# MSO 201a: Probability and Statistics
## 2016-2017-II Semester
## Assignment-IX

### A. Illustrative Discussion Problems

1. Let $\underline{X} = (X_1, X_2, X_3)'$ be a discrete random vector with p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{x_1 x_2 x_3}{72}, & \text{if } (x_1, x_2, x_3) \in \{1,2\} \times \{1,2,3\} \times \{1,3\} \\ 0, & \text{otherwise} \end{cases}.$$

(i) Let $Y_1 = 2X_1 - X_2 + 3X_3$ and $Y_2 = X_1 - 2X_2 + X_3$. Find the correlation coefficient between $Y_1$ and $Y_2$;
(ii) For a fixed $x_2 \in \{1, 2, 3\}$, find $E(Y|X_2 = x_2)$and $\mathrm{Var}(Y|X_2 = x_2)$,where $Y = X_1X_3$.
(iii) Find $E(Y)$ and $\mathrm{Var}(Y)$, where $Y = X_1X_3$.

2. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{x_1+2x_2}{18}, & \text{if } x_1 = 1, 2,\ x_2 = 1, 2 \\ 0, & \text{otherwise} \end{cases}.$$

Determine the conditional mean and conditional variance of $X_2$ given $X_1 = x_1$; $x_1 = 1, 2$.

3. Let the r.v. $N$ denote the number of customers visiting a departmental store on a given day. Assume that customers visit the departmental store independent of each other. Each customer visiting the store spends a random amount of money (independent of number of customers visiting the store) having mean 1000 and variance 400. Suppose that $E(N) = 100$, $\mathrm{Var}(N) = 20$ and let $Z$ denote the random variable denoting the total sale of the store on a given day. Find the mean and the variance of $Z$.

4. Let $(X, Y)$ be a random vector such that the p.d.f. of $X$ is

$$f_X(x) = \begin{cases} 4x(1 - x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0, 1)$, the conditional p.d.f. of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2}, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find $E(X|Y=\frac{1}{2})$ and $\mathrm{Var}(X|Y=\frac{1}{2})$.

5. Let $X$ and $Y$ be random variables with means 0 and variances 1 and $\mathrm{Corr}(X,Y)=\rho$. Show that $E\left(\max(X^2,Y^2)\right)\le 1+\sqrt{1-\rho^2}$ (Hint: $\max(a,b)=\frac{a+b+|a-b|}{2}$).

6. Let $X_1,\ldots,X_n$ be random variables and let $p_1,\ldots,p_n$ be positive real numbers with $\sum_{i=1}^n p_i=1$. Prove that:
(a) $\sqrt{\mathrm{Var}(\sum_{i=1}^n p_iX_i)}\le\sum_{i=1}^n p_i\sqrt{\mathrm{Var}(X_i)}\le\sqrt{\sum_{i=1}^n p_i\mathrm{Var}(X_i)}$;
(b) $\mathrm{Var}(\frac{\sum_{i=1}^n X_i}{n})\le\frac{1}{n}\sum_{i=1}^n\mathrm{Var}(X_i)$.

7. Let the r.v. $\underline{X}=(X_1,X_2)'$ have the p.d.f.

$$f(x_1,x_2)=\begin{cases}\frac{1}{2x_1^2x_2}, & \text{if } 1<x_1<\infty,\ \frac{1}{x_1}<x_2<x_1\\ 0, & \text{otherwise}\end{cases}.$$

(i) Find the marginal p.d.f.s of $X_1$ and $X_2$;
(ii) For a fixed $x_2>0$, find the conditional mean and conditional variance of $X_1$ given $X_2=x_2$;

8. Suppose that the random vector $(Y,Z)$ has m.g.f.

$$M_{Y,Z}(t_1,t_2)=\frac{e^{\frac{t_1^2}{1-2t_2}}}{1-2t_2},\quad -\infty<t_1<\infty,\ -\infty<t_2<\frac{1}{2}.$$

Find $\mathrm{Corr}(Y,Z)$. Are $Y$ and $Z$ independent?

9. Let $\underline{X}=(X_1,X_2)'$ have the joint m.g.f.

$$M(t_1,t_2)=e^{t_1+2t_2+\frac{t_1^2+t_2^2+2\rho t_1t_2}{2}},\quad (t_1,t_2)\in\mathbb{R}^2.$$

(a) If $\mathrm{Corr}(X_1,X_2)=0$, can you say that $X_1$ and $X_2$ are independent?
(b) Let $Y_1=X_1+X_2$ and $Y_2=X_1-X_2$. Are $Y_1$ and $Y_2$ independent?

10. Let $X_1,\ldots,X_n$ be a random sample and, for $r=1,\ldots,n$, let $X_{r:n}$ denote the $r$-th smallest of $\{X_1,\ldots,X_n\}$ so that $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ are order statistics corresponding to random sample $X_1,\ldots,X_n$.
(a) If $X_1$ is absolutely continuous type then show that $P(\{X_1<X_2<\cdots<X_n\})=P(\{X_{\beta_1}<X_{\beta_2}<\cdots<X_{\beta_n}\})=\frac{1}{n!}$, for any permutation $(\beta_1,\ldots,\beta_n)$ of $(1,\ldots,n)$;
(b) If $X_1$ is absolutely continuous then show that $P(\{X_i=X_{r:n}\})=\frac{1}{n},\ i=1,\ldots,n$;
(c) Show that $E(X_i|\sum_{j=1}^n X_j=t)=\frac{t}{n},\ i=1,\ldots,n$.

11. Let $X_1,\ldots,X_n$ be a random sample of absolutely continuous random variables. If the expectation of $X_1$ is finite and the distribution of $X_1$ is symmetric about

$\mu \in (-\infty, \infty)$ then show that:

(a) $X_{r:n} - \mu \stackrel{d}{=} \mu - X_{n-r+1:n},\ r = 1, \ldots, n$;

(b) $E(X_{r:n} + X_{n-r+1:n}) = 2\mu,\ r = 1, \ldots, n$;

(c) $E(X_{\frac{n+1}{2}:n}) = \mu$, if $n$ is odd;

(d) $P(X_{\frac{n+1}{2}:n} > \mu) = \frac{1}{2}$, if $n$ is odd.

### B. Practice Problems from the Text Book

**Chapter 2: Multivariate Distributions, Problem Nos.:** 3.1, 3.7, 3.8, 3.12, 4.2, 4.3, 4.11, 5.2, 5.6, 5.12, 5.13, 6.1, 6.3, 6.9

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Solutions

$\boxed{\text{Problem No. 1}}$ (i) Clearly $f_{\underline{X}}(x_1, x_2, x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)$, $\forall\, \underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ with

$$f_{X_1}(x) = \begin{cases} \frac{x}{3}, & \text{if } x = 1, 2 \\ 0, & \text{otherwise} \end{cases}; \quad f_{X_2}(x) = \begin{cases} \frac{x}{6}, & \text{if } x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \quad \& \quad f_{X_3}(x) = \begin{cases} \frac{x}{4}, & \text{if } x = 1, 3 \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow X_1, X_2, X_3$ are independent $\Rightarrow \mathrm{Cov}(X_i, X_j) = 0 \;\forall\, i \neq j$

$$\mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(2X_1 - X_2 + 3X_3,\; X_1 - 2X_2 + X_3)$$
$$= 2\,\mathrm{Var}(X_1) + 2\,\mathrm{Var}(X_2) + 3\,\mathrm{Var}(X_3)$$

$$\mathrm{Var}(Y_1) = \mathrm{Var}(2X_1 - X_2 + 3X_3) = 4\,\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 9\,\mathrm{Var}(X_3).$$

$$\mathrm{Var}(Y_2) = \mathrm{Var}(X_1 - 2X_2 + X_3) = \mathrm{Var}(X_1) + 4\,\mathrm{Var}(X_2) + \mathrm{Var}(X_3)$$

$$E(X_1) = \sum_{x=1,2} \frac{x^2}{3} = \frac{5}{3};\; E(X_1^2) = \sum_{x=1,2} \frac{x^3}{3} = 3;\; \mathrm{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{2}{9}$$

$$E(X_2^2) = \sum_{x=1,2,3} \frac{x^2}{6} = \frac{7}{3};\; E(X_2^2) = \sum_{x=1,2,3} \frac{x^3}{6} = 6,\; \mathrm{Var}(X_2) = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}$$

$$E(X_3^2) = \sum_{x=1,3} \frac{x^2}{4} = \frac{5}{2};\; E(X_3^2) = \sum_{x=1,3} \frac{x^3}{4} = 7,\; \mathrm{Var}(X_3) = 7 - \left(\frac{5}{2}\right)^2 = \frac{3}{4}$$

$$\mathrm{Cov}(Y_1, Y_2) = \frac{4}{9} + \frac{10}{9} + \frac{9}{4} = \frac{137}{36};\; \mathrm{Var}(Y_1) = \frac{8}{9} + \frac{5}{9} + \frac{27}{4} = \frac{295}{36}$$

$$\mathrm{Var}(Y_2) = \frac{2}{9} + \frac{20}{9} + \frac{3}{4} = \frac{115}{36}$$

$$\mathrm{Corr}(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\,\mathrm{Var}(Y_2)}} = \frac{137}{\sqrt{295}\,\sqrt{115}} = 0.7438\ldots$$

(ii) & (iii) Since $X_1, X_2, X_3$ are independent

$$E(Y \mid X_2 = x_2) = E(X_1 X_3 \mid X_2 = x_2) = E(X_1 X_3) = E(X_1)\,E(X_3) = \frac{25}{6}$$

$$\mathrm{Var}(Y \mid X_2 = x_2) = \mathrm{Var}(X_1 X_3 \mid X_2 = x_2) = \mathrm{Var}(X_1 X_3) = \mathrm{Var}(Y)$$

$$E(Y^2) = E(X_1^2 X_3^2) = E(X_1^2)\,E(X_3^2) = 21$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = 21 - \frac{625}{36} = \frac{131}{36}$$

$$\mathrm{Var}(Y \mid X_2 = x_2) = \mathrm{Var}(Y) = \frac{131}{136}.$$

$\boxed{\text{Problem No. 2}}$ $P(X_1 = 1) = \frac{4}{9}$, $P(X_1 = 2) = \frac{5}{9}$; $P(X_2 = 1) = \frac{7}{18}$; $P(X_2 = 2) = \frac{11}{18}$

$$P(X_2 = x_2 \mid X_1 = 1) = \begin{cases} 3/8, & \text{if } x_2 = 1 \\ 5/8, & \text{if } x_2 = 2 \\ 0, & \text{otherwise} \end{cases}; \quad P(X_2 = x_2 \mid X_1 = 2) = \begin{cases} 2/5, & \text{if } x_2 = 1 \\ 3/5, & \text{if } x_2 = 2 \\ 0, & \text{otherwise} \end{cases}$$

$E(X_2 \mid X_1 = 1) = \frac{13}{8}$, $E(X_2^2 \mid X_1 = 1) = \frac{23}{8}$, $\mathrm{Var}(X_2 \mid X_1 = 1) = \frac{15}{64}$

$E(X_2 \mid X_1 = 2) = \frac{8}{15}$, $E(X_2^2 \mid X_1 = 2) = \frac{14}{5}$, $\mathrm{Var}(X_2 \mid X_1 = 2) = \frac{6}{25}$.

**Problem No. 3** Let $X_i$ denote the money spent by $i$-th customer visiting the store, $i=1,2,\ldots$

Then $X_1, X_2, \ldots$ are i.i.d. and $N, X_1, X_2, \ldots$ are independent. We have $Z = \sum_{i=1}^{N} X_i$. Thus

$$E(Z) = E\left(\sum_{i=1}^{N} X_i\right) = E\left(E\left(\sum_{i=1}^{N} X_i \mid N\right)\right) = E\left(E\left(\sum_{i=1}^{N} X_i \mid N\right)\right)$$

$= E\left(\sum_{i=1}^{N} E(X_i)\right)$ (since $X_i$ and $N$ are independent)

$= E(1000 N)$ ($X_i$'s are i.i.d. with $E(X_1)=1000$)

$= 1000\, E(N) = 1000 \times 100 = 100000.$

$Var(Z) = Var\left(\sum_{i=1}^{N} X_i\right) = Var\left(E\left(\sum_{i=1}^{N} X_i \mid N\right)\right) + E\left(Var\left(\sum_{i=1}^{N} X_i \mid N\right)\right)$

$= Var\left(\sum_{i=1}^{N} E(X_i \mid N)\right) + E\left(\sum_{i=1}^{N} Var(X_i \mid N)\right)$ (given $N$, $X_i$'s are independent)

$= Var\left(\sum_{i=1}^{N} E(X_i)\right) + E\left(\sum_{i=1}^{N} Var(X_i)\right)$ ($N$ and $X_i$ are independent)

$= Var(1000 N) + E(400 N)$ ($X_i$'s are identically distributed with $E(X_1)=1000$ and $Var(X_1)=400$)

$= 1000^2\, Var(N) + 400\, E(N) = 1000^2 \times 20 + 400 \times 100.$

**Problem No. 4** The joint p.d.f. of $(X, Y)$ is: $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$

$$= \begin{cases} 8xy, & \text{if } 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal p.d.f. of $Y$ is $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^y 8xy\,dx & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$$= \begin{cases} 4y^3 & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

For fixed $y \in (0,1)$, the conditional p.d.f. of $X$ given $Y=y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{y^2}, & \text{if } 0<x<y \\ 0, & \text{otherwise} \end{cases}$$

[Alternatively: For fixed $y \in (0,1)$, $f_{X|Y}(x|y) \propto f_{X,Y}(x,y)$ (as a function of $x$)

$= \begin{cases} c(y)\, x, & \text{if } 0<x<y \\ 0, & \text{otherwise} \end{cases}$ ; $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = 1 \Rightarrow c(y) = \frac{2}{y^2}$.]

$E(X^r \mid Y=y) = \frac{2}{y^2} \int_0^y x^{r+1}\,dx = \frac{2y^r}{r+2}, \quad r>-1$

$E(X \mid Y=y) = \frac{2y}{3} \Rightarrow E(X \mid Y=\tfrac{1}{2}) = \frac{1}{3}$

$E(X^2 \mid Y=y) = \frac{y^2}{2} \Rightarrow E(X \mid Y=\tfrac{1}{2}) = \frac{1}{8}$; $Var(X \mid Y=y) = E(X^2 \mid Y=y) - (E(X \mid Y=y))^2$

$= \frac{y^2}{2} - \frac{4y^2}{9} = \frac{y^2}{18} \Rightarrow Var(X \mid Y=\tfrac{1}{2}) = \frac{1}{72}.$

$\boxed{2/6}$

$\boxed{\text{Problem No. 5}}$ $E(X^2) = E(Y^2) = 1$, $\rho = E(XY)$

$$\max(X^2, Y^2) = \frac{X^2+Y^2+|X^2-Y^2|}{2}$$

$$E(\max(X^2, Y^2)) = 1 + \frac{1}{2} E(|X^2-Y^2|)$$

$$E(|X^2-Y^2|) = E(|X-Y|\,|X+Y|) \le \sqrt{E(|X-Y|^2)\,E(|X+Y|^2)}$$

(Cauchy- Schwarz Inequality)

$$E(|X \pm Y|^2) = E((X\pm Y)^2) = E(X^2) + E(Y^2) \pm 2E(XY) = 2(1 \pm \rho)$$

$$\Rightarrow E(|X^2-Y^2|) \le 2\sqrt{1-\rho^2} \Rightarrow E(\max(X^2, Y^2)) \le 1 + \sqrt{1-\rho^2}.$$

$\boxed{\text{Problem No. 6}}$ (a) Note that $\mathrm{Corr}(X_i, X_j) \le 1 \quad \forall i \ne j$

$$\Rightarrow \mathrm{Cov}(X_i, X_j) \le \sqrt{\mathrm{Var}(X_i)\,\mathrm{Var}(X_j)}, \quad \forall i \ne j$$

$$\mathrm{Var}\left(\sum_{i=1}^n p_i X_i\right) = \sum_{i=1}^n p_i^2 \mathrm{Var}(X_i) + 2 \sum\sum_{1\le i<j\le n} p_i p_j \mathrm{Cov}(X_i, X_j)$$

$$\le \sum_{i=1}^n p_i^2 \mathrm{Var}(X_i) + 2 \sum\sum_{1\le i<j\le n} p_i p_j \sqrt{\mathrm{Var}(X_i)}\sqrt{\mathrm{Var}(X_j)}$$

$$= \left(\sum_{i=1}^n p_i \sqrt{\mathrm{Var}(X_i)}\right)^2$$

$$\Rightarrow \sqrt{\mathrm{Var}\left(\sum_{i=1}^n p_i X_i\right)} \le \sum_{i=1}^n p_i \sqrt{\mathrm{Var}(X_i)}$$

For proving the other inequality consider a r.v. with p.m.f.

$$f_Y(y) = \begin{cases} p_i & \text{if } y = a_i \\ 0, & \text{otherwise} \end{cases} \quad i = 1, 2, \ldots, n, \quad \left(p_i > 0, \ \sum_{i=1}^n p_i = 1 \Rightarrow f_Y \text{ is a p.m.f.}\right)$$

for some non-negative constants $a_1, a_2, \ldots, a_n$

Then

$$\mathrm{Var}(\sqrt{Y}) \ge 0 \Rightarrow E(Y) \ge (E(\sqrt{Y}))^2 \Rightarrow \sum_{i=1}^n a_i p_i \ge \left(\sum_{i=1}^n \sqrt{a_i}\, p_i\right)^2.$$

Now take $a_i = \mathrm{Var}(X_i)$ $i = 1, \ldots, n$ to get the desired inequality.

(b) Take $p_i = \frac{1}{n}$, $i = 1, \ldots, n$ in (a).

$\boxed{\text{Problem No. 7}}$

(a) The marginal p.m.f. of $X_1$ is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 = \begin{cases} \int_{1/x_1}^{x_1} \frac{1}{2x_1^2 x_2} dx_2, & \text{if } x_1 > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{\ln x_1}{x_1^2}, & \text{if } x_1 > 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal p.d.f. of $X_2$ is

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1 = \begin{cases} \int_{\max\{x_2, \frac{1}{x_2}\}}^{\infty} \frac{1}{2x_1^2 x_2}\,dx_1, & \text{if } x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2\max(x_2^2, 1)}, & \text{if } x_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) For fixed $x_2 > 0$, the conditional p.d.f of $X_1$ given $X_2 = x_2$ is

$$f_{X_1|X_2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \begin{cases} \frac{1}{x_1^2}\max(x_2, \frac{1}{x_2}), & \text{if } x_1 > \max(x_2, \frac{1}{x_2}) \\ 0, & \text{otherwise} \end{cases}$$

$E(X_1 | X_2 = x_2)$ and $E(X_1^2 | X_2 = x_2)$ are therefore not finite.
Hence $Var(X_1|X_2 = x_2)$ also does not exist.

**Problem No. 8** $\psi(t_1, t_2) = \ln M_{Y,Z}(t_1, t_2) = \frac{t_1^2}{1-2t_2} - \ln(1-2t_2)$, $t_1 \in \mathbb{R}$, $t_2 < \frac{1}{2}$

$$\frac{\partial}{\partial t_1}\psi(t_1, t_2) = \frac{2t_1}{1-2t_2}, \quad \frac{\partial^2}{\partial t_2 \partial t_1}\psi(t_1, t_2) = \frac{4t_1}{(1-2t_2)^2}, \quad t_1 \in \mathbb{R}, \ t_2 < \frac{1}{2}$$

$$E(Y) = \left[\frac{\partial}{\partial t_1}\psi(t_1, t_2)\right]_{(t_1, t_2) = (0, 0)} = 0$$

$$Cov(Y, Z) = \left[\frac{\partial^2}{\partial t_2 \partial t_1}\psi(t_1, t_2)\right]_{(t_1, t_2) = (0, 0)} = 0$$

$\Rightarrow Corr(Y, Z) = 0.$

The marginal m.g.f.s of $Y$ and $Z$ are

$$M_Y(t_1) = M_{Y,Z}(t_1, 0) = e^{t_1^2}, \quad -\infty < t_1 < \infty$$

$$M_Z(t_2) = M_{Y,Z}(0, t_2) = \frac{1}{1-2t_2}, \quad t_2 < \frac{1}{2}$$

Since

$$M_{Y,Z}(t_1, t_2) \neq M_Y(t_1) M_Z(t_2), \quad \forall\, t_1 \in \mathbb{R}, \ t_2 < \frac{1}{2}$$

$Y$ and $Z$ are not independent.

**Problem No. 9** $\psi(t_1, t_2) = \ln M(t_1, t_2) = t_1 + 2t_2 + \frac{t_1^2 + t_2^2 + 2\rho t_1 t_2}{2}$, $(t_1, t_2) \in \mathbb{R}^2$

$$\frac{\partial}{\partial t_1}\psi(t_1, t_2) = 1 + t_1 + \rho t_2, \quad \frac{\partial^2}{\partial t_2 \partial t_1}\psi(t_1, t_2) = \rho$$

$$Cov(X_1, X_2) = \left[\frac{\partial^2}{\partial t_2 \partial t_1}\psi(t_1, t_2)\right]_{(t_1, t_2) = (0, 0)} = \rho$$

(a) $Corr(X_1, X_2) = 0 \Rightarrow Cov(X_1, X_2) = 0 \Rightarrow \rho = 0$

The marginal m.g.f.s of $X_1$ and $X_2$ are

$M_{X_1}(t_1) = M(t_1, 0) = e^{t_1 + \frac{t_1^2}{2}}$, $t_1 \in \mathbb{R}$

$M_{X_2}(t_2) = M(0, t_2) = e^{2t_2 + \frac{t_2^2}{2}}$, $t_2 \in \mathbb{R}$

Since

$M(t_1, t_2) \neq M_{X_1}(t_1)\, M_{X_2}(t_2)$, $\forall (t_1, t_2) \in \mathbb{R}$,

$X_1$ and $X_2$ are not independent.

(b) The joint m.g.f. of $(Y_1, Y_2)$ is

$$M_{Y_1,Y_2}(t_1, t_2) = E(e^{t_1Y_1 + t_2Y_2}) = E(e^{(t_1+t_2)X_1 + (t_1-t_2)X_2})$$

$$= M(t_1+t_2,\ t_1-t_2)$$

$$= e^{3t_1 + (1+\rho)t_1^2} \times e^{-t_2 + (1-\rho)t_2^2}, \quad (t_1, t_2) \in \mathbb{R}$$

$$= M_{Y_1,Y_2}(t_1, 0) \times M_{Y_1,Y_2}(0, t_2)$$

$$= M_{Y_1}(t_1)\, M_{Y_2}(t_2), \quad (t_1, t_2) \in \mathbb{R}$$

$\Rightarrow Y_1$ and $Y_2$ are independent.

**Problem 10** Let $S_n$ denote the set of all permutations of $(1, \ldots, n)$. Then

$X_1, \ldots, X_n$ are iid $\Rightarrow (X_1, \ldots, X_n) \stackrel{d}{=} (X_{\beta_1}, \ldots, X_{\beta_n})$, $\forall (\beta_1, \ldots, \beta_n) \in S_n$. .... $(A_1)$

(a) Using $(A_1)$ we have

$P(X_1 < \cdots < X_n) = P(X_{\beta_1} < \cdots < X_{\beta_n})$, $\forall (\beta_1, \ldots, \beta_n) \in S_n$.

$X_1$ is A.c. $\Rightarrow \underline{X} = (X_1, \ldots, X_n)$ is A.c. (since $X_1, \ldots, X_n$ are iid)

$\Rightarrow P(X_{1:n} < X_{2:n} < \cdots < X_{n:n}) = 1$

$$\Rightarrow \sum_{(\beta_1, \ldots, \beta_n) \in S_n} P(X_{\beta_1} < \cdots < X_{\beta_n}) = 1 \quad \ldots\ldots (A_2)$$

$\Rightarrow P(X_1 < \cdots < X_n) = P(X_{\beta_1} < \cdots < X_{\beta_n}) = \frac{1}{\underline{|n}}$, $\forall (\beta_1, \ldots, \beta_n) \in S_n$

(using $(A_1)$ and $(A_2)$)

(b) Since $(A_1)$ holds,

$P(X_i = X_{r:n}) = P(X_i = r\text{-th smallest of } X_1, \ldots, X_n)$

$= P(X_{\beta_i} = r\text{-th smallest of } X_{\beta_1}, \ldots, X_{\beta_n})$

$= P(X_{\beta_i} = r\text{-th smallest of } X_1 \ldots X_n)$ ($r$-th smallest of $X_{\beta_1}, \ldots, X_{\beta_n}$ $= r$-th smallest of $X_1 \ldots X_n$)

$= P(X_{\beta_i} = X_{r:n})$, $i = 1 \ldots n$, $\forall \underline{\beta} \in S_n$

$\Rightarrow P(X_i = X_{r:n}) = P(X_j = X_{r:n})$, $\forall i, j = 1 \ldots n$ .... $(A_3)$

Since $\underline{X}$ is A.C.

$$\sum_{i=1}^{n} P(X_i = X_{r:n}) = 1$$

$$\Rightarrow \quad P(X_i = X_{r:n}) = \frac{1}{n}, \quad i=1 \ldots n \quad \text{(using } (A_3)\text{)}$$

(c) Since $(A_1)$ holds

$$E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = E\left(X_{\beta_i} \,\middle|\, \sum_{j=1}^{n} X_{\beta_j} = t\right) = E\left(X_{\beta_i} \,\middle|\, \sum_{j=1}^{n} X_j = t\right), \ \forall \underline{\beta}$$

$$\left(\text{since } \sum_{j=1}^{n} X_{\beta_j} = \sum_{j=1}^{n} X_j\right)$$

$$\Rightarrow E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = E\left(X_1 \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = c(t), \text{ say}$$

$$\Rightarrow \quad \sum_{i=1}^{n} E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = n\,c(t)$$

$$\Rightarrow \quad E\left(\sum_{i=1}^{n} X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = n\,c(t) \quad \Rightarrow \quad t = n\,c(t)$$

$$\Rightarrow E\left(X_i \,\middle|\, \sum_{j=1}^{n} X_j = t\right) = c(t) = \frac{t}{n}, \quad i=1, \ldots, n$$

$\boxed{\text{Problem No. 11}}$ $X_1, X_2, \ldots, X_n$ are i.i.d and $X_1 - \mu \stackrel{d}{=} \mu - X_1$

$$\Rightarrow (X_1-\mu, \ldots, X_n-\mu) \stackrel{d}{=} (\mu - X_1, \ldots, \mu - X_n)$$

$\Rightarrow$ $r$-th smallest of $\{X_1-\mu, \ldots, X_n-\mu\}$

$\stackrel{d}{=}$ $r$-th smallest of $\{\mu-X_1, \ldots, \mu-X_n\}$

$$\Rightarrow \quad X_{r:n} - \mu \stackrel{d}{=} \mu - X_{n-r+1:n} \quad r=1 \ldots n$$

(b) By (a), for $r=1 \ldots n$, we have

$$E(X_{r:n}-\mu) = E(\mu - X_{n-r+1:n}) \Rightarrow E(X_{r:n} + X_{n-r+1:n}) = 2\mu.$$

(c) Take $r = \frac{n+1}{2}$ in (b)

(d) Taking $r=\frac{n+1}{2}$ in (a) we have $X_{\frac{n+1}{2}:n} - \mu \stackrel{d}{=} \mu - X_{\frac{n+1}{2}:n}$

$$\Rightarrow P(X_{\frac{n+1}{2}:n} - \mu > 0) = P(\mu - X_{\frac{n+1}{2}:n} > 0)$$

$$\Rightarrow P(X_{\frac{n+1}{2}:n} > \mu) = P(X_{\frac{n+1}{2}:n} < \mu)$$

Since $\underline{X}$ is A.C., $P(X_{\frac{n+1}{2}:n} = \mu) = 0$. Thus we have

$$P(X_{\frac{n+1}{2}:n} > \mu) = P(X_{\frac{n+1}{2}:n} < \mu) = \frac{1}{2}.$$

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