ASSIGNMENT-6

1. From D'Alembert we have

$$u(x,t) = \frac{f(x+ct) - f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$$

where, $u(x,0) = f(x)$ & $u_t(x,0) = g(x)$.

Given, $u(x,0) = \sin x$ and $u_t(x,0) = \cos x$

$$\therefore u(x,t) = \frac{\sin(x+ct) - \sin(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\xi \, d\xi.$$

$$= \frac{\sin x \cancel{\cos ct} + \sin ct \cos x - \sin x \cancel{\cos ct} + \sin ct \cos x}{2}$$

$$+ \frac{1}{2c}\left(+\sin\xi\right)\Big|_{x-ct}^{x+ct}$$

$$= \sin ct \cos x + \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right]$$

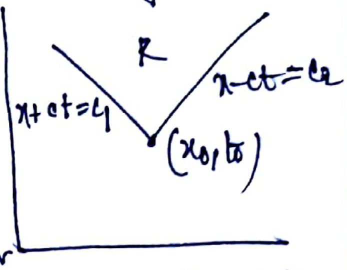
$$= \sin ct \cos x + \frac{1}{c}\sin ct \cos x. \quad = \left(1+\frac{1}{c}\right)\cos x \sin ct.$$

$$\cancel{= (1-1/c)\cos x \sin ct}.$$

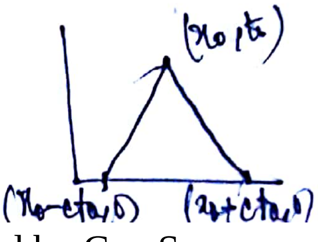
Domain of Dependence :- $u(x_0,t_0)$ depends on sin and cos in the interval $[x_0-ct_0, x_0+ct_0]$.

$\therefore$ Domain of Dependence is the interval $I = [x_0-ct_0, x_0+ct_0]$.

Range of Influence :- The initial displacement or the initial velocity can only influence the soln in the area $t \geq t_0$ bounded by the characteristic $x \pm ct = \text{constant}$. Hence it is the region 'R' in the graph.



Region of determinancy :- $u(x,t)$ is completely determined by the Cauchy data on $[x_0-ct_0, x_0+ct_0]$. D = The triangular area in the graph is the region of determinancy



② $u_{tt} = c^2 u_{xx}$; $u(x,t) = f(x)$ on $x+ct=0$ & $u(x,t) = g(x)$ on $x-ct=0$

The general soln is $u(x,t) = \phi(x+ct) + \psi(x-ct)$ for ϕ & ψ are C^2.

From the initial conditions we have,

$u(x,t) = \phi(2x) + \psi(0) = f(x)$ on $x+ct=0$ ——— ①

& $f(0) = g(0)$

$u(x,t) = \phi(0) + \psi(2x) = g(x)$ on $x-ct=0$ ——— ②

Replace, x with $\frac{x+ct}{2}$ in ① and $\frac{x-ct}{2}$ in ②

$\therefore \phi(x+ct) = f\left(\frac{x+ct}{2}\right) - \psi(0)$

and, $\phi(x-ct) = g\left(\frac{x-ct}{2}\right) - \phi(0)$.

$\therefore$ the soln $u(x,y) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - f(0)$.

3. No Maximum Principle for wave Eqn $\S$

(i) $u_{tt} - u_{xx} = 0$ on $(0,\pi) \times (0,\pi) := \Omega$ } — (1) (Bdd Case)

$u|_{\partial\Omega} = 0$

$u(x,t) = \sin x \sin t$ is a soln of (1).

$\operatorname{Max} u(x,t) = 1$ occurs at $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2}) \in$ Interior of Ω.

(ii) $u_{tt} - u_{xx} = 0$ for $x \in \mathbb{R}, t > 0$
$u(x,0) = 0$ for $x \in \mathbb{R}$
$u_t(x,0) = 1$ for $x \in \mathbb{R}$ } — (2) (Unbounded Case).

$u(x,t) = t$ is the Soln of (2)

Since, $u(x,t) = t$ is an increasing fxn hence the max of 'u' can't be acheived on the line $\{x=0\}$.

(4) $u_t - Ku_{xx} = 0$, on $(0,l) \times \mathbb{R}$
$u(0,t) = 0$; $u(l,t) = 1$, $t > 0$
$u(x,0) = f(x)$

Soln:- Define,
$$v(x,t) = u(x,t) - \frac{x}{l} \quad : [0,l] \times \mathbb{R}.$$

Since, $u \in C^{2,1}([0,l] \times \mathbb{R})$ we have $v \in C^{2,1}([0,l] \times \mathbb{R})$.

$\therefore$ $v_t = u_t$ and $v_{xx} = u_{xx}$.

So, $v_t - Kv_{xx} = 0$
$v(0,t) = 0$; $v(l,t) = 0$
$v(x,0) = f(x) - \frac{x}{L} [:= g(x)]$ } — (ii)

Using Seperation of variable,
$$v(x,t) = X(x)T(t)$$
transform (ii) as: $X'' \pm \lambda X$; $X(0) = X(L) = 0$ (iii) & $T' = \lambda T$ (IV)

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From (III),

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$$

and $\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2$.

and, $T_n(t) = \exp\left(-\frac{n^2\pi^2 k}{\ell^2}t\right)$

So, $v_n(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\exp\left(-\frac{kn^2\pi^2}{\ell^2}t\right)$

$$\therefore v(x,t) = \sum_{n=1}^{\infty} c_n v_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right)\exp\left(-\frac{kn^2\pi^2}{\ell^2}t\right)$$

Now, $g(x) = v(x,0) = \sum_1^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) \left(= f(x) - \frac{x}{L}\right)$

Hence, $c_n = \frac{2}{\ell}\int_0^{\ell}\left(f(x) - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$.

The final solⁿ is $u(x,t) = \left[\sum_1^{\infty}\left[\frac{2}{\ell}\int_0^{\ell}\left(f(x) - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx\right]\sin\left(\frac{n\pi x}{\ell}\right)\exp\left(-\frac{n^2\pi^2 k}{\ell^2}t\right)\right] + \frac{x}{\ell}$

5. $$\left.\begin{aligned} &u_t - K u_{xx} = 2 \\ &u(0,t) = u(l,t) = 0 \\ &u(x,0) = f(x) \end{aligned}\right\} \text{—(1)}$$

Step 1:- Let us solve $-KW_{xx} = 2$

$\Rightarrow W_{xx} = -2/K \ (:= \alpha)$.

$\Rightarrow W_x = \alpha x + c_1 \quad (c_1 = \text{constant})$

$\Rightarrow W(x) = \frac{\alpha}{2}x^2 + c_1 x + c_2 \quad (c_1, c_2 = \text{constant})$. $\boxed{\text{Choose, } W(x) = -\frac{x^2}{K} + x}$

Step 2:- Define, $v(x,t) = u(x,t) - W(x)$

So, $v_t - kv_{xx} = u_t - K u_{xx} + kW'' \quad (W'' = W_{xx})$.

$= u_t - K u_{xx} - 2 = 0 \quad (\because kW_{xx} = -2)$

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so, $v_t - Kv_{xx} = 0$.

$v(0,t) = u(0,t) - w(0)$

$= 0 - 0 = 0$.

$v(l,t) = u(l,t) - w(l)$

$= \frac{l^2}{K} - l \cdot (= \beta \text{ say})$

and, $v(x,0) = u(x,0) - w(x)$

$= f(x) + \frac{x^2}{K} - x \; (= g(x) \text{ say})$

From Problem 4 we know,

$$u(x,t) = \frac{x\beta}{l} + \sum_{1}^{\infty} \left[\frac{2}{l} \int_0^l \left(g(x) - \frac{x\beta}{l} \right) \sin\left(\frac{n\pi x}{l}\right) dx \right] \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\frac{n^2\pi^2 K}{l^2} t\right).$$

$$= \left(\frac{l}{K} - 1\right)x + \sum_{1}^{\infty} \left[\frac{2}{l} \int_0^l \left(g(x) - \frac{x\beta}{l} \right) \sin\left(\frac{n\pi x}{l}\right) dx \right] \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\frac{n^2\pi^2 K}{l^2} t\right).$$

(6) Let $v(x,t) = e^{-t}\sin x$

$\therefore v_t = -e^{-t}\sin x$ & $v_{xx} = -e^{-t}\sin x$

Hence, $v_t - v_{xx} = 0$.

Also, $v(0,t) = v(\pi,t) = 0$.

and, $v(x,0) = \sin x$.

Recall by comparison principle if $u_1, u_2 \in C^{2,1}(\Omega_T)$ be two solutions of the heat eqn with $u_1 \leq u_2$ on Γ_T (The parabolic boundary) then $u_1 \leq u_2$ on $\bar{\Omega}_T$.

Hence comparing with our eqn we have,

$u(x,t) \leq v(x,t)$

e.g, $u(x,t) \leq e^{-t}\sin x$ —— (i)

Again from maximum principle the $\min\limits_{\bar{\Omega}_T} u \geq \min\limits_{\Gamma_T} u$

Hence, $u(x,t) \geq 0$ in Ω_T —— (ii)

Combining (i) & (ii) we have

$0 \leq u(x,t) \leq e^{-t}\sin x$