Solution: Tutorial 01

1. Truncation error

$$f(x) = e^x \cos(x)$$
$$f'(x) = e^x[\cos(x) - \sin(x)]$$
$$f''(x) = -2e^x \sin(x)$$
$$f'''(x) = -2e^x\left[\sin(x) + \cos(x)\right]$$

Second order Taylor series

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + R(\xi) = 1 + x + R(\xi)$$

$$\text{where, } R(\xi) = f'''(\xi)\frac{x^3}{3!} \text{ and } 0 \le \xi \le x$$

(i) Second order approximation of the function $\tilde{f}(x)$

$$x = 0.5 \quad \tilde{f}(0.5) = 1 + 0.5 = 1.5$$
$$x = \overset{1.0}{\cancel{0.5}} \quad \tilde{f}(1.0) = 1 + 1.0 = 2.0$$

(ii) True values of the function

$$f(0.5) = 0.14469 \times 10^1$$
$$f(1.0) = 0.14687 \times 10^1$$

True error

$$e = f(0.5) - \tilde{f}(0.5) = -0.531 \times 10^{-1}$$
$$e = f(1.0) - \tilde{f}(1.0) = -0.531 \times 10^0$$

Error bound

Since the magnitude of $f'''(x)$ monotonically increases and attains maximum value at $\pi/2$, hence the error bounds (ζ) will be

$$\text{for } x = 0.5 \quad \zeta \le \left|R(\xi = 0.5)\right| = 0.93222 \times 10^{-1}$$
$$x = 1.0 \quad \zeta \le \left|R(\xi = 1.0)\right| = 0.12520 \times 10^1$$

(iii) True values of the integral

$$I = \int_0^1 f(x)\mathrm{d}x = \int_0^1 \left[1 + x + R(\xi)\right]\mathrm{d}x = 1 + 0.5 + \int_0^1 R(\xi)\mathrm{d}x$$

Excluding the residual term

$$\tilde{I} = 1.5$$

True value of the integral

$$I = \int_0^1 e^x \cos(x)\mathrm{d}x = \frac{e^x}{2}\left[\cos(x) + \sin(x)\right]\Big|_0^1 = 0.13780 \times 10^1$$

True error

$$e = I - \tilde{I} = -0.1220$$

Error Bound

As argued in (ii)

$$\zeta \le \left|\int_0^1 R(\xi = 1.0)\,\mathrm{d}x\right| = \left|\int_0^1 1.2520x^3\mathrm{d}x\right| = 0.31300 \times 10^0$$

2. <u>Propagation of data error</u>

$$z = f(x,y) = x^2y - xy^2; \;\; x = 3.0,\; y = 2.0,\; \delta x = \delta y = 0.1$$

$$\frac{\partial f}{\partial x} = 2xy - y^2 = 8 \qquad \frac{\partial^2 f}{\partial x^2} = 2y = 4$$

$$\frac{\partial f}{\partial y} = x^2 - 2xy = -3 \qquad \frac{\partial^2 f}{\partial y^2} = -2x = -6$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x - 2y = 2$$

$$\begin{aligned}\delta z &= f(x+\delta x, y+\delta y) - f(x,y) \\ &= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\delta x)^2 + \frac{\partial^2 f}{\partial y^2}(\delta y)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\delta x \delta y)\right] + R\end{aligned}$$

(i) First order error analysis

$$\delta z \approx \left|\frac{\partial z}{\partial x}\delta x\right| + \left|\frac{\partial z}{\partial y}\delta y\right| = 0.8 + 0.3 = 1.1$$

(ii) Similarly, by second order error analysis

$$\delta z \approx 1.1 + \frac{1}{2}(0.04 + 0.06 + 0.04) = \cancel{1.8}\;\; 1.17$$

3. <u>Round-off error</u>

$f(x) = \sqrt{(x+1)} - \sqrt{x}$

(i) Condition number: For large values of x

$$C_p = \left| \frac{f'(x)x}{f(x)} \right| = \left| \frac{1}{2} \frac{\left[1/\sqrt{(x+1)} - 1/\sqrt{x}\right] x}{\sqrt{(x+1)} - \sqrt{x}} \right| = \frac{1}{2} \frac{x}{\sqrt{(x+1)}\sqrt{x}} \approx \frac{1}{2}$$

The problem is well-conditioned

(ii) Relative error

$x = 208208$

$\tilde{f}(x) = 0.456299 \times 10^3 - 0.456298 \times 10^3 = 0.1 \times 10^{-2}$

$f(x) = 0.1095774 \times 10^{-2}$

$$e_r = \frac{f(x) - \tilde{f}(x)}{f(x)} \times 100 = 8.74\%$$

(iii) Modified algorithm

Multiply and divide $f(x)$ by $\sqrt{(x+1)} + \sqrt{x}$

$$f(x) = \frac{1}{\sqrt{(x+1)} + \sqrt{x}}$$

$$\tilde{f}(x) = \frac{1}{0.456299 \times 10^3 + 0.456298 \times 10^3} = 0.109577 \times 10^{-2}$$

$$e_r = \frac{f(x) - \tilde{f}(x)}{f(x)} \times 100 = 0.4 \times 10^{-5}\%$$