Module 16
EQUALITY IN DISTRIBUTION

Definition 1: Random variables X and Y are said to have the same distribution (written as $X \stackrel{d}{=} Y$) if they have the same d.f., i.e., if $F_X(x) = F_Y(x),\ \forall\ x \in \mathbb{R}$.

Result 1: Let X and Y be r.v.s with p.m.f.s/p.d.f.s $f_X(\cdot)$ and $f_Y(\cdot)$, respectively. Then

(a) $f_X(x) = f_Y(x), \forall\ x \in \mathbb{R} \ \Rightarrow\ X \stackrel{d}{=} Y$;

(b) For some $h > 0$, $M_X(t) = M_Y(t), \forall\ t \in (-h, h) \ \Rightarrow\ X \stackrel{d}{=} Y$;

(c) $X \stackrel{d}{=} Y \ \Rightarrow\ h(X) \stackrel{d}{=} h(Y)$, for any function $h : \mathbb{R} \to \mathbb{R}$;

(d) $X \stackrel{d}{=} Y \ \Rightarrow\ E(h(X)) = E(h(Y))$, for any function $h : \mathbb{R} \to \mathbb{R}$ for which the expectation exists.

Proof: Proofs of (a), (c) and (d) are straightforward and hence omitted. Proof of (b) is based on uniqueness of m.g.f.s.

Example 1:

For $p \in (0,1)$, let Y_p be a r.v. having p.m.f.

$$f_{Y_p}(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y}, & \text{if } y \in \{0,1,\ldots,n\} \\ \\ 0, & \text{otherwise} \end{cases},$$

where $n \in \mathbb{N}$ (set of positive integers) is a fixed constant. Find the m.g.f. of Y_p and show that $n - Y_p \stackrel{d}{=} Y_{1-p}$.

Solution

$$\begin{aligned} M_{Y_p}(t) &= E(e^{tY_p}) \\ &= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y (1-p)^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{n-y} \\ &= (1-p+pe^t)^n, \quad t \in \mathbb{R}. \end{aligned}$$

Let $Z_p = n - Y_p,\ p \in (0,1)$. Then

$$\begin{aligned} M_{Z_p}(t) &= E(e^{t(n-Y_p)}) \\ &= e^{nt}E(e^{-tY_p}) \\ &= e^{nt}M_{Y_p}(-t) \\ &= e^{nt}(1-p+pe^{-t})^n \\ &= (p+(1-p)e^t)^n \\ &= M_{Y_{1-p}}(t), \quad \forall\ t \in \mathbb{R} \\ \Rightarrow \quad Z_p &\stackrel{d}{=} Y_{1-p}. \end{aligned}$$

Definition 2: A r.v. X is said to have a symmetric distribution about a point $\mu \in \mathbb{R}$ if $X - \mu \stackrel{d}{=} \mu - X$.

Remark 1: In Example 1

$$\begin{aligned} n - Y_{\frac{1}{2}} &\stackrel{d}{=} Y_{\frac{1}{2}} \\ \Rightarrow \quad \frac{n}{2} - Y_{\frac{1}{2}} &\stackrel{d}{=} Y_{\frac{1}{2}} - \frac{n}{2}, \end{aligned}$$

implying that the distribution of $Y_{\frac{1}{2}}$ is symmetric about $\frac{n}{2}$. Clearly

$$\begin{aligned} E\left(\frac{n}{2} - Y_{\frac{1}{2}}\right) &= E\left(Y_{\frac{1}{2}} - \frac{n}{2}\right) \\ \Rightarrow \quad E\left(Y_{\frac{1}{2}}\right) = \frac{1}{2}. \end{aligned}$$

Result 2:

Let X be a r.v. with p.m.f./p.d.f. $f_X(\cdot)$ and $d.f.$ $F_X(\cdot)$. Let $\mu \in \mathbb{R}$.

(a) If $f_X(\mu - x) = f_X(\mu + x)$, $\forall\ x \in \mathbb{R}$, then the distribution of X is symmetric about μ;

(b) Distribution of X is symmetric about μ iff $F_X(\mu + x) + F_X((\mu - x)-) = 1$;

(c) Distribution of X is symmetric about μ iff the distribution of $Y = X - \mu$ is symmetric about 0;

(d) If distribution of X is symmetric about μ and $E(X)$ exists then $\mu = E(X)$;

(e) If distribution of X is symmetric about μ then $F_X(\mu-) \leq \frac{1}{2} \leq F_X(\mu)$; ($F_X(\mu) = \frac{1}{2}$, if $F_X(\cdot)$ is continuous at μ);

(f) If distribution of X is symmetric about μ then $E((X - \mu)^{2m-1}) = 0,\ m \in \{1, 2, \ldots\}$, provided the expectations exist.

Proof.

(a) Let $Y_1 = X - \mu$ and $Y_2 = X - \mu$. Then

$$\begin{aligned} f_{Y_1}(y) = f_X(\mu + y) &= f_X(\mu - y) = f_{Y_2}(y), \quad \forall\, y \in \mathbb{R} \\ \Rightarrow \quad Y_1 &\overset{d}{=} Y_2. \end{aligned}$$

(b)

$$\begin{aligned} & X - \mu \overset{d}{=} \mu - X \\ \Leftrightarrow\ & P\left(\{X - \mu \le x\}\right) = P\left(\{\mu - X \le x\}\right), \quad \forall\, x \in \mathbb{R} \\ \Leftrightarrow\ & P\left(\{X \le \mu + x\}\right) = P\left(\{X \ge \mu - x\}\right), \quad \forall\, x \in \mathbb{R} \\ \Leftrightarrow\ & F_X(\mu + x) + F_X((\mu - x)-) = 1, \quad \forall\, x \in \mathbb{R}. \end{aligned}$$

(c) Let $Y_1 = X - \mu$. Then

$$\begin{aligned} X - \mu &\overset{d}{=} \mu - X \\ \Leftrightarrow\ X - \mu &\overset{d}{=} -(X - \mu) \\ \Leftrightarrow\ Y_1 - 0 &\overset{d}{=} 0 - Y_1. \end{aligned}$$

(d)

$$\begin{aligned} X-\mu &\stackrel{d}{=} \mu - X \\ \Rightarrow\ E(X-\mu) &= E(\mu-X) \\ \Leftrightarrow\ E(X) &= \mu. \end{aligned}$$

(e) By (b)

$$F_X(\mu+x)+F_X((\mu-x)-)=1 \quad \forall\, x\in\mathbb{R}$$

$$\Rightarrow\ F_X(\mu)+F_X(\mu-)=1$$

$$\Rightarrow\ F_X(\mu-)\le \frac{1}{2}\le F_X(\mu)\ \ (\text{since } F_X(\mu-)\le F_X(\mu)).$$

(f)

$$\begin{aligned} X-\mu &\stackrel{d}{=} \mu - X \\ \Rightarrow\ E((X-\mu)^{(2m-1)}) &= E((\mu-X)^{(2m-1)}) \\ \Rightarrow\ E((X-\mu)^{(2m-1)}) &= -E((X-\mu)^{(2m-1)}) \\ \Rightarrow\ E((X-\mu)^{(2m-1)}) &= 0, \quad m=1,2,\ldots. \end{aligned}$$

Example 2:

Let X be a r.v. having the p.d.f.

$$f_X(x) = \frac{1}{\pi}.\frac{1}{1+(x-\mu)^2}, \quad -\infty < x < \infty.$$

Clearly,

$$\begin{aligned} f_X(\mu + x) &= f_X(\mu - x), \ \forall \ x \in \mathbb{R} \\ \Rightarrow \ X - \mu &\stackrel{d}{=} \mu - X. \end{aligned}$$

However $E(X)$ does not exists.

Take Home Problems

1. Let X be a r.v. having a p.d.f.

$$f_X(x) = \frac{1}{2} e^{-|x|}, \ -\infty < x < \infty.$$

Show that the distribution of X is symmetric about zero. Hence find $E(X)$ (Does it exists?).

2. Let X be a r.v. having the m.g.f.

$$M_X(t) = e^{\frac{t^2}{2}}, \ -\infty < t < \infty.$$

Show that $X \stackrel{d}{=} -X$ (i.e., distribution of X is symmetric about zero);

$$E(X^{2r-1}) = 0, \ r \in \{1, 2, \dots\},$$

and

$$E(X^{2r}) = \frac{(2r)!}{2^r r!}, \ r \in \{1, 2, \dots\}.$$

Abstract of Next Module

Let $A \subseteq \mathbb{R}$, $g = \mathbb{R} \to \mathbb{R}$ and let X be a r.v. In many situations $P(\{X \in A\})$ or $E(g(X))$ can not be evaluated precisely. In such situations some approximations of $P(\{X \in A\})$ or $E(g(X))$ may be useful. Some useful approximations can be provided in form of inequalities.

Thank you for your patience