Basics of Parameter Estimation in Probabilistic Models

Piyush Rai

Probabilistic Machine Learning (CS772A)

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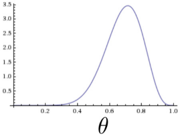
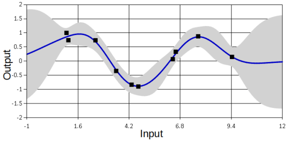
Recap: Why a Probabilistic Approach to ML?

- Gives a principle way to model the uncertainty in data, parameters, and in future predictions





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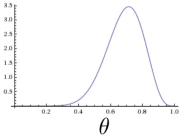

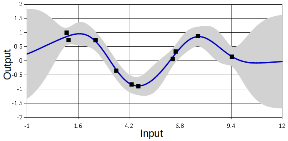

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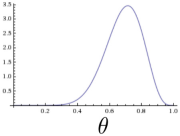

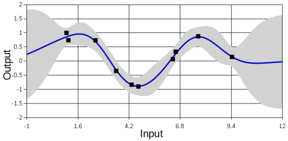

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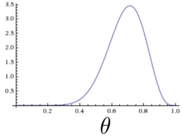

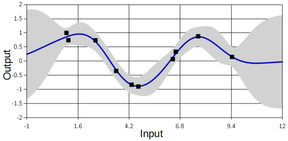

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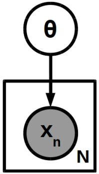
- Can construct complex models in "modular" fashion by combining simpler probabilistic models
- Probabilistic modeling gives us a "language" to do such things naturally

Probabilistic Modeling

- Assume data $\mathbf{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ generated from a probabilistic model with unknown parameters θ

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \sim p(\boldsymbol{x}|\theta)$$



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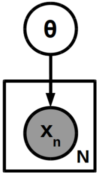

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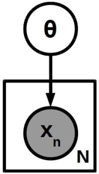

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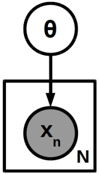

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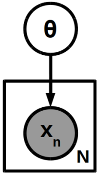

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- Can use the Bayes rule to compute the posterior **distribution** over parameters

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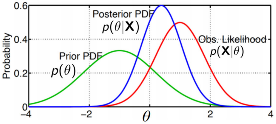

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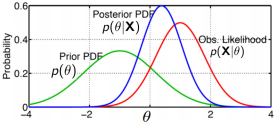

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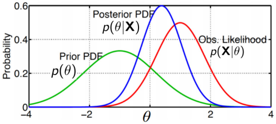

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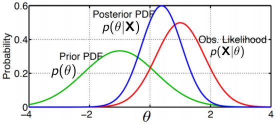

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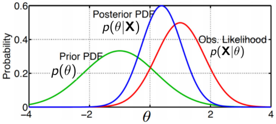

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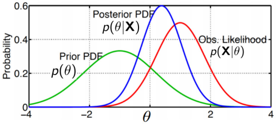

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- Note: It's easy if the likelihood and prior are "conjugate" to each other (we will see it later)

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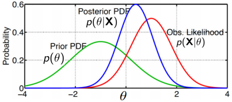

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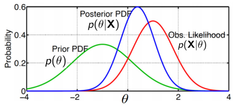

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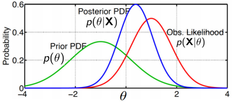

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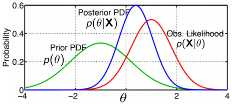

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  - .. and some other methods such as "method of moments"

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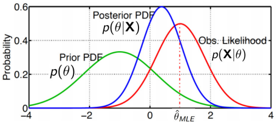

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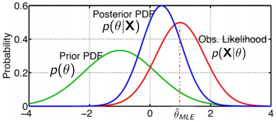

- Note: MLE is "consistent", i.e., as $N \longrightarrow \infty$, $\hat{\theta}$ converges to the true θ (will see it later)

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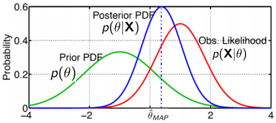

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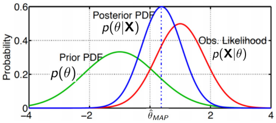

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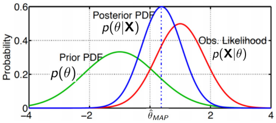

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- Despite using the prior, MAP is NOT considered a Bayesian approach (still gives a point estimate)

Point Estimation (MLE/MAP) vs Loss Function Minimization

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- Note that NLL is like a loss function and $-\log p(\theta)$ is like a regularizer on θ

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$$\boxed{\hat{\theta}_{MAP} = \arg\min_{\theta} NLL(\theta) - \log p(\theta)}$$

- Note that NLL is like a loss function and $-\log p(\theta)$ is like a regularizer on θ
- Thus MLE is like empirical loss/risk minization (ERM) and MAP is like regularized ERM

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- MLE doesn't have a way to express our prior belief about θ. Can be problematic especially when the number of observations is very small (e.g., suppose very few or zero heads when N is small).

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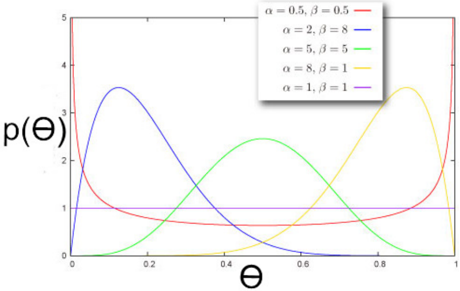

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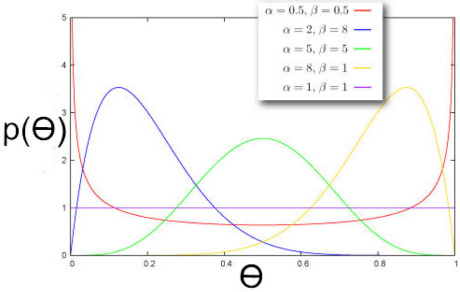

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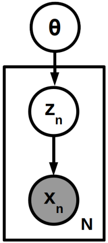
Parameter Estimation in Models with Latent Variables

- Becomes harder if there are latent variables in the model



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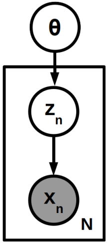

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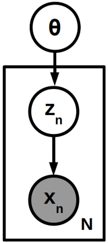

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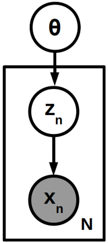

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- More on this when we will look at latent variable models

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- MLE and MAP only provide us with a best "point estimate" of θ
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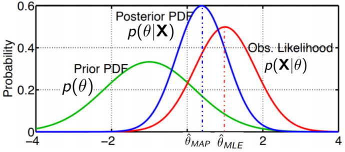

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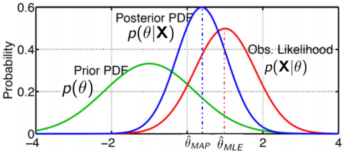

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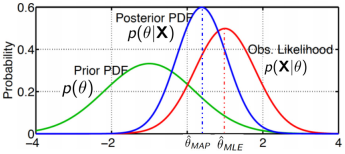

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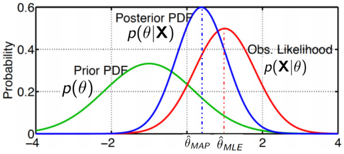

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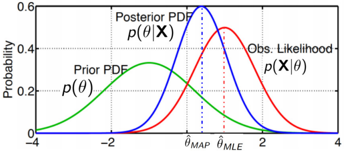

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- In other cases, it can be approximated via **approximate Bayesian inference** methods such as MCMC and variational inference (more on this later during the semester)

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- Recall that each likelihood term was Bernoulli: $p(\boldsymbol{x}_n|\theta) = \theta^{\boldsymbol{x}_n}(1-\theta)^{1-\boldsymbol{x}_n}$
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- Here, the posterior has the same form as the prior (both Beta): property of **conjugate priors**.

Conjugate Priors

- Many pairs of distributions are conjugate to each other. E.g.,
  - Bernoulli (likelihood) + Beta (prior) $\Rightarrow$ Beta posterior
  - Binomial (likelihood) + Beta (prior) $\Rightarrow$ Beta posterior
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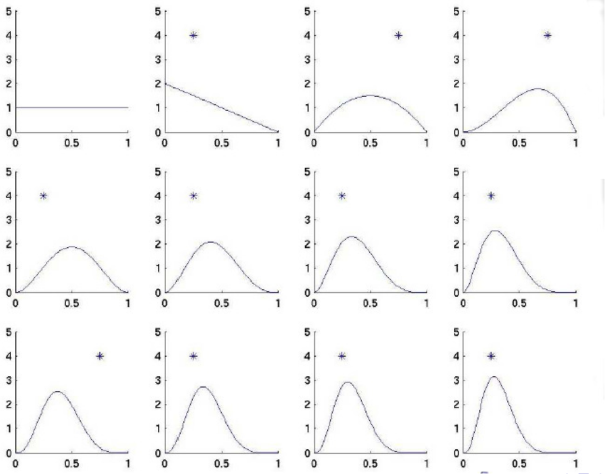
- More on conjugate priors when we will look at exponential family distributions

Posterior Evolution with Observed Data

- Assume starting with a uniform prior (equivalent to Beta(1,1)) in the coin-toss example and observing a sequence of heads and tails



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- Note that the predictive distribution doesn't depend on a single value of θ (θ_{MLE} or θ_{MAP})