Assignment - 2

1. (i) Given the ODE

$$P(x)y'' + Q(x)y' + R(x)y + \lambda y = 0 \quad ; \lambda \in \mathbb{C}. \longrightarrow (1)$$

If we want (1) to look like

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0$$

ie, $py'' + p'y' + qy + \lambda r y = 0 \longrightarrow$ (ii)

then, we multiply (1) with $\mu(x)$ and observe that

$$p(x) = \mu(x)P(x) \quad \text{and} \quad p'(x) = \mu(x)Q(x)$$

(Comparing with (ii))

Hence,

$$\mu(x)Q(x) = [\mu(x)P(x)]' = \mu P' + P\mu'$$

$$\Rightarrow \mu' P = \mu(-P' + Q) = (Q - P')\mu.$$

$$\Rightarrow \mu' = \frac{Q-P'}{P}\mu.$$

$$\Rightarrow \mu = \exp\left(\int_x \frac{Q-P'}{P}\,dx\right).$$

Thus if we multiply the I.F $\mu(x) = \exp\left(\int_x \frac{Q-P'}{P}\,dx\right)$ we can turn (1) into a self-adjoint form.

(ii) Given, $y'' + xy' + \lambda y = 0 \longrightarrow$ (iii)

Here $P(x) = 1$, $Q(x) = x$

Hence, to convert (iii) into self-adjoint form we calculate

$$\mu(x) = \exp\left(\int_x \frac{Q-P'}{P}\,dx\right)$$

ie $\mu(x) = \exp\left(\int_x x\,dx\right) = e^{x^2/2} \quad (e \equiv \exp)$

Hence the self-adjoint form of (11) is given by

$$e^{x^2/2} y'' + x e^{x^2/2} y' + \lambda e^{x^2/2} y = 0.$$

ie, $\left[y' e^{x^2/2}\right]' + \lambda e^{x^2/2} y = 0$.

2. Given the ODE

$$(py')' + qy + \lambda r y = 0$$

with eigenpair (λ_n, ϕ_n). Hence we have,

$$(p\phi_n')' + q\phi_n + \lambda r\phi_n = 0$$

$$\Rightarrow \int_a^b \phi_n (p\phi_n')' + \int_a^b q\phi_n^2 + \lambda_n \int_a^b r\phi_n^2 = 0$$

$$\Rightarrow -\int_a^b (p\phi_n')\phi_n' + p\phi_n'\phi_n \Big|_a^b + \int_a^b q\phi_n^2 + \lambda_n \int_a^b r\phi_n^2 = 0 \quad \text{(Integration by parts).}$$

$$\Rightarrow \lambda_n = \frac{-p\phi_n'\phi_n \Big|_a^b + \int_a^b \left[p\phi_n'^2 - q\phi_n^2\right] dx}{\int_a^b r\phi_n^2 \, dx}.$$

~~3. $(xy')' + y/x = 1/x$; $x \in [1, e]$~~

~~$y(1) = y(e) = 0$~~

~~The eqn can be written as~~

~~$xy'' + y' + \dots$~~

~~3b $2x^2y'' + x$~~

3. $(xy')' + \frac{y}{x} = \frac{1}{x}$; $x \in [1, e]$

$y(1) = y(e) = 0$

i)

Firstly we solve for

$$(xy')' + \frac{y}{x} = -\lambda r y \ ; \ y(1) = y(e) = 0$$

$$\Rightarrow x^2y'' + xy' + (1 + \lambda r x) y = 0$$

Choose the weight fxn $r(x) = \frac{1}{x}$. Hence

$$x^2y'' + xy' + (1+\lambda) y = 0$$

The char eqn is

$$r^2 + (1+\lambda) = 0.$$

Assuming $\lambda > -1$ we have

$\lambda_n = n^2\pi^2 - 1$ and $y_n(x) = C \sin(n\pi \ln x)$; $n = 1, 2, \ldots$

ii) To normalize the eigenfunction we have,

$$\int_1^e y_n^2(x)\, r(x)\, dx = 1$$

$$\Rightarrow c^2 \int_1^e \sin(n\pi \ln x) \frac{1}{x}\, dx = 1$$

$$\Rightarrow c^2/2 = 1. \Rightarrow C = \sqrt{2}.$$

$\therefore$ The orthonormal set is given by $\left\{ \sqrt{2} \sin(n\pi \ln x) : n \in \mathbb{N} \right\}$

iii) To solve the eqn $Ly = \frac{1}{x}$ write

$$y(x) = \sum_1^\infty c_n \sqrt{2} \sin(n\pi \ln x)$$

Substituting in the equation we have,

$$\frac{1}{x} = Ly = -\sum_{n=1}^\infty c_n \lambda_n \sqrt{2} \sin(n\pi \ln x) \cdot \frac{1}{x}$$

Multiplying both sides with $y_m(x) = \sqrt{2} \sin(m\pi \ln x)$;

$$\lambda_m c_m = \int_1^e \sqrt{2} \sin(m\pi \ln x) \cdot \frac{1}{x}\, dx = \frac{\sqrt{2}}{m\pi} \left[(-1)^m - 1 \right]$$

$$\Rightarrow C_m = \frac{\sqrt{2}}{m\pi} \frac{[(-1)^{m}-1]}{m^2\pi^2-1}$$

Hence, $$y(x) = \sum_{1}^{\infty} \frac{2}{n\pi} \frac{[(-1)^n-1]}{n^2\pi^2-1} \sin(n\pi \ln x).$$

4. (i) f is a piecewise continuous fn on $[-\pi,\pi]$ s.t f' also piecewise continuous.

Given, $$f(x) = a_0 + \sum_{1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

WLOG, we may assume f' is continuous on $[-\pi,\pi]$.

$$\Rightarrow f'(x) = \sum_{1}^{\infty} [-n a_n \sin(nx) + n b_n \cos(nx)]$$

[For pts where f' is not continuous we can't talk about the derivative of 'f'. Since the set of discontinuity of 'f'' is finite the Fourier series representation of f' is valid where f' exists]

(ii) $f: [-\pi,\pi] \to \mathbb{R}$ is piecewise continuous.

Hence, $$f(x) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

Integrating 'f' in the interval $[-\pi, x]$ we have.

$$g(x) := \int_{-\pi}^{x} f(x)\,dx = a_0(x+\pi) + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{x} \cos(nx)\,dx + b_n \int_{-\pi}^{x} \sin nx\,dx\right]$$

$$\Rightarrow g(x) = a_0(x+\pi) + \sum_{n=1}^{\infty} \left[\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx) - \frac{b_n}{n}\right].$$

$\blacksquare$

5. $\sum_{1}^{\infty} \frac{(-1)^n}{n^2} = ?$

Let us start by finding ~~the~~ the Fourier series of
$f(x) = x^2$ on $[-\pi, \pi]$.

$\because f$ is an even function hence we may write $f(x) = a_0 + \sum_{1}^{\infty} a_n \cos(nx)$

~~the~~ where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

ie, $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$.

and, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx. = (-1)^n . \frac{4}{n^2}$

$\because f$ is continuous hence we have,

$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$ for all $x \in [-\pi, \pi]$

Set, $x = 0$,

$0 = \frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{(-1)^n}{n^2}$

$\Rightarrow \sum_{1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.