CS201A/201: Math for CS I/Discrete Mathematics midsem

Max marks: 100
Time: 120 mins. 20-Sep-2014

1. *Answer all 4 questions. The paper has 2 pages. Hints are part of the question. Use them.*
2. **Please start each answer to a question on a fresh page. And keep answers of parts of a question together.**
3. *Just writing a number/final value will not get you credit. You must calculate/justify your answers.*
4. *You can consult* **only your own handwritten notes and my notes that I put up on the course site**. *Nothing else is allowed.*

1. (a) Term T_n is given by the following equation:

$$T_n = \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!}$$

First conjecture a simpler formula for T_n only in terms of n and then prove that it calculates T_n using induction.

Solution:
We first calculate T_n for a few values of n.

n	T_n
1	$\frac{1}{2}$
2	$\frac{5}{6}$
3	$\frac{23}{24}$
4	$\frac{119}{120}$

We see that the numerator is 1 less than the denominator so a reasonable conjecture is:

$$T_n = \frac{(n+1)! - 1}{(n+1)!}.$$

We now prove this by induction.
It is clearly true for $n = 1$,(acutally from the table also for $n = 2, 3, 4$ as well). Assume true for n. That is: $T_n = \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = \frac{(n+1)!-1}{(n+1)!}$.

For $(n+1)$ we get:

$$\begin{aligned}
T_{n+1} &= \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} \\
&= \frac{(n+1)! - 1}{(n+1)!} + \frac{n+1}{(n+2)!} \\
&= \frac{(n+2)! - (n+2) + (n+1)}{(n+2)!} \\
&= \frac{(n+2)! - 1}{(n+2)!}
\end{aligned}$$

(b) Consider the sequence defined by $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with $a_1 = 1,\ a_2 = 2,\ a_3 = 3$. i) Argue using induction that $a_n < 2^n$ clearly stating the inductive hypothesis. ii) What is the difference between the inductive hypothesis in part (a) and the current hypothesis?

Solution:
The hypothesis is clearly true for $n = 1, 2, 3$ and for $n = 4$ we get $a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 < 2^4$.
Inductive hypothesis: Assume hypothesis is true for $n,\ (n-1),\ (n-2)$. Then for $n+1$ we get:

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} + a_{n-2} \\
&< 2^n + 2^{n-1} + 2^{n-2} \\
&< 2^{n-2}(2^2 + 2 + 1) \\
&< 2^{n-2}7 \\
&< 2^{n-2}2^3 \\
&< 2^{n+1}
\end{aligned}$$

The inductive hypothesis is a version of the strong inductive hypothesis since it is assumed for $n,\ n-1,\ n-2$ unlike the normal case when it is assumed to hold only for n.

(c) i. Argue that the sum of a rational number and an irrational number is always irrational.

Solution:
Let r be the irrational number and $q = \frac{m}{n}$ the rational number. Proof is by contradiction. Assume $r + q$ is rational. So, $r + q = \frac{a}{b}$. Then we get $r = \frac{a}{b} - \frac{m}{n} = \frac{an - bm}{bn}$. Since $a, b, m.n \in \mathbb{Z}$ $r = \frac{c}{d}$ where $c, d \in \mathbb{Z}$. This is a contradiction since r is irrational.

ii. On the other hand, show that the sum of two irrational numbers can be rational.

Solution:
Consider $x = \frac{1}{4} + \sqrt{2}$ and $y = \frac{1}{4} - \sqrt{2}$. Then by (i) both $x,\ y$ are irrational but $x + y = \frac{1}{2}$ a rational. On the other hand $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is irrational since if it was rational then $\sqrt{2}$ would be rational. So, a sum of irrationals can be either rational or irrational.

(d) $a, b, c, d \in \mathbb{N}$. Both $a,\ b$ are divisible by both $c,\ d$ (written, $c|a, c|b$ and $d|a,\ d|b$). Also, $c \not| d$ (c does not divide d). Someone gives the following proof that cd divides both a and b.

Proof. Since $d|a$ and $d|b$ there exisit $x,\ y$ such that:

$$a = xd \text{ and } b = yd \qquad (1)$$

. Also, c divides both $a,\ b$ so we have $c|dx$ and $c|dy$. But since $c \not| d$ it must be the case that $c|x$ and $c|y$ that is $x = uc$ and $y = vc$ for some $u, v \in \mathbb{N}$. Substituting for x and y in (1) we get $a = ucd$ and $b = vcd$. So, cd divides both $a,\ b$. □

i. Is the above proof correct? If not where and what precisely is the error?

Solution:
The proof is wrong. The error is in the claim: if $c|xd$ and $c \not| d$ then $c|x$. This is easliy seen to be false. Consider a case where $c > d$ and $c > x$, for example $c = 12$ $d = 3,\ x = 8$. While $c|xd$ and $c \not| d$ it is not the case that $c|x$.

ii. Produce a counterexample for the result stated.

Solution:
A counter example can be constructed from the above observation: Let $a = 16,\ b = 24$ $c = 8,\ d = 4$. Then $x = 4,\ y = 6$ and $c \not| x,\ c \not| y$ though both $c,\ d$ divide both $a,\ b$ and $c \not| d$.

[(2,3),(6,2),(4,3),(3,2)=25]

2. (a) Let S be a set. How many binary relations are possible if the cardinality of S is 3?

Solution:
Since $|S| = 3,\ |S \times S| = 9$. A relation is any subset of $S \times S$. So the number of relations is $|\mathcal{P}(S \times S)| = 2^9 = 512$.

(b) Let $f : S_1 \to S_2,\ g : S_2 \to S_3,\ h : S_1 \to S_3$ are functions and $h = g \circ f$ that is $h(x) = g(f(x)),\ x \in S_1$.

i. Argue that if h is surjective and f is total and injective, then g must be surjective.

Solution:
Since h is surjective any $z \in S_3$ has one or more pre-image elements in S_1. Let these be $x_1, \ldots, x_m$, i.e. $h(x_i) = z,\ i = 1..m$. Since $h(x) = g(f(x))$ and f is total and injective $y_i = f(x_i),\ i = 1..m$ and $y_i \neq y_j$ when $i \neq j$. Clearly, one of these y_is must be such that $g(y_i) = z$ due to the definition of h as $h = g(f(x))$ and therefore g is a surjection since z is any element in S_3.

ii. Let h be injective and g total. Show that f must be injective. Give a counterexample to show how this claim can fail if g is not total.

Solution:
For h to be injective distinct elements in S_1 must map to distinct elements in S_3. This is impossible if distinct elements in S_1 map to the same element in S_2 since $h(x) = g(f(x))$ so distinct elements in S_2 map to the same element in S_3 and h fails to be injective. So, f must be injective.
It is also clear that if g is not total f can map just two elements, say $x_1, x_2 \in S_1$, to the same element $y \in S_2$ and all others to distinct elements of S_2 and let g be partial with no image for y and mapping the rest to distinct elements of S_3. Then h is injective while f is not. So, for f to be injective it is necessary that g is total.

(c) Argue that every subset S of disjoint open intervals in $\mathbb{R}$ is countable. (*Hint:* $\mathbb{Q}$)

Solution:
Results for infinitie sets are generally counter intuitive. It would seem that since $\mathbb{R}$ is uncountable, all kinds of subsets of it would also be uncountable but that is not true. Since S is a set of disjoint open intervals so every element of S is of the form $(a, b),\ a < b$. Every such interval will contain some rationals (actually infinitely many of them). Choose any one of them and associate it with the interval. This allows us to define a bijection $f : S \rightarrow P \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable any subset of $\mathbb{Q}$ is also countable. So, S is countable.

Notice that since the intervals are disjoint the rational points in P are distinct and can never overlap.

(d) Show that

$$f(x) = \frac{2x - 1}{2x(1 - x)}$$

is a bijection between $f : (0, 1) \rightleftarrows \mathbb{R}$ by arguing that f is an injection and a surjection. (*Note: Cannot use calculus.*) (*Hint: For injection show f is monotonically increasing and for surjection that every $y \in \mathbb{R}$ has a pre-image.*)

Solution:

There are multiple ways to show a bijection between $(0, 1)$ and $\mathbb{R}$. The common one is to use some form of the *tan* function. Here we are able to do the same with a function that is a ratio of two simple polynomials.

To show that f is injective it is enough to show that f is a monotonically increasing function. Then distinct values from $(0, 1)$ will map to distinct values of $\mathbb{R}$. We can break up f into two fractions (partial fractions method): $f = \frac{1}{1-x} + \left(\frac{-1}{2x}\right)$.

First note that $f(x) \to -\infty$ as $x \to 0$ and $f(x) \to \infty$ as $x \to 1$. The fraction $\frac{1}{1-x}$ has its least value when $x = 0$ and increases monotonically as $x \to 1$ since denominator decreases montonically. Similarly, the fraciton $\frac{-1}{2x}$ grows from larger negative values to smaller negative values as x goes from 0 to 1 - that is it is also a monotonically increasing function. The sum of two monotonically increasing functions is monotonically increasing so f is an injection.

To show that f is also a surjection let us calculate the pre-image x for some $f(x) = y$. So, we get $y = \frac{2x-1}{2x(1-x)}$. We solve for x in terms of y. Simplifying we get $2yx^2 + (2 - 2y)x - 1 = 0$ a quadratic in x. Solving the quadratic gives:

$$x = \frac{-(2-2y) \pm \sqrt{(2-2y)^2 + 8y}}{4y}$$

Simplifying the above we get:

$$x = \frac{-1 + y \pm \sqrt{1+y^2}}{2y}$$

We have to a) determine whether to take the positive or negative sign for the square root and b) since y is in the denominator we have to see if we can get a better form for the equation. Note that because $y \in \mathbb{R}$ it can be 0 which is not admissible in the formula above.

For a) let us put $y = 1$ and we see that choosing the negative sign gives $\frac{-1}{\sqrt{2}}$ which is outside $(0, 1)$. Choosing the positive sign gives a value $\frac{1}{\sqrt{2}}$ which is within the interval. So, we get:

$$x = \frac{-1 + y + \sqrt{1+y^2}}{2y}$$

For b) to get rid of y in the denominator let us multiply both numerator and denominator by the conjugate:

$$
\begin{aligned}
x &= \left(\frac{-1+y+\sqrt{1+y^2}}{2y}\right)\left(\frac{-1+y-\sqrt{1+y^2}}{-1+y-\sqrt{1+y^2}}\right) \\
&= \frac{1-2y+y^2-1-y^2}{2y(-1+y-\sqrt{1+y^2})} \\
&= \frac{-2y}{2y(-1+y-\sqrt{1+y^2})} \\
&= \frac{1}{\sqrt{1+y^2}-y+1}
\end{aligned}
$$

For any y note that $\sqrt{1+y^2}-y>0$ and $\sqrt{1+y^2}-y+1>1$ so the value of x is always between 0 and 1 for any value of y and this x is our pre-image. So, f is a surjection. Since we have already shown it is an injection it is, therefore, a bijection.

[3,(4,4),5,(4,5)=25]

3. (a) Five distinct boys and four distinct girls have to line up in a row such that no two girls are next to each other. How many such arrangements are possible?

Solution:

Imagine we arrange the 5 boys as below where E stands for empty.

E B1 E B2 E B3 E B4 E B5 E

The girls can occupy any of 6 E positions so we have ${}^6P_4$ arrangements of girls for each arrangement of boys. So, the total number is: ${}^5P_5.{}^6P_4 = 120 \times 360 = 43200$.

(b) Given a standard deck of cards (52 cards, 4 suits - $\heartsuit$, $\clubsuit$, $\diamondsuit$, $\spadesuit$) what is the minimum number of cards that must be drawn to ensure:

i. Two cards each from any two suits in the selection.

ii. Three $\spadesuit$ cards in the selection.

Solution:

i. Simple pigeonhole application. Worst case: one card of each suit (i.e. 4). Fifth card will give us 2 cards of one suit. Now worst case is we get 11 more cards of the same suit. Giving us a total of 16 cards. The 17th card will make at least 2 cards from 2 suits. So we must select 17 cards.

Another way to count: 13 cards of one suit followed by one card from each of the other suits gives us 16 cards. The 17th card will ensure at least 2 cards from 2 suits.

ii. You may not draw a spade for the first 39 cards after which the next 3 cards will be $\spadesuit$. So a selection of 42 cards is enough to ensure you have at least 3 $\spadesuit$. Note that it does not matter if we draw 1 or 2 ♠ cards in between. The minimum cards needed to ensure 3 ♠ cards remains the same.

(c) Consider the ordered sequence $(x_1, x_2, \ldots, x_m)$ of positive integers (i.e. $x_i \in \mathbb{N},\ i = 1..m$) that satisfy the equation $x_1 + x_2 + \ldots + x_m = n$. How many such sequences are possible? (*Hint: Bijection principle.*)

Solution:
Let us write n as a sequence of n 1s with an empty space between them shown by e and one 0 each at the two ends. So, this is how it will look:

$$01e1e1e1e\ldots1e10$$

Now imagine distributing $m-1$ 0s in the $n-1$ empty places marked by e and adding the 1s between any two 0s. This will give us m positive integers that add up to n. So, all possible ways of distributing $m-1$ 0s in the $n-1$ slots marked by e will give the total number of sequences that are possible. This number is ${}^{n-1}C_{m-1}$. Note that there is a bijection between the binary string and the sequence. Since 0 cannot occur consecutively all x_i are positive.

(d) Using the *principle of inclusion and exclusion* calculate the number of integers from 1 to 100 that are not divisible by 2, 3, or 5.

Solution:
Let $S = \{1, 2, \ldots, 100\}$, S_i be the set of numbers in S divisible by i, $S_{i,j}$ be the set of numbers for S that are divisible by i and by j, similarly, $S_{i,j,k}$. Then by PIE

$$S_{none} = |S| - (|S_2| + |S_3| + S_5) + (|S_{2,3}| + |S_{2,5}| + |S_{3,5}|) - |S_{2,3,5}|$$

Then by simple counting $|S_2| = 50$, $|S_3| = 33$, $|S_5| = 20$. For $|S_{i,j}|$ we have to count how many are divisible by $i \times j$. So, $|S_{2,3}| = 16$, $|S_{2,5}| = 10$, $|S_{3,5}| = 6$, $|S_{2,3,5}| = 3$. So,

$$S_{none} = 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

[3,(3,3),10,6=25]

4. (a) Solve the recurrence relation: $a_n = 9a_{n-1} - 26a_{n-2} + 24a_{n-3}$ for $n > 2$ with $a_0 = 0,\ a_1 = 1,\ a_2 = 10$ to obtain an expression for a_n.

Solution:
This requires a systematic application of the rules to solve such recurrences. Let

$A(s) = \sum_{n=0}^{\infty} a_n x^n$. Multiply recurrence by x^n and sum for $n > 2$:

$$\sum_{n=3}^{\infty} a_n x^n - 9\sum_{n=3}^{\infty} a_{n-1}x^n + 26\sum_{n=3}^{\infty} a_{n-2}x^n - 24\sum_{n=3}^{\infty} a_{n-3}x^n = 0.$$

Write in terms of $A(x)$:

$$(A(x) - a_0 - a_1x - a_2x^2) - 9x(A(x) - a_0 - a_1x) + 26x^2(A(x) - a_0) - 24x^3A(x).$$

Simplify:

$$A(x)(1 - 9x + 26x^2 + 24x^3) = a_0 + a_1x + a_2x^2 - 9a_0x - 9a_1x^2 + 26a_0x^2$$

$$A(x) = \frac{a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 26a_0)x^2}{1 - 9x + 26x^2 + 24x^3} \qquad (1)$$

Break into partial fractions:

$$\begin{aligned} A(x) &= \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{1-4x} \\ &= \sum_{n=0}^{\infty} c_1 2^n x^n + \sum_{n=0}^{\infty} c_2 3^n x^n + \sum_{n=0}^{\infty} c_3 4^n x^n \\ &= \sum_{n=0}^{\infty} (c_1 2^n + c_2 3^n + c_3 4^n)x^n \end{aligned}$$

So, $a_n = c_1 2^n + c_2 3^n + c_3 4^n$. To calculate c_1, c_2, c_3 use the given initial conditions. Substituting $a_= 0$, $a_1 = 1$ and $a_2 = 10$ in (1) above:

$$A(x) = \frac{x + x^2}{(1-2x)(1-3x)(1-4x)} = \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{1-4x}$$

Comparing coefficients the simultaneous equations we get are:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 7c_1 + 6c_2 + 5c_3 &= -1 \\ 12c_1 + 8c_2 + 6c_3 &= 1 \end{aligned}$$

Solving the above: $c_1 = \frac{3}{2},\ c_2 = -4,\ c_3 = \frac{5}{2}$.

(b) The Bubble sort algorithm sorts a sequence S of length n by making $(n-1)$ bubble passes over the sequence. In a single bubble pass the algorithm compares $S(i), S(i+1)$ and switches the elements if $S(i) > S(i+1)$ for $n = 1..(n-1)$. Note $S(i)$ stands for the i^{th} element in the sequence.

Derive a recurrence for the number of comparisons in the Bubble sort algorithm and solve the recurrence.

Solution:
Let $T(n)$ be the number of comparisons required to Bubble sort a sequence of size n. The difference between the number of comparisons for $T(n-1)$ and $T(n)$ is one bubble pass which can take $(n-1)$ comparisons in the worst case. So, the recurrence is: $T(n) = T(n-1) + n - 1$, with $T(1) = 0$. This is most easily solved by unfolding and substitution:

$$\begin{aligned}
T(n) &= T(n-1) + n - 1 \\
&= T(n-2) + n - 2 + n - 1 = T(n-2) + 2n - (1+2) \\
&\ldots \\
&= T(1) + (n-1)n - (1 + 2 + \ldots + (n-1) \\
&= 0 + n(n-1) - \frac{n(n-1)}{2} \\
&= \frac{n(n-1)}{2}
\end{aligned}$$

So, solution is $\frac{n(n-1)}{2}$

(c) Many algorithms have the following recurrence structure: $a(n) = d.a(n/d) + e$ where $d, e \in \mathbb{N}$, $d > 1$ and $e > 0$. Assuming $n = d^k$, $k \in \mathbb{N}$ what is the solution to the recurrence.

Solution:
We can solve this by undfolding and substitition.

$$\begin{aligned}
a(n) &= d.a(\frac{n}{d}) + e \\
&= d^2 a(\frac{n}{d^2}) + 2e \\
&= d^3 a(\frac{n}{d^3}) + 3e \\
&\ldots \\
&= d^{log_d(n)} a(1) + log_d(n).e \\
&= a(1)n + log_d(n)e
\end{aligned}$$

[10,(4,4),7=25]