Probabilistic Models for Classification: Generative Classification

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Probabilistic Machine Learning (CS772A)

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Recap

Parameter Estimation in Probabilistic Models

- Fully Bayesian inference (what we would ideally like to do)

$$p(\theta|\mathbf{X}) = \frac{p(\mathbf{X}|\theta)p(\theta)}{p(\mathbf{X})} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Marginal likelihood}} \qquad \text{(easy when we have conjugacy!)}$$

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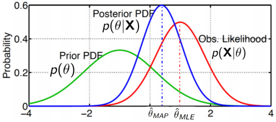

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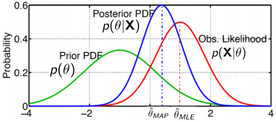

- In some cases, we may want (or may *have* to) to do a "hybrid" sort of inference (fully Bayesian inference for some parameters and MLE/MAP for other parameters)

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- Marginal likelihood can be hard to compute in general, but is a very useful quantity

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- Can use marginal likelihood to compare and choose from a set of models $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_K$
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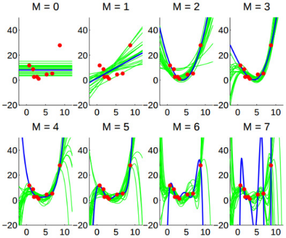

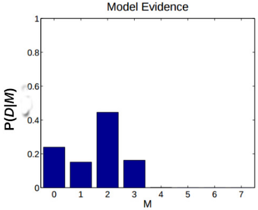

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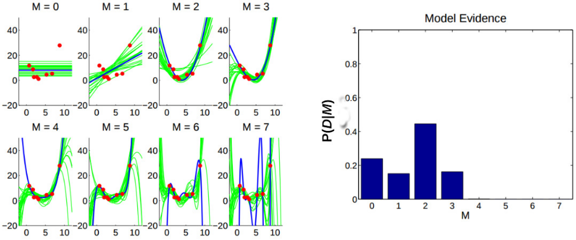

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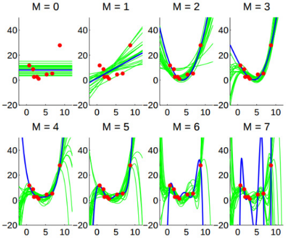

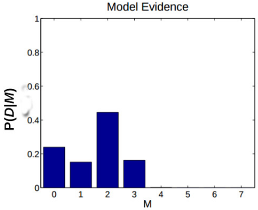

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- Important: This approach doesn't require a separate held-out data (therefore especially useful when there is very little training data, or in unsupervised learning where cross-validation is impossible)

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Probabilistic Models for Classification

$$p(y = k|x) = ?, \quad k = 1, 2, \ldots, K$$

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  - Some examples: Logistic regression, softmax classification, Gaussian Process classification, etc.

Two Approaches to Probabilistic Classification

- Generative classification
  - Use training data to learn class-conditional distribution $p(\boldsymbol{x}|y)$ of inputs from each class $y = 1, \ldots, K$
  - Use training data to learn class prior probability distribution $p(y)$ for each class $y = 1, \ldots, K$
  - Note: We can use MLE/MAP or a fully Bayesian procedure to estimate $p(\boldsymbol{x}|y)$ and $p(y)$
  - Finally, use these distributions to compute $p(y = k|\boldsymbol{x})$ for an input $\boldsymbol{x}$ by applying Bayes rule

$$p(y = k|\boldsymbol{x}) = \frac{p(y = k)p(\boldsymbol{x}|y = k)}{p(\boldsymbol{x})} = \frac{p(y = k)p(\boldsymbol{x}|y = k)}{\sum_{k=1}^{K} p(y = k)p(\boldsymbol{x}|y = k)}$$

  - Some examples: Naïve Bayes, Linear/Quadratic *Discriminant* Analysis (name **a misnomer**), etc.
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  - Some examples: Logistic regression, softmax classification, Gaussian Process classification, etc.
- Important: Generative approach models the inputs $\boldsymbol{x}$. Discriminative approach treats $\boldsymbol{x}$ as fixed

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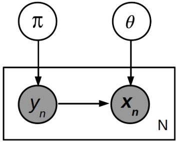
- Basic idea: Each input $\boldsymbol{x}_n$ is generated conditioned on the value of the corresponding label y_n



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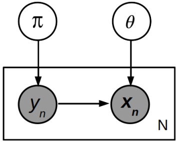

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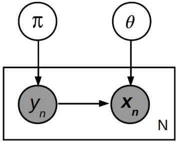

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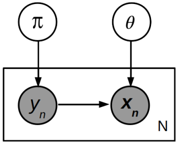

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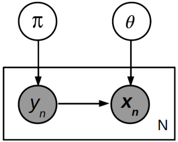

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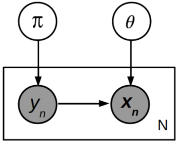

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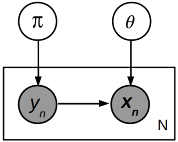

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  - Now draw ("generate") the input $\boldsymbol{x}_n$ from a distribution specific to the value y_n takes

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- An Important Note: The above rule is not truly Bayesian (it WILL be if we inferred the full posterior distribution of the parameters and averaged $p(y=k|\boldsymbol{x},\theta,\pi)$ over that posterior distribution)

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- We can now do MLE for the parameters $\Theta = \{\pi_k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}_{k=1}^K$

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- Note: Can also do MAP or fully Bayesian inference for these parameters

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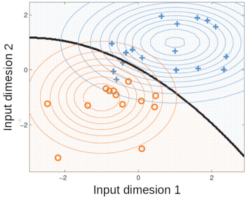

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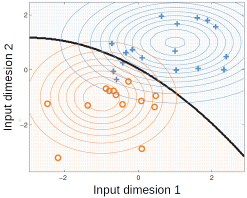

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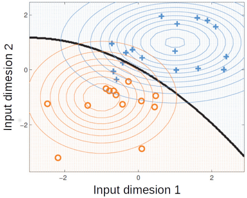

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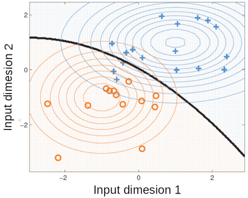

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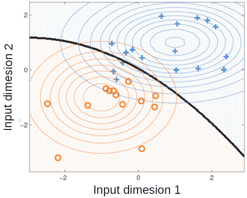

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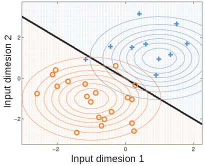

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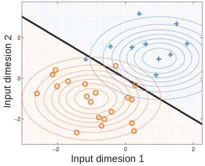

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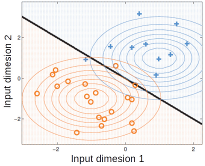

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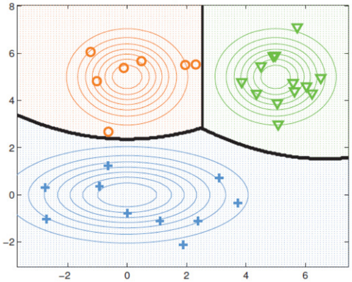
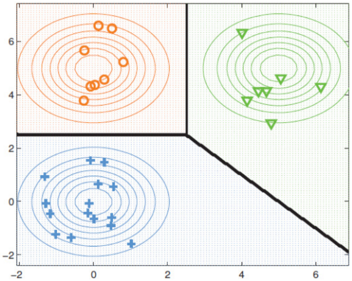
Decision Boundaries

- Depending on the form of the covariance matrices, the boundaries can be quadratic/linear





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- Interestingly, this has exactly the same form as the softmax classification model, which is a discriminative model (will look at these later), as opposed to a generative model.

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- This is equivalent to assigning $\boldsymbol{x}$ to the "closest" class in terms of a Mahalanobis distance

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- This is equivalent to assigning $\boldsymbol{x}$ to the "closest" class in terms of a Mahalanobis distance
  - The covariance matrix "modulate" how the distances are computed

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- Generative models are also useful for semi-supervised learning (will look at later)

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- A good density estimation model is necessary for generative classification model to work well