# Module 12

## TRANSFORMATION OF DISCRETE R.V.S

- $X$: a discrete r.v. on some probability space $(\Omega, \mathcal{P}(\Omega), P)$;
- $F_X(\cdot)$: d.f. of $X$;
- $f_X(\cdot)$: p.m.f. of $X$;
- $S_X$ $(= D_X)$: support of $X$;
- $h : \mathbb{R} \to \mathbb{R}$: a given function;
- **Goal:** To find probability distribution (i.e., d.f./p.m.f./p.d.f.) of r.v. $Y = h(X)$.

Define

$$h(S_X) = \{h(x) : x \in S_X\} \quad \text{(image of } S_X \text{ under } h\text{)}.$$

For $y \in h(S_X)$, define

$$h^{-1}(y) = \{x \in S_X : h(x) = y\}.$$

Then

- $S_X$ is countable $\Rightarrow h(S_X)$ is countable;
- Let $y \in h(S_X)$ (so that $h(x_0) = y$, for some $x_0 \in S_X$). Then

$$\begin{aligned} P(\{Y = y\}) &= P(\{h(X) = y\}) \\ &= P(\{h(X) = h(x_0)\}) \\ &\geq P(\{X = x_0\}) > 0. \\ &\qquad\qquad \text{(since } x_0 \in S_X\text{)} \end{aligned}$$

- Also

$$
\begin{aligned}
P\left(\{Y \in h\left(S_X\right)\}\right) &= P\left(\{h\left(X\right) \in h\left(S_X\right)\}\right) \\
&= P\left(\{X \in S_X\}\right) = 1.
\end{aligned}
$$

- Thus there exists a countable set $h(S_X)$ such that

$$
\begin{aligned}
P\left(\{Y = y\}\right) &> 0, \quad \forall\, y \in h(S_X) \\
\text{and} \quad P\left(\{Y \in h\left(S_X\right)\}\right) &= 1.
\end{aligned}
$$

- It follows that $Y$ is a discrete r.v. with support $S_Y = h\left(S_X\right)$.

- For $y \in S_Y = h(S_X)$

$$
\begin{aligned}
P(\{Y = y\}) &= P(\{h(X) = y\}) \\
&= P\left(\{X \in h^{-1}(y)\}\right) \\
&= \sum_{x \in h^{-1}(y)} f_X(x).
\end{aligned}
$$

**Result 1:** The r.v. $Y = h(X)$ is discrete with support $S_Y = h(S_X)$ and p.m.f.

$$f_Y(y) = \begin{cases} \sum_{x \in h^{-1}(y)} f_X(x), & \text{if } y \in S_Y \\ 0, & \text{otherwise} \end{cases}.$$

**Example 1:** Let $X$ be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{4}, & \text{if } -1 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}.$$

and let $Y = 2X^2 + 1$.

(a) Find the d.f. of $Y$ and hence find the p.m.f. of $Y$;

(b) Find the p.m.f. of $Y$ and hence find the d.f. of $Y$.

**Solution:** $D_X = \{-1, 0, 1\}$ and $P(\{X \in D_X\}) = 1$. Thus $X$ is a discrete r.v. $Y = h(X) = 2X^2 + 1$ and $S_Y = h\left(S_X\right) = \{1, 3\}$. By last result $Y$ is a discrete r.v. with support $S_Y = \{1, 3\}$.

(a) For $y < 1$, $F_Y(y) = 0$ and for $y \geq 3$, $F_Y(y) = 1$. For $1 \leq y < 3$

$$
\begin{aligned}
F_Y(y) &= P\left(\{Y \leq y\}\right) \\
&= P\left(\{2X^2 + 1 \leq y\}\right) \\
&= P\left(\{X^2 \leq \frac{y-1}{2}\}\right) \\
&= P\left(\{-\sqrt{\frac{y-1}{2}} \leq X \leq \sqrt{\frac{y-1}{2}}\}\right) \\
&= F_X\left(\sqrt{\frac{y-1}{2}}\right) - F_X\left(-\sqrt{\frac{y-1}{2}}-\right) \\
&= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\end{aligned}
$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 1 \\ \frac{1}{4}, & \text{if } 1 \leq y < 3 \\ 1, & \text{if } y \geq 3 \end{cases},$$

and the p.m.f. of $Y$ is

$$\begin{aligned} f_Y(y) &= P(\{Y = y\}) \\ &= F_Y(y) - F_Y(y-) \\ &= \begin{cases} \frac{1}{4}, & \text{if } y = 1 \\ \frac{3}{4}, & \text{if } y = 3 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

(b) The p.m.f. of $X$ is

$$f_X(x) = P(\{X = x\}) = \begin{cases} \frac{1}{4}, & \text{if } x = -1 \\ \frac{1}{4}, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} .$$

$$
\begin{aligned}
f_Y(y) &= P(\{Y = y\}) \\
&= P\left(\{2X^2 + 1 = y\}\right) \\
&= P\left(\{X^2 = \frac{y-1}{2}\}\right) \\
&= P\left(\left\{X \in \left\{-\sqrt{\frac{y-1}{2}}, \sqrt{\frac{y-1}{2}}\right\}\right\}\right) \\
&= \begin{cases} P(\{X = 0\}), & \text{if } y = 1 \\ P(\{X \in \{-1, 1\}\}), & \text{if } y = 3 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{4}, & \text{if } y = 1 \\ \frac{3}{4}, & \text{if } y = 3 \\ 0, & \text{otherwise} \end{cases} .
\end{aligned}
$$

The d.f. of $Y$ is

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 1 \\ \frac{1}{4}, & \text{if } 1 \leq y < 3 \\ 1, & \text{if } y \geq 3 \end{cases}.$$

**Example 2:** Let $X$ be a r.v. with p.m.f.

$$f_X(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & \text{if } x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = X - 1$. Find the p.m.f. of $Y$.

**Solution:** $S_X = \{1, 2, 3, \ldots\}$, $Y = h(X) = X - 1$.

$$\begin{aligned}
S_Y &= h(S_X) = \{0, 1, 2, \ldots\} \\
f_Y(y) &= P(\{Y = y\}) \\
&= P(\{X - 1 = y\}) \\
&= P(\{X = y + 1\}) \\
&= f_X(y + 1) \\
&= \begin{cases} \left(\frac{1}{2}\right)^{y+1}, & \text{if } y = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases} .
\end{aligned}$$

## Take Home Problem

Let $X$ be a r.v. with d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{4}, & \text{if } -2 \leq x < 0 \\ \frac{1}{2}, & \text{if } 0 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}.$$

(i) Determine whether or not $X$ is discrete;

(ii) Let $Y = X^2 + |X|$. Find the d.f. of $Y$ and hence find the p.m.f. of $Y$;

(iii) Find the p.m.f. of $Y$ and hence find the d.f. of $Y$.

## Abstract of Next Module

- $X$: an A.C. r.v. with d.f. $F_X(\cdot)$ and p.d.f. $f_X(\cdot)$
- $Y = h(X)$, for some function $h : \mathbb{R} \to \mathbb{R}$;
- Under what conditions r.v. $Y$ is A.C.? In that case what is a p.d.f. of $Y$?

# Thank you for your patience