MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-II

A. Illustrative Discussion Problems

1. Consider a random experiment of two independent tosses of a coin so that the sample space is $\Omega = \{HH, HT, TH, TT\}$ with obvious interpretations of outcomes HH, HT, TH and TT. Let $P(\cdot)$ be a probability function defined on $\mathcal{P}(\Omega)$ such that $P(\{HH\}) = p^2,\ P(\{HT\}) = P(\{TH\}) = p(1-p)$ and $P(\{TT\}) = (1-p)^2$, where $p \in (0,1)$. Define the r.v. $X : \Omega \to \mathbb{R}$ by $X(\{HH\}) = 2, X(\{HT\}) = X(\{TH\}) = 1$ and $X(\{TT\}) = 0$, i.e., $X(\omega)$ denotes the number of Hs (heads) in ω. Find the probability function induced by X and hence find the d.f. of X.

2. Consider the probability space $(\mathbb{R}, \mathcal{P}(\mathbb{R}), P)$, where
$$P(A) = \int_A \frac{e^{-|t|}}{2} dt = \int_{-\infty}^{\infty} \frac{e^{-|t|}}{2} I_A(t)dt,\ A \in \mathcal{P}(\mathbb{R}),$$
and, for $B \subseteq \mathbb{R}, I_B(\cdot)$ denotes the indicator function of B (i.e., $I_B(t) = 1$, if $t \in B$, $= 0$, if $t \notin B$). Define the r.v. $X : \mathbb{R} \to \mathbb{R}$ by $X(\omega) = \omega^2,\ \omega \in \mathbb{R}$. Find the probability function induced by X and hence find the d.f. of X.

3. Do the following functions define distributions functions?
$$(a) \quad F_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x \le \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2} \end{cases}; \qquad (b) \quad F_2(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases};$$
and

(c) $F_3(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}, -\infty < x < \infty$.

4. Let $F(\cdot)$ and $G(\cdot)$ be two distribution functions. Verify whether or not the following functions are distributions functions:

(a) $H(x) = F(x)+G(x)$; (b) $H(x) = \max(F(x), G(x))$; (c) $H(x) = \min(F(x), G(x))$.

5. Let $F_1(\cdot), \ldots, F_n(\cdot)$ be distribution functions and let $a_1, \ldots, a_n$ be positive real numbers satisfying $\sum_{i=1}^n a_i = 1$. Show that $G(x) = \sum_{i=1}^n a_i F_i(x)$ is also a distribution function.

6. Do there exist real numbers α, β, γ and δ such that the following functions become a distribution function?

$$\text{(a) } F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } 0 \le x < 1 \\ \frac{1}{2} + \alpha(x-1)^2, & \text{if } 1 \le x \le 2 \\ \beta + \frac{(x-2)^4}{7}, & \text{if } 2 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}; \quad \text{(b) } G(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \gamma + \delta e^{-\frac{x^2}{2}}, & \text{if } x > 0 \end{cases}.$$

7. Let X be a random variables with distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{3}, & \text{if } -2 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 5 \\ \frac{1}{2} + \frac{(x-5)^2}{2}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}.$$

Sketch the graph of $F(x)$ and compute $P(\{X = c\})$, $c \in \mathbb{R}$, $P(\{-1 < X \le 4\})$, $P(\{-2 \le X < 5\})$, $P(\{0 < X < 6\})$, $P(\{0 \le X \le 5\})$ and the conditional probability $P(\{1 < X < \frac{11}{2}\}|\{X \ge 5\})$.

8. A random variable X has the distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{2}{3}, & \text{if } 2 \le x < 5 \\ \frac{7-6k}{6}, & \text{if } 5 \le x < 9 \\ \frac{3k^2-6k+7}{6}, & \text{if } 9 \le x < 14 \\ \frac{16k^2-16k+19}{16}, & \text{if } 14 \le x \le 20 \\ 1, & \text{if } x > 20 \end{cases},$$

where $k \in \mathbb{R}$.

(i) Find the value of constant k;

(ii) Find $P(\{X \in \{2, 3, 5, 8, 9, 11, 14, 18, 20\}\})$.

(iii) Find $P(\{2 < X \le 9\})$, $P(\{5 < X < 14\})$, $P(\{9 \le X \le 20\})$ and $P(\{2 \le X < 14\})$.

B. Practice Problems from the Text Book

Chapter 1: Probability and Distributions, Problem Nos.: 5.1, 5.6, 5.7, 5.8.

MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-II
Solutions

Problem No. 1

$$P_X(\{x\}) = P(\{X = x\}) = \begin{cases} (1-p)^2 & \text{if } x = 0 \\ 2p(1-p) & \text{if } x = 1 \\ p^2, & \text{if } x = 2 \end{cases}$$

For $A \in \mathcal{B}(\mathbb{R})$

$$P_X(A) = \sum_{\substack{x=0 \\ x \in A}}^{2} P_X(\{x\}).$$

$$F_X(x) = P_X((-\infty, x]) = \sum_{\substack{i=0 \\ i \le x}}^{2} P_X(\{i\}) = \begin{cases} 0, & x < 0 \\ (1-p)^2, & 0 \le x < 1 \\ 1-p^2, & 1 \le x < 2 \\ 1, & x \ge 1 \end{cases}.$$

Problem No. 2 For $B \in \mathcal{B}(\mathbb{R})$

$$P_X(B) = P(\{\omega \in \Omega : X(\omega) \in B\})$$

$$= P(\{\omega \in \Omega : \omega^2 \in B\})$$

$$= \int_{-\infty}^{\infty} \frac{e^{-|t|}}{2} I_B(t^2)\, dt = \int_0^{\infty} e^{-t} I_B(t^2)\, dt$$

$$= \int_0^{\infty} \frac{e^{-\sqrt{z}}}{2\sqrt{z}} I_B(z)\, dz = \int_{B \cap [0,\infty)} \frac{e^{-\sqrt{z}}}{2\sqrt{z}}\, dz$$

$$F_X(x) = P_X((-\infty, x])$$

$$= \int_{(-\infty, x] \cap [0,\infty)} \frac{e^{-\sqrt{z}}}{2\sqrt{z}}\, dz = \begin{cases} 0, & \text{if } x < 0 \\ \int_0^x \frac{e^{-\sqrt{z}}}{2\sqrt{z}}\, dz, & \text{if } x \ge 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\sqrt{x}}, & \text{if } x \ge 0 \end{cases}.$$

Problem No. 3 (a) $F_1(\frac{1}{2}+) = 1 \ne \frac{1}{2} = F_1(\frac{1}{2}) \Rightarrow F_1$ is not right continuous $\Rightarrow F_1$ is not a d.f.

(b) F_2 is non-decreasing, continuous, $F_2(-\infty) = 0$ and $F_2(\infty) = 1$. So F_2 is a d.f.

(c) Same as (b).

Problem No. 4. (a) $H(+\infty) = F(+\infty) + G(+\infty) = 1 + 1 = 2$
$\Rightarrow H$ is not a d.f.

(b) Let $-\infty < x < y < \infty$. Then

$$F(x) \le F(y) \le \max\{F(y), G(y)\} = H(y)$$
$$G(x) \le G(y) \le \max\{F(y), G(y)\} = H(y)$$

$$\Rightarrow \max\{F(x), G(x)\} \le H(y)$$
$$\Rightarrow H(x) \le H(y) \Rightarrow H(\cdot) \uparrow$$

$$H(-\infty) = \max\{F(-\infty), G(-\infty)\} = \max\{0, 0\} = 0$$
$$H(+\infty) = \max\{F(\infty), G(\infty)\} = \max\{1, 1\} = 1$$

$$\lim_{n\to\infty} H(x+\tfrac{1}{n}) = \lim_{n\to\infty} \max\{F(x+\tfrac{1}{n}), G(x+\tfrac{1}{n})\}$$
$$= \lim_{n\to\infty} \frac{F(x+\frac{1}{n}) + G(x+\frac{1}{n}) + |F(x+\frac{1}{n}) - G(x+\frac{1}{n})|}{2}$$

$$\left(\max\{a, b\} = \frac{a+b+|a-b|}{2}\right)$$

$$= \frac{F(x) + G(x) + |F(x) - G(x)|}{2} \qquad [h(x) = |x|,\ x \in \mathbb{R} \text{ is continuous}]$$

$$= \max\{F(x), G(x)\} = H(x)$$

$\Rightarrow$ H is right continuous. Thus H is a d.f.

(c) Let $-\infty < x < y < \infty$. Then

$$F(y) \ge F(x) \ge \min\{F(x), G(x)\} = H(x)$$
$$G(y) \ge G(x) \ge \min\{F(x), G(x)\} = H(x)$$

$$\Rightarrow \min\{F(y), G(y)\} \ge H(x)$$
$$\Rightarrow H(y) \ge H(x) \Rightarrow H(\cdot) \uparrow$$

$$H(-\infty) = \min\{F(-\infty), G(-\infty)\} = \min\{0, 0\} = 0$$
$$H(\infty) = \max\{F(\infty), G(\infty)\} = \max\{1, 1\} = 1$$

$$\lim_{n\to\infty} H(x+\tfrac{1}{n}) = \lim_{n\to\infty} \min\{F(x+\tfrac{1}{n}), G(x+\tfrac{1}{n})\} =$$
$$= \lim_{n\to\infty} \frac{F(x+\frac{1}{n}) + G(x+\frac{1}{n}) - |F(x+\frac{1}{n}) - G(x+\frac{1}{n})|}{2} \qquad \min\{a, b\} = \frac{a+b-|a-b|}{2}$$

$$= \frac{F(x) + G(x) - |F(x) - G(x)|}{2}$$
$$= \min\{F(x), G(x)\} = H(x).$$ Thus H is a d.f.

Problem No. 5 Clearly G is non-decreasing & right continuous. Moreover $G(-\infty) = \sum_{i=1}^{n} a_i F_i(-\infty) = 0$ and $G(\infty) = \sum_{i=1}^{n} a_i F_i(\infty) = \sum_{i=1}^{n} a_i = 1$

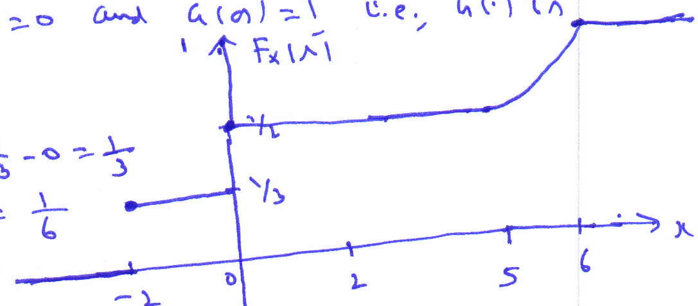
Thus $G(\cdot)$ is a d.f.

Problem No. 6 (a) $F(2) = F(2+) \Rightarrow \frac{1}{2} + \alpha = \beta$; $F(3) = F(3+) \Rightarrow \beta + \frac{1}{7} = 1 \Rightarrow \beta = \frac{6}{7}$

$\Rightarrow \alpha = \frac{5}{14}$ and $\beta = \frac{6}{7}$. For these values of α and β, F is non-decreasing, right continuous, $F(-\infty) = 0$ and $F(\infty) = 1$ i.e. F is a d.f.

(b) $G(\infty) = 1 \Rightarrow \theta = 1$, $G(0) = G(0+) \Rightarrow 0 = \theta + \delta \Rightarrow \delta = -1$.

For these values of θ and δ, G is non-decreasing, right continuous, $G(-\infty) = 0$ and $G(\infty) = 1$ i.e. $G(\cdot)$ is a d.f.



Problem No. 7 $P(\{X = -2\}) = F(-2) - F(-2-) = \frac{1}{3} - 0 = \frac{1}{3}$

$P(\{X = 0\}) = F(0) - F(0-)\ \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$P(\{X = 5\}) = F(5) - F(5-) = \frac{1}{2} - \frac{1}{2} = 0$

$P(\{X = 6\}) = F(6) - F(6-) = 1 - 1 = 0$

For $x \notin \{-2, 0\}$ $F(x)$ is continuous at x, thus $P(\{X = x\}) = 0$

$P(\{-1 < X \leq 4\}) = F(4) - F(-1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$P(\{-2 \leq X < 5\}) = F(5-) - F(-2-) = \frac{1}{2} - 0 = \frac{1}{2}$

$P(\{0 < X < 6\}) = F(6-) - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(\{0 \leq X \leq 5\}) = F(5) - F(0-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$$P(\{1 < X < \tfrac{11}{2}\} \mid \{X \geq 5\}) = \frac{P(\{1 < X < \frac{11}{2}, X \geq 5\})}{P(\{X \geq 5\})} = \frac{P(\{5 \leq X < \frac{11}{2}\})}{1 - P(\{X < 5\})}$$

$$= \frac{F(\frac{11}{2}-) - F(5-)}{1 - F(5-)} = \frac{(\frac{1}{2} + \frac{1}{8}) - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{4}.$$

Problem No. 8 $F_X(20) = F_X(20+) \Rightarrow 16k^2 - 16k + 3 = 0 \Rightarrow k = \frac{1}{4}, \frac{3}{4}$

$F_X(5-) \leq F_X(5) \Rightarrow \frac{2}{3} \leq \frac{7 - 6k}{6} \Rightarrow k \leq \frac{1}{2} \Rightarrow k = \frac{1}{4}$

Thus
$$F_X(x) = \begin{cases} 0, & x < 2 \\ \frac{2}{3}, & 2 \leq x < 5 \\ \frac{11}{12}, & 5 \leq x < 9 \\ \frac{91}{96}, & 9 \leq x < 14 \\ 1, & x \geq 14 \end{cases}$$

F_X is continuous everywhere except on set $D_X = \{2, 5, 9, 14\}$

$P(\{X = x\}) = 0$ $\forall$ $x \notin D_X \{2, 5, 9, 14\}$

$P(\{X = 2\}) = F_X(2) - F_X(2-) = \frac{2}{3}$

$P(\{X = 5\}) = F_X(5) - F_X(5-) = \frac{1}{4}$

$P(\{X = 9\}) = F_X(9) - F_X(9-) = \frac{1}{32}$

$P(\{X = 14\}) = F_X(14) - F_X(14-) = \frac{5}{96}$

$P(\{X \in \{2, 3, 5, 8, 9, 11, 14, 18, 20\}\}) = \frac{2}{3} + \frac{1}{4} + \frac{1}{32} + \frac{5}{96} = 1$

$P(\{2 < X \leq 9\}) = F_X(9) - F_X(2)$, $P(\{5 < X < 14\}) = F_X(14-) - F_X(5)$

$P(\{9 \leq X \leq 20\}) = F_X(20) - F_X(9-)$; $P(\{2 \leq X < 14\}) = F_X(14-) - F_X(2-)$.