CS203B, Assignment 1

Prof. Manindra Agarwal

August 2016

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1. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, i.e. a natural number modulo n. Define an operation ($\oplus$) on this set as $(a + b)$ modulo n $\forall a, b \in \mathbb{Z}_n$. Prove that $\mathbb{Z}_n$ under $\oplus$ is a group.[2]

2. Let S be a maximal subset of $\mathbb{Z}_n$ such that S forms a group w.r.t multiplication $(modn)$. What are the possible elements of S ? Elaborate and prove why S, constructed by your method, is a group.[8]

3. Prove that $(\mathbb{R}/\{0\}, *) \ncong (\mathbb{R}, +)$[5]

4. In the class we discussed about the direct product of groups under obvious operation. Prove the following:

   (a) Any group of order 4 is either cyclic or isomorphic to $\mathbb{Z}_2 \mathrm{X} \mathbb{Z}_2$, Where $\mathbb{Z}_2$ is the residue group $mod2$. Look up the definition of cyclic group, it's more or less intuitive. [8]

   (b) Prove that all groups of order ≤ 5 are abelian.[7]

5. In the class you were taught about quotienting with a subgroup. It is important to note that quotienting with a subgroup will not always give you a group. In this exercise we will figure out when does this happen.

   (a) Let H be a subgroup of G and $\frac{G}{H}$ be the quotient set. Prove that for $\frac{G}{H}$ to be a group under $\oplus$ defined by $(a+H) \oplus (b+H) = (a+b)+H$, H should have the following property

   $$a + H = H + a \ \forall a \in G$$

   We call such a group to be normal subgroup of G.[8]

   (b) Prove $Z(G) = \{a \in G | ab = ba \forall b \in G\}$. Prove that $Z(G)$ is a normal subgroup of G. $Z(G)$ is called the center of the group.[5]

   (c) Let ϕ be a homomorphism from a group G to another group H. Define

$$K = \{g \in G | \phi(G) = 1\}$$

Now prove that K is normal subgroup of G. Also show that $K = \{1\}$ iff ϕ is an isomorphism (recall your MTH102)[7]

6. Define $Q = \{-1, i, j, k | (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1\}$ where 1 is identity and -1 commutes with other elements of the group. Prove that Q is isomorphic to $H = \{x, y | x^4 = y^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\}$. Find all the elements of the group Q and also determine their order.
   This group is known as a group of Quaternions and a notable fact is that its all subgroups are normal but the group is still not abelian (check!!).[20]