MSO 203 B - PDE Lecture 14 (POISSON EQUATION-2)

$$(1)\ \begin{cases} \Delta u = f & \text{in } (0,a)\times(0,b) := \Omega. \\ u = g & \text{on } \partial\Omega. \end{cases}$$

— f and g are continuous.

By soln $u \in \underline{C^2(\Omega)} \cap C(\bar{\Omega})$ which satisfies (1).

$$A \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad + \quad B \begin{cases} \Delta u = f & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We start with the problem :-

$$\left.\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } \Omega. \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \text{— E.V.P}$$

Let, $u(x,y) = X(x)Y(y)$. is the soln of E.V.P

$$\Delta u = X''Y + Y''X$$

$$X''Y + Y''X + \lambda XY = 0 \implies \frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \implies \frac{X''}{X} = -\lambda - \frac{Y''}{Y} := \mu.$$

$$\boxed{X''Y + XY'' = \cancel{f(x,y)}}\ 0$$

$$= \cancel{\phi(x)\,\psi(y)}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\Delta u + \lambda u = 0$$

$$\Delta(cu) + \lambda(cu) = 0$$

$$\implies c(\Delta u + \lambda u) = 0$$

$$\implies$$

$X'' = \mu X$ — (I) $\quad X(0) = X(a) = 0.$

$Y'' + (\mu+\lambda)Y = 0$ — (II) $\quad Y(0) = Y(b) = 0.$

$\mu_n = -\left(\frac{n\pi}{a}\right)^2$; $\quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$; $n \in \mathbb{N}$

From (II)

$Y'' + \sigma Y = 0 \quad (\sigma = \mu + \lambda).$

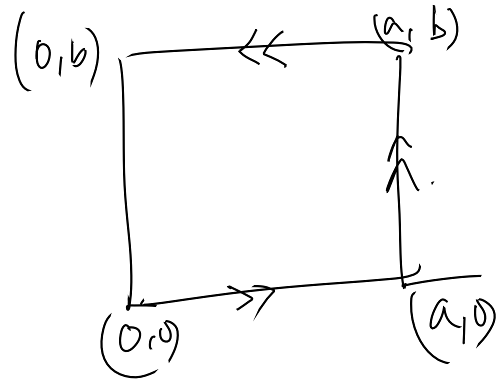
$Y(0) = Y(b) = 0.$

$\sigma_n = \left(\frac{m\pi}{b}\right)^2$; $m \in \mathbb{N}$, $\quad Y_n(y) = \sin\left(\frac{m\pi y}{b}\right).$

Eigenvalues :- $\quad \lambda_n = \sigma_n - \mu_n = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2.$

$u_{n,m}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$

$\therefore u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$



$-\lambda - \frac{Y''}{Y} = \mu$

$-\frac{Y''}{Y} = \mu + \lambda$

$Y'' = -(\mu+\lambda)Y$

(i) $u(x,0) = 0$

$X(x)Y(0) = 0 \Rightarrow Y(0) = 0$

(ii) $u(a,y) = 0$

$\Rightarrow X(a) = 0$

(iii) $Y(b) = 0$

(iv) $X(0) = 0$

Rough

$\int X''X = \int \mu X^2.$

$\Rightarrow -\int (X')^2 + \cancel{XX'\big|_0^a} = \mu\int X^2$

$\Rightarrow$

(B)
$$\left.\begin{array}{l}\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega\end{array}\right\} - (1)$$

Assume $\exists u$ solving (1).

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right).$$

$$u_{xx} = -\sum_{1}^{\infty}\sum_{1}^{\infty} u_{m,n}\left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

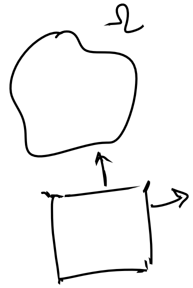
$$u_{yy} = -\sum_{1}^{\infty}\sum_{1}^{\infty} u_{m,n}\left(\frac{m\pi}{b}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

$$-\sum_{1}^{\infty}\sum_{1}^{\infty}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] u_{m,n} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) = f.$$

$\Delta u = f$ in Ω

$\frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega$

$\nabla u \cdot \eta = 0$



Double Fourier Series.

We want 'f' to be represented as double Fourier sine series.

$$f(x,y) = \sum_{1}^{\infty}\sum_{1}^{\infty} f_{m,n} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

To find $f_{m',n'}$:-

$$\int_0^a\int_0^b f(x,y)\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{b}\right)dx\,dy = \sum_{1}^{\infty}\sum_{1}^{\infty} f_{m,n}\int_0^a\int_0^b \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m'\pi y}{b}\right)dx\,dy$$

$$\frac{a}{2}\cdot\frac{b}{2}\cdot f_{m'n'} = \int_0^a\int_0^b f(x,y)\underbrace{\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{b}\right)}\,dx\,dy$$

$$\Downarrow$$

$$f_{m'n'} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\, U_{m'n'}\,dx\,dy$$

$$-\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]u_{m,n} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\,\underbrace{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)}\,dx\,dy$$

$$\Rightarrow u_{m,n} = -\frac{4}{ab\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]}\cdot\int_0^a\int_0^b f(x,y)\,\psi_{m,n}(x,y)\,dx\,dy$$