MSO 201a: Probability and Statistics

2016-2017-II Semester
Assignment-V

A. Illustrative Discussion Problems

1. (a) Find the moments of the random variable that has the m.g.f. $M(t) = (1-t)^{-3}$, $t < 1$.
   (b) Let the random variable X have the m.g.f.
   $$M(t) = \frac{e^{-t}}{8} + \frac{e^{t}}{4} + \frac{e^{2t}}{8} + \frac{e^{3t}}{2},\ t \in \mathbb{R}.$$
   Find the distribution function of X and find $P(\{X^2 = 1\})$.
   (c) If the m.g.f. of a random variable X is
   $$M(t) = \frac{e^{t} - e^{-2t}}{3t},\ \text{for } t \neq 0,$$
   find the p.d.f. of $Y = X^2$.

2. Let X be a r.v. with m.g.f.
   $$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}},\quad t \in \mathbb{R},$$
   where $\mu \in \mathbb{R}$ and $\sigma > 0$ are fixed constants.
   (a) Show that the distribution of X is symmetric about μ (i.e., $X - \mu \overset{d}{=} \mu - X$);
   (b) Find the mean, the variance, the third central moment and the fourth central moment of X.

3. Let the random variable X have the m.g.f.
   $$M(t) = e^{\lambda(e^{t}-1)},\ t \in \mathbb{R},$$
   where $\lambda > 0$ is a fixed constant.
   (a) Let $Y = Xe^{X}$. Find the mean and the variance of X;
   (b) Let $Z = 2^{X}$. Find the mean and the variance of Z.

4. Let X be a r.v. whose distribution is symmetric about μ ($\in \mathbb{R}$).
   (a) If X is discrete and $P(\{X = \mu\}) = \frac{2}{3}$, find $P(\{X > \mu\})$;
   (b) If X is A.C., find $P(\{X > \mu\})$.

5. (a) Let X be a discrete/continuous r.v. with d.f. $F(\cdot)$. For $u \in (0,1)$, define the function $Q(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}$ (called quantile function of X or of $F(\cdot)$).

Let U be uniformly distributed on unit interval $(0,1)$, i.e., U has a p.d.f.

$$g(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Show that $Q(U) \stackrel{d}{=} X$.

(b) Let X be a continuous r.v. with d.f. $F(\cdot)$ and let U be uniformly distributed on unit interval $(0,1)$. Show that $F(X) \stackrel{d}{=} U$.

B. Practice Problems from the Text Book

Chapter 1: Probability and Distributions, Problem Nos.: 9.25, 9.26, 10.4, 10.5.

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Assignment-V (Solutions)

Problem 1 (a) $M(t)$ is finite in $t \in (-\frac{1}{2}, \frac{1}{2})$ (a neighborhood of 0)

Let $M(t)$ be the m.g.f. of r.v. X. Then $M^{(1)}(t) = 3(1-t)^{-4}$

$M^{(2)}(t) = 3\times 4(1-t)^{-5}, \ldots \quad M^{(r)}(t) = 3\times 4 \times \cdots \times (r+2)(1-t)^{-(r+3)}$

$= \frac{\lfloor r+2}{2}(1-t)^{-(r+3)}, \quad -\frac{1}{2} < t < \frac{1}{2}.$

$E(X^r) = M^{(r)}(0) = \frac{\lfloor r+2}{2}, \quad r = 1, 2, \ldots$

(b) Clearly $M(t)$ is the m.g.f. of r.v. X having p.m.f.

$$f(x) = \begin{cases} \frac{1}{8}, & \text{if } x = -1, 2 \\ \frac{1}{4}, & \text{if } x = 1 \\ \frac{1}{2}, & \text{if } x = 3 \end{cases}$$

(We have used the uniqueness of m.g.f.).

Then the d.f. of X is

$$F(x) \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{8}, & \text{if } -1 \le x < 1 \\ \frac{3}{8}, & \text{if } 1 \le x < 2 \\ \frac{1}{2}, & \text{if } 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$P(X^2 = 1) = P(X \in \{-1, 1\}) = P(X=-1) + P(X=1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$

(c) Clearly $M(t)$ is the m.g.f. of r.v. X having p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } -2 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$Y = h(X) = X^2$, where $h(x) = x^2$, $x \in \mathbb{R}$. $S_X = [-2, 1] = [-2, 0) \cup [0, 1] = S_{1,X} \cup S_{2,X}$ (say)

$h(x) = x^2$ is 1-1 in $S_{1,X}$ and $S_{2,X}$.

$S_{1,X} = [-2, 0)$ $\qquad$ $S_{2,X} = [0, 1]$

$h(S_{1,X}) = (0, 4]$ $\qquad$ $h(S_{2,X}) = [0, 1]$

$h_1^{-1}(y) = -\sqrt{y}$ $\qquad$ $h_2^{-1}(y) = \sqrt{y}$

Thus Y is A.C. with p.d.f.

$g(y) = f(h_1^{-1}(y)) \left|\frac{d}{dy} h_1^{-1}(y)\right| I_{h(S_{1,X})}(y) + f(h_2^{-1}(y)) \left|\frac{d}{dy} h_2^{-1}(y)\right| I_{h(S_{2,X})}(y)$

$= f(-\sqrt{y}) \left|-\frac{1}{2\sqrt{y}}\right| I_{(0,4]}(y) + f(\sqrt{y}) \left|\frac{1}{2\sqrt{y}}\right| I_{[0,1]}(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & \text{if } 0 < y < 1 \\ 1/6\sqrt{y}, & \text{if } 1 \le y < 4 \\ 0, & \text{otherwise} \end{cases}$

Problem 2 (a) Let $Y = X-\mu$ and $Z = \mu - X$. Then

$$M_Y(t) = E(e^{t(X-\mu)}) = e^{-\mu t}E(e^{tX}) = e^{-\mu t}M(t) = e^{-\mu t}e^{\mu t+\frac{\sigma^2t^2}{2}} = e^{\frac{\sigma^2t^2}{2}}, t\in\mathbb{R}$$

$$M_Z(t) = E(e^{t(\mu-X)}) = e^{\mu t}E(e^{-tX}) = e^{\mu t}M(-t) = e^{\mu t}e^{-\mu t+\frac{\sigma^2t^2}{2}} = e^{\frac{\sigma^2t^2}{2}}, t\in\mathbb{R}$$

Thus

$$M_Y(t) = M_Z(t), \forall t\in\mathbb{R} \Rightarrow Y \overset{d}{=} Z, \text{ i.e., } X-\mu \overset{d}{=} \mu - X.$$

(b) Let $Y = X-\mu$. Then, from (a),

$$M_Y(t) = e^{\frac{\sigma^2t^2}{2}} = \sum_{r=0}^{\infty}\frac{1}{\underline{|r}}\left(\frac{\sigma^2t^2}{2}\right)^r = \sum_{r=0}^{\infty}\frac{\underline{|2r}}{2^r\,\underline{|r}}\sigma^{2r}\frac{t^{2r}}{\underline{|2r}}, t\in\mathbb{R}$$

$\Rightarrow E(Y^{2m+1}) =$ Coefficient of $\frac{t^{2m+1}}{\underline{|2m+1}}$ in Maclaurin's series expansion of $M_Y(t)$ $= 0$, $m=0,1,2,\ldots$

$E(Y^{2m}) =$ Coefficient of $\frac{t^{2m}}{\underline{|2m}}$ in Maclaurin's series expansion of $M_Y(t) = \frac{\underline{|2m}}{2^m\,\underline{|m}}\sigma^{2m}$, $m=1,2,\ldots$

$\Rightarrow E(Y)=0 \Rightarrow E(X-\mu)=0 \Rightarrow E(X)=\mu$

$\mu_2 = Var(X) = E(Y^2) = \sigma^2$

$\mu_3 = E(Y^3) = 0$; $\mu_4 = E(Y^4) = \frac{\underline{|4}}{2^2\,\underline{|2}}\sigma^4 = 3\sigma^4$.

Problem 3 (a) $M(t) = E(e^{tX}) \Rightarrow M^{(1)}(t) = E(Xe^{tX})$ and $M^{(2)}(t) = E(X^2e^{tX})$, $t\in\mathbb{R}$

$\Rightarrow E(Y) = E(Xe^X) = M^{(1)}(1)$. $E(Y^2) = E(X^2e^{2X}) = M^{(2)}(2)$

$M^{(1)}(t) = \lambda e^t e^{\lambda(e^t-1)} \Rightarrow E(Y) = M^{(1)}(1) = \lambda e\, e^{\lambda(e-1)}$;

$M^{(2)}(t) = \lambda^2e^{2t}e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$

$$\Rightarrow E(Y^2) = M^{(2)}(2) = \lambda^2e^4e^{\lambda(e^2-1)} + \lambda e^2 e^{\lambda(e^2-1)}$$
$$= \lambda e^2(\lambda e^2+1)e^{\lambda(e^2-1)}$$

$Var(Y) = E(Y^2) - (E(Y))^2$.

(b) $M(t) = E(e^{tX}) \Rightarrow M(\ln t) = E(e^{X\ln t}) = E(t^X)$, $t>0$

$E(Z) = E(2^X) = M(\ln 2) = e^{\lambda}$

$E(Z^2) = E(4^X) = M(\ln 4) = e^{3\lambda}$

$Var(Z) = E(Z^2) - (E(Z))^2 = e^{3\lambda} - e^{2\lambda} = e^{2\lambda}(e^{\lambda}-1)$.

Problem 4 We are given that $X-\mu \overset{d}{=} \mu - X$

$$\Rightarrow P(X-\mu>0) = P(\mu - X > 0)$$
$$\Rightarrow P(X>\mu) = P(X<\mu)$$

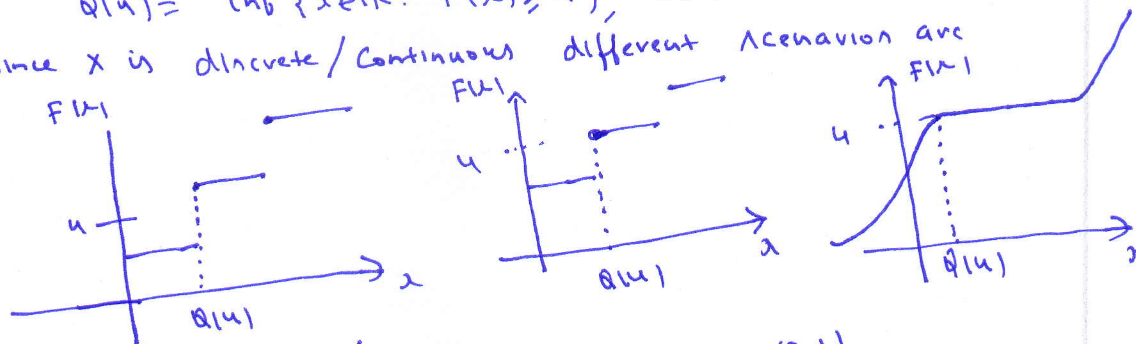
Also we have
$$P(X>\mu) + P(X=\mu) + P(X<\mu) = 1$$
$$\Rightarrow \quad 2P(X>\mu) + P(X=\mu) = 1$$

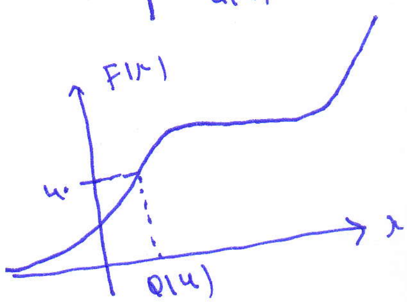
(a) $$2P(X>\mu) = 1-\frac{2}{3} = \frac{1}{3} \quad \Rightarrow \quad P(X>\mu) = \frac{1}{6}.$$

(b) In this case $P(X=\mu)=0$. Therefore
$$P(X>\mu) = \frac{1}{2}.$$

Problem 5 (a) $Q(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad 0<u<1.$

Since X is discrete / continuous different scenarios are





Clearly, for $u \in (0,1)$,
$$F(x) \geq u \iff x \geq Q(u)$$
Let $Y = Q(U)$ and let $H(\cdot)$ be the d.f. of Y.

Then
$$H(x) = P(Y \leq x) = P(Q(U) \leq x) = P(U \leq F(x))$$
$$= \int_{-\infty}^{F(x)} g(u)\,du = \int_0^{F(x)} du \quad (0 \leq F(x) \leq 1)$$
$$= F(x) \quad \forall x \in \mathbb{R}$$

Thus
$$H(x) = F(x) \quad \forall x \in \mathbb{R} \Rightarrow Y \overset{d}{=} X, \text{ i.e., } Q(U) \overset{d}{=} X.$$

(b) Let $Q(\cdot)$ be as defined in (a) and let $Z = F(X)$. Then for $u \in (0,1)$

$$P(Z \geq u) = P(F(X) \geq u) = P(X \geq Q(u))$$
$$\Rightarrow P(Z<u) = P(X<Q(u)) = F(Q(u)) = u \quad \left[\begin{array}{l} F(\cdot) \text{ is continuous} \\ \text{and } F(Q(u)) = u \end{array}\right]$$
$$\Rightarrow P(Z \leq u) = \lim_{n\to\infty} P\left(Z < u+\frac{1}{n}\right) = \lim_{n\to\infty}\left(u+\frac{1}{n}\right) = u$$
$$\Rightarrow P(Z \leq u) = \begin{cases} 0, & u<0 \\ u, & 0 \leq u < 1 \\ 1, & u \geq 1 \end{cases} = G(u) \quad (\text{the d.f. of } U)$$
$$\Rightarrow Z \overset{d}{=} U.$$

$$\left[\text{Note: } G(u) = \int_{-\infty}^{u} g(t)\,dt = \begin{cases} 0, & u<0 \\ u, & 0 \leq u < 1. \\ 1, & u \geq 1 \end{cases}\right]$$