Module 29

Some Special Absolutely Continuous Distributions

- X: an A.C. r.v. with support S_X, d.f. $F_X(\cdot)$ and p.d.f. $f_X(\cdot)$;
- $$\mu = E(X) = \int_{-\infty}^{\infty} x \, f_X(x)dx \qquad \text{(Mean)}$$

  $$\sigma^2 = \text{Var}(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, f_X(x)dx \qquad \text{(Variance)}$$

- For any function $h(\cdot)$,

  $$E(h(X)) = \int_{-\infty}^{\infty} h(x) \, f_X(x)dx. \qquad \text{(provided integral is finite)}$$

- M.G.F.

  $$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} \, f_X(x)dx, \; t \in \mathbb{R}.$$

I. Uniform or Rectangular Distribution

- Let α and β be real numbers such that $\alpha < \beta$. An A.C. r.v. X is said to have uniform (or rectangular) distribution over the interval (α, β) (written as $X \sim \mathrm{U}(\alpha, \beta)$) if the p.d.f. of X is given by

$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}.$$

- Clearly

$$f_X\left(\frac{\alpha+\beta}{2} - x\right) = f_X\left(\frac{\alpha+\beta}{2} + x\right) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } -\frac{\beta-\alpha}{2} < x < \frac{\beta-\alpha}{2} \\ 0, & \text{otherwise} \end{cases}.$$

- Thus, the distribution of X is symmetric about $\frac{\alpha+\beta}{2}$, i.e. $X - \frac{\alpha+\beta}{2} \stackrel{d}{=} \frac{\alpha+\beta}{2} - X$.
- For $m \in \{1, 2, \ldots\}$,

$$\begin{aligned} E(X^m) &= \int_{-\infty}^{\infty} x^m \ f_X(x) dx \\ &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x^m \ dx \\ &= \frac{\beta^{m+1} - \alpha^{m+1}}{(m+1)(\beta - \alpha)}. \end{aligned}$$

- Mean $= \mu = E(X) = \frac{\alpha+\beta}{2} =$ Median.

- $E(X^2) = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\beta\alpha+\alpha^2}{3}$.

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$$\begin{aligned}\text{Variance} = \sigma^2 = \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \frac{(\beta-\alpha)^2}{12}.\end{aligned}$$

- The m.g.f. of $X \sim \text{U}(\alpha, \beta)$ is given by

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} e^{tX}\, dx \\ &= \begin{cases} \frac{e^{t\beta}-e^{t\alpha}}{t(\beta-\alpha)}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0 \end{cases}.\end{aligned}$$

- The d.f. of $X \sim \mathrm{U}(\alpha, \beta)$ is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \begin{cases} 0, & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x < \beta \\ 1, & \text{if } x \geq \beta \end{cases}.$$

Result 1.

(i) Let $-\infty < \alpha < \beta < \infty$. Then

$$X \sim \mathrm{U}(\alpha, \beta) \Rightarrow Y = \frac{X - \alpha}{\beta - \alpha} \sim \mathrm{U}(0, 1).$$

(ii) Let X be a r.v. with d.f. $F(\cdot)$. Define the quantile function $Q : (0, 1) \to \mathbb{R}$ by

$$Q(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}, \ 0 < p < 1.$$

(a) If X is continuous, show that $Y = F(X) \sim \mathrm{U}(0, 1)$;

(a) If $U \sim \mathrm{U}(0, 1)$, show that $Q(U) \stackrel{d}{=} X$.

Proof.

(i) We have

$$F_X(x) = \begin{cases} 0, & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \le x < \beta \\ 1, & \text{otherwise} \end{cases}.$$

Thus,

$$F_Y(y) = P\left(\frac{X-\alpha}{\beta-\alpha} \le y\right) = P(X \le \alpha + (\beta-\alpha)y)$$

$$= F_X\Big(\alpha + (\beta-\alpha)y\Big) = \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } 0 \le y < 1 \\ 1, & \text{if } y \ge 1 \end{cases},$$

which is the d.f. of U(0, 1) distribution.

(ii) See Assignment V, Problem 5.

Remark 1.

Uniform distributions are used in modeling experiments whose outcomes are number X chosen at random from an interval $[\alpha, \beta]$ in the sense that if $I \subseteq [\alpha, \beta]$ is any sub-interval then $P(X \in I)$ depends only on length of I and not on location of I in $[\alpha, \beta]$.

II. Gamma Distribution

- The gamma function $\Gamma : (0, \infty) \to (0, \infty)$ is defined by

$$\Gamma(\alpha) \; = \; \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

- It can be shown that the above integral is finite for any $\alpha > 0$.

- **Properties of Gamma Function:**
  - $\Gamma(\alpha + 1) \; = \; \alpha\Gamma(\alpha), \;\; \alpha > 0$ (integration by parts).
  - $\Gamma(\alpha) = (\alpha - 1)!$, if α is a positive integer.
  - $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = \sqrt{\pi}$.

- A r.v. X is said to have gamma distribution with parameters $\alpha > 0$ (called shape parameter) and $\theta > 0$ (called scale parameter) if its p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{\theta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\theta}} x^{\alpha-1}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

  and we write $X \sim \mathrm{G}(\alpha, \beta)$.

- Clearly $f_X(x) \geq 0$, $\forall\ x \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty} f_X(x)\ dx = \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\frac{x}{\theta}} x^{\alpha-1}\ dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1}\ dt$$

$$= 1.$$

- $$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tX} f_X(x)dx \\ &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^\infty e^{tX} e^{-\frac{x}{\theta}} x^{\alpha-1} dx \\ &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\frac{1-t\theta}{\theta}x} x^{\alpha-1} dx \\ &= \frac{\Gamma(\alpha)\left(\frac{1-t\theta}{\theta}\right)^{-\alpha}}{\theta^\alpha \Gamma(\alpha)} = (1-t\theta)^{-\alpha}, \; t < \frac{1}{\theta}. \end{aligned}$$

- We have

$$M_X(t) = 1 \ + \ \alpha t\theta \ + \ \frac{\alpha(\alpha+1)}{2!}(t\theta)^2 \ + \ \frac{\alpha(\alpha+1)(\alpha+2)}{3!}(t\theta)^3 + \ldots, \ t < \frac{1}{\theta}.$$

-

$$\begin{aligned}
\text{Mean} \ &= \ \mu \ = \ E(X) \ = \ \text{coefficient of } t \text{ in } M_X(t) \ = \ \alpha\theta \\
E(X^2) \ &= \ \text{coefficient of } \frac{t^2}{2!} \text{ in } M_X(t) \ = \ \alpha(\alpha+1)\theta^2 \\
\text{Variance} \ &= \ \sigma^2 \ = \ \text{Var}(X) \ = \ E(X^2) - (E(X))^2 \ = \ \alpha\theta^2.
\end{aligned}$$

- A $\mathrm{G}(1, \theta)$ distribution is called exponential distribution with mean $\theta > 0$ (denoted by $\mathrm{Exp}(\theta)$, $\theta > 0$). If $X_1 \sim \mathrm{Exp}(\theta)$, then

$$f_{X_1}(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

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$$\text{Mean} \;=\; \mu = E(X_1) = \theta, \;\text{Variance} \;=\; \sigma^2 = \mathrm{Var}(X_1) = \theta^2,$$
$$\text{and} \quad M_{X_1}(t) \;=\; (1 - t\theta)^{-1}, \; t < \frac{1}{\theta}.$$

- For a positive integer n, a $G(\frac{n}{2}, 2)$ distribution is called Chi-squared distribution with n degrees of freedom (d.f.) and is denoted by χ_n^2. If $X_2 \sim \chi_n^2$, then

$$f_{X_2}(x) = \begin{cases} \frac{e^{-\frac{x}{2}} x^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

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$$\begin{aligned} \text{Mean} &= \mu = E(X_2) = n; \\ \text{Variance} &= \sigma^2 = \mathrm{Var}(X_2) = 2n; \\ \text{and} \quad M_{X_2}(t) &= (1-2t)^{-\frac{n}{2}}, \ t < \frac{1}{2}. \end{aligned}$$

- Quantiles of chi-squared distributions (for various values of degrees of freedom) are tabulated in various textbooks.

Result 2.

Let $X_1, \ldots, X_k$ be independent r.v.s with $X_i \sim \mathrm{G}(\alpha_i, \theta)$, $\alpha_i > 0$, $\theta > 0$, $i = 1, \ldots, k$. Then $Y = \sum\limits_{i=1}^{k} X_i \sim \mathrm{G}\Big(\sum\limits_{i=1}^{k} \alpha_i, \theta\Big)$.

Proof. We have

$$\begin{aligned} M_Y(t) = E(e^{tY}) &= \prod_{i=1}^{k} M_{X_i}(t) \\ &= \prod_{i=1}^{k} (1-\theta t)^{-\alpha_i} \\ &= (1-\theta t)^{-\sum\limits_{i=1}^{k} \alpha_i}, \; t < \frac{1}{\theta}, \end{aligned}$$

which is the m.g.f. of $\mathrm{G}(\sum\limits_{i=1}^{k} \alpha_i, \theta)$. The result now follows by uniqueness of m.g.f.s.

Corollary 1.

Let $X_1, \ldots, X_k$ be independent r.v.s and let $Y = \sum\limits_{i=1}^{k} X_i$.

(i) $X_i \sim \text{Exp}(\theta),\ i = 1, \ldots, k \Rightarrow Y \sim \text{G}(k, \theta)$.

(ii) $X_i \sim \chi^2_{n_i},\ i = 1, \ldots, k \Rightarrow Y \sim \chi^2_{\sum\limits_{i=1}^{k} n_i}$.

- Suppose that $X \sim \text{Exp}(\theta),\ \theta > 0$. Then

$$P(X > s) = \int_s^\infty \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = e^{-\frac{s}{\theta}},\ \ s > 0$$

$$P(X > s + t | X > s) = \frac{P(X > s + t)}{P(X > s)} = e^{-\frac{t}{\theta}} = P(X > t),\ \forall s, t > 0.$$

- Thus for exponential distribution, $\forall\ s, t > 0$,

$$P(X > s + t | X > s) = P(X > t) \tag{1}$$

$$\text{i.e.,} \quad P(X > s + t) = P(X > S)P(X > t). \tag{2}$$

- Let $X \sim \text{Exp}(\theta)$ denote the lifetime of a component. Then the property (1) (equivalently property (2)) about lifetime X of the component has the following interesting interpretation: Given that the component has survived s units of time, the probability that it will survive additional t units of time is the same as the probability that a fresh unit (of age 0) will survive t units of time. In other words, the component is not aging with time, i.e., the used component is the same as the new one. This property of a continuous r.v. is also known as the lack of memory or memory less property (at each state component forgets its age and behaves like a fresh component).

Result 3.

Let Y be a continuous r.v. with $F_Y(0) = 0$. Then Y has the lack of memory property (1) if and only if $Y \sim \text{Exp}(\theta)$, for some $\theta > 0$.

Proof. Clearly if $Y \sim \text{Exp}(\theta)$, for some $\theta > 0$, then Y has lack of memory property . Conversely, suppose that Y has lack of memory property (1). Let

$$\bar{F}_Y(t) \; = \; 1 - F_Y(t); \; t \in \mathbb{R}.$$

Then,

$$\begin{aligned} \bar{F}_Y(s+t) \; &= \bar{F}_Y(s) \; \bar{F}_Y(t), \; \forall s, t > 0 \\ \Rightarrow \bar{F}_Y(s_1 + \cdots + s_m) \; &= \bar{F}_Y(s_1) \cdots \bar{F}_Y(s_m), \; s_i > 0, \; i = 1, \ldots, m \end{aligned}$$

$$\bar{F}_Y\left(\frac{m}{n}\right) = \left[\bar{F}_Y\left(\frac{1}{n}\right)\right]^m, \; m \in \mathbb{N}$$

$$\bar{F}_Y(1) = \left[\bar{F}_Y\left(\frac{1}{n}\right)\right]^n, \; n \in \mathbb{N}$$

$$\bar{F}_Y\left(\frac{m}{n}\right) = \left[\bar{F}_Y(1)\right]^{\frac{m}{n}}, \; m, n \in \mathbb{N}$$

Let $\lambda = \bar{F}_Y(1)$, so that $0 \leq \lambda \leq 1$.

$$\begin{aligned}
\lambda = 0 \Rightarrow \bar{F}_Y\left(\frac{1}{n}\right) &= 0, \; \forall n = 1, 2, \ldots \\
\Rightarrow \lim_{n\to\infty} \bar{F}_Y\left(\frac{1}{n}\right) &= 0 \\
\Rightarrow \bar{F}_Y(0) &= 0 \\
\Rightarrow F_Y(0) &= 1 \qquad \text{(contradiction)}
\end{aligned}$$

$$\begin{aligned}
\lambda = 1 \Rightarrow \bar{F}_Y(n) = [\bar{F}_Y(1)]^n &= 1, \; \forall n = 1, 2, \ldots \\
\Rightarrow \bar{F}_Y(\infty) &= 1 \\
\Rightarrow F_Y(\infty) &= 0 \qquad \text{(contradiction)}
\end{aligned}$$

Thus $0 < \lambda < 1$. Let $\lambda = e^{-\frac{1}{\theta}}, \theta > 0$. Then

$$\bar{F}_Y(r) \; = \; [\bar{F}_Y(1)]^r \; = \; e^{-\frac{r}{\theta}}, \; \forall \; r \in \mathbb{Q},$$

where $\mathbb{Q}$ denotes the set of positive rational numbers. Now let $x > 0$. Then there exists a sequence $\{r_n\}_{n \geq 1} \subseteq \mathbb{Q}$ such that $r_n \to x$. Then, since F_Y is continuous

$$\bar{F}_Y(x) = \lim_{n \to \infty} \bar{F}_Y(r_n) = \lim_{n \to \infty} e^{-\frac{r_n}{\theta}} = e^{-\frac{x}{\theta}},$$

implying that $X \sim \text{Exp}(\theta)$.

III. Beta Distribution

- The beta function $\beta : (0, \infty) \times (0, \infty) \to (0, \infty)$ is defined by

$$\beta(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \ a > 0, \ b > 0.$$

- It can be shown that

$$\beta(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \ a > 0, \ b > 0.$$

- A r.v. X is said to have beta distribution with parameters $a > 0$ and $b > 0$ (written as $X \sim \mathrm{B}(a, b)$) if its p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{\beta(a,b)} x^{a-1} (1-x)^{b-1}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

- Note that $B(1,1)$ distribution is nothing but $\mathrm{U}(0,1)$ distribution.

- Suppose that $X \sim \mathrm{B}(a,b)$, $a>0$, $b>0$. Then, for $r>-a$,

$$
\begin{aligned}
E(X^r) &= \frac{1}{\beta(a,b)} \int_0^1 x^r \; x^{a-1} \; (1-x)^{b-1} dx \\
&= \frac{1}{\beta(a,b)} \int_0^1 x^{a+r-1} \; (1-x)^{b-1} dx \\
&= \frac{\beta(a+r,b)}{\beta(a,b)} \; = \; \frac{\Gamma(a+r)\;\Gamma(a+b)}{\Gamma(a)\;\Gamma(a+b+r)}, \;\; r>-a.
\end{aligned}
$$

- $$\begin{aligned} \text{Mean} &= \mu &= E(X) = \frac{a}{a+b}; \\ E(X^2) &= \frac{a(a+1)}{(a+b)(a+b+1)}; \\ \text{Variance} &= \sigma^2 &= \mathrm{Var}(X) = E(X^2) - (E(X))^2 \\ &= \frac{a\ b}{(a+b)^2\ (a+b+1)}. \end{aligned}$$

- If $X \sim \mathrm{B}(a, a)$, $a > 0$, then

$$\begin{aligned} f_X\left(\frac{1}{2} - x\right) &= f_X\left(\frac{1}{2} + x\right) \\ &= \begin{cases} \frac{1}{\beta(a,a)}\left(\frac{1}{2} - x\right)^{a-1}\left(\frac{1}{2} + x\right)^{a-1}, & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

i.e., distribution of $X \sim \mathrm{B}(a, a)$ is symmetric about $\frac{1}{2}$.

- If $X \sim \mathrm{B}(a, b)$, $a > 0$, $b > 0$, then

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\,\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+r)} \frac{t^r}{r!}, \ t \in \mathbb{R}. \end{aligned}$$

IV. Normal (or Gaussian) Distribution

- We know that

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt \\ &= 2\int_{0}^{\infty} e^{-\frac{t^2}{2}}\, dt \\ &= \frac{2}{\sqrt{2}}\int_{0}^{\infty} \frac{e^{-z}}{\sqrt{z}}\, dz \\ &= \sqrt{2}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi} \end{aligned}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt = 1$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1,\ \forall\ \mu\ \in \mathbb{R},\ \sigma > 0.$$

- A r.v. X is said to have the normal distribution with parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$ (written as $X \sim \mathrm{N}(\mu, \sigma^2)$) if its p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, & \text{if } -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}.$$

- Clearly if $X \sim \mathrm{N}(\mu, \sigma^2)$, then

$$f_X(\mu - x) = f_X(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}},\ \forall -\infty < x < \infty,$$

  i.e., $X - \mu \stackrel{d}{=} \mu - X$, and distribution of X is symmetric about μ.

- The $\mathrm{N}(0, 1)$ distribution is called the standard normal distribution.

- The p.d.f. and d.f. of N(0, 1) distribution will be denoted by $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, i.e.,

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}},\; -\infty < z < \infty$$

$$\text{and } \Phi(z) = \int_{-\infty}^{z} \phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.$$

- Clearly N(0, 1) distribution is symmetric about 0.

- Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$, for some $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Then,

$$
\begin{aligned}
X - \mu &\overset{d}{=} \mu - X \\
P(X - \mu \le x) &= P(\mu - X \le x) \\
P(X \le \mu + x) &= P(X \ge \mu - x) \\
\Rightarrow F_X(\mu + x) &= 1 - F_X(\mu - x) \\
\Rightarrow F_X(\mu + x) + F_X(\mu - x) &= 1.
\end{aligned}
$$

In particular, $\Phi(x) + \Phi(-x) = 1, \forall\ x \in \mathbb{R}$.

Result 4.

(a) Let $X \sim \mathrm{N}(\mu, \sigma^2)$, for some $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Then,

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}(0, 1).$$

(b) If $Z \sim \mathrm{N}(0, 1)$, then $Y = Z^2 \sim \chi_1^2$.

Proof.

(a) Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Then, the p.d.f. of $Z = \frac{X-\mu}{\sigma}$ is

$$\begin{aligned} f_Z(z) &= f_X(\mu + \sigma z)\ |\sigma|\ I_{(-\infty,\infty)}(z) \\ &= \frac{1}{\sqrt{2\pi}}\ e^{-\frac{z^2}{2}},\ -\infty < z < \infty, \end{aligned}$$

i.e., $Z \sim \mathrm{N}(0, 1)$.

(a) Let $Z \sim \mathrm{N}(0,1)$ and $Y = Z^2$. Then, for $t < 0$, $F_Y(t) = 0$. For $t \geq 0$,

$$\begin{aligned} F_Y(t) &= P(Z^2 \leq t) \\ &= P(-\sqrt{t} \leq Z \leq \sqrt{t}) \\ &= \int\limits_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} dz \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{t}} e^{-\frac{z^2}{2}} dz \\ &= \int_0^t \frac{z^{\frac{1}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} dz. \end{aligned}$$

Thus,

$$F_Y(t) = \begin{cases} 0, & \text{if } t < 0 \\ \int_0^t \frac{z^{\frac{1}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} dz, & \text{if } t \geq 0 \end{cases}$$

$$= \int_{-\infty}^{t} f_Y(z) dz,$$

where $$f_Y(z) = \begin{cases} \frac{z^{\frac{1}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} dz, & \text{if } z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow Y \sim \chi_1^2.$$

Result 4.

Let $X \sim \text{N}(\mu, \sigma^2)$, for some $\mu \in (-\infty, \infty)$ and $\sigma > 0$.

(a) Then $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, $t \in \mathbb{R}$.

(b) Then $E(X) = \mu = \text{Median}$ and $\text{Var}(X) = \sigma^2$.

(c) Let $Y = aX + b$, where $a \neq 0$ and $b \in \mathbb{R}$ are fixed real constants. Then $Y \sim \text{N}(a\mu + b, a^2\,\sigma^2)$.

(d) Let $Z = \frac{X-\mu}{\sigma}$ (so that $Z \sim \text{N}(0,1)$). Then

$$E(Z^r) = \begin{cases} 0, & \text{if } r = 1, 3, 5, \ldots \\ \frac{r!}{2^{\frac{r}{2}}\,\left(\frac{r}{2}\right)!}, & \text{if } r = 2, 4, 6, \ldots \end{cases}.$$

(e) Then, Co-efficient of skewness $= \beta_1 = 0$ and Kurtosis $= \gamma_1 = 3$.

Proof.

(a) For $t \in \mathbb{R}$

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}} dz \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t\sigma)^2}{2}} dz \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}}, \ t \in \mathbb{R}. \end{aligned}$$

- Let

$$\psi_X(t) = \ln(M_X(t)) = \mu t + \frac{\sigma^2 t^2}{2}, \quad t \in \mathbb{R}.$$

Then,

$$\psi_X^{(1)}(t) = \mu + t\sigma^2, t \in \mathbb{R}$$

$$\psi_X^{(2)}(t) = \sigma^2, t \in \mathbb{R}$$

$$\Rightarrow \text{ Mean } = \psi_X^{(1)}(0) = \mu$$

$$\text{Variance } = \psi_X^{(2)}(0) = \sigma^2.$$

$$\text{Also, } X - \mu \overset{d}{=} \mu - X \text{ implies that}$$

$$\mu = \text{Median.}$$

(c) The m.g.f. of $Y = aX + b$ is

$$\begin{aligned} M_Y(t) &= E(e^{t(aX+b)}) \\ &= e^{tb} E(e^{atX}) \\ &= e^{tb} \mathrm{M}_X(at) \\ &= e^{tb} e^{\mu at + \frac{\sigma^2 a^2 t^2}{2}} \\ &= e^{(a\mu+b)t + \frac{a^2\sigma^2 t^2}{2}}, \ t \in \mathbb{R}, \end{aligned}$$

which is the m.g.f. of $\mathrm{N}(a\mu + b, a^2\sigma^2)$ distribution.

(d) By (c) we have

$$\begin{aligned} M_Z(t) &= e^{\frac{t^2}{2}} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}, \ \ t \in \mathbb{R}. \end{aligned}$$

For $r \in \{1, 2, \ldots\}$

$$\begin{aligned} E(Z^r) &= \text{coefficient of } \frac{t^r}{r!} \text{ in expansion of } M_Z(t) \\ &= \begin{cases} 0, & \text{if } r = 1, 3, 5, \ldots \\ \frac{r!}{2^{\frac{r}{2}}(\frac{r}{2})!}, & \text{if } r = 2, 4, 6, \ldots \end{cases}. \end{aligned}$$

(e)

$$\begin{aligned} E(Z^3) &= 0 \\ \Rightarrow E((X-\mu)^3) &= 0 \\ \Rightarrow \beta_1 &= 0. \end{aligned}$$

Also

$$\begin{aligned} E(Z^4) &= \frac{4!}{4 \times 2!} = 3 \\ \Rightarrow E((X-\mu)^4) &= 3\sigma^4 \\ \Rightarrow \text{Kurtosis} &= \gamma_1 = \frac{\mu_4}{\mu_2^2} = 3. \end{aligned}$$

- If $X \sim \mathrm{N}(\mu, \sigma^2)$. Then, for $Z = \frac{X-\mu}{\sigma}$ (so that $Z \sim \mathrm{N}(0,1)$)

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(Z \leq \frac{x-\mu}{\sigma}) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right), \; x \in \mathbb{R}. \end{aligned}$$

- Let τ_α be the $(1-\alpha)$th quantile of $\mathrm{N}(0,1)$ distribution, i.e.,

$$\Phi(\tau_\alpha) = 1 - \Phi(-\tau_\alpha) = 1 - \alpha.$$

- If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $F_X(\mu) = \frac{1}{2}$.
- $\Phi(0) = \frac{1}{2}$.

- The following table provides various quantiles of N$(0, 1)$ distribution.

α	.001	.005	.01	.025	.05	.1	.25
τ_α	3.092	2.5758	2.326	1.96	1.6499	1.282	.675

- Tables of $\Phi(z)$ (for various values of z) are available in various text books.

Example 1:

Let $X \sim \mathrm{N}(10, 4)$. Find $P(X \leq 6.08), P(X > 13.3), P(X > 7.44)$ and $P(X \leq 11.35)$.

Solution

$$P(X \leq 6.08) = \Phi\left(\frac{6.08 - 10}{2}\right) = \Phi(-1.96) = 1 - \Phi(1.96) = 1 - .975 = .025;$$

$$P(X > 13.3) = 1 - \Phi\left(\frac{13.3 - 10}{2}\right) = 1 - \Phi(1.65) = .05;$$

$$P(X > 7.44) = 1 - \Phi\left(\frac{7.44 - 10}{2}\right) = 1 - \Phi(-1.28) = \Phi(1.28) = .9;$$

$$P(X \leq 11.35) = \Phi\left(\frac{11.35 - 10}{2}\right) = \Phi(.675) = .75.$$

Result 6 :

Let $X_1, X_2, \ldots, X_k$ be independent random variables with $X_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, $\mu_i \in (-\infty, \infty), \sigma_i > 0, i = 1, \ldots, k$. Let $a_1, a_2, \ldots, a_k$ be real constants such that $\sum_{i=1}^{k} a_i^2 > 0$. Then

(a) $\sum_{i=1}^{k} a_i X_i \sim \mathrm{N}\left(\sum_{i=1}^{k} a_i \mu_i, \sum_{i=1}^{k} a_i^2 \sigma_i^2\right)$;

(b) $\sum_{i=1}^{k} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi_k^2$.

Proof.

(a) Let $Y = \sum_{i=1}^{k} a_i X_i$. Then

$$
\begin{aligned}
M_Y(t) &= E\left(e^{t\sum\limits_{i=1}^{k} a_i X_i}\right) \\
&= E\left(\prod_{i=1}^{k} e^{t a_i X_i}\right) \\
&= \prod_{i=1}^{k} E\left(e^{t a_i X_i}\right) \\
&= \prod_{i=1}^{k} M_{X_i}(t a_i) \\
&= \prod_{i=1}^{k} e^{a_i \mu_i t + \frac{\sigma_i^2 a_i^2 t^2}{2}} \\
&= e^{\left(\sum\limits_{i=1}^{k} a_i \mu_i\right) t + \frac{\left(\sum\limits_{i=1}^{k} a_i^2 \sigma_i^2\right) t^2}{2}},
\end{aligned}
$$

which is the m.g.f. of $\mathrm{N}\left(\sum_{i=1}^{k} a_i\mu_i, \sum_{i=1}^{k} a_i^2\sigma_i^2\right)$.

(b) Let $Z_i = \frac{X_i-\mu_i}{\sigma_i}, i = 1, \ldots, k$. Then

$$Z_1, Z_2, \ldots, Z_k \text{ are i.i.d. N}(0,1) \text{ r.v.s.}$$

$$\Rightarrow Z_1^2, Z_2^2, \ldots, Z_k^2 \text{ are i.i.d. } \chi_1^2 \text{ r.v.s.}$$

$$\Rightarrow \sum_{i=1}^{k} Z_i^2 \sim \chi_k^2.$$

Result 7 :

Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample (i.i.d.) from $\mathrm{N}(\mu, \sigma^2)$ distribution, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $\bar{X} = \frac{1}{n} \sum\limits_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum\limits_{i=1}^{n} (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively. Then

(i) $\bar{X} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n})$;

(ii) $\bar{X}$ and S^2 are independently distributed;

(iii) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$;

(iv) $E(S^2) = \sigma^2$ and $\mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$.

Proof.

(i) Follows from Result 6 (a).

(ii) Let $Y_i = X_i - \bar{X}, i = 1, \ldots, n$ and $\underline{Y} = (Y_1, \ldots, Y_n)$. Then

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n}(X_i - \bar{X}) = \sum_{i=1}^{n} X_i - n\bar{X} = 0$$

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n} Y_i^2.$$

The joint m.g.f. of $(\underline{Y}, \bar{X})$ is given by

$$M_{\underline{Y},\bar{X}}(\underline{u}, v) = E\left(e^{\sum_{i=1}^{n} u_i Y_i + v\bar{X}}\right), \ \underline{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n, \ v \in \mathbb{R}.$$

$$
\begin{aligned}
\sum_{i=1}^{n} u_i Y_i + v\bar{X} &= \sum_{i=1}^{n} u_i (X_i - \bar{X}) + v\bar{X} \\
&= \sum_{j=1}^{n} u_j X_j + \frac{\left(v - \sum_{i=1}^{n} u_i\right)}{n} \sum_{j=1}^{n} X_j \\
&= \sum_{j=1}^{n} \left(u_j - \bar{u} + \frac{v}{n}\right) X_j \\
&= \sum_{j=1}^{n} t_j X_j,
\end{aligned}
$$

where $\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$, and $t_j = u_j - \bar{u} + \frac{v}{n}$, $j = 1, \ldots, n$. Note that $\sum_{i=1}^{n} (u_i - \bar{u}) = 0$ and therefore

$$\sum_{j=1}^{n} t_j = \sum_{j=1}^{n}\left(u_j - \bar{u} + \frac{v}{n}\right) = v$$

$$\sum_{j=1}^{n} t_j^2 = \sum_{j=1}^{n}\left(u_j - \bar{u} + \frac{v}{n}\right)^2 = \sum_{i=1}^{n}(u_i - \bar{u})^2 + \frac{v^2}{n}.$$

Consequently

$$\begin{aligned}
M_{\underline{Y},\bar{X}}(\underline{u}, v) &= E\left(e^{\sum\limits_{i=1}^{n} u_i Y_i + v\bar{X}}\right) \\
&= E\left(e^{\sum\limits_{j=1}^{n} t_j X_j}\right) \\
&= \prod_{j=1}^{n} M_{X_j}(t_j) \\
&= \prod_{j=1}^{n} e^{\mu t_j + \frac{\sigma^2 t_j^2}{2}}
\end{aligned}$$

$$= \quad e^{\mu \sum_{j=1}^{n} t_j + \frac{\sigma^2}{2} \sum_{j=1}^{n} t_j^2}$$

$$= \quad e^{\mu v + \frac{\sigma^2 v^2}{2n} + \frac{\sigma^2}{2} \sum_{i=1}^{n} (u_i - \bar{u})^2}, \ \underline{u} \in \mathbb{R}^n, \ v \in \mathbb{R}$$

$$M_{\underline{Y}}(\underline{u}) = M_{\underline{Y},\bar{X}}(\underline{u}, 0) = e^{\frac{\sigma^2}{2} \sum_{i=1}^{n} (u_i - \bar{u})^2}, \ \underline{u} \in \mathbb{R}^n$$

$$M_{\bar{X}}(v) = \mathrm{M}_{\underline{Y},\bar{X}}(0, v) = e^{\mu v + \frac{\sigma^2 v^2}{2n}}, \ v \in \mathbb{R}.$$

Clearly

$$M_{\underline{Y},\bar{X}}(\underline{u}, v) = M_{\underline{Y}}(\underline{y}) M_{\bar{X}}(v), \forall \ (\underline{u}, v) \in \mathbb{R}^{n+1}$$

$$\Rightarrow \underline{Y} = (X_1 - \bar{X}, \ldots, X_n - \bar{X}) \text{ and } \bar{X} \text{ are independent.}$$

$$\Rightarrow \sum_{i=1}^{n} (X_1 - \bar{X})^2 \text{ and } \bar{X} \text{ are independent.}$$

$$\Rightarrow S^2 \text{ and } \bar{X} \text{ are independent.}$$

(iii) Let $Z_i = \frac{X_i - \mu}{\sigma}, i = 1, \ldots, n,\ Z = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}$ and
$Y = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2}$.
Then $Z_1, \ldots, Z_n$ are i.i.d $\mathrm{N}(0, 1)$ r.v.s and $Z \sim \mathrm{N}(0, 1)$. Let

$$W = Z^2 = \frac{n(\bar{X} - \mu)^2}{\sigma^2} \quad (\text{so that } W \sim \chi_1^2)$$

and

$$T = \sum_{i=1}^{n} Z_i^2 \quad (\text{so that } T \sim \chi_n^2).$$

Then

$$\begin{aligned}
T &= \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)^2 \\
&= \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{\sigma^2}+\frac{n(\bar{X}-\mu)^2}{\sigma^2} \\
&= Y+W
\end{aligned}$$

By (ii) Y and W are independently distributed. Thus

$$\begin{aligned}
M_T(t) &= M_Y(t)M_W(t) \\
(1-2t)^{-\frac{n}{2}} &= M_Y(t)\times(1-2t)^{-\frac{1}{2}},\ t<\frac{1}{2} \\
\Rightarrow\ M_Y(t) &= 1-2t)^{-\frac{(n-1)}{2}},\ t<\frac{1}{2},
\end{aligned}$$

which is the m.g.f.of χ^2_{n-1} r.v.

(iv)

$$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = E(\chi^2_{n-1}) = n-1$$

$$\Rightarrow E(S^2) = \sigma^2.$$

Moreover

$$\mathrm{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\Rightarrow \mathrm{Var}(S^2) = \frac{2}{n-1}\sigma^4.$$

Take Home Problems

1. Let $-\infty < \alpha < \beta < \infty$ and let X be an A.C. r.v. such that $P(\alpha \leq X \leq \beta) = 1$. Show that $X \sim \mathrm{U}(\alpha, \beta)$ iff $P(X \in I) = P(X \in J)$ for any pair of intervals $I, J \subseteq [\alpha, \beta)$ having the same length.
2. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample (i.i.d.) from a population (distribution) having mean $\mu \in (-\infty, \infty)$ and variance $\sigma^2 > 0$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively. Show that $E(\bar{X}) = \mu$ and $E(S^2) = \sigma^2$.

Thank you for your patience