MSO 203B - PDE (Canonical Form - Lecture 13)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad \text{— (1)}$$

Based on $B^2 - 4AC$ we know the nature of the eqn

$\exists\ (x,y) \mapsto (\xi(x,y), \eta(x,y))$ and define $w(\xi,\eta) := u(x,y)$

$$\tilde{A}w_{\xi\xi} + \tilde{B}w_{\xi\eta} + \tilde{C}w_{\eta\eta} + \tilde{D}w_\xi + \tilde{E}w_\eta + \tilde{F}w = \tilde{G} \quad \text{— (2)}$$

$\tilde{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$.

$\tilde{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$.

$\tilde{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2B\xi_y\eta_y$.

Case 3 :- Given $B^2 - 4AC < 0 \Rightarrow \tilde{B}^2 - 4\tilde{A}\tilde{C} < 0$.

Choose $\tilde{B} = 0$ & $\tilde{A} = \tilde{C}$

$\tilde{A}[w_{\xi\xi} + w_{\eta\eta}] + \ell[w_\xi, w_\eta, w, \xi, \eta] = 0$

$\Rightarrow w_{\xi\xi} + w_{\eta\eta} + \ell[w_\xi, w_\eta, w, \xi, \eta] = 0 \longleftarrow$ CANONICAL FORM FOR ELLIPTIC EQN

$u_{xx} + u_{yy} = 0$

How to find ξ & η :-

$$2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\eta_y\xi_y = 0 \quad \text{--- (3)} \times i$$

$$A(\xi_x^2 - \eta_x^2) + B(\xi_x\xi_y - \eta_x\eta_y) + C(\xi_y^2 - \eta_y^2) = 0 \quad \text{--- (4)}$$

Define $\theta = \xi + i\eta$.

Multiply (3) with (i) and add with (4).

$$A\theta_x^2 + C\theta_y^2 + B\theta_x\theta_y = 0$$

$\Rightarrow$ Char Eqns are $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{B^2 - 4AC}}{2A}$

This gives us two complex (distinct) curves along which θ is constant

Ex1 :- $u_{xx} + x^2 u_{yy} = 0$. $(x \neq 0)$

$B^2 - 4AC = 0 - 4 \cdot x^2 \cdot 1 = -4x^2 < 0$.

Elliptic -

$a u_x + b u_y = c$

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{-4x^2}}{2 \cdot 1}$$

$$= \pm i x$$

$\Downarrow$

$$2y \pm i x^2 = c$$

$\exists$ c.o.v $(x,y) \mapsto (\xi(x,y), \eta(x,y))$.

Define $w(\xi,\eta) := u(x,y)$

Canonical Form $\quad w_{\xi\xi} + w_{\eta\eta} + \ell[w_\xi, w_\eta, w, \xi, \eta] = 0$

$u_x = w_\xi \xi_x + w_\eta \eta_x$

$u_{xx} = \xi_{xx} w_\xi + \xi_x [w_{\xi\xi} \xi_x + w_{\xi\eta} \eta_x] + w_\eta \eta_{xx} + \eta_x [w_{\eta\xi} \xi_x + w_{\eta\eta} \eta_x]$. —— (5)

$u_y = w_\xi \xi_y + w_\eta \eta_y$.

$u_{yy} = \xi_{yy} w_\xi + \xi_y [w_{\xi\xi} \xi_y + w_{\xi\eta} \eta_y] + w_\eta \eta_{yy} + \eta_y [w_{\eta\xi} \xi_y + w_{\eta\eta} \eta_y]$. —— (6)

$$\boxed{u_{xx} + x^2 u_{yy} = 0}$$

$$\Rightarrow \left(\xi_x^2 + x^2 \xi_y^2\right) w_{\xi\xi} + \left(\eta_x^2 + x^2 \eta_y^2\right) w_{\eta\eta} + 2\left(\xi_x \eta_x + x^2 \xi_y \eta_y\right) w_{\eta\xi} + \xi_{xx} w_\xi + x^2 \xi_{yy} w_\xi + w_\eta \eta_{xx} + x^2 w_\eta \eta_{yy} = 0$$

$\xi_x^2 + x^2 \xi_y^2 = \eta_x^2 + x^2 \eta_y^2$ —— (7)

$2(\xi_x \eta_x + x^2 \xi_y \eta_y) = 0$ —— (8)

$\theta = \xi + i\eta$.

$$\boxed{\theta_x^2 + x^2\theta_y^2 = 0} \leftarrow \text{Char Eqn}$$

$$\frac{\theta_x^2}{\theta_y^2} + x^2 = 0.$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \pm ix} \Rightarrow 2y \pm ix^2 = C$$

It means " θ is constant along $2y \pm ix^2 = C$ ".

Choose, $\xi(x,y) = 2y$ & $\eta(x,y) = x^2$

$\xi_x = 0$; $\xi_{xx} = 0$.

$\eta_x = 2x$; $\eta_{xx} = 2$.

$\eta_{xy} = 0$; $\xi_{xy} = 0$.

$\eta_y = 0$.

$\eta_{yy} = 0$.

$\xi_y = 2$.

$\xi_{yy} = 0$.

Canonical Form

$$4x^2 w_{\xi\xi} + 4x^2 w_{\eta\eta} + 2w_\eta = 0$$

$$\boxed{\Rightarrow w_{\xi\xi} + w_{\eta\eta} + \frac{w_\eta}{2\eta} = 0}$$

$w_{\xi\xi} + w_{\eta\eta} = f \sim Lw = f$

From 102:

$Lw = 0 \Rightarrow w_h$ is the soln of homogeneous eqn.

$Lw = f \Rightarrow w_p$ is a particular soln

Define

$w := w_h + w_p$.

$Lw = Lw_h + Lw_p$

$= 0 + f = f$