Module 17

INEQUALITIES

- X: a given r.v. with d.f. $F_X(\cdot)$ and p.m.f/p.d.f. $f_X(\cdot)$;
- $g : \mathbb{R} \to \mathbb{R}$: a given function;
- Inequalities provide useful estimates of probabilities (or moments) when they can not be evaluated precisely;
- In this module we will derive some useful inequalities.

Result 1:

Let $g : \mathbb{R} \to [0, \infty)$ be a non-negative function such that $E(g(X)) < \infty$. Then, for any $c > 0$,

$$P(\{g(X) > c\}) \leq \frac{E(g(X))}{c}.$$

Proof: (For A.C. Case.) Let $A = \{x \in \mathbb{R} : g(x) > c\}$. Then

$$\begin{aligned}
E(g(X)) &= \int_{-\infty}^{\infty} g(x) f_X(x) dx \\
&= \int_A g(x) f_X(x) dx + \int_{A^c} g(x) f_X(x) dx \\
&\geq \int_A g(x) f_X(x) dx \quad (g(\cdot) \geq 0,\ f_X(\cdot) \geq 0) \\
&= \int_{-\infty}^{\infty} g(x) I_A(x) f_X(x) dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \quad c\int_A f_X(x)dx \quad (g(x)I_A(x) \geq cI_A(x), \forall\ x \in \mathbb{R}) \\
&= \quad c\ P(A) \\
&= \quad c\ P(\{g(X) > c\}) \\
\Rightarrow\ P(\{g(X) > c\}) &\leq \quad \frac{E(g(X))}{c}.
\end{aligned}
$$

Notation: For any set $B \subseteq \mathbb{R}$ and any integrable function $h(\cdot)$

$$\int_B h(x)dx = \int_{-\infty}^{\infty} h(x)I_B(x)dx.$$

Corollary 1:

Let $g : [0, \infty) \to [0, \infty)$ be a non-negative and strictly $\uparrow$ function such that $E(g(|X|)) < \infty$. Then for any $c > 0$ such that $g(c) > 0$

$$P(\{|X| > c\}) \leq \frac{E(g(|X|))}{g(c)}.$$

Proof:

$$\begin{aligned} P(\{|X| > c\}) &= P(\{g(|X|) > g(c)\}) \\ &\leq \frac{E(g(|X|))}{g(c)}. \quad \text{(using Result 1)} \end{aligned}$$

Corollary 2: Let $r > 0$ and $c > 0$. Then

$$P(\{|X| > c\}) \leq \frac{E(|X|^r)}{c^r},$$

provided $E(|X|^r) < \infty$.

Corollary 3 (Markov Inequality): Suppose that $E(|X|) < \infty$. Then

$$P(\{|X| > c\}) \leq \frac{E(|X|)}{c}.$$

Proof: Take $r = 1$ in Corollary 2.

Result 2 (Chebyshev Inequality): Let X be a r.v. with finite mean $\mu = E(X)$ and finite variance $\sigma^2 = E((X-\mu)^2)$. Then for any $\epsilon > 0$

$$P\left(\{|X-\mu| > \epsilon\sigma\}\right) \leq \frac{1}{\epsilon^2}$$

or equivalently

$$P\left(\{|X-\mu| \leq \epsilon\sigma\}\right) \geq 1 - \frac{1}{\epsilon^2}.$$

Proof: Using Corollary 2 for $r = 2$, we have

$$\begin{aligned} P\left(\{|X-\mu| > \epsilon\sigma\}\right) &\leq \frac{E(|X-\mu|^2)}{\epsilon^2\sigma^2} \\ &= \frac{Var(X)}{\epsilon^2\sigma^2} = \frac{1}{\epsilon^2}. \end{aligned}$$

Remark 1:

For any probability distribution

$$P\left(\{\mu - 2\sigma \le X \le \mu + 2\sigma\}\right) \ge 1 - \frac{1}{2^2} = 0.75;$$

$$P\left(\{\mu - 3\sigma \le X \le \mu + 3\sigma\}\right) \ge 1 - \frac{1}{3^2} \ge 0.88.$$

Example 1 (Chebyshev's bound is sharp):

Let X be a r.v. with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{8}, & \text{if } x \in \{-1, 1\} \\ \frac{3}{4}, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases} .$$

Then

$$\mu = E(X) = \frac{1}{8} \times -1 + \frac{1}{8} \times 1 + \frac{3}{4} \times 0 = 0;$$

$$\sigma^2 = Var(x) = E(X^2) = \frac{1}{8} \times (-1)^2 + \frac{1}{8} \times 1^2 + \frac{3}{4} \times 0 = \frac{1}{4}.$$

For $\epsilon = \frac{199}{100}$, Chebyshev inequality gives the following inequality

$$P(\{|X - \mu| > \epsilon\sigma\}) \leq \frac{100^2}{199^2} \approx 0.25252.$$

Actual probability is

$$\begin{aligned} P(\{|X-\mu| > \epsilon\sigma\}) &= P\left(\left\{|X| > \frac{199}{200}\right\}\right) \\ &= P(\{X \in \{-1, 1\}\}) \\ &= \frac{1}{4} = 0.25. \end{aligned}$$

Example 2:

Let X be a r.v. with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & \text{if } -\sqrt{3} < x < \sqrt{3} \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$\mu = E(X) = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x}{2\sqrt{3}} dx = 0;$$

$$\sigma^2 = E(X^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \times \frac{1}{2\sqrt{3}} dx = 1.$$

Chebyshev inequality for $\epsilon = \frac{3}{2}$ gives

$$P\left(\left\{|X| > \frac{3}{2}\right\}\right) \leq \frac{4}{9} = 0.444\cdots.$$

Actual probability is

$$\begin{aligned}
P\left(\left\{|X| > \frac{3}{2}\right\}\right) &= 1 - P\left(\left\{-\frac{3}{2} \leq X \leq \frac{3}{2}\right\}\right) \\
&= 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx \\
&= 1 - \frac{\sqrt{3}}{2} = 0.134\cdots
\end{aligned}$$

Here the Chebyshev bound is not that sharp.

Definition 1: Let $-\infty \leq a < b \leq \infty$. A function $\phi : (a, b) \to \mathbb{R}$ is said to be convex (concave) on (a,b) if

$$\phi(\alpha x + (1-\alpha)y) \ \leq \ (\geq)\ \alpha\phi(x) + (1-\alpha)\phi(y),\ \forall\ x, y \in (a, b),\ \alpha \in (0, 1).$$

The function ϕ is said to be strictly convex (strictly concave) if the above inequality is strict.

We state the following standard result without providing its proof.

Result 3: Let $\phi : (a, b) \to \mathbb{R}$ be a given function.

(a) Then, ϕ is convex $\Leftrightarrow -\phi$ is concave;

(b) Let ϕ be differentiable on (a,b) and let ϕ' denote the derivative function . Then ϕ is convex (concave) on (a, b) iff ϕ' is $\uparrow$ $(\downarrow)$ on (a, b);

(c) Let ϕ be twice differentiable on (a, b) and let ϕ'' denote the second derivative function. Then ϕ is convex (concave) on (a, b) iff $\phi''(x) \geq (\leq\ )\ 0,\ \forall\ x \in (a, b)$.

Result 4 (Jensen Inequality):

Let X be a r.v. with support $S_X \subseteq (a,b)$ and let $\phi:(a,b) \to \mathbb{R}$ be a convex (concave) function; here $-\infty \leq a < b \leq \infty$. Then

$$E(\phi(X)) \;\geq\; (\leq)\; \phi(E(X)),$$

provided the expectations exist.

Proof: For simplicity assume that ϕ is differentiable on (a,b). Then $\phi' \uparrow$ on (a,b) and for $x \in S_X$ (note that $\mu = E(X) \in (a,b)$)

$$\phi(x) = \phi(\mu) + (x-\mu)\phi'(\xi),$$

for some ξ between x and μ. Since $\phi' \uparrow$ it follows that $(x-\mu)\phi'(\xi) \geq (x-\mu)\phi'(\mu)$. Therefore

$$\begin{aligned} & \phi(x) \geq \phi(\mu) + (x-\mu)\phi'(\mu), \;\forall\; x \in (a,b) \\ \Rightarrow\quad & \phi(X) \geq \phi(\mu) + (X-\mu)\phi'(\mu) \\ \Rightarrow\quad & E[\phi(X)] \geq \phi(\mu) = \phi(E(X)). \end{aligned}$$

Example 3

(a) $E(|X|) \geq |E(X)|$, ($\phi(x) = |x|$ is convex on $\mathbb{R}$);

(b) $E(X^2) \geq (E(X))^2$, ($\phi(x) = x^2$ is convex on $\mathbb{R}$);

(c) If $P(X \geq 0) = 1$ and $r > 1$, then $E(X^r) \geq (E(X))^r$ (for $r > 1$, $\phi(x) = x^r$ is convex on $[0, \infty)$);

(d) If $P(X > 0) = 1$ and $r < 1$, then $E(X^r) \leq (E(X))^r$ (for $r < 1$, $\phi(x) = x^r$ is concave on $(0, \infty)$);

(e) If $P(X > 0) = 1$, then $E(\ln X) \leq \ln E(X)$ ($\phi(x) = \ln x$ is concave on $(0, \infty)$);

(f) $E(e^X) \geq e^{E(X)}$; ($\phi(x) = e^x$ is convex on $\mathbb{R}$).

Example 4:

For $0 < p < q < \infty$, show that

$$(E(|X|^p))^{\frac{1}{p}} \leq (E(|X|^q))^{\frac{1}{q}}.$$

Solution: Using Example 3 (c)

$$\begin{aligned} E(|Y|^{\frac{q}{p}}) &\geq (E(|Y|))^{\frac{q}{p}} \\ \Rightarrow (E(|X|^q)) &\geq (E(|X|^p))^{\frac{q}{p}} \quad (\text{taking } Y = X^p) \\ \Rightarrow (E(|X|^q))^{\frac{1}{q}} &\geq (E(|X|^p))^{\frac{1}{p}}. \end{aligned}$$

Remark 2: If $E(|X|^q) < \infty$ for some $q > 0$, then $E(|X|^p) < \infty, \forall\ 0 < p < q$.

Take Home Problems

(1) Suppose that $P(X > 0) = 1$. Show that

$$E(X)E(\frac{1}{X}) \geq 1.$$

(2) Let $\omega_i > 0,\ a_i > 0,\ i = 1, \ldots, n$ and let $\sum_{i=1}^{n} \omega_i = 1$. Show that

$$\sum_{i=1}^{n} a_i \omega_i \geq \prod_{i=1}^{n} a_i^{\omega_i} \geq \frac{1}{\sum_{i=1}^{n} \frac{\omega_i}{a_i}}.$$

(AM $\geq$ GM $\geq$ HM).

Abstract of Next Module

- We will introduce a set of numerical measures that provide a summary of prominent features of a probability distribution.

Thank you for your patience