Module 21

INDEPENDENT RANDOM VARIABLES

- $\underline{X} = (X_1, \cdots, X_p)$: a p-dimensional r.v. with joint d.f. $F_{\underline{X}}(\cdot)$;

- $F_{X_i}(\cdot)$: marginal d.f. of X_i, $i = 1, \cdots, p$, i.e.,

$$F_{X_i}(x) = \lim_{\substack{x_j \to \infty \\ j \neq i}} F_{\underline{X}}(x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_p), \;\; i = 1, \cdots, p.$$

Definition 1:

(a) Random variables $X_1, \ldots, X_p$ are said to be independent if, for any subcollection $\{X_{\lambda_1}, \ldots, X_{\lambda_q}\} \subseteq \{X_1, \ldots, X_p\}$ $(2 \leq q \leq p,\ \{\lambda_1, \ldots, \lambda_q\} \subseteq \{1, \ldots, p\})$, we have

$$F_{X_{\lambda_1}, \ldots, X_{\lambda_q}}(x_1, \ldots, x_q) = \prod_{i=1}^{q} F_{X_{\lambda_i}}(x_i),\ \forall\ \underline{x} = (x_1, \ldots, x_q) \in \mathbb{R}^q.$$

(b) Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of random variables. The r.v.s in the family $\{X_\lambda : \lambda \in \Lambda\}$ are said to be independent if those in any finite subcollection of $\{X_\lambda : \lambda \in \Lambda\}$ are independent.

Remark 1:

(a) If $\Lambda_1 \subseteq \Lambda_2$ and the r.v.s $\{X_\lambda : \lambda \in \Lambda_2\}$ are independent then r.v.s $\{X_\lambda : \lambda \in \Lambda_1\}$ are also independent.

(b) The above definition of independent r.v.s can be extended to independence of random vectors (of possibly different dimensions) in an obvious manner. For example, let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional r.v. with joint d.f. $F(\cdot)$, $\underline{Y} = (Y_1, \ldots, Y_q)$ is a q-dimensional r.v. with joint d.f. $G(\cdot)$, and let $\underline{Z} = (X_1, \ldots, X_p, Y_1, \ldots, Y_q) = (\underline{X}, \underline{Y})$ (a $(p+q)$-dimensional r.v.) have joint d.f. $H(\cdot)$. Then $\underline{X}$ and $\underline{Y}$ are said to be independent iff

$$H(\underline{x}, \underline{y}) = F(\underline{x})G(\underline{y}), \ \forall \ \underline{z} = (\underline{x}, \underline{y}) \in \mathbb{R}^{p+q}.$$

Result 1 :

R.V.s $X_1, \dots X_p$ are independent iff

$$F_{X_1,\dots,X_p}(x_1,\dots,x_p) = \prod_{i=1}^{p} F_{X_i}(x_i), \ \forall \ \underline{x} = (x_1,\dots,x_p) \in \mathbb{R}^p. \qquad (1)$$

Proof: Suppose that $X_1, \dots, X_p$ are independent. Then, by definition, (1) holds.

Conversely suppose that (1) holds. Then, for $\underline{x} = (x_1, \dots, x_{p-1}) \in \mathbb{R}^{p-1}$,

$$\begin{aligned} F_{X_1,\dots,X_{p-1}}(x_1,\dots,x_{p-1}) &= \lim_{x_p \to \infty} F_{X_1,\dots,X_{p-1},X_p}(x_1,\dots,x_{p-1},x_p) \\ &= \lim_{x_p \to \infty} \prod_{j=1}^{p} F_{X_j}(x_j) \\ &= \Big[\lim_{x_p \to \infty} F_{X_p}(x_p)\Big] \prod_{j=1}^{p-1} F_{X_j}(x_j)\Big] \end{aligned}$$

$$= \prod_{j=1}^{p-1} F_{X_j}(x_j).$$

In general, for $2 \leq r \leq p$, $\{X_{\lambda_1}, \ldots, X_{\lambda_r}\} \subseteq \{X_1, \ldots, X_p\}$ and $\underline{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$

$$F_{X_{\lambda_1}, \ldots, X_{\lambda_r}}(x_1, \ldots, x_r) = \prod_{i=1}^{r} F_{X_{\lambda_i}}(x_i).$$

Remark 2: The above results remain valid if random variables $X_1, \ldots, X_p$ are replaced by random vectors $\underline{X}_1, \ldots, \underline{X}_p$ of (possibly) different dimensions.

Example 1: Let (X, Y) be a r.v. with joint d.f.

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \\ y^2, & \text{if } 0 \le y < 1 \text{ and } x \ge y \\ x(2y - x), & \text{if } 0 \le x < y < 1 \\ x(2 - x), & \text{if } 0 \le x < 1,\ y \ge 1 \\ 1, & \text{if } x \ge 1,\ y \ge 1 \end{cases}.$$

The marginal d.f.s of X and Y are

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{if } x < 0 \\ x(2 - x), & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases},$$

and

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = \begin{cases} 0, & \text{if } y < 0 \\ y^2, & \text{if } 0 \le y < 1 \\ 1, & \text{if } y \ge 1. \end{cases},$$

respectively.

Clearly,

$$F_{X,Y}(x,y) \neq F_X(x)F_Y(y), \forall\ (x,y) \in \mathbb{R}^2$$

$$\Rightarrow \quad X \text{ and } Y \text{are not independent.}$$

Example 2: Let the r.v. (X, Y) have the joint d.f.

$$F_{X,Y}(x,y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{e^x(1-(1+y)e^{-y})}{2}, & \text{if } x < 0,\ y \geq 0 \\ \frac{(2-e^{-x})(1-(1+y)e^{-y})}{2}, & \text{if } x \geq 0,\ y \geq 0 \end{cases}.$$

The marginal d.f.s of X and Y are

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \begin{cases} \frac{e^x}{2}, & \text{if } x < 0 \\ \frac{2-e^{-x}}{2}, & \text{if } x \geq 0 \end{cases}$$

and

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = \begin{cases} 0, & \text{if } y < 0 \\ 1 - (1 + y)e^{-y}, & \text{if } y \geq 0 \end{cases},$$

respectively. Clearly X and Y are independent, as

$$F_{X,Y}(x, y) = F_X(x)F_Y(y), \forall\ (x, y) \in \mathbb{R}^2.$$

Take Home Problems

- Let F and G be d.f.s on $\mathbb{R}$. Define,

$$H_1(x, y) = \min\{F(x), G(y)\}, \ (x, y) \in \mathbb{R}^2$$

and

$$H_2(x, y) = \max\{F(x) + G(y) - 1, 0\}, (x, y) \in \mathbb{R}^2.$$

(a) Show that H_1 and H_2 are d.f.s on $\mathbb{R}^2$.
(b) Find marginal d.f.s of H_1. Check independence.
(c) Find marginal d.f.s of H_2. Check independence.

Abstract of Next Module

- We will introduce discrete random vectors and study their probability distributions.

Thank you for your patience