CS201: MATHEMATICS FOR COMPUTER SCIENCE - I
NITIN SAXENA

ASSIGNMENT 5

POINTS: 30

DATE GIVEN: 28-OCT-2016 DUE: 07-NOV-2016(7PM)

Rules:

- You are strongly encouraged to work *independently*.
- Write the solutions on your own and honorably *acknowledge* the sources if any.
  `http://cse.iitk.ac.in/pages/AntiCheatingPolicy.html`
- The "0 point" questions are optional.
- Submit your solutions, before time, to your TAs as per the roll numbers: Amit Sinhababu (12000–150130), Pranav Bisht (150131–150365), Ashish Dwivedi (150366–150600), Pulkit Kariryaa (150601–150840).

Question 1: *[6 points]* For events $A_1, A_2, \cdots, A_n$, prove,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) + \cdots + (-1)^{n-1} P(\bigcap_{i=1}^{n} A_i)\,.$$

Question 2: *[4 points]* An event A is *positively correlated* to B if $P(A \mid B) \ge P(A)$. Suppose A is positively correlated to B, then show that,

- B is positively correlated to A.
- B^c is negatively correlated to A. What will be the definition of "negatively correlated" ?

Question 3: *[6 points]* Let X_i, $i \in [n]$, be mutually independent random variables. Prove that,

$$E\left[\prod_{i\in[n]} X_i\right] = \prod_{i\in[n]} E[X_i]\,.$$

Question 4: *[5 points]* Prove Chebyshev's inequality: for *any* random variable X and $a \in \mathbb{R}_{>0}$,

$$P\left(|X - E[X]| \geq a\right) \leq \frac{Var(X)}{a^2},$$

where $Var(X) := E[X^2] - E[X]^2$ is called the *variance* of X.

Question 5: *[9 points]* Consider any graph $G = (V, E)$. Show that G contains some independent set of size at least $\sum_{v \in V} \frac{1}{deg(v)+1}$.

(Hint: Use the probabilistic method and a random ordering of the vertices.)

Question 6: *[0 points]* There are three boxes: B1 contains two gold coins, B2 contains two silver coins, and B3 contains a gold and a silver coin.

Suppose Bertrand picks a box at random and then picks a random coin in it. If that happens to be gold then what is the probability of the next being gold as well?

Question 7: *[0 points]* Show that there are infinite primes of the form $3k + 2$.

Question 8: *[0 points]* Let X be the random variable which counts the number of fixed points (i maps to i) in a random permutation. What is the expected value of X?

Question 9: *[0 points]* For the sticker collection problem in the linearity of expectation discussion, let us look at a different solution. Say, T_i is the random variable which counts the chewing-gums needed to collect the i^{th} sticker. Since the probability to pick i is $1/n$, we deduce that $E[T_i] = n$. Thus, $E[T] = \sum_{i \in [n]} E[T_i] = n^2$. Is this reasoning correct, if not, what is wrong here?

Question 10: *[0 points]* Read about Stirling's bound on factorial and binomial coefficients.

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