Module 8

RANDOM VARIABLES

and

INDUCED PROBABILITY SPACE

- $\mathcal{E}$: given random experiment;
- $(\Omega, \mathcal{P}(\Omega), P)$: probability space associated with $\mathcal{E}$;
- In many situations we may not be directly interested in sample space Ω; rather we may be interested in some numerical aspect of sample space (i.e., we may be interested in a real-valued function defined on sample space Ω).

Example 1:

- $\mathcal{E}$: Tossing a fair can three times independently;
- Sample space $\Omega = \{(\omega_1, \omega_2, \omega_3) : \ \omega_i \in \{H, T\}, \ i = 1, 2, 3\}$; here, in $(\omega_1, \omega_2, \omega_3)$, ω_i $(i = 1, 2, 3)$ indicates the outcome of i^{th} toss. Clearly the sample space has $2^3 = 8$ elements;
- Suppose we are interested in number of heads obtained in three tosses, i.e., we are interested the function $X : \Omega \to \mathbb{R}$, where

$$X(w_1, w_2, w_3) = \begin{cases} 0, & \text{if } (\omega_1, \omega_2, \omega_3) = (T, T, T) \\ 1, & \text{if } (\omega_1, \omega_2, \omega_3) \in \{(H, T, T), (T, H, T), (T, T, H)\} \\ 2, & \text{if } (\omega_1, \omega_2, \omega_3) \in \{(H, H, T), (H, T, H), (T, H, H)\} \\ 3, & \text{if } (\omega_1, \omega_2, \omega_3) = (H, H, H) \end{cases}.$$

Definition 1: A real valued function $X : \Omega \to \mathbb{R}$ is called a random variable (r.v.).

Notations:

- $\mathcal{P}(\mathbb{R})$: power set of the real line $\mathbb{R}$;
- For a r.v. X

$$\{X \in A\} \doteq X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}, \quad A \in \mathcal{P}(\mathbb{R}).$$

For example, for a real constant c,

$$\{X = c\} = X^{-1}(\{c\}) = \{\omega \in \Omega : X(\omega) = c\};$$

$$\{X \leq c\} = X^{-1}((-\infty, c]) = \{\omega \in \Omega : X(\omega) \leq c\};$$

$$\{X > c\} = X^{-1}((c, \infty)) = \{\omega \in \Omega : X(\omega) > c\}.$$

Result 1: Let X be a r.v. Then

(a)

$$X^{-1}(\bigcap_{\alpha\in\Lambda} A_\alpha) = \bigcap_{\alpha\in\Lambda} X^{-1}(A_\alpha);$$

(b)

$$X^{-1}(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} X^{-1}(A_\alpha);$$

(c)

$$X^{-1}(A^c) = (X^{-1}(A))^c$$

(d)

$$A \cap B = \phi \;\;\Rightarrow\;\; X^{-1}(A)\bigcap X^{-1}(B) = \phi.$$

Proof: Left as an exercise.

Induced probability space

- X: a given r.v. on probability space $(\Omega, \mathcal{P}(\Omega), P)$;
- Define the set function $P_X : \mathcal{P}(\Omega) \to [0,1]$ as

$$\begin{aligned} P_X(A) &= P(X^{-1}(A)) \\ &\doteqdot P(\{\omega \in \Omega : X(\omega) \in A\}), \quad A \in \mathcal{P}(\Omega). \end{aligned}$$

Result 2: The set function $P_X(.)$ defined above is a probability function on $\mathcal{P}(\mathbb{R})$, i.e., $(\mathbb{R}, \mathcal{P}(\mathbb{R})), P_X)$ is a probability space.

Proof: Since, $P(\cdot)$ is a probability function

$$P_X(A) = P(X^{-1}(A)) \geq 0, \quad \forall\, A \in \mathcal{P}(\mathbb{R}).$$

Let $\{A_i : i \in S\}$ be a countable collection of disjoint events in $\mathcal{P}(\mathbb{R})$. Then

$$
\begin{aligned}
P_X(\bigcup_{i\in S} A_i) &= P(X^{-1}(\bigcup_{i\in S} A_i)) \\
&= P(\bigcup_{i\in S} X^{-1}(A_i)) \quad \text{(using Result 1 (b))} \\
&= \sum_{i\in S} P(X^{-1}(A_i)) \quad (X^{-1}(A_i)\text{s are disjoint}) \\
&= \sum_{i\in S} P_X(A_i).
\end{aligned}
$$

Also

$$
\begin{aligned}
P_X(\mathbb{R}) &= P(X^{-1}(\mathbb{R})) \\
&= P(\Omega) \\
&= 1.
\end{aligned}
$$

Remark 1:

(a) The probability space $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$ is called the probability space induced by r.v. X and the probability function $P_X(\cdot)$ is called the probability function induced by r.v. X.

(b) Given a r.v. X, we are generally no longer interested in the original probability space $(\Omega, \mathcal{P}(\Omega), P)$; rather we are then, interested in induced probability space $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$. We have

$$X : (\Omega, \mathcal{P}(\Omega), P) \quad \rightarrow \quad (\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X),$$

where

$$\begin{aligned} P_X(A) &= P(X^{-1}(A)) \\ &= P(\{\omega \in \Omega : \quad X(\omega) \in A\}), \quad A \in \mathcal{P}(\mathbb{R}). \end{aligned}$$

Example 2:

- $\mathcal{E}$: a fair coin is tossed three times independently;
- Sample space $\Omega = \{(\omega_1, \omega_2, \omega_3) : \ \omega_i \in \{H, T\}, \quad i = 1, 2, 3\}$;
- Suppose we are interested in number of heads in three tosses of coin, i.e., we are interested in r.v. $X : \Omega \to \mathbb{R}$, defined by

$$X(\omega) = \begin{cases} 0, & \text{if } \ \omega = (T, T, T) \\ 1, & \text{if } \ \omega \in \{(H, T, T), (T, H, T), (T, T, H)\} \\ 2, & \text{if } \ \omega \in \{(H, H, T), (H, T, H), (T, H, H)\} \\ 3, & \text{if } \ \omega = (H, H, H) \end{cases}.$$

- We have

$$
\begin{aligned}
P_X(\{0\}) &= P(X^{-1}(\{0\})) \\
&= P(\{(T,T,T)\}) \\
&= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}; \\
P_X(\{1\}) &= P(X^{-1}(\{1\})) \\
&= P(\{(H,T,T),(T,H,T),(T,T,H)\}) \\
&= \frac{3}{8}; \\
P_X(\{2\}) &= P(X^{-1}(\{2\})) \\
&= P(\{(H,H,T),(H,T,H),(T,H,H)\}) \\
&= \frac{3}{8}; \\
P_X(\{3\}) &= P(X^{-1}(\{3\})) \\
&= P(\{(H,H,H)\}) \\
&= \frac{1}{8}.
\end{aligned}
$$

- For $A \subseteq \mathcal{P}(\mathbb{R})$

$$\begin{aligned} P_X(A) &= P(X^{-1}(A)) \\ &= \sum_{\omega \in A \cap \{0,1,2,3\}} P_X(\{\omega\}) \end{aligned}$$

Take Home Problem:

Prove result 1.

Abstract of Next Module

- We will introduce the concept of distribution function (d.f):

$$
\begin{aligned}
F_X(x) &= P_X((-\infty, x]) \\
&= P(\{\omega \in \Omega : \ X(\omega) \leq x\}), \quad x \in \mathbb{R}.
\end{aligned}
$$

- One can study the induced probability space $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P_X)$ through d.f. F_X.

Thank you for your patience