Module 24

EXPECTATIONS OF RANDOM VECTORS

- $\underline{X} = (X_1, \ldots, X_p)'$: a p-dimensional random vector of either discrete or of A.C. type;

- $S_{\underline{X}}$: support of $\underline{X}$;

- $f_{\underline{X}}(\cdot)$: p.m.f./ p.d.f. of $\underline{X}$;

- S_{X_i}: support of $X_i, i = 1, \ldots, p$;

- $f_{X_i}(\cdot)$: marginal p.m.f./ p.d.f. of $X_i, i = 1, \ldots, p$.

Result 1:

Let $\psi : \mathbb{R}^p \to \mathbb{R}$ be a function such that $E(\psi(\underline{X}))$ is finite.

(i) If $\underline{X}$ is finite then

$$E(\psi(\underline{X})) = \sum_{\underline{x} \in S_{\underline{X}}} \psi(\underline{x}) f_{\underline{X}}(\underline{x}).$$

(ii) If $\underline{X}$ is A.C. then

$$E(\psi(\underline{X})) = \int_{\mathbb{R}^p} \psi(\underline{x}) f_{\underline{X}}(\underline{x}) d\underline{x}.$$

Definition 1:

Let X and Y be two random variables.

(a) The quantity

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))],$$

provided the expectations exist, is called the covariance between r.v.s X and Y.

(b) Suppose that $\text{Var}(X) > 0$ and $\text{Var}(Y) > 0$. The quantity

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}},$$

provided the expectations exist, is called the correlation between X and Y.

(c) Random variables X and Y are called uncorrelated (correlated) if $\rho(X, Y) = 0$ ($\rho(X, Y) \neq 0$).

Remark 1 :

(a) $\mathrm{Cov}(X,X) = E[(X-E(X))^2] = \mathrm{Var}(X)$;

(b) $\rho(X,X) = 1$;

(c)

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E[(X-E(X))(Y-E(Y))] \\
&= E[XY - E(Y)X - E(X)Y + E(X)E(Y)] \\
&= E(XY) - E(Y)E(X) - E(X)E(Y) + E(X)E(Y) \\
&= E(XY) - E(X)E(Y).
\end{aligned}$$

(d) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$ and $\rho(X,Y) = \rho(Y,X)$;

(e) X and Y are independent $\Rightarrow \mathrm{Cov}(X,Y) = 0 \Leftrightarrow \rho(X,Y) = 0$. Converse may not be true.

Example 1 :

Let (X, Y) be an A.C. bivariate r.v. with joint p.d.f.

$$f(x,y) = \begin{cases} 1, & \text{if } 0 < \mid y \mid \leq x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Show that X and Y are uncorrelated but not independent.

Solution.

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f(x,y)dxdy \\ &= \int_{0}^{1}\int_{-x}^{x} xy\ dydx = 0 \\ E(Y) &= \int_{0}^{1}\int_{-x}^{x} y\ dydx = 0 \\ \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) = 0 \\ \Rightarrow \rho(X,Y) &= 0. \end{aligned}$$

$$S_X = [0,1],\ S_Y = [-1,1]$$

$$\begin{aligned} S_{X,Y} &= \{(x,y) \in \mathbb{R}^2 : 0 \le |\, y \,| \le |\, x \,| \le 1\} \\ &\ne S_X \times S_Y \end{aligned}$$

$\Rightarrow X$ and Y are not independent.

Result 2: Let $\underline{X} = (X_1, \ldots, X_{p_1})'$ and $\underline{Y} = (Y_1, \ldots, Y_{p_2})'$ be r.v.s and let $a_1, \ldots, a_{p_1}$, $b_1, \ldots, b_{p_2}$ be real constants. Then, provided the involved expectations are finite,

(a) $E\Big(\sum_{i=1}^{p_1} a_i X_i\Big) = \sum_{i=1}^{p_1} a_i E(X_i)$;

(b) $\mathrm{Cov}\Big(\sum_{i=1}^{p_1} a_i X_i, \sum_{j=1}^{p_2} b_j Y_j\Big) = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} a_i b_j \mathrm{Cov}(X_i, Y_j)$;

(c)

$$\text{Var}\Big(\sum_{i=1}^{p_1} a_i X_i\Big) = \sum_{i=1}^{p_1} {a_i}^2 \text{Var}(X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{p_1} \sum_{j=1}^{p_1} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{p_1} {a_i}^2 \text{Var}(X_i) + 2 \sum\sum_{1 \leq i < j \leq p_1} a_i a_j \text{Cov}(X_i, X_j)$$

Proof. For A.C. case

(a)

$$E\Big(\sum_{i=1}^{p_1} a_i X_i\Big) = \int_{\mathbb{R}^p} \Big(\sum_{i=1}^{p_1} a_i x_i\Big) f_{\underline{X}}(\underline{x}) d\underline{x}$$

$$= \sum_{i=1}^{p_1} a_i \int_{\mathbb{R}^p} x_i f_{\underline{X}}(\underline{x}) d\underline{x}$$

$$= \sum_{i=1}^{p_1} a_i E(X_i).$$

(b)

$$\sum_{i=1}^{p_1} a_i X_i - E\left(\sum_{i=1}^{p_1} a_i X_i\right)$$
$$= \sum_{i=1}^{p_1} a_i X_i - \sum_{i=1}^{p_1} a_i E(X_i)$$
$$= \sum_{i=1}^{p_1} a_i (X_i - E(X_i))$$

Similarly,

$$\sum_{j=1}^{p_2} b_j Y_j - E\left(\sum_{j=1}^{p_2} b_j Y_j\right)$$
$$= \sum_{j=1}^{p_2} b_j (Y_j - E(Y_j))$$

Then,

$$
\begin{aligned}
\mathrm{Cov}\left(\sum_{i=1}^{p_1} a_i X_i, \sum_{j=1}^{p_2} b_j Y_j\right) &= E\left(\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} a_i b_j (X_i - E(X_i))(Y_j - E(Y_j))\right)\\
&= \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} a_i b_j E\Big((X_i - E(X_i))(Y_j - E(Y_j))\Big)\\
&= \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} a_i b_j \mathrm{Cov}(X_i, Y_j).
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathrm{Var}\Big(\sum_{i=1}^{p_1} a_i X_i\Big) &= \mathrm{Cov}\Big(\sum_{i=1}^{p_1} a_i X_i, \sum_{i=1}^{p_1} a_i X_i\Big)\\
&= \sum_{i=1}^{p_1}\sum_{j=1}^{p_1} a_i a_j \mathrm{Cov}(X_i, X_j)
\end{aligned}
$$

$$= \sum_{i=1}^{p_1} {a_i}^2 \text{Cov}(X_i, X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{p_1} \sum_{j=1}^{p_1} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{p_1} {a_i}^2 \text{Var}(X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{p_1} \sum_{j=1}^{p_1} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{p_1} {a_i}^2 \text{Var}(X_i) + 2 \sum\sum_{1 \leq i < j \leq p_1} a_i a_j \text{Cov}(X_i, X_j)$$

$$(\text{Cov}(X_i, X_j) = \text{Cov}(X_j, X_i), i \neq j)$$

Result 3 : Let $\underline{X}_1, \ldots, \underline{X}_p$ be independent r.v.s, where $\underline{X}_i$ is r_i-dimensional, $i = 1, \ldots, p$.

(i) Let $\psi_i : \mathbb{R}^{r_i} \to \mathbb{R}$, $i = 1, \ldots, p$ be given functions. Then

$$E\left(\prod_{i=1}^{p} \psi_i(\underline{X}_i)\right) = \prod_{i=1}^{p} E\big(\psi_i(\underline{X}_i)\big),$$

(ii) For $A_i \subseteq \mathbb{R}^{r_i}$, $i = 1, \ldots, p$,

$$P(\{\underline{X}_i \in A_i, i = 1, \ldots, p\}) = \prod_{i=1}^{p} P(\{\underline{X}_i \in A_i\}).$$

Proof. Let $r = \sum_{i=1}^{p} r_i$ and $\underline{X} = (\underline{X}_1, \ldots, \underline{X}_p)$. Then, we have

$$f_{\underline{X}}(\underline{x}_1, \ldots, \underline{x}_p) = \prod_{i=1}^{p} f_{\underline{X}_i}(\underline{x}_i),\ \underline{x}_i \in \mathbb{R}^{r_i},\ i = 1, \ldots, p.$$

(i)

$$E\Big(\prod_{i=1}^{p} \psi_i(\underline{X}_i)\Big) = \int_{\mathbb{R}^r} \Big(\prod_{i=1}^{p} \psi_i(\underline{x}_i)\Big) f_{\underline{X}}(\underline{x}) d\underline{x}$$
$$= \int_{\mathbb{R}^{r_1}} \cdots \int_{\mathbb{R}^{r_p}} \Big(\prod_{i=1}^{p} \psi_i(\underline{x}_i)\Big)\Big(\prod_{i=1}^{p} f_{\underline{X}_i}(\underline{x}_i)\Big) d\underline{x}_p, \ldots, d\underline{x}_1$$

$$= \int\limits_{\mathbb{R}^{r_1}} \cdots \int\limits_{\mathbb{R}^{r_p}} \left(\prod_{i=1}^{p} \psi_i(\underline{x}_i) f_{\underline{X}_i}(\underline{x}_i) \right) d\underline{x}_p, \ldots, d\underline{x}_1$$

$$= \prod_{i=1}^{p} \int\limits_{\mathbb{R}^{r_i}} \psi_i(\underline{x}_i) f_{\underline{X}_i}(\underline{x}_i) d\underline{x}$$

$$= \prod_{i=1}^{p} E\Big(\psi_i(\underline{X}_i)\Big).$$

(ii) Follows from (i) by taking

$$\psi_i(\underline{x}_i) = \begin{cases} 1, & \text{if } \underline{x}_i \in A_i \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, p.$$

Result 4 (Cauchy-Schwarz Inequality for random variables) :

Let (X, Y) be a bivariate r.v. Then, provided the involved expectations are finite,

(a)
$$(E(XY))^2 \leq E(X^2)E(Y^2). \tag{1}$$

The equality is attained iff $P(\{Y = cX\}) = 1$ (or $P(\{X = cY\}) = 1$), for some real constant c.

(b) Let $E(X) = \mu_X \in (-\infty, \infty)$, $E(Y) = \mu_Y \in (-\infty, \infty)$, $\mathrm{Var}(X) = \sigma_X^2 \in (0, \infty)$ and $\mathrm{Var}(Y) = \sigma_Y^2 \in (0, \infty)$ be finite. Then

$$-1 \leq \rho(X, Y) \leq 1$$

and

$$\rho(X, Y) = \pm 1 \Leftrightarrow \frac{X - \mu_X}{\sigma_X} = \pm \frac{Y - \mu_Y}{\sigma_Y},$$

with probability one.

Proof.

(a) Consider the following two cases:
Case 1: $E(X^2) = 0$.
In this case $P(\{X = 0\}) = 1$ and therefore $P(\{XY = 0\}) = 1$.
It follows that $E(XY) = 0$, $E(X) = 0$, $P(X = cY) = 1$ (for $c = 0$) and the equality in (1) is attained.
Case 2: $E(X^2) > 0$.
Then

$$0 \leq E((Y - \lambda X)^2) = \lambda^2 E(X^2) - 2\lambda E(XY) + E(Y^2)$$
$$\text{i.e., } \lambda^2 E(X^2) - 2\lambda E(XY) + E(Y^2) \geq 0, \ \forall \lambda \in \mathbb{R}$$

This implies that the discriminant of the quadratic equation

$$\lambda^2 E(X^2) - 2\lambda E(XY) + E(Y^2) = 0$$

is non-negative, i.e.,

$$(4E(XY))^2 \leq 4E(X^2)E(Y^2)$$

$$\Rightarrow (E(XY))^2 \leq E(X^2)E(Y^2)$$

and the equality is attained iff

$$E((Y - cX)^2) = 0, \text{for some } c \in \mathbb{R}$$
$$\Leftrightarrow P(Y = cX) = 1, \text{for some } c \in \mathbb{R}$$

(b) Let $Z_1 = \frac{X-\mu_X}{\sigma_X}$ and $Z_2 = \frac{Y-\mu_Y}{\sigma_Y}$ so that $E(Z_1) = E(Z_2) = 0$, $\text{Var}(Z_1) = E(Z_1^2)$, $\text{Var}(Z_2) = E(Z_2^2)$, $\text{Var}(Z_1) = \text{Var}(Z_2) = 1$ and

$$\begin{aligned} E(Z_1 Z_2) &= E\left(\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right) \\ &= \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X\sigma_Y} \\ &= \rho(X, Y). \end{aligned}$$

By C-S inequality

$$
\begin{aligned}
&(E(Z_1 Z_2))^2 \leq (E(Z_1^2))(E(Z_2^2)) \\
&\Leftrightarrow (\rho(X, Y))^2 \leq 1.
\end{aligned}
$$

By (a) equality is attained iff

$$
\begin{aligned}
&P(\{Z_1 = cZ_2\}) = 1, \text{ for some c} \in \mathbb{R} \\
&\qquad \Leftrightarrow \; P\left(\{\frac{X-\mu_X}{\sigma_X} = c\frac{Y-\mu_Y}{\sigma_Y}\}\right) = 1, \text{ for some c} \in \mathbb{R}
\end{aligned}
$$

Since $\mathrm{Var}\left(\frac{X-\mu_X}{\sigma_X}\right) = \mathrm{Var}\left(\frac{Y-\mu_Y}{\sigma_Y}\right) = 1$, we have $c^2 = 1$.

Take Home Problem

Let (X, Y) be a bivariate discrete r.v. with p.m.f. given by:

(x, y)	(-1, 1)	(0, 0)	(1, 1)
$f(x, y)$	p_1	p_2	p_1

where $p_i \in (0, 1), i = 1, 2$ and $2p_1 + p_2 = 1$.

(a) Find $\rho(X, Y)$;

(b) Are X and Y independent?

Abstract of Next Module

- We will discuss the concept of conditional expectation of A.C. r.v.s.

Thank you for your patience