MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-XI

A. Illustrative Discussion Problems

1. Suppose that the lifetime of electric bulbs manufactured by a manufacturer follows exponential distribution with mean of 50 hours. Eight such bulbs are chosen at random.

   (a) Find the probability that, among eight chosen bulbs, 2 will last less than 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours;

   (b) Find the expected number of bulbs in the lot of chosen 8 bulbs with lifetime between 60 and 80 hours;

   (c) Find the expected number of bulbs in the lot of 8 chosen bulbs with lifetime between 60 and 80 hours, given that the number of bulbs in the lot with lifetime anywhere between 40 and 60 hours is 2.

2. Suppose that $\underline{X} = (X_1, X_2, X_3, X_4) \sim \text{Mult}(30, \theta_1, \theta_2, \theta_3, \theta_4)$. Find the conditional probability mass function of (X_1, X_2, X_3, X_4) given that $\sum_{i=1}^{4} X_i = 28$.

3. Suppose that $\underline{X} \sim N_2(0, 0, 1, 1, 0)$. Find c_1 such that $P(-c_1 \leq X_1 \leq c_1, -c_1 \leq X_2 \leq c_1) = 0.95$.

4. Let $\underline{X} = (X_1, X_2)' \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and, for constants a_1, a_2, a_3 and a_4 ($a_i \neq 0$, $i = 1, 2, 3, 4$, $a_1 a_4 \neq a_2 a_3$), let $Y = a_1 X_1 + a_2 X_2$ and $Z = a_3 X_1 + a_4 X_2$.

   (a) Find the joint p.d.f. of (Y, Z);

   (b) Find the marginal p.d.f.s. of Y and Z.

5. (a) Let $(X, Y)' \sim N_2(5, 8, 16, 9, 0.6)$. Find $P(\{5 < Y < 11\}|\{X = 2\})$, $P(\{4 < X < 6\})$ and $P(\{7 < Y < 9\})$;

   (b) Let $(X, Y)' \sim N_2(5, 10, 1, 25, \rho)$, where $\rho > 0$. If $P(\{4 < Y < 16\}|\{X = 5\}) = 0.954$, determine ρ.

6. Let $\underline{X} = (X_1, X_2)' \sim N_2(0, 0, 1, 1, \rho)$.

   (a) Find the m.g.f. of $Y = X_1 X_2$;

   (b) Using (a), find $E(X_1^2 X_2^2)$;

   (c) Using conditional expectation, find $E(X_1^2 X_2^2)$.

7. Let $f_r(\cdot,\cdot)$, $-1 < r < 1$, denote the pdf of $N_2(0,0,1,1,r)$ and, for a fixed $\rho \in (-1,1)$, let the random variable (X,Y) have the joint p.d.f.

$$g_\rho(x,y) = \frac{1}{2}[f_\rho(x,y) + f_{-\rho}(x,y)].$$

   (a) Show that X and Y are normally distributed but the distribution of (X,Y) is not bivariate normal;

   (b) Find $\mathrm{Corr}(X,Y)$;

   (c) Are X and Y independent?

8. Let X and Y be i.i.d. $N(0,1)$ random variables. Define the random variables R and Θ by $X = R\cos\Theta$, $Y = R\sin\Theta$.
   (a) Show that R and Θ are independent with $\frac{R^2}{2} \sim \mathrm{Exp}(1)$ and $\Theta \sim U(0,2\pi)$.
   (b) Show that X^2+Y^2 and $\frac{X}{Y}$ are independently distributed.
   (c) Show that $\sin\Theta$ and $\sin 2\Theta$ are identically distributed and hence find the pdf of $T = \frac{XY}{\sqrt{X^2+Y^2}}$.
   (d) Find the distribution of $U = \frac{3X^2Y-Y^3}{X^2+Y^2}$.
   (e) Let U_1 and U_2 be i.i.d. $U(0,1)$ r.v.s. Then show that $X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ are i.i.d. $N(0,1)$ r.v.s. (This is known as the Box-Muller transformation).

9. Let $(X,Y)' \sim N_2(0,0,1,1,\rho)$.
   (a) Show that $P(X>0,Y>0) = \frac{1}{4} + \frac{\arcsin\rho}{2\pi}$. Also find $P(X<0,Y<0)$, $P(X>0,Y<0)$ and $P(X<0,Y>0)$.
   (b) Show that $P(XY>0) = \frac{1}{2} + \frac{\arcsin\rho}{\pi}$ and $P(XY<0) = \frac{1}{2} - \frac{\arcsin\rho}{\pi}$.

B. Practice Problems

1. Consider a sample of size 3 drawn with replacement from an urn containing 3 white, 2 black and 3 red balls. Let the random variables X_1 and X_2 respectively denote the number of white balls and the number of black balls in the sample.

   (a) Find the correlation between X_1 and X_2. Are X_1 and X_2 independent?;

   (b) Find the marginal p.m.f.s of X_1 and $Z = X_1 + X_2$;

   (c) Find $\mathrm{Var}(X_1+X_2)$;

2. Let $\underline{X} = (X_1,X_2)'$ have the joint p.d.f.

$$f(x,y) = \begin{cases} \frac{1}{\pi}e^{-\frac{1}{2}(x^2+y^2)}, & \text{if } xy > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Show that marginals of $f(\cdot,\cdot)$ are each $N(0,1)$ but $f(\cdot,\cdot)$ is not the p.d.f. of a bivariate normal distribution.

3. Let $\underline{X} = (X_1, X_2)'$ have the joint p.d.f.

$$f_{\underline{X}}(x_1, x_2) = \phi(x_1)\phi(x_2)[1+\alpha(2\Phi(x_1)-1)(2\Phi(x_2)-1)], \;\; -\infty < x_i < \infty,\; i = 1, 2,\;\; |\alpha| \leq 1.$$

(a) Verify that $f_{\underline{X}}(x_1, x_2)$ is a p.d.f.;

(b) Find the marginal p.d.f.s of X_1 and X_2;

(c) Is (X_1, X_2) jointly normal?

4. Let $(X, Y)' \sim N_2(0, 0, 1, 1, \rho)$.

(a) Show that $X + Y$ and $X - Y$ are independent. Find their distributions.

(b) Find $P(X + Y > 2|X < Y)$.

MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment XI (Solutions)

$\boxed{\text{Problem No. 1}}$ X: r.v. denoting the lifetime of electric bulb $\sim Exp(50)$

$$f_X(x) = \begin{cases} \frac{1}{50} e^{-\frac{x}{50}}, & \text{if } x>0 \\ 0, & \text{otherwise} \end{cases};\quad F_X(x) = P(X \le x) = \begin{cases} 1-e^{-\frac{x}{50}}, & \text{if } x>0 \\ 0, & \text{o.w.} \end{cases}$$

X_1 = # of bulbs out of 8 having lifetime < 40 hrs
X_2 = " " " " " " " $\in [40, 60)$ hrs
X_3 = " " " " " " " $\in [60, 80)$ hrs
X_4 = # " " " " " " $\in [80, \infty)$ hrs

$(X_1, X_2, X_3) \sim Mult(8, \theta_1, \theta_2, \theta_3)$,

where $\theta_1 = P(X < 40) = 1-e^{-40/50}$

$\theta_2 = P(40 \le X < 60) = e^{-40/50} - e^{-60/50}$

$\theta_3 = P(60 \le X < 80) = e^{-60/50} - e^{-80/50}$

Let $\theta_4 = 1 - \sum_{i=1}^{3} \theta_i = e^{-80/50}$

(a) Required prob $= P(X_1 = 2, X_2 = 3, X_3 = 2)$

$$= \frac{\underline{|8}}{\underline{|2}\ \underline{|3}\ \underline{|2}\ \underline{|1}} \theta_1^2 \theta_2^3 \theta_3^2 \theta_4^1$$

(b) $X_3 \sim Bin(8, \theta_3) \Rightarrow E(X) = 8\theta_3$

(c) It is easy to verify that $X_3 | X_2 = x_2 \sim Bin\left(8-x_2, \frac{\theta_3}{1-\theta_2}\right)$

$E(X_3 | X_2 = 2) = \frac{6\theta_3}{1-\theta_2}$

$\boxed{\text{Problem No. 2}}$ For $\sum_{i=1}^{4} x_i = 28$, $0 \le x_i \le 28$, $i = 1(1)4$

$P(X_1 = x_1, \ldots, X_4 = x_4 | \sum_{i=1}^{4} X_i = 28)$

$$= \frac{P(X_1 = x_1, \ldots, X_4 = x_4)}{P(\sum_{i=1}^{4} X_i = 28)} = \frac{\frac{\underline{|30}}{\underline{|x_1}\cdots\underline{|x_4}\ \underline{|2}} \theta_1^{x_1}\cdots\theta_4^{x_4} (1-\sum_{i=1}^{4}\theta_i)^2}{\binom{30}{28} (\sum_{i=1}^{4}\theta_i)^{28} (1-\sum_{i=1}^{4}\theta_i)^2} \quad \left(\sum_{i=1}^{4} X_i \sim Bin(30, \sum_{i=1}^{4}\theta_i)\right)$$

$$= \frac{\underline{|28}}{\underline{|x_1}\cdots\underline{|x_4}} \left(\frac{\theta_1}{1-\theta_5}\right)^{x_1} \cdots \left(\frac{\theta_4}{1-\theta_5}\right)^{x_4}, \quad \text{where } \theta_5 = 1 - \sum_{i=1}^{4}\theta_i.$$

$\Rightarrow (X_1, X_2, X_3) | \sum_{i=1}^{4} X_i = 28 \sim Mult\left(28, \frac{\theta_1}{1-\theta_5}, \frac{\theta_2}{1-\theta_5}, \frac{\theta_3}{1-\theta_5}\right)$.

Problem No. 3 $\rho=0 \Rightarrow X_1$ and X_2 are i.i.d. $N(0,1)$.

Therefore

$$P(-c_1 \le X_1 \le c_1,\ -c_1 \le X_2 \le c_1) = 0.95 \iff [\Phi(c_1) - \Phi(-c_1)]^2 = 0.95$$

$$\Rightarrow [2\Phi(c_1)-1]^2 = 0.95 \Rightarrow \Phi(c_1) = \frac{1+\sqrt{0.95}}{2}$$

$$\Rightarrow c_1 = \Phi^{-1}\left(\frac{1+\sqrt{0.95}}{2}\right).$$

Problem No. 4 We know that $(Y_1, Y_2)' \sim N_2 \iff$ every linear combination of Y_1 and Y_2 is univariate normal.

(a) Consider $t_1 Y + t_2 Z = (t_1 a_1 + t_2 a_3) X_1 + (t_1 a_2 + t_2 a_4) X_2$
$\sim N_1$ (since $(X_1, X_2)' \sim N_2$) $\Rightarrow (Y, Z)' \sim N_2$

(b) $(Y, Z)' \sim N_2 \Rightarrow Y \sim N_1,\ Z \sim N_1 \Rightarrow E(Y) = a_1\mu_1 + a_2\mu_2 = \mu_Y$, say.
$E(Z) = a_3\mu_1 + a_4\mu_2 = \mu_Z$, say. $Var(Y) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\rho\sigma_1\sigma_2$
$= \sigma_Y^2$, say. $Var(Z) = a_3^2\sigma_1^2 + a_4^2\sigma_2^2 + 2a_3a_4\rho\sigma_1\sigma_2 = \sigma_Z^2$, say.
Then $Y \sim N_1(\mu_Y, \sigma_Y^2)$, $Z \sim N_1(\mu_Z, \sigma_Z^2)$.

Problem No. 5 (a) $Y|X=2 \sim N_1\left(8 + \frac{0.6\times 3}{4}(2-5),\ 9(1-(0.6)^2)\right)$
$= N_1(6.65, 5.76)$

$$P(5<Y<11|X=2) = \Phi\left(\frac{11-6.65}{\sqrt{5.76}}\right) - \Phi\left(\frac{5-6.65}{\sqrt{5.76}}\right)$$

$$= \Phi(1.8125) - \Phi(-0.6875) \approx \Phi(1.8) + \Phi(.7) - 1 = .964 + .758 - 1$$
$$= .722$$

(b) $Y|X=5 \sim N_1\left(10 + \frac{\rho\times 5}{1}(5-5),\ 25(1-\rho^2)\right) = N_1(0, 25(1-\rho^2))$

$$P(4<Y<16|X=5) = 0.954 \Rightarrow \Phi\left(\frac{16-0}{5\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{4-10}{5\sqrt{1-\rho^2}}\right) = 0.954$$

$$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{-6}{5\sqrt{1-\rho^2}}\right) = 0.954 \Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = \frac{1+.954}{2} = .977$$

$$\Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow \rho^2 = \frac{16}{25} \Rightarrow \rho = \frac{4}{5} = 0.8\ (\rho>0).$$

Problem No. 6 (a) $M_Y(t) = E(e^{tX_1X_2}) = E\left(E(e^{tX_1X_2}|X_1)\right)$

$X_2|X_1 = x \sim N(\rho x, 1-\rho^2) \Rightarrow E(e^{tX_1X_2}|X_1) = e^{\rho t X_1^2 + \frac{(1-\rho^2)t^2 X_1^2}{2}}$
$= e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}$

$X_1^2 \sim \chi_1^2 \Rightarrow M_Y(t) = E\left(e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}\right) = [1-2\rho t-(1-\rho^2)t^2]^{-\frac{1}{2}}$,
$-\frac{1}{1-\rho} < t < \frac{1}{1+\rho}$.

(b) $M_Y'(t) = \frac{1}{2}\left[1-2\rho t-(1-\rho^2)t^2\right]^{-\frac{3}{2}}\left[2\rho+2(1-\rho^2)t\right]$

$$M_Y''(t) = \frac{3}{4}\left[1-2\rho t-(1-\rho^2)t^2\right]^{-5/2}\left[2\rho+2(1-\rho^2)t\right]^2$$
$$+\left[1-2\rho t-(1-\rho^2)t^2\right]^{-3/2}(1-\rho^2)$$

$E(Y^2) = M_Y''(0) = 1+2\rho^2.$

(c) $E(X_1^2X_2^2) = E(X_1^2E(X_2^2|X_1)) = E(X_1^2(1-\rho^2+\rho^2X_1^2))$

$= 1-\rho^2+\rho^2E(X_1^4) = 1-\rho^2+3\rho^2 = 1+2\rho^2 \quad (X_1\sim N(0,1))$

$\boxed{\text{Problem No. 7}}$ (a) $f_X(x) = \int_{-\infty}^{\infty} g_\rho(x,y)\,dy = \frac{1}{2}\left[\int_{-\infty}^{\infty} f_\rho(x,y)\,dy + \int_{-\infty}^{\infty} f_{-\rho}(x,y)\,dy\right]$

$= \frac{1}{2}$ [pdf of $N(0,1)$ at point x + pdf of $N(0,1)$ at point x]

$= \frac{1}{2}[\phi(x)+\phi(x)] = \phi(x), \quad -\infty<x<\infty \quad \Rightarrow X\sim N(0,1)$

By symmetry $Y\sim N(0,1)$

Clearly $g_\rho(x,y)$ is not the p.d.f. of bivariate normal distribution.

(b) Since $X, Y\sim N(0,1)$, $E(X)=E(Y)=0$, $Var(X)=Var(Y)=1$

$$E(XY) = \frac{1}{2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f_\rho(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f_{-\rho}(x,y)\,dx\,dy\right]$$

$= \frac{1}{2}[\rho+(-\rho)] = 0 \quad \Rightarrow \quad Corr(X,Y)=0$

(c) Clearly X and Y are not independent.

$\boxed{\text{Problem No. 8}}$ $f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}, \quad (x,y)\in\mathbb{R}^2 = S_{X,Y}.$

(a) Consider the 1-1 transformation $(X,Y)\to(R,\Theta)$ defined by $X=R\cos\Theta$ $Y=R\sin\Theta$

$S_{R,\Theta} = \{(r,\theta): r\geq 0, 0\leq\theta\leq 2\pi\}$, $J = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r$

Thus

$$f_{R,\Theta}(r,\theta) = \begin{cases}\frac{1}{2\pi}e^{-\frac{r^2}{2}}|r|, & \text{if } r\geq 0,\ 0\leq\theta\leq 2\pi\\ 0, & \text{o.w.}\end{cases}$$

Clearly R and Θ are independent with

$$f_R(r) = \begin{cases} re^{-r^2/2}, & r\geq 0\\ 0, & \text{o.w.}\end{cases}; \quad f_\Theta(\theta) = \begin{cases}\frac{1}{2\pi}, & 0\leq\theta\leq 2\pi\\ 0, & \text{o.w.}\end{cases}$$

Clearly $\frac{R^2}{2} \sim \text{Exp}(1)$ and $\Theta \sim U(0, 2\pi)$

(b) $X^2+Y^2=R^2$, $\frac{X}{Y}=\cot\Theta$

Since R and Θ are independent, it follows that R^2 and $\cot\Theta$ are independent i.e. X^2+Y^2 and $\frac{X}{Y}$ are independent.

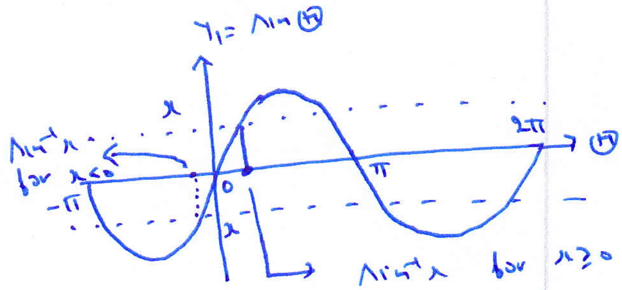
(c) $Y_1=\sin\Theta \in [-1, 1]$

For $-1 \le x < 0$

$$F_{Y_1}(x) = P(Y_1 \le x) = P(\sin\Theta \le x)$$

$$= P(\pi - \sin^{-1}x \le \Theta \le 2\pi + \sin^{-1}x)$$

$$= \frac{1}{2\pi}(\pi + 2\sin^{-1}x) = \frac{1}{2} + \frac{\sin^{-1}x}{\pi}$$



For $0 \le x \le 1$

$$F_{Y_1}(x) = P(Y_1 \le x) = 1 - P(\sin^{-1}x \le \Theta \le \pi - \sin^{-1}x)$$

$$= 1 - \frac{1}{2\pi}[\pi - 2\sin^{-1}x] = \frac{1}{2} + \frac{\sin^{-1}x}{\pi}.$$

Thus
$$F_{Y_1}(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{2} + \frac{\sin^{-1}x}{\pi}, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}$$

Now let $Y_2 = \sin 2\Theta \in [-1, 1]$.

For $-1 \le x < 0$

$$F_{Y_2}(x) = P(\sin 2\Theta \le x) = P(\pi - \sin^{-1}x \le 2\Theta \le 2\pi + \sin^{-1}x) + P(3\pi - \sin^{-1}x \le 2\Theta \le 4\pi + \sin^{-1}x)$$

$$= P\left(\frac{\pi}{2} - \frac{\sin^{-1}x}{2} \le \Theta \le \pi + \frac{\sin^{-1}x}{2}\right) + P\left(\frac{3\pi}{2} - \frac{\sin^{-1}x}{2} \le \Theta \le 2\pi + \frac{\sin^{-1}x}{2}\right)$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{2} + \sin^{-1}x\right] + \frac{1}{2\pi}\left[\frac{\pi}{2} + \sin^{-1}x\right] = \frac{1}{2} + \frac{\sin^{-1}x}{\pi}$$

For $0 \le x \le 1$

$$F_{Y_2}(x) = 1 - \left[P(\sin^{-1}x \le 2\Theta \le \pi - \sin^{-1}x) + P(2\pi + \sin^{-1}x \le 2\Theta \le 3\pi - \sin^{-1}x)\right]$$

$$= 1 - \left[P\left(\frac{\sin^{-1}x}{2} \le \Theta \le \frac{\pi}{2} - \frac{\sin^{-1}x}{2}\right) + P\left(\pi + \frac{\sin^{-1}x}{2} \le \Theta \le \frac{3\pi}{2} - \frac{\sin^{-1}x}{2}\right)\right]$$

$$= 1 - \frac{1}{2\pi}\left(\frac{\pi}{2} - \sin^{-1}x\right) - \frac{1}{2\pi}\left(\frac{\pi}{2} - \sin^{-1}x\right) = \frac{1}{2} + \frac{\sin^{-1}x}{\pi}.$$

Thus

$$F_{Y_2}(x) = \begin{cases} 0, & \text{if } x<-1 \\ \frac{1}{2} + \frac{\sin^{-1}x}{\pi}, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases} = F_{Y_1}(x) \quad \forall x \in \mathbb{R}$$

$\Rightarrow Y_1 \stackrel{d}{=} Y_2 \Rightarrow \sin\Theta \stackrel{d}{=} \sin 2\Theta$

$$T = \frac{XY}{\sqrt{X^2+Y^2}} = \frac{R^2 \cos\Theta \sin\Theta}{R} = \frac{R\sin 2\Theta}{2} = \frac{RY_2}{2} \stackrel{d}{=} \frac{RY_1}{2}$$

$\stackrel{d}{=} \frac{R\sin\Theta}{2} = \frac{Y}{2} \sim N(0, \frac{1}{4})$ $\left(\begin{array}{l} R \text{ and } \Theta \text{ are independent and} \\ Y_1 \stackrel{d}{=} Y_2 \Rightarrow (R, Y_1) \stackrel{d}{=} (R, Y_2) \end{array}\right)$

$\Rightarrow f_T(t) = \sqrt{\frac{2}{\pi}}\, e^{-2t^2}, \quad -\infty < t < \infty.$

(d) $U = \frac{3R^3\cos^2\Theta\sin\Theta - R^3\sin^3\Theta}{R^2} = R\sin\Theta(3\cos^2\Theta - \sin^2\Theta)$

$= R\sin 3\Theta$

As in (c) $\sin\Theta \stackrel{d}{=} \sin 3\Theta$ and thus using independence of R and Θ, $(R, \sin\Theta) \stackrel{d}{=} (R, \sin 3\Theta) \Rightarrow R\sin 3\Theta \stackrel{d}{=}$ $R\sin\Theta = Y \sim N(0,1)$

(e) Clearly $-\ln U_1 \sim Exp(1)$ and $2\pi U_2 \sim U(0, 2\pi)$

$\Rightarrow -\ln U_1 \sim Exp(1)$ and $2\pi U_2 \sim U(0, 2\pi)$ are independent

$\Rightarrow (-\ln U_1, 2\pi U_2) \stackrel{d}{=} (\frac{R^2}{2}, \Theta)$ (by (a))

$\Rightarrow (X, Y) \stackrel{d}{=} (R\cos\Theta, R\sin\Theta) \stackrel{d}{=} (X_1, X_2)$

$\Rightarrow X_1$ and X_2 are i.i.d. $N(0,1)$ r.v.s.

$\boxed{\text{Problem No. 9}}$ The joint p.d.f of (X, Y) is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}, \quad -\infty < x, y < \infty.$$

(a) Consider the transformation $(X, Y) \to (R, \Theta)$ defined by $X = R\cos\Theta$, $Y = R\sin\Theta$. Then $S_{R,\Theta} = [0,\infty) \times [0, 2\pi)$ and $J = r$. Thus

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{r^2}{2(1-\rho^2)}(1-2\rho\sin\theta\cos\theta)}, \quad r \ge 0,\ 0 \le \theta \le 2\pi$$

For $0 \le \theta \le 2\pi$

$$f_{\Theta}(\theta) = \int_0^\infty \frac{r}{2\pi\sqrt{1-\rho^2}} e^{-\frac{r^2}{2(1-\rho^2)}(1-2\rho\cos\theta\sin\theta)} \, d\theta$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi\,(1-2\rho\sin\theta\cos\theta)}, \quad 0 \le \theta \le 2\pi$$

Thus

$$P(X>0, Y>0) = P\left(0 \le \Theta \le \frac{\pi}{2}\right) = \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\pi/2} \frac{d\theta}{1-2\rho\sin\theta\cos\theta}$$

$$= \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^{\pi/2} \frac{\sec^2\theta}{\sec^2\theta - 2\rho\tan\theta}\,d\theta = \frac{\sqrt{1-\rho^2}}{2\pi}\int_0^\infty \frac{dz}{1+z^2-2\rho z}$$

$$= \frac{1}{\sqrt{1-\rho^2}\;2\pi}\int_0^\infty \frac{dz}{1+\left(\frac{z-\rho}{\sqrt{1-\rho^2}}\right)^2} = \frac{1}{2\pi}\int_{-\frac{\rho}{1-\rho^2}}^{\infty} \frac{dt}{1+t^2}\,dt =$$

$$\frac{1}{2\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)\right] = \frac{1}{4} - \frac{1}{2\pi}\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

Let $\tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right) = v \Rightarrow \tan v = \frac{-\rho}{\sqrt{1-\rho^2}} \Rightarrow \sec^2 v =$

$1+\tan^2 v = \frac{1}{1-\rho^2} \Rightarrow \sin^2 v = \rho^2 \Rightarrow |\sin v| = |\rho|$

Case I: $-1 < \rho \le 0$

Then $v \in [0, \frac{\pi}{2}] \Rightarrow \sin v = -\rho \Rightarrow v = \sin^{-1}(-\rho) = -\sin^{-1}\rho$

$\Rightarrow P(X>0, Y>0) = \frac{1}{4} + \frac{\sin^{-1}\rho}{2\pi}$

Case II $0 < \rho < 1$

Then $v \in (-\frac{\pi}{2}, 0) \Rightarrow -\sin v = \rho \Rightarrow v = \sin^{-1}(-\rho) = -\sin^{-1}\rho$

$\Rightarrow P(X>0, Y>0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho.$

Thus $P(X>0, Y>0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho; \quad -1 < \rho < 1.$

Note that $(-X, -Y) \sim N_2(0, 0, 1, 1, \rho)$, $(-X, Y) \sim N_2(0, 0, 1, 1, -\rho)$,

$(X, -Y) \sim N_2(0, 0, 1, 1, -\rho)$. Thus

$P(X<0, Y<0) = P(-X>0, -Y>0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho$

$P(X>0, Y<0) = P(X>0, -Y>0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-\rho) = \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho$

$P(X<0, Y>0) = P(-X>0, Y>0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-\rho) = \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho$

$P(XY>0) = P(X>0, Y>0) + P(X<0, Y<0)$ $\boxed{6/6}$

$= \frac{1}{2} + \frac{\sin^{-1}\rho}{\pi}$

$P(XY<0) = 1 - P(XY>0) = \frac{1}{2} - \frac{\sin^{-1}\rho}{\pi}.$