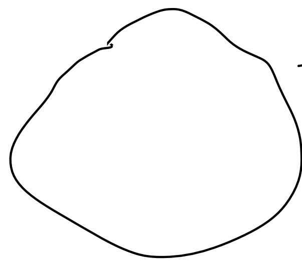
MSO 203B (PDE) $\boxed{\text{Lecture 13 : Seperation of Variable}}$

Poisson Equation :- $(\Omega \subseteq \mathbb{R}^2)$

$$\left.\begin{aligned} \Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega . \end{aligned}\right\} - (1)$$

where Ω is an open set in $\mathbb{R}^2$ and f, g are smooth.



$\Omega (\subseteq \mathbb{R}^2)$

$\Delta u = u_{xx} + u_{yy}$

$w_{\xi\eta} = l[w_\xi, w_\eta, \xi, \eta, u]$

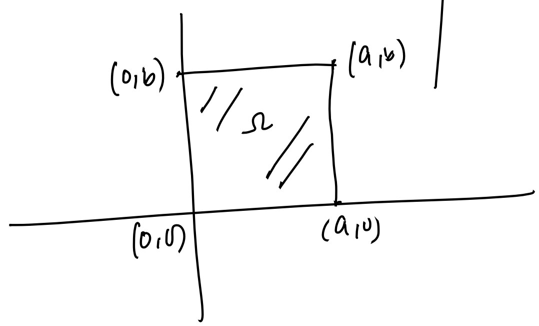
$w_{\eta\eta} = \quad ''$

$w_{\xi\xi} + w_{\eta\eta} = \underline{\underline{\quad}}$

$\bar{\Omega} := \Omega \cup \partial\Omega$

Soln :- We say $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a soln of (1) if u satisfies (1) for all $x \in \Omega$.

For now assume, $\Omega = \overset{x}{\overset{\uparrow}{(0,a)}} \times \overset{y}{\overset{\uparrow}{(0,b)}}$.



$$P_A : \quad \begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega . \end{aligned} \quad \leftarrow u_1$$

$$P_B : \quad \begin{aligned} \Delta u &= f \quad \text{in } \Omega . \\ u &= 0 \quad \text{on } \partial\Omega . \end{aligned} \quad \leftarrow u_2$$

Define $u := u_1 + u_2$.

$\Delta u = \Delta(u_1 + u_2)$

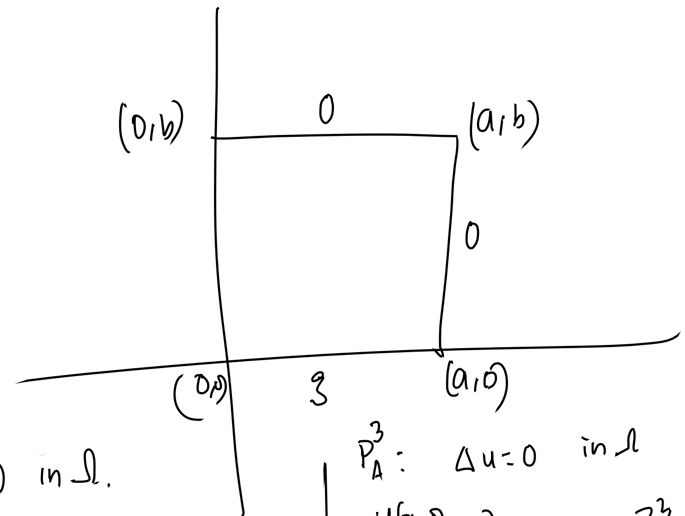
$= \Delta u_1 + \Delta u_2$

$= f$ in Ω.

$u = g$ on $\partial\Omega$.

P_A : $\Delta u = 0$ in $(0,a)\times(0,b) = \Omega$.

$u = g$ on $\partial\Omega$.



P_A^1 : $\Delta u = 0$ in Ω u_{11}

$u(x,0) = g$; $0 \le x \le a$

$u(a,y) = 0$; $0 \le y \le b$.

$u(x,b) = 0$; $0 \le x \le a$

$u(0,y) = 0$; $0 \le y \le b$.

$u_A = u^{11} + u^{22} + u^{33} + u^{44}$

P_A^2 : $\Delta u = 0$ in Ω. u^{22}

$u(x,0) = 0$; $0 \le x \le a$

$u(a,y) = g$; $0 \le y \le b$

$u(x,b) = 0$; $0 \le x \le a$

$u(0,y) = 0$; $0 \le y \le b$.

P_A^3 : $\Delta u = 0$ in Ω u^{33}

$u(x,0) = 0$

$u(a,y) = 0$

$u(x,b) = g$

$u(0,y) = 0$

P_A^4 : $\Delta u = 0$ u^{44}

$u(x,0) = 0$

$u(a,y) = 0$

$u(x,b) = 0$

$u(0,y) = g$

P_A^1 : $\Delta u = 0$ in $(0,a)\times(0,b)$. — (*)

$u(x,0) = g$ — (i)

$u(a,y) = 0$ — (ii)

$u(x,b) = 0$ — (iii)

$u(0,y) = 0$ — (iv)

Let us assume, $u(x,y) = X(x)\,Y(y)$. $\left(X, Y \in C^2(0,a) \text{ \& } C^2(0,b) \text{ resp}\right)$.

$u_{xx} = X''Y$, $u_{yy} = XY''$.

Putting it in (*)

$$X''Y + XY'' = 0.$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

$$\Rightarrow \begin{cases} X'' - \lambda X = 0 \;;\; X(a) = X(0) = 0. \\ Y'' + \lambda Y = 0 \;;\; Y(b) = 0 \end{cases}$$

Interpreting the B.C

$u(x,0) = g$

$\boxed{X(x)\,Y(0) = g}$ — (i)

(ii) $u(a,y) = 0$

$X(a)\,Y(y) = 0 \Rightarrow X(a) = 0.$

(iii) $Y(b) = 0$

(iv) $X(0) = 0$.

$X'' - \lambda X = 0$; $X(0) = X(a) = 0$. $\lambda < 0$

$\lambda_n = -\left(\frac{n\pi}{a}\right)^2$ $n \in \mathbb{N}$

$X_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right)$.

A_n are constant.

$Y'' + \lambda Y = 0$

$\Rightarrow Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0$.

$\Rightarrow Y_n(y) = A_n e^{-\left(\frac{n\pi}{a}\right)y} + B_n e^{\left(\frac{n\pi}{a}\right)y}$.

$0 = Y(b) = A_n e^{-\left(\frac{n\pi}{a}\right)b} + B_n e^{\left(\frac{n\pi}{a}\right)b}$.

$\Rightarrow B_n = -A_n e^{-2\left(\frac{n\pi}{a}\right)b}$

$\therefore Y_n(y) = A_n\left[e^{-\left(\frac{n\pi}{a}\right)y} - e^{-\left(\frac{2n\pi}{a}\right)b} e^{\left(\frac{n\pi}{a}\right)y}\right] := A_n \phi_n(y)$

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$\int x x'' - \int \lambda x^2 = 0$

$-\int (x')^2 + \cancel{[x x']_0^a} - \lambda \int x^2 = 0$

$-\int (x')^2 = \lambda \int x^2$

$$u_n(x,y) = \tilde{A}_n \sin\left(\frac{n\pi x}{a}\right) \cdot A_n \phi_n(y).$$
$$= \tilde{B}_n \sin\left(\frac{n\pi x}{a}\right) \phi_n(y).$$

$$u(x,y) = \sum_{n=1}^{\infty} \tilde{B}_n \sin\left(\frac{n\pi x}{a}\right) \phi_n(y).$$

$$g = u(x,0) = \sum_{n=1}^{\infty} \tilde{B}_n \sin\left(\frac{n\pi x}{a}\right) \phi_n(0).$$

$$\tilde{B}_n = \frac{2}{a} \int_0^a \sin\left(\frac{n\pi \xi}{a}\right) g(\xi)\, d\xi$$