2-2}, \text{ if } n_2 \in \{3,4,\ldots\} \\
\mu_2 &= \mathrm{Var}(X) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}, \text{ if } n_2 \in \{5,6,\ldots\} \\
\beta_1 &= \text{coefficient of skewness} \\
&= \frac{2(2n_1+n_2-2)}{n_2-6}\sqrt{\frac{2(n_2-4)}{n_1(n_1+n_2-2)}}, \text{ if } n_2 \in \{7,8,\ldots\} \\
\text{and } \gamma_1 &= \text{kurtosis} \\
&= \frac{12[(n_2-2)^2(n_2-4)+n_1(n_1+n_2-2)n_1(5n_2-22)]}{n_1(n_2-6)(n_2-8)(n_1+n_2-2)} + 3, \\
&\qquad \text{if } n_2 \in \{9,10,\ldots\}.
\end{aligned}$$

Proof :

(i) The joint p.d.f. of (Y, Z) is given by

$$\begin{aligned} f_{Y,Z}(y,z) &= f_Y(y) f_Z(z) \\ &= \begin{cases} \frac{1}{2^{\frac{m+1}{2}}\Gamma(\frac{m}{2})\sqrt{\pi}} e^{-\frac{y+z^2}{2}} y^{\frac{m}{2}-1}, & \text{if } (y,z) \in (0,\infty) \times \mathbb{R} \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Clearly $S_{Y,Z} = (0, \infty] \times \mathbb{R}$. Consider the transformation $\underline{h} = (h_1, h_2) : S_{Y,Z} \to \mathbb{R}^2$ defined by $h_1(y,z) = \frac{z}{\sqrt{\frac{y}{m}}}$ and $h_2(y,z) = \sqrt{\frac{y}{m}}$. Then $T = h_1(Y,Z) = \frac{Z}{\sqrt{\frac{Y}{m}}}$. Let $U = h_2(Y,Z) = \sqrt{\frac{Y}{m}}$.

Clearly the transformation $\underline{h} = (h_1, h_2) : S_{Y,Z} \to \mathbb{R}^2$ is one-to-one with inverse transformation $\underline{h}^{-1} = (h_1^{-1}, h_2^{-1})$, where for $(t, u) \in h(S_{Y,Z})$,

$$h_1^{-1}(t, u) = mu^2 \text{ and } h_2^{-1}(t, u) = tu.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial t} & \frac{\partial h_1^{-1}}{\partial u} \\ \frac{\partial h_2^{-1}}{\partial t} & \frac{\partial h_2^{-1}}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 2mu \\ u & t \end{vmatrix} = -2mu^2.$$

Also

$$\begin{aligned} \underline{h}(S_{Y,Z}) &= \{(t, u) \in \mathbb{R}^2 : (h_1^{-1}(t, u), h_2^{-1}(t, u)) \in S_{Y,Z}\} \\ &= \{(t, u) \in \mathbb{R}^2 : mu^2 \in [0, \infty),\ u > 0,\ tu \in \mathbb{R}\} \\ &= \{(t, u) \in \mathbb{R}^2 : t \in \mathbb{R},\ u > 0\} \\ &= \mathbb{R} \times (0, \infty) \\ &= A, \text{ say.} \end{aligned}$$

Therefore the joint p.d.f. of (T, U) is given by

$$
\begin{aligned}
f_{T,U}(t,u) &= f_{Y,Z}(h_1^{-1}(t,u), h_2^{-1}(t,u))|J| I_{\underline{h}(S_{Y,Z})}(t,u) \\
&= f_{Y,Z}(mu^2, tu)|-2mu^2| I_A(t,u) \\
&= \begin{cases} \frac{m^{\frac{m}{2}}}{\sqrt{\pi}2^{\frac{m-1}{2}}\Gamma(\frac{m}{2})} u^m e^{-\frac{(m+t^2)u^2}{2}}, & \text{if } (t,u) \in \mathbb{R} \times (0,\infty) \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

Consequently the p.d.f. of T is given by

$$
\begin{aligned}
f_T(t) &= \int_{-\infty}^{\infty} f_{T,U}(t,u)du \\
&= \frac{m^{\frac{m}{2}}}{\sqrt{\pi}2^{\frac{m-1}{2}}\Gamma(\frac{m}{2})} \int_0^{\infty} u^m e^{-\frac{(m+t^2)u^2}{2}} du,\ \ t \in \mathbb{R} \\
&= \frac{1}{\sqrt{m\pi}\Gamma(\frac{m}{2})\left(1+\frac{t^2}{m}\right)^{\frac{m+1}{2}}} \int_0^{\infty} y^{\frac{m-1}{2}} e^{-y} dy \\
&= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\Gamma(\frac{m}{2})} \cdot \frac{1}{\left(1+\frac{t^2}{m}\right)^{\frac{m+1}{2}}},\ \ t \in \mathbb{R},
\end{aligned}
$$

which is the p.d.f. of Student's t-distribution with m degrees of freedom.

(ii) The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is given by

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) \\ &= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2})} e^{-\frac{x_1+x_2}{2}} x_1^{\frac{n_1}{2}-1} x_2^{\frac{n_2}{2}-1} I_{(0,\infty)^2}(x_1, x_2). \end{aligned}$$

We have $S_{X_1,X_2} = [0, \infty)^2$. Consider the one-to-one transformation $\underline{h} = (h_1, h_2) : S_{X_1,X_2} \to \mathbb{R}^2$ given by

$$h_1(x_1, x_2) = \frac{n_2}{n_1} \frac{x_1}{x_2} \text{ and } h_2(x_1, x_2) = \frac{x_2}{n_2}.$$

Define $U = h_1(X_1, X_2) = \frac{X_1/n_1}{X_2/n_2}$ and $V = h_2(X_1, X_2) = \frac{X_2}{n_2}$. Then the inverse of transformation $\underline{h} = (h_1, h_2) : S_{X_1,X_2} \to \mathbb{R}$ is $\underline{h}^{-1} = (h_1^{-1}, h_2^{-1})$, where for $(u, v) \in \underline{h}(S_{X_1,X_2})$,

$$h_1^{-1}(u,v) = n_1 uv \text{ and } h_2^{-1}(u,v) = n_2 v.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \begin{vmatrix} n_1 v & n_1 u \\ 0 & n_2 \end{vmatrix} = n_1 n_2 v.$$

Also

$$\begin{aligned} \underline{h}(S_{X_1,X_2}) &= \{(u,v) \in \mathbb{R}^2 : (h_1^{-1}(u,v), h_2^{-1}(u,v)) \in S_{X_1,X_2}\} \\ &= \{(u,v) \in \mathbb{R}^2 : n_1 uv > 0,\ n_2 v > 0\} \\ &= \{(t,u) \in \mathbb{R}^2 : t > 0,\ u > 0\} \\ &= (0,\infty)^2, \end{aligned}$$

and therefore, the joint p.d.f. of (U, V) is given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(h_1^{-1}(u,v), h_2^{-1}(u,v))|J| I_{\underline{h}(S_{X_1,X_2})}(u,v) \\ &= f_{X_1,X_2}(n_1 uv, n_2 v)|n_1 n_2 v| I_{(0,\infty]^2}(u,v) \\ &= \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}{2^{(\frac{n_1+n_2}{2})}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} u^{\frac{n_1}{2}-1} v^{\frac{n_1+n_2}{2}-1} e^{-\frac{(n_2+n_1 u)v}{2}} I_{(0,\infty)^2}(u,v). \end{aligned}$$

Consequently the p.d.f. of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)dv.$$

Clearly $f_U(u) = 0$, if $u \leq 0$. For $u > 0$

$$
\begin{aligned}
f_U(u) &= \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}{2^{\frac{n_1+n_2}{2}}\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} u^{\frac{n_1}{2}-1} \int_0^\infty v^{\frac{n_1+n_2}{2}-1} e^{-\frac{(n_2+n_1u)v}{2}} dv \\
&= \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\frac{n_1}{n_2}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\left(\frac{n_1}{n_2}u\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}u\right)^{\frac{n_1+n_2}{2}}}, \ 0 < u < \infty.
\end{aligned}
$$

Therefore

$$
U = \frac{X_1/n_1}{X_2/n_2} \sim F_{n_1,n_2}.
$$

(iii) For $m \in \{1, 2, \ldots\}$, by (i),

$$X \overset{d}{=} \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim \mathrm{N}(0, 1)$ and $Y \sim \chi^2_m$ are independent random variables. Thus, for $m \in \{1, 2, \ldots\}$ and $r > 0$,

$$E(X^r) = m^{\frac{r}{2}} E(Z^r Y^{-\frac{r}{2}}) = m^{\frac{r}{2}} E(Z^r) E(Y^{-\frac{r}{2}}),$$

(since Y and Z are independent)

provided the expectations are finite. We have,

$$E(Z^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \frac{r!}{2^{\frac{r}{2}} \left(\frac{r}{2}\right)!}, & \text{if } r \text{ is even} \end{cases}.$$

Moreover, for $r \in \{1, 2, \ldots\}$,

$$E(Y^{-\frac{r}{2}}) = \frac{1}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} \int_0^\infty y^{\frac{m-r}{2}-1} e^{-\frac{y}{2}} dy,$$

which is finite if, and only if, $m > r$. Also, for $m > r$

$$E(Y^{-\frac{r}{2}}) = \frac{2^{\frac{m-r}{2}}\Gamma\left(\frac{m-r}{2}\right)}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} = \frac{\Gamma\left(\frac{m-r}{2}\right)}{2^{\frac{r}{2}}\Gamma\left(\frac{m}{2}\right)}.$$

Thus $E(X^r)$ is finite if $r \in \{1, 2, \ldots, m-1\}$. For $r \in \{1, 2, \ldots, m-1\}$ and $m \geq r+1$

$$E(X^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \dfrac{m^{\frac{r}{2}} r! \Gamma\left(\frac{m-r}{2}\right)}{2^r \left(\frac{r}{2}\right)! \Gamma\left(\frac{m}{2}\right)}, & \text{if } r \text{ is even} \end{cases}.$$

(iv) Using (iii), we have

$$
\begin{aligned}
\mu_1' &= E(X) = 0, \text{ if } m \in \{2, 3, \dots\} \\
\mu_2 &= \mu_2' = E(X^2) = \frac{m}{m-2}, \text{ if } m \in \{3, 4, \dots\} \\
\mu_3 &= \mu_3' = E(X^3) = 0 \text{ if } m \in \{4, 5, \dots\} \\
\text{and } \mu_4 &= \mu_4' = E(X^4) = \frac{3m^2}{(m-2)(m-4)}, \text{ if } m \in \{5, 6, \dots\}.
\end{aligned}
$$

Consequently

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = 0, \text{ if } m \in \{4, 5, \dots\}$$

and

$$\gamma_1 = \frac{\mu_4}{\mu_2^2} = \frac{3(m-2)}{m-4}, \text{ if } m \in \{5, 6, \dots\}.$$

(v) Using (ii), we have

$$X \stackrel{d}{=} \frac{n_2}{n_1}\frac{X_1}{X_2},$$

where $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ are independent random variables. Fix $r \in \{1, 2, \ldots\}$. Then

$$E(X^r) = \left(\frac{n_2}{n_1}\right)^r E(X_1^r X_2^{-r}) = \left(\frac{n_2}{n_1}\right)^r E(X_1^r)E(X_2^{-r}),$$

(since X_1 and X_2 are independent)

provided the expectations are finite. Since $X_1 \sim \chi^2_{n_1}$, $E(X_1^r)$ is finite for any $r > 0$ and

$$\begin{aligned} E(X_1^r) &= \frac{1}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)}\int_0^\infty x^{\frac{n_1}{2}+r-1}e^{-\frac{x}{2}}\,dx \\ &= \frac{2^{\frac{n_1}{2}+r}\Gamma\left(\frac{n_1}{2}+r\right)}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)} \end{aligned}$$

$$
\begin{aligned}
&= 2^r\Big(\frac{n_1}{2}+r-1\Big)\Big(\frac{n_1}{2}+r-2\Big)\ldots\frac{n_1}{2} \\
&= (n_1+2(r-1))(n_1+2(r-2))\ldots n_1 \\
&= \prod_{i=1}^{r}(n_1+2(i-1)),\ \ r\in\{1,2,\ldots\}.
\end{aligned}
$$

Since $X_2\sim\chi^2_{n_2}$, $E(X_2^{-r})$ is finite if, and only if, $n_2>2r$. For $n_2>2r$

$$
E(X_2^{-r})=\frac{2^{\frac{n_2}{2}-r}\Gamma\big(\frac{n_2}{2}-r\big)}{2^{\frac{n_2}{2}}\Gamma\big(\frac{n_2}{2}\big)}=\frac{1}{\prod_{i=1}^{r}(n_2-2i)}.
$$

It follows that, for $n_2\in\{1,2,\ldots,2r\}$ and $r\geq\frac{n_2}{2}$, $E(X^r)$ is not finite. For $n_2\in\{2r+1,2r+2,\ldots\}$

$$
E(X^r)=\left(\frac{n_2}{n_1}\right)^r\prod_{i=1}^{r}\left(\frac{n_1+2(i-1)}{n_2-2i}\right).
$$

(vi) Follows on using (v) after some tedious calculations.

Corollary 1:

Let $X_1, \ldots, X_n$ ($n \geq 2$) be a random sample from $\mathrm{N}(\mu, \sigma^2)$ distribution, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively. Then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim \mathrm{N}(0, 1)$$

and

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}.$$

Proof.

$\bar{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n})$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ are independent random variables.

$$\Rightarrow \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim \mathrm{N}(0,1) \text{ and } \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

are independent random variables.

$$\Rightarrow \frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)S^2/\sigma^2}{n-1}}} \sim t_{n-1},$$

i.e.,

$$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}.$$

Corollary 2:

Let $X_1, \ldots, X_m$ $(m \geq 2)$ be a random sample from $\mathrm{N}(\mu_1, \sigma_1^2)$ distribution and let $Y_1, \ldots, Y_n$ $(n \geq 2)$ be a random sample from $\mathrm{N}(\mu_2, \sigma_2^2)$ distribution, where $\mu_i \in (-\infty, \infty)$ and $\sigma_i > 0, i = 1, 2$. Further suppose that $\underline{X} = (X_1, \ldots, X_m)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ are independent. Let $S_1^2 = \frac{1}{m-1} \sum\limits_{i=1}^{m} (X_i - \bar{X})^2$ and $S_2^2 = \frac{1}{n-1} \sum\limits_{i=1}^{n} (Y_i - \bar{Y})^2$ be the sample variances based on random samples $\underline{X} = (X_1, \ldots, X_m)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$, respectively; here $\bar{X} = \frac{1}{m} \sum\limits_{i=1}^{m} X_i$ and $\bar{Y} = \frac{1}{n} \sum\limits_{i=1}^{n} Y_i$ are the sample means based on two random samples. Then

$$\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim \mathrm{F}_{m-1, n-1}.$$

Proof :

We have

$$\frac{(m-1)S_1^2}{\sigma_1^2} \sim \chi^2_{m-1} \text{ and } \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}.$$

Also the independence of $\underline{X}$ and $\underline{Y}$ implies that $\frac{(m-1)S_1^2}{\sigma_1^2}$ (a function of $\underline{X}$ alone) and $\frac{(n-1)S_2^2}{\sigma_2^2}$ (a function of $\underline{Y}$ alone) are independent. Thus

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim \mathrm{F}_{m-1,n-1},$$

i.e.,

$$\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim \mathrm{F}_{m-1,n-1}.$$

Remark 2 :

(i) Suppose that $X \sim t_m$. Then

$$X \stackrel{d}{=} \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim \mathrm{N}(0,1)$ and $Y \sim \chi^2_m$ are independent random variables. Therefore

$$X^2 \stackrel{d}{=} \frac{Z^2}{Y/m} = \frac{\chi^2_1/1}{\chi^2_m/m} \quad \Big\rangle \text{ independent}$$

Thus

$$X \sim t_m, \Rightarrow X^2 \sim \mathrm{F}_{1,m}.$$

(ii) Suppose that $X \sim \mathrm{F}_{n_1,n_2}$. Then,

$$X \stackrel{d}{=} \frac{\chi^2_{n_1}/n_1}{\chi^2_{n_2}/n_2} \Bigg\rangle \text{independent}$$

$$\Rightarrow \frac{1}{X} \stackrel{d}{=} \frac{\chi^2_{n_2}/n_2}{\chi^2_{n_1}/n_1} \Bigg\rangle \text{independent} \ \sim \mathrm{F}_{n_2,n_1}$$

Thus,

$$X \sim \mathrm{F}_{n_1,n_2} \Rightarrow \frac{1}{X} \sim \mathrm{F}_{n_2,n_1}.$$

- Note that if $X \sim t_m$ then, the distribution of X is symmetric about 0 and its kurtosis is

$$\gamma_1 = \frac{3(m-2)}{m-4} > 3, \; m > 4.$$

- Thus a t-distribution with m (> 4) degrees of freedom is symmetric and leptokurtic (i.e., it has shaper peak and longer fatter tails).
- Note that the kurtosis ν_1 decreases as m increases and $\gamma_1 \to 3$, as $m \to \infty$. This suggests that, for large degrees of freedom, Studentś t-distribution behaves like $\mathrm{N}(0,1)$. distribution. This is infact true.

- Suppose that $X \sim t_m$ and, for a fixed $\alpha \in (0,1)$, let $t_{m,\alpha}$ be the $(1-\alpha)$-th quantile of X, i.e.,

$$F_X(t_{m,\alpha}) = P(X \leq t_{m,\alpha}) = 1 - \alpha.$$

  Then

$$F_X(-t_{m,\alpha}) = 1 - F_X(t_{m,\alpha}) = \alpha \text{ (since } X \stackrel{d}{=} -X).$$

- Now suppose that $X \sim \mathrm{F}_{n_1,n_2}$ and, for a fixed $\alpha \in (0,1)$, let $f_{n_1,n_2,\alpha}$ be the $(1-\alpha)$-th quantile of X, i.e.,

$$F_X(f_{n_1,n_2,\alpha}) = P(\{X \leq f_{n_1,n_2,\alpha}\}) = 1 - \alpha.$$

- Since $\frac{1}{X} \sim F_{n_2,n_1}$ and $P(\{X > 0\}) = 1$, follows that

$$P\left(\left\{\frac{1}{X} \geq \frac{1}{f_{n_1,n_2,\alpha}}\right\}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left\{\frac{1}{X} \leq \frac{1}{f_{n_1,n_2,\alpha}}\right\}\right) = \alpha = 1 - (1 - \alpha)$$

$$\Rightarrow f_{n_2,n_1,1-\alpha} = \frac{1}{f_{n_1,n_2,\alpha}}.$$

i.e.,

$$f_{n_1,n_2,\alpha} \times f_{n_2,n_1,1-\alpha} = 1.$$

Take Home Problems

(1) Let Z_1 and Z_2 be i.i.d. $\mathrm{N}(0,1)$ r.v.s. Show that

$$\frac{Z_1}{Z_2} \stackrel{d}{=} \frac{Z_1}{|Z_2|}.$$

Hence show that $Z = \frac{Z_1}{Z_2}$ follows Cauchy distribution (i.e., $Z \sim t_1$).

(2) Let X_1 and X_2 be i.i.d. $\mathrm{N}(\mu, \sigma^2)$ r.v.s. Show that $X_1 + X_2$ and $X_1 - X_2$ are independent. Find the p.d.f. of

$$Y = \frac{X_1 + X_2 - 2\mu}{\sqrt{2}|X_1 - X_2|}.$$

Thank you for your patience