Module 27

DISTRIBUTION OF FUNCTIONS OF RANDOM VARIABLES

- $\underline{X} = (X_1, \ldots, X_p)'$: a p-dimensional discrete or A.C. random vector;
- $f_{\underline{X}}(\cdot)$: p.m.f./ p.d.f. of $\underline{X}$;
- $g : \mathbb{R}^p \to \mathbb{R}$;
- In many situations, we may be interested in knowing the probability distribution of r.v. $Y = g(\underline{X})$.

Example 1.

- A company manufactures electric bulbs.
- Past life testing experiments on electric bulbs suggest that the lifetime of randomly chosen electric bulbs manufactured by company can be described by r.v. X having p.d.f.

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0$$

However $\theta > 0$ is unknown.

- To obtain information about unknown θ, a life-testing experiment was conducted on n bulbs independently.
- X_i: lifetime of i-th bulb, $i = 1, \ldots, n$ (a r.v.)

- $X_1, \ldots, X_n$: a collection of independently and identically distributed (i.i.d.) r.v.s. We call a collection of i.i.d. r.v.s as a random sample.

- Since $E(X) = \theta$, a natural estimator of θ is the sample mean $\overline{X} = \frac{1}{n} \sum_{1}^{n} X_i$.

- To study theoretical properties of the estimator $\overline{X}$, we may need the probability distribution of $\overline{X}$.

Definition 1.

(i) A function of one or more r.v.s that does not depend on any unknown parameter involved in joint p.m.f./ p.d.f. of r.v.s is called a statistic.

(ii) A collection of i.i.d. r.v.s is called a random sample.

Example 2.

In the above example $\overline{X}$, X_1, $\frac{X_1+X_2}{2}$ are statistics whereas, $\overline{X} - \theta$ or X_1/θ are not statistics.

Now we will discuss various techniques to find the probability distribution of $Y = g(\underline{X})$.

Method 1: Distribution Function Technique

The distribution of $Y = g(X_1, \ldots, X_p)$ can be determined by computing the distribution function of $Y = g(\underline{X})$.

Example 3. Let X_1, X_2 be a random sample from a distribution having p.d.f.

$$f(x) = \begin{cases} 2x, & 0 < \mathrm{x} < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X_1 + X_2$ and hence find p.m.f./ p.d.f. of Y.
Solution. Joint p.d.f. of $\underline{X} = (X_1, X_2)$ is

$$\begin{aligned} f_{\underline{X}}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) \\ &= f(x_1) f(x_2) \\ &= \begin{cases} 4x_1x_2, & \text{if } 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly support of $\underline{X}$ is $S_{\underline{X}} = [0, \infty) \times [0, \infty)$. Also support of $\underline{Y}$ is $S_{\underline{Y}} = [0, 2]$. We have

$$F_Y(x) = P(X_1 + X_2 \leq x), \; x \in \mathbb{R}.$$

Clearly, for $x < 0$, $F_Y(x) = 0$ and, for $x \geq 2$, $F_Y(x) = 1$. For $0 \leq x < 2$,

$$\begin{aligned}
F_Y(x) &= \underset{x_1 + x_2 \leq x}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}} f_{\underline{X}}(x_1, x_2) d\underline{x} \\
&= \underset{x_1 + x_2 \leq x}{\int_0^1 \int_0^1} 4x_1 x_2 \; dx_1 dx_2 \\
&= \int_0^{\min\{x,1\}} \int_0^{\min\{1, x - x_1\}} 4x_1 x_2 \; dx_2 dx_1
\end{aligned}$$

For $0 \leq x < 1$,

$$F_Y(x) = \int_0^{x} \int_0^{x-x_1} 4x_1x_2 \; dx_2 dx_1 = \frac{x^4}{6}.$$

For $1 \leq x < 2$,

$$\begin{aligned} F_Y(x) &= \int_0^{1} \int_0^{\min\{1, x-x_1\}} 4x_1x_2 \; dx_2 dx_1 \\ &= \int_0^{x-1} \int_0^{1} 4x_1x_2 \; dx_2 dx_1 + \int_{x-1}^{1} \int_0^{x-x_1} 4x_1x_2 \; dx_2 dx_1 \\ &= (x-1)^2 + \frac{(4x-3) - (x+3)(x-1)^3}{6} \end{aligned}$$

Thus,

$$F_Y(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^4}{6}, & \text{if } 0 \leq x < 1 \\ (x-1)^2 + \frac{(4x-3)-(x+3)(x-1)^3}{6}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

Clearly, Y is of A.C. type with a p.d.f.

$$f_Y(x) = \begin{cases} \frac{2}{3}x^3, & \text{if } 0 < x < 1 \\ 2(x-1) + \frac{2}{3}(1-(x+2)(x-1))^2, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.

Let X_1, X_2, X_3 be a random sample from a distribution having p.m.f.

$$f(x) = \begin{cases} \frac{1}{6}, & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \min\{X_1, X_2, X_3\}$. Find the d.f. and hence p.m.f. of Y.
Solution. We have, for $x \in \mathbb{R}$,

$$\begin{aligned} F_Y(x) &= P(Y \leq x) \\ &= P(\min\{X_1, X_2, X_3\} \leq x) \\ &= 1 - P(\min\{X_1, X_2, X_3\} > x) \\ &= 1 - P(X_1 > x, X_2 > x, X_3 > x) \\ &= 1 - P(X_1 > x)P(X_2 > x)P(X_3 > x) \\ &= 1 - (1 - F(x))^3 \end{aligned}$$

where F is the d.f. of X_1, given by,

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{i}{6}, & \text{if } i \le x < i+1, i = 1, \ldots, 5 \\ 1, & \text{if } x \ge 6 \end{cases}$$

Thus,

$$F_Y(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1 - (1 - \frac{i}{6})^3, & \text{if } i \le x < i+1, i = 1, \ldots, 5 \\ 1, & \text{if } x \ge 6 \end{cases}$$

The support of Y is $S_Y = \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned} f_Y(x) &= F_Y(x) - F_Y(x-) \\ &= \begin{cases} (1 - \frac{x-1}{6})^3 - (1 - \frac{x}{6})^3, & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(A.) For Discrete case

Result 1. Let $\underline{X} = (X_1, \ldots, X_p)'$ be a discrete r.v. with support $S_{\underline{X}}$ and p.m.f. $f_{\underline{X}}(\cdot)$. Let $g_i : \mathbb{R}^p \to \mathbb{R}$, $i = 1, \ldots, k$ be k (≥ 1) given functions and let $Y_i = g_i(X)$, $i = 1, \ldots, k$. Define
$S_{\underline{Y}} = \{\underline{y} \in \mathbb{R}^k : \underline{y} = (g_1(\underline{x}), \ldots, g_k(\underline{x})), \text{for some } \underline{x} \in S_{\underline{X}}\}$. Also, for each $\underline{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, define

$$A_{\underline{y}} = \{\underline{x} \in S_{\underline{x}} : g_1(\underline{x}) \leq y_1, \ldots, g_k(\underline{x}) \leq y_k\}$$

and

$$B_{\underline{y}} = \{\underline{x} \in S_{\underline{x}} : g_1(\underline{x}) = y_1, \ldots, g_k(\underline{x}) = y_k\}.$$

Then the r.v. $\underline{Y} = (Y_1, \ldots, Y_k)$ is of discrete type with support $S_{\underline{Y}}$, d.f.

$$F_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in A_{\underline{y}}} f_{\underline{X}}(\underline{x}), \;\; \underline{y} \in \mathbb{R}^k$$

and p.m.f.

$$f_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in B_{\underline{y}}} f_{\underline{X}}(\underline{x}), \ \underline{y} \in \mathbb{R}^k.$$

Example 5. Let X_1 and X_2 be i.i.d. r.v.s with common p.m.f.

$$f(x) = \begin{cases} \theta(1-\theta)^{x-1}, & \text{if } x \in \{1, 2, \ldots\} \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in (0, 1)$. Let $Y_1 = \min\{X_1, X_2\}$ and
$Y_2 = \max\{X_1, X_2\} - \min\{X_1, X_2\}$.

(i) Find the marginal p.m.f. of Y_1 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;

(ii) Find the marginal p.m.f. of Y_2 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;

(iii) Find joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;

(iv) Are Y_1 and Y_2 independent?

(iiv) Using (iii), find marginal p.m.f.s of Y_1 and Y_2.

Solution. The joint p.m.f. of $\underline{X} = (X_1, X_2)$ is

$$f_{\underline{X}}(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} \theta^2(1-\theta)^{x_1+x_2-2}, & \text{if } (x_1, x_2) \in \mathbb{N} \times \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

(i) Clearly

$$S_{Y_1} = \text{support of} Y_1 = \{1, 2, \ldots\} = \mathbb{N}$$

For $y \in S_{Y_1}$,

$$\begin{aligned} f_{Y_1}(y) &= P(Y_1 = y) \\ &= P(\min\{X_1, X_2\} = y) \end{aligned}$$

$$
\begin{aligned}
&= P(\min\{X_1, X_2\} = y, X_1 < X_2) + P(\min\{X_1, X_2\} = y, X_1 = X_2) \\
&\qquad\qquad\qquad\qquad\qquad + P(\min\{X_1, X_2\} = y, X_1 > X_2) \\
&= 2P(\min\{X_1, X_2\} = y, X_1 < X_2) + P(\min\{X_1, X_2\} = y, X_1 = X_2) \\
&= 2P(X_1 = y, y < X_2) + P(X_1 = y, X_2 = y) \\
&= 2P(X_1 = y)P(X_2 > y) + P(X_1 = y)P(X_2 = y) \\
&= 2\theta(1-\theta)^{y-1} \sum_{x=y+1}^{\infty} \theta(1-\theta)^{x-1} + \theta^2(1-\theta)^{2y-1} \\
&= 2\theta^2 \frac{(1-\theta)^{y-1}(1-\theta)^{y}}{\theta} + \theta^2(1-\theta)^{2y-2} \\
&= \theta(1-\theta)^{2y-2}(2-\theta).
\end{aligned}
$$

Thus, p.m.f. of Y_1 is

$$f_Y(y) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y-2}, & \text{if } y \in \{1, 2, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Clearly

$$S_{Y_2} = \{0, 1, 2, \ldots\}.$$

For $y \in S_{Y_2}$,

$$\begin{aligned} P(Y_2 = y) &= P(\max\{X_1, X_2\} - \min\{X_1, X_2\} = y) \\ &= P(X_2 - X_1 = y, X_1 < X_2) + P(0 = y, X_1 = X_2) \\ &\qquad + P(X_1 - X_2 = y, X_1 > X_2) \\ &= 2P(X_2 - X_1 = y, X_1 < X_2) + P(0 = y, X_1 = X_2) \end{aligned}$$

For $y = 0$,

$$\begin{aligned} P(Y_2 = y) &= P(X_1 = X_2) \\ &= \sum_{x=1}^{\infty} P(X_1 = x, X_2 = x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x=1}^{\infty} P(X_1 = x)P(X_2 = x) \\
&= \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x-2} \\
&= \frac{\theta^2}{1-(1-\theta)^2} \\
&= \frac{\theta}{2-\theta}.
\end{aligned}
$$

For $y \in \{1, 2, \ldots\}$

$$
\begin{aligned}
P(Y_2 = y) &= 2P(X_2 - X_1 = y, X_1 < X_2) \\
&= 2P(X_2 = X_1 + y, X_1 < X_2) \\
&= 2P(X_2 = X_1 + y) \\
&= 2\sum_{x=1}^{\infty} P(X_2 = X_1 + y, X_1 = x)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{x=1}^{\infty} P(X_1 = x, X_2 = x + y) \\
&= 2\sum_{x=1}^{\infty} \theta^2(1-\theta)^{2x+y-2} \\
&= \frac{2\theta^2(1-\theta)^y}{1-(1-\theta)^2} \\
&= \frac{2\theta(1-\theta)^y}{2-\theta}.
\end{aligned}
$$

Thus,

$$
f_{Y_2}(y) = \begin{cases} \frac{\theta}{2-\theta}, & \text{if } y = 0 \\ \frac{2\theta(1-\theta)^y}{2-\theta}, & \text{if } y = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}
$$

(iii) Clearly support of $\underline{Y} = (Y_1, Y_2)$ is

$$\begin{aligned} S_{\underline{Y}} &= \{(y_1, y_2) : 1 \leq y_1 \leq y_1 + y_2\} \\ &= \{(y_1, y_2) : y_1 \geq 1,\ \ y_2 \geq 0\} \\ &= \mathbb{N} \times \{0, 1, 2, \ldots\}. \end{aligned}$$

For $\underline{y} = (y_1, y_2) \in S_{\underline{Y}}$

$$\begin{aligned} f_{\underline{Y}}(\underline{y}) &= P(Y_1 = y_1, Y_2 = y_2) \\ &= P(\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} = y_1 + y_2) \\ &= P(X_1 = y_1, X_2 = y_1 + y_2, X_1 < X_2) \\ &\qquad\qquad + P(X_1 = y_1, X_1 = y_1 + y_2, X_1 = X_2) \\ &\qquad\qquad + P(X_2 = y_1, X_1 = y_1 + y_2, X_1 > X_2) \\ &= 2P(X_1 = y_1, X_2 = y_1 + y_2, X_1 < X_2) \\ &\qquad\qquad + P(X_1 = y_1, X_1 = y_1 + y_2, X_1 = X_2) \end{aligned}$$

For $y_2 = 0$,

$$\begin{aligned} f_{\underline{Y}}(\underline{y}) &= P(X_1 = y_1, X_2 = y_1) \\ &= \theta^2(1-\theta)^{2y_1-2}. \end{aligned}$$

For $y_2 \in \{1, 2, \ldots\}$,

$$\begin{aligned} f_{\underline{Y}}(\underline{y}) &= 2P(X_1 = y_1, X_2 = y_1 + y_2) \\ &= 2\theta^2(1-\theta)^{2y_1+y_2-2}. \end{aligned}$$

Thus,

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} 2\theta^2(1-\theta)^{2y_1+y_2-2}, & \text{if } (y_1, y_2) \in \mathbb{N} \times \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

(iv) Clearly

$$f_{\underline{Y}}(\underline{y}) = f_{Y_1}(y_1)\, f_{Y_2}(y_2),\ \forall (y_1, y_2) \in \mathbb{R}^2$$

and thus Y_1 and Y_2 are independent.

(v) For $y_1 \in \{1, 2, \ldots\}$,

$$\begin{aligned} f_{Y_1}(y_1) &= \sum_{y_2:(y_1,y_2)\in S_{\underline{Y}}} f_{Y_1,Y_2}(y_1, y_2) \\ &= \sum_{y_2=0}^{\infty} 2\theta^2(1-\theta)^{2y_1+y_2-2} \\ &= \theta(2-\theta)(1-\theta)^{2y_1-2}. \end{aligned}$$

Thus,

$$f_{Y_1}(y) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y_1-2}, & \text{if } y \in \{1, 2, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

For $y_2 \in \{0, 1, 2, \ldots\}$

$$f_{Y_2}(y_2) = \sum_{y_1:(y_1,y_2)\in S_{\underline{Y}}} f_{Y_1,Y_2}(y_1, y_2)$$

For $y_2 = 0$,

$$f_{Y_2}(y_2) = \sum_{y_1=1}^{\infty} \theta^2(1-\theta)^{2y_1-2} = \frac{\theta}{2-\theta}$$

For $y_2 \in \{1, 2, \ldots\}$

$$\begin{aligned} f_{Y_2}(y_2) &= \sum_{y_1=1}^{\infty} 2\theta^2(1-\theta)^{2y_1+y_2-2} \\ &= \frac{2\theta(1-\theta)^{y_2}}{2-\theta} \end{aligned}$$

Thus,

$$f_{Y_2}(y_2) = \begin{cases} \frac{\theta}{2-\theta}, & \text{if } y_2 = 0 \\ \frac{2\theta(1-\theta)^{y_2}}{2-\theta}, & \text{if } y_2 \in \{1, 2, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

(B.) For A.C. case

Result 2. Let $\underline{X} = (X_1, \ldots, X_p)$ be an A.C. r.v. with joint p.d.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}}$. Let $S_1, \ldots, S_k$ be open subsets of $\mathbb{R}^p$ such that $S_i \cap S_j = \phi$, if $i \neq j$ and $\bigcup\limits_{i=1}^{k} S_i = S_{\underline{X}}^0$, where $S_{\underline{X}}^0$ is the interior of $S_{\underline{X}}$. Suppose $h_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, \ldots, p$, are p functions such that on each S_i, $\underline{h} = (h_1, \ldots, h_p) : S_i \to \mathbb{R}^p$ is one-to-one with inverse transformation $h_i^{-1}(\underline{t}) = (h_{1,i}^{-1}(\underline{t}), \ldots, h_{p,i}^{-1}(\underline{t}))$ (say), $i = 1, \ldots, k$. Further suppose that $h_{j,i}^{-1}(\underline{t})$, $j = 1, \ldots, p$, $i = 1, \ldots, k$ have continuous partial derivatives and the Jacobian determinants

$$J_i = \begin{vmatrix} \frac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_p} \\ \frac{\partial h_{2,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_{2,i}^{-1}(\underline{t})}{\partial t_p} \\ \vdots & & \vdots \\ \frac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0, \;\; i = 1, \ldots, p.$$

Define $\underline{h}(S_j) = \{\underline{h}(\underline{x}) = (h_1(\underline{x}), \ldots, h_p(\underline{x})) \in \mathbb{R}^p : \underline{x} \in S_j\}$, $j = 1, \ldots, k$, and $T_j = h_j(X_1, \ldots, X_p)$, $j = 1, \ldots, p$. Then the r.v. $\underline{T} = (T_1, \ldots, T_p)$ is A.C. with joint p.d.f.

$$f_{\underline{T}}(\underline{t}) = \sum_{j=1}^{k} f_{\underline{X}}(h_{1,j}^{-1}(\underline{t}), \ldots, h_{p,j}^{-1}(\underline{t})) \; |J_j| \; I_{h(S_j)}(\underline{t}).$$

Corollary 1.

Let $\underline{X} = (X_1, \ldots, X_p)$ be an A.C. r.v. with joint p.d.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}}$. Suppose that $h_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, \ldots, p$, are p functions such that $\underline{h} = (h_1, \ldots, h_p) : S^0_{\underline{X}} \to \mathbb{R}^p$ is one-to-one with inverse transformation $h^{-1}(\underline{t}) = (h_1^{-1}(\underline{t}), \ldots, h_p^{-1}(\underline{t}))$ (say). Further suppose that h_i^{-1}, $i = 1, \ldots, p$, have continuous partial derivatives and the Jacobian determinants

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_1^{-1}(\underline{t})}{\partial t_p} \\ \frac{\partial h_2^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_2^{-1}(\underline{t})}{\partial t_p} \\ \vdots & & \vdots \\ \frac{\partial h_p^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_p^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0.$$

Define $\underline{h}(S^0_{\underline{X}}) = \{\underline{h}(\underline{x}) = (h_1(\underline{x}), \ldots, h_p(\underline{x})) \in \mathbb{R}^p : \underline{x} \in S^0_{\underline{X}}\}$, and $T_j = h_j(X_1, \ldots, X_p)$, $j = 1, \ldots, p$. Then the r.v. $\underline{T} = (T_1, \ldots, T_p)$ is A.C. with joint p.m.f.

$$f_{\underline{T}}(\underline{t}) = f_{\underline{X}}(h_1^{-1}(\underline{t}), \ldots, h_p^{-1}(\underline{t}))\ |J|\ I_{\underline{h}(S^0_{\underline{X}})}(\underline{t}).$$

Example 6. Let X_1 and X_2 be i.i.d. r.v.s having p.d.f.

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the d.f. of $Y = X_1 + X_2$ and hence find the p.d.f. of Y;

(b) Find the p.d.f. of Y directly using the Jacobian method.

Solution.

(a) The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\ f_{X_2}(x_2) \quad = \begin{cases} 4x_1x_2, & \text{if } 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly support of Y is $S_{\underline{Y}} = [0, 2]$.
For $\underline{y} \in \mathbb{R}$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X_1 + X_2 \leq y) \\ &= \underset{x_1+x_2 \leq y}{\int_0^1 \int_0^1} 4x_1x_2\ d\underline{x} \end{aligned}$$

Clearly, for $y < 0$, $F_Y(y) = 0$ and, for $y \geq 2$, $F_Y(y) = 1$.
For $0 \leq y < 2$,

$$F_Y(y) = 4 \int_0^{\min\{1,y\}} \int_0^{\min\{1,y-x_1\}} x_1 x_2 \ dx_2 dx_1$$

For $0 \leq y < 1$,

$$\begin{aligned} F_Y(y) &= 4 \int_0^{x_1} \int_0^{y-x_1} x_1 x_2 \ dx_2 dx_1 \\ &= \frac{y^4}{6}. \end{aligned}$$

For $1 \le y < 2$,

$$F_Y(y) = 4\int_0^{y-1}\int_0^1 x_1x_2 \; dx_2dx_1 + 4\int_{y-1}^1\int_0^{y-x_1} x_1x_2 \; dx_2dx_1$$
$$= (x-1)^2 + \frac{(4x-3)-(x+3)(x-1)^3}{6}.$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{y^4}{6}, & \text{if } 0 \le y < 1 \\ (y-1)^2 + \frac{(4y-3)-(y+3)(y-1)^3}{6}, & \text{if } 1 \le y < 2 \\ 1, & \text{if } y \ge 2 \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{2}{3}y^3, & \text{if } 0 < y < 1 \\ 2(y-1) + \frac{2}{3}(1-(y+2)(y-1)^2), & \text{if } 1 < y < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Clearly $S_{\underline{X}} = [0,1] \times [0,1]$ and $S_{\underline{X}}^0 = (0,1) \times (0,1)$. Define $Z = X_2$. Then the transformation $(Y, Z) = \underline{h}(X_1, X_2) = (X_1 + X_2, X_2) \to \mathbb{R}^2$ is 1-1 on $S_{\underline{X}}^0$ with inverse transformations

$$\begin{aligned} h_1^{-1}(y,z) &= x_1 &= y - z \\ h_2^{-1}(y,z) &= x_2 &= z, \end{aligned}$$

the Jacobian

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

$$\underline{h}(S_{\underline{X}}^0) \ = \ \{(y,z) \in \mathbb{R}^2 : (y-z,z) \in \overset{\circ}{S}_{\underline{X}}\}.$$

We have

$$\begin{aligned}(y-z,z) \in \overset{\circ}{S}_{\underline{X}} &\Leftrightarrow 0 < y-z < 1,\ 0 < z < 1 \\ &\Leftrightarrow z < y < 1+z,\ 0 < z < 1\end{aligned}$$

Thus

$$\underline{h}(S_{\underline{X}}^0) \ = \ \{(y,z) \in \mathbb{R}^2 : z < y < 1+z,\ 0 < z < 1\}$$

and

$$
\begin{aligned}
f_{Y,Z}(y,z) &= f_{X_1,X_2}(y-z,z)\ I_{\underline{h}(S_{\underline{X}})}(y,z) \\
&= \begin{cases} 4z(y-z), & \text{if } z < y < 1+z,\ 0 < z < 1 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Then $S_Y = [0,2]$. For $y \in (0,2)$

$$
\begin{aligned}
f_Y(y) &= \int_{\max\{0,y-1\}}^{\min\{1,y\}} 4z(y-z)dz \\
&= \begin{cases} \int_0^y 4z(y-z)dz, & \text{if } 0 < y < 1 \\ \int_{y-1}^1 4z(y-z)dz, & \text{if } 1 < y < 2 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Method 3: Moment Generating Function Technique

Let $M(\cdot)$ be the m.g.f. of some known distribution say D_1. If we can show that a r.v. (say X) has m.g.f. $M(\cdot)$ in a neighborhood of zero, then using uniqueness of m.g.f., we can conclude that X has distribution D_1.

Example 7. let X be a r.v. with m.g.f.

$$M(t) = \frac{e^{-t}}{4} + \frac{1}{2} + \frac{e^{t}}{4}, \ t \in \mathbb{R}.$$

Find the distribution of $Y = X^2$.
Solution. Clearly

$$M(t) \ = \ E(e^{tX}) \ = \ \sum_{x \in S_X} e^{tX} f(x)$$

is the m.g.f. of r.v. X having p.m.f.

$$f(x) = P(\{X = x\}) = \begin{cases} \frac{1}{2}, & \text{if } x = 0 \\ \frac{1}{4}, & \text{if } x = -1, 1. \end{cases}$$

The support of Y is $S_Y = \{0, 1\}$ and the p.m.f. of Y is

$$f_Y(y) = P(\{X^2 = y\}) = \begin{cases} \frac{1}{2}, & \text{if } y = 0 \\ \frac{1}{2}, & \text{if } y = 1. \end{cases}$$

Take Home Problem

(1) Let X_1, X_2, X_3 be i.i.d. r.v.s with common p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ -\infty < x < \infty.$$

Let $T = X_1^2 + X_2^2 + X_3^2$.

(a) Find the d.f. of Y and hence find the p.d.f. of Y;

(b) Find the p.d.f. of Y directly. (Hint: Use spherical coordinates transformation $x_1 = r\sin\theta_1 \sin\theta_2, x_2 = r\sin\theta_1\cos\theta_2, x_3 = r\cos\theta_1$, $r \geq 0, 0 < \theta_1 \leq \pi, 0 < \theta_2 \leq \pi$).

(2) Let $X_1, \ldots, X_k$ be independent r.v.s with X_i having p.m.f.

$$f_i(x) = \begin{cases} \binom{n_i}{x} p^x (1-p)^{n_i - x}, & \text{if } x = 0, 1, \ldots, n_i \\ 0, & \text{otherwise} \end{cases},$$

where $n_i \in \mathbb{N}$, $i = 1, \ldots, k$, and $p \in (0,1)$ are fixed. Show that the random variable $Y = \sum_{i=1}^{p} X_i$ has p.m.f.

$$f_Y(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y}, & \text{if } y \in \{0, 1, \ldots, n\} \\ 0, & \text{otherwise}, \end{cases}$$

where $n = \sum_{i=1}^{k} n_i$.

Thank you for your patience