Name: Rollno:

CSE340: Theory of Computation (Homework Assignment 3)

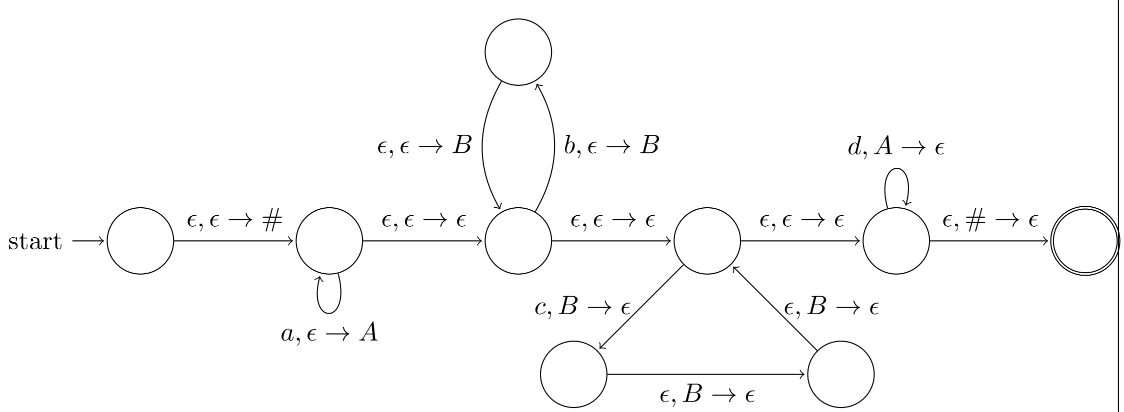
Due Date: 24th October, 2017 (in class)

Total Number of Pages: 1 Total Points 50

Question 1. Design PDAs for the following languages (give the transition diagram only).

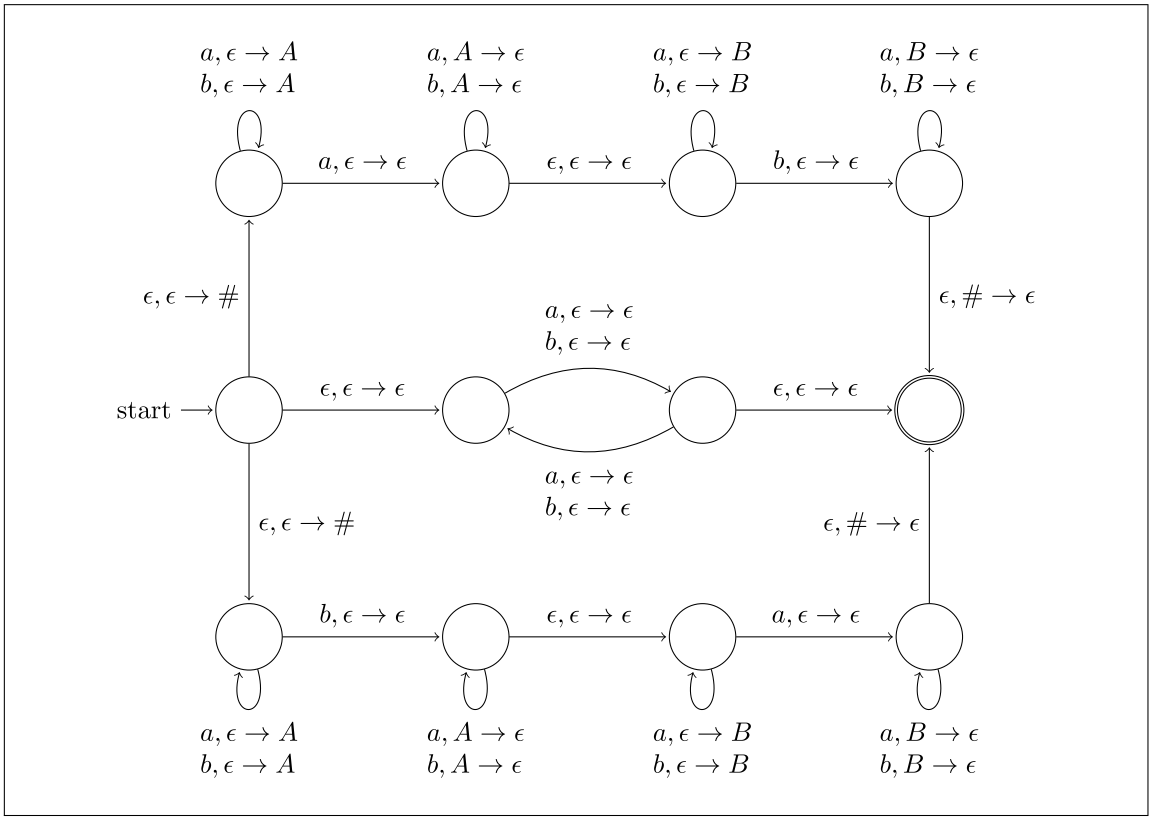
(a) (6 points) $L_1 = \{a^i b^j c^k d^l \mid i = l \text{ and } i + 2j = 3k + l\}$.

Solution:



(b) (8 points) $L_2 = \Sigma^* \setminus \{ww \mid w \in \Sigma^*\}$. Assume that $\Sigma = \{a, b\}$.

Solution: Observe that any string in L_2 either has odd length or is of the form $xayubv$ or $xbyuav$, where $|x| = |y|$ and $|u| = |v|$.



Question 2. One of the following two languages is context-free and one is not.

$$
\begin{aligned}
L &= \{a^i b^j c^k d^l \mid i = k \text{ and } j = 2l\} \\
M &= \{a^i b^j c^k d^l \mid i = k \text{ or } j = 2l\}
\end{aligned}
$$

(a) (2 points) Which of the above two languages is context-free?

Solution: M is context-free.

(b) (6 points) Give a CFG for the language which is context-free

Solution: CFG for M with start variable S.

$$
\begin{aligned}
S &\longrightarrow T_1 D \mid A T_2 \\
T_1 &\longrightarrow a T_1 c \mid B \\
T_2 &\longrightarrow bb T_2 d \mid C \\
A &\longrightarrow aA \mid \epsilon \\
B &\longrightarrow bB \mid \epsilon \\
C &\longrightarrow cC \mid \epsilon \\
D &\longrightarrow dD \mid \epsilon
\end{aligned}
$$

(c) (6 points) Show that the other language is not context-free.

Solution: Consider the language

$$L' = \{a^i b^j c^k d^l \mid i = k \text{ and } j = l\}.$$

In Lecture Notes 10, Exrecise 1(a) it was asked to show that L' is not CFL. I will use the fact that L' is not a CFL and show that L is not a CFL.

Consider the homomorphism h defined as $h(a) = a$, $h(b) = bb$, $h(c) = c$ and $h(d) = d$. Now consider a string $w = a^i b^{2j} c^i d^j \in L$. The only preimage of w is $a^i b^j c^i d^j$ which is in L'. Conversely every string in L' is preimage of exactly string in L. Therefore $L' = h^{-1}(L)$. Since CFLs are closed under inverse homomorphism and $h^{-1}(L)$ is not a CFL therefore L is not a CFL as well.

Question 3. Show that the following languages are decidable.

(a) (7 points) $L_1 = \{\langle M \rangle \mid M \text{ is a DFA which does not accept any string that contains 101 as a substring}\}$

Solution: The language

$$A = \{w \mid w \text{ does not contain 101 as a substring}\}$$

is a regular language. Therefore let D be a DFA for A. Now we design an algorithm for L_1 where the algorithm has a description of D hardcoded in it. Observe that $\langle M \rangle \in L_1$ if and only if $L(M) = A$. Moreover since we know that EQ_{DFA} is decidable it is sufficient for us to reduce L_1 to EQ_{DFA}.

Input: $\langle M \rangle$, where M is a DFA.

(i) Accept if and only if $L(M) = L(D)$.

(b) (7 points) $L_2 = \{\langle R, S \rangle \mid R, S \text{ are regular expressions and } L(R) \subseteq L(S)\}$

Solution: We will use the fact that $A \subseteq B$ if and only if $A \cap \overline{B} = \emptyset$. Also since we know that E_{DFA} is decidable it is sufficient for us to reduce L_2 to E_{DFA}.

Input: $\langle R, S \rangle$, where R, S are regular expressions.

(i) Convert R and S to DFAs D_R and D_S respectively.

(ii) Build a DFA D' for the language $L(D_R) \cap \overline{L(D_S)}$.

(iii) Accept if and only if $L(D') = \emptyset$.

Question 4. (8 points) Show that the following language is decidable

$$L = \{\langle G \rangle \mid G \text{ is a CFG over } \{0,1\}^* \text{ and } 1^* \subseteq L(G)\}.$$

Solution: Take G and remove all rules that contain a terminal other than 1 on the RHS of the rule. Say the new grammar is G'. Now observe that for a string $w \in 1^*$, $w \in L(G)$ if and only if $w \in L(G')$. Moreover, $L(G') \subseteq 1^*$ since 1 is the only terminal variable in G'. Therefore,

$$1^* \subseteq L(G) \Longleftrightarrow L(G') = 1^*.$$

Consider the grammar G'. The Pumping Lemma for CFLs gives a pumping constant, say n for $L(G')$, such that for all $w \in L(G')$ with $|w| \geq n$, there exists a partition $w = uvxyz$ with $|vxy| \leq n$ and $|vy| > 0$, such that for all $i \geq 0$, $uv^ixy^iz \in L(G')$ as well. Since $L(G') \subseteq 1^*$, therefore $uv^ixy^iz = 1^{|uxz|+i|vy|}$. Recall that n depends only on G'.

Based on this, the algorithm to check whether $L(G') = 1^*$ is as follows:

(i) For all $0 \leq i \leq 2n$ check whether $1^i \in L(G')$. This can be done using the algorithm for A_{CFG}

(ii) If for all i, $1^i \in L(G')$, then *accept* else *reject*

Observe that even if one string 1^i $(0 \leq i \leq 2n)$ is not in $L(G')$ then $L(G') \neq 1^*$.

Consider any string 1^m, for $m > 2n$. For every $0 < t \leq n$, we can write m as $m = qt + r$, where $r < t$. So there is some string of the form $1^{m'}$, where $n \leq m' \leq 2n$, and some $0 < t \leq n$ (which is essentially the value of $|vy|$ for that particular string), such that $m - m' = qt$. Since we assume that all strings upto length $2n$ are in $L(G')$, therefore by Pumping Lemma $1^m \in L(G')$ as well.