Nonparametric Bayesian Models for Unsupervised Learning

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Probabilistic Machine Learning (CS772A)

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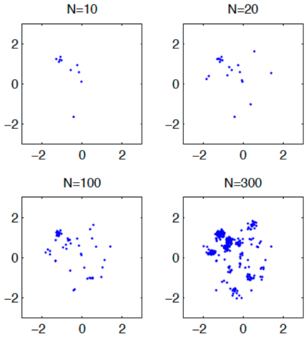
Nonparametric Bayesian Models: Motivation

- Often more/newer structures may appear as we observe more and more data



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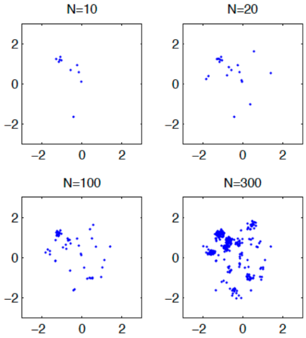

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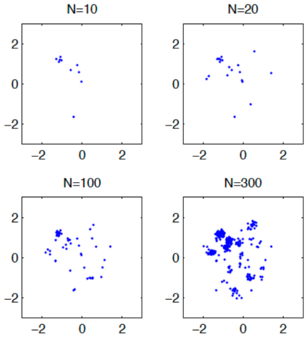

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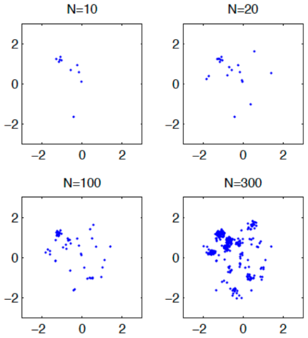

- Would like to have models that don't assume a fixed-sized structure in advance, e.g.,
  - Would like to have a mixture model s.t. number of clusters can grow/adapt with data
  - Would like to have a neural net which can grow/adapt in size (number/width of layers) with data

Nonparametric Bayesian Models for Clustering

Finite Mixture Model

- Data $\mathbf{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N]$, cluster assignments $\mathbf{Z} = [\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N]$, K clusters

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- Integrating out $\boldsymbol{\pi}$, the marginal prior probability of cluster assignments $\mathbf{Z}$

$$
p(\mathbf{Z}|\alpha) = \int p(\mathbf{Z}|\boldsymbol{\pi}) p(\boldsymbol{\pi}|\boldsymbol{\alpha}) d\boldsymbol{\pi}
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$$p(\mathbf{Z}|\alpha) = \frac{1}{B(\frac{\alpha}{K}, \frac{\alpha}{K}, \ldots, \frac{\alpha}{K})} \int \prod_{k=1}^{K} \pi_k^{m_k + \frac{\alpha}{K} - 1} d\boldsymbol{\pi}$$

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where $m_k = \sum_{n=1}^{N} z_{nk}$, $B(\alpha_1, \alpha_2, \ldots, \alpha_K) = \frac{\prod_{k=1}^{K}\Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{K}\alpha_k)}$, Γ is gamma func. $(\Gamma(x+1) = x\Gamma(x))$

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$$p(\mathbf{Z}|\alpha) = \frac{1}{B(\frac{\alpha}{K}, \frac{\alpha}{K}, \dots, \frac{\alpha}{K})} \int \prod_{k=1}^{K} \pi_k^{m_k+\frac{\alpha}{K}-1} d\boldsymbol{\pi} = \frac{B(m_1 + \frac{\alpha}{K}, m_2 + \frac{\alpha}{K}, \dots, m_K + \frac{\alpha}{K})}{B(\frac{\alpha}{K}, \frac{\alpha}{K}, \dots, \frac{\alpha}{K})}$$

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- Using the above, the $p(\mathbf{Z}|\alpha) = p(\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_N|\alpha)$ can be written as

$$p(\mathbf{Z}|\alpha) = \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} \frac{\prod_{k=1}^{K}\Gamma(m_k + \frac{\alpha}{K})}{\Gamma(\frac{\alpha}{K})^K} \qquad \text{(verify)}$$

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$$p(\boldsymbol{z}_n = j|\mathbf{Z}_{-n},\alpha) \quad = \quad \frac{\color{red}{p(\boldsymbol{z}_n = j,\mathbf{Z}_{-n}|\alpha)}}{\color{blue}{p(\mathbf{Z}_{-n}|\alpha)}}$$

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&= \frac{m_{-n,j}+\frac{\alpha}{K}}{N-1+\alpha} \qquad (m_{-n,j} = m_j - 1 \text{ denotes no. of other examples in cluster } j)
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- Thus prior prob. of $\mathbf{z}_n = j$ is prop. to $m_{-n,j}$, i.e., number of other examples assigned to cluster j (this is like a "rich gets richer" phenomenon; a popular cluster will attract more examples)

Finite Mixture Model

- Note that with $\boldsymbol{\pi}$ integrated out, the cluster assignments are not i.i.d. anymore

$$p(z_n = j | \mathbf{Z}_{-n}, \alpha) \quad = \quad \frac{m_{-n,j} + \frac{\alpha}{K}}{N - 1 + \alpha}$$

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- Note: We can also derive the above result as

$$p(\boldsymbol{z}_n = j|\mathbf{Z}_{-n}, \alpha) = \int p(\boldsymbol{z}_n|\boldsymbol{\pi})p(\boldsymbol{\pi}|\mathbf{Z}_{-n}, \alpha)d\boldsymbol{\pi} \qquad \text{(complete the exercise!)}$$

Taking the Infinite Limit..

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- Note that the probability of starting a new cluster is proportional to Dirichlet hyperparam. α

A Metaphor: Chinese Restaurant Process (CRP)

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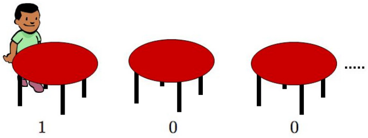

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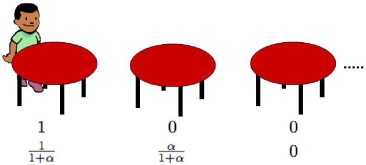

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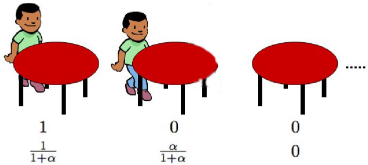

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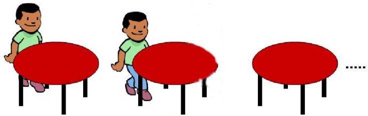

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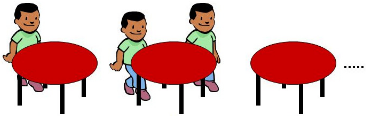

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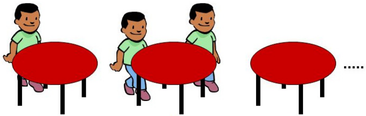

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- Also possible to do collaposed Gibbs sampling by integrating out $\{\mu_j, \Sigma_j\}_{j=1}^{K_+}$ (Neal, 2000)

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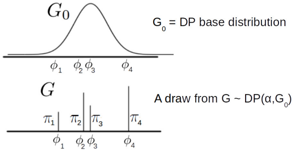

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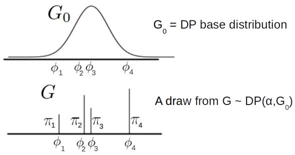

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$$\beta_k \sim \text{Beta}(1, \alpha) \qquad k = 1, \ldots, \infty$$

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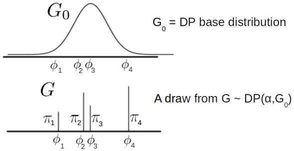

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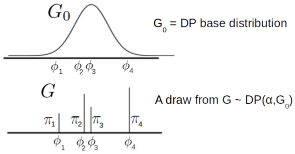

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May refer to "Dirichlet Process" by Teh (2010) for some of these connections

Nonparametric Bayesian Models for Latent Feature Learning

Learning Binary Latent Features

- Suppose we have N observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ and each $\boldsymbol{x}_n$ can be written as

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- Note that each $\boldsymbol{x}_n$ is a binary weighted sum of K basis vectors $\mathbf{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_K]$

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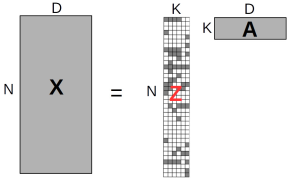

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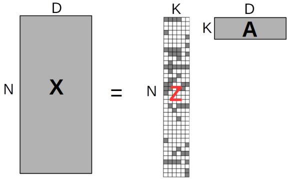

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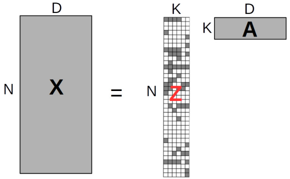

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Binary Matrices with Unbounded Number of Columns

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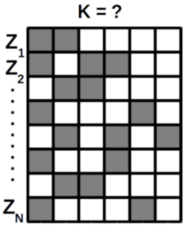

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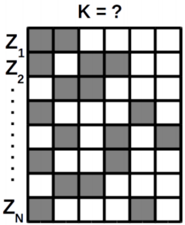

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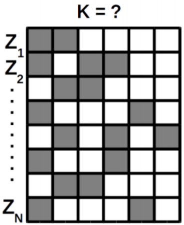

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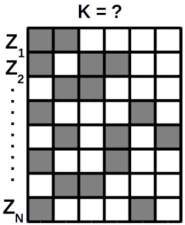

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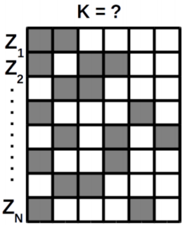

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Modeling Binary Matrices with Finite Many Columns

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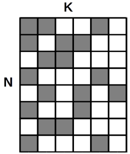

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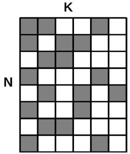

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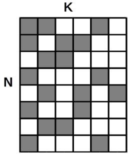

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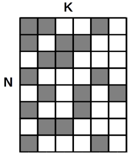

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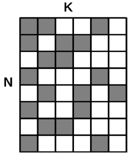

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where $m_{-n,k} = \sum_{i \neq n} z_{ik}$ denotes how many other entries in column k are equal to 1

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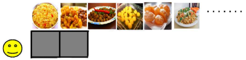

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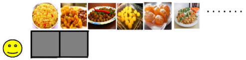

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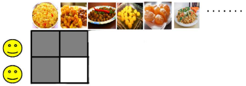

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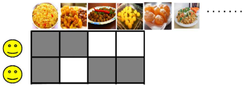

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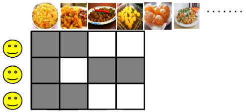

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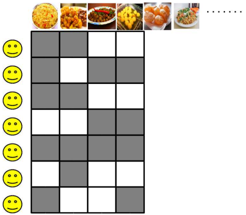

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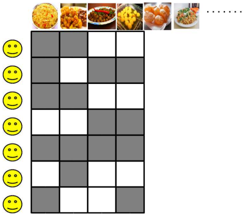

- The above can be used as a prior for $\mathbf{Z}$. Refer to (Griffiths and Ghahramani, 2011) for examples and other theoretical details of the model. Also has connections to Beta Processes

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- Again, remember that "nonparametric" doesn't mean no params but unbounded number of params

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- Very elegant and rich theory based on stocahstic processes (beyond the score of this course)