Lecture 15 : MSO 203B (PDE) $\boxed{\text{Well posedness of the Poisson Equation}}$

$$\Delta u = f \quad \text{in } \Omega.$$
$$u = g \quad \text{on } \partial\Omega$$

Given f, g continuous. and $\Omega = (0,a)\times(0,b)$ then

$$\left.\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}\right\} - (1)$$

(i) Existence : $\exists$ atleast one soln,

(ii) Uniqueness : The obtained soln is unique

(iii) Stability : For small change in initial data the change in soln is small.

$$\max_{\Omega} |u_1 - u_2| \leq C \max_{\partial\Omega} |g_1 - g_2|$$

where u_i are the soln of (1) corresponding to g_i.

Maximum Principle for the Harmonic Function :- (Ω = Open, bdd).

Lemma :- Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be such that $\Delta u > 0$ in Ω. Then the fxn u doesnot attain an maximum in Ω.

Defines $v^y(x) = u(x,y)$ and let $(x_0, y_0) \in \Omega$ be s.t $u(x_0,y_0) = \max_{\Omega} (u(x,y))$.

$$v_x^y(x_0) = (v^y)'(x_0) = 0.$$

$$v_{xx}^y(x_0) = u_{xx}(x_0,y_0) \leq 0 \qquad |||^{ly} \quad u_{yy}(x_0,y_0) \leq 0.$$

$$\Rightarrow \Delta u(x_0,y_0) \leq 0.$$

$\|^{ly}$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\Delta u < 0$ does not attain a minima in the interior of Ω.

Th: Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\Delta u = 0$ then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ & $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.

Proof: Define, $u^{\varepsilon}(x,y) = u(x,y) + \varepsilon(x^2+y^2)$. ; $(x,y) \in \Omega$; $\varepsilon > 0$

$$\Delta u^{\varepsilon} = \Delta u + \varepsilon \Delta(x^2+y^2)$$
$$= 0 + 4\varepsilon$$
$$= 4\varepsilon > 0$$

$\exists R > 0$ s.t. $x^2+y^2 \leq R^2$. (Bddness).

$$\Rightarrow \max_{\bar{\Omega}} u_{\varepsilon} = \max_{\partial\Omega} u_{\varepsilon}$$

$$\therefore \max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} u_{\varepsilon} = \max_{\partial\Omega} u_{\varepsilon} = \max_{\partial\Omega} \left[u + \varepsilon(x^2+y^2)\right] \leq \max_{\partial\Omega} \left[u + \varepsilon R^2\right] = \max_{\partial\Omega} u + \varepsilon R^2.$$

$$\max_{\partial\Omega} u \leq \max_{\bar{\Omega}} u$$

$$\Rightarrow \max_{\partial\Omega} u \leq \max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u + \varepsilon R^2 \Rightarrow \max_{\partial\Omega} u = \max_{\bar{\Omega}} u.$$

Uniqueness :-

$$\left.\begin{aligned} \Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega . \end{aligned}\right\} \text{ — (2)}$$

Let u_1 and u_2 are soln of (2)

$\tilde{u} := u_1 - u_2$. then $\tilde{u} \in C^2(\Omega) \cap C(\bar{\Omega})$.

RTP:- $\tilde{u} \equiv 0$ in Ω.

$$\Delta \tilde{u} = \Delta u_1 - \Delta u_2 = 0 \quad \text{in } \Omega$$

$$\tilde{u} = g - g = 0 \quad \text{on } \partial\Omega .$$

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } \Omega . \\ \tilde{u} = 0 & \text{on } \partial\Omega . \end{cases}$$

From max Principle $\Rightarrow \tilde{u} \equiv 0 \Rightarrow u_1 = u_2$ in Ω.

Stability :- (Ω is open, bdd).

$$\left.\begin{array}{l}\Delta u = f \text{ in } \Omega \\ u = g_1 \text{ on } \partial\Omega .\end{array}\right\} — (1)$$

$$\left.\begin{array}{l}\Delta u = f \text{ in } \Omega \\ u = g_2 \text{ on } \partial\Omega .\end{array}\right\} — (2)$$

RTP :- $\max_{\Omega} |u_1 - u_2| \leq C \max_{\partial\Omega} |g_1 - g_2|$.

Assume, $M = \max_{\partial\Omega} |g_1 - g_2|$.

Define $v = u_1 - u_2$.

$$\Delta v = 0 \text{ in } \Omega.$$
$$v = g_1 - g_2 \text{ on } \partial\Omega .$$

From max principle, $\sup_{\Omega} v \leq \sup_{\partial\Omega} v = \sup_{\partial\Omega} (g_1 - g_2) \leq M$

$$\inf_{\Omega}(v) \geq \inf_{\partial\Omega} v = \inf_{\partial\Omega} (g_1 - g_2) \Rightarrow \sup(-v) \leq -\inf(g_1 - g_2) = +\sup(g_2 - g_1) \leq M$$

$\sup |v|$

$$|v| = \begin{cases} v, & v > 0 \\ -v, & v < 0 \end{cases}$$

$\sup(-v) = \underline{\underline{-\inf v}}$