Module 26

JOINT MOMENT GENERATING FUNCTION OF R.V.'s AND EQUALITY IN DISTRIBUTION

- $\underline{X} = (X_1, \ldots, X_p)'$: a p-dimensional random vector;
- $A = \{\underline{t} = (t_1, \ldots, t_p) \in \mathbb{R}^p : E\left(|e^{\sum_{i=1}^{p} t_i X_i}|\right) = E\left(e^{\sum_{i=1}^{p} t_i X_i}\right) < \infty\}$;
- Define the function $M_{\underline{X}} : A \to \mathbb{R}$ by

$$M_{\underline{X}}(\underline{t}) = E\left(e^{\sum_{i=1}^{p} t_i X_i}\right), \; \underline{t} = (t_1, \ldots, t_p) \in A. \tag{1}$$

Definition 1: The function $M_{\underline{X}} : A \to \mathbb{R}$, defined by (1), is called the joint moment generating function (m.g.f.) of $\underline{X}$.

Remark 1:

(a) $M_{\underline{X}}(\underline{0}) = 1$. Therefore, $\underline{0} \in A$ and $A \neq \phi$;

(b) Let $\underline{X} = (X_1, \ldots, X_p)$, where $X_1, \ldots, X_p$ are independent. For $\underline{a} = (a_1, \ldots, a_p) \in \mathbb{R}^p$, let $Y = \sum_{i=1}^{p} a_i X_i$. Then

$$\begin{aligned} M_Y(\underline{t}) &= E\left(e^{t\sum_{i=1}^{p} a_i X_i}\right) \\ &= E\left(\prod_{i=1}^{p} e^{ta_i X_i}\right) \\ &= \prod_{i=1}^{p} E\left(e^{ta_i X_i}\right) \\ &= \prod_{i=1}^{p} M_{X_i}(ta_i), \;\; t \in \mathbb{R}. \end{aligned}$$

In particular, if $Z = \sum_{i=1}^{p} X_i$, then

$$M_Z(t) = \prod_{i=1}^{p} M_{X_i}(t), t \in \mathbb{R},$$

i.e. m.g.f. of sum of independent r.v.s is same as the product of marginal m.g.f.s.

(c) Let $\underline{X} = (X_1, \ldots, X_p)$ and $\underline{Y} = (X_1, \ldots, X_{p-1})$. Then

$$M_{\underline{Y}}(t_1, \ldots, t_{p-1}) = M_{\underline{X}}(t_1, \ldots, t_{p-1}, 0),$$

provided the involved expectations are finite. Thus, one can obtain the marginal m.g.f.s from the joint m.g.f. by taking the argument of unwanted variable in the joint m.g.f to be 0. In particular,

$$M_{\underline{X}}(0, \ldots, 0, t_i, 0, \ldots, 0) = M_{X_i}(t_i), \ i = 1, \ldots, p,$$
$$M_{\underline{X}}(0, \ldots, 0, t_i, 0, \ldots, 0, t_j, 0, \ldots, 0) = M_{X_i, X_j}(t_i, t_j), \ i, j \in \{1, \ldots, p\},$$

provided the expectations exist.

(d) If $X_1, \ldots, X_p$ are independent, then

$$M_{\underline{X}}(\underline{t}) = E\left(e^{t\sum_{i=1}^{p} X_i}\right) = E\left(\prod_{i=1}^{p} e^{tX_i}\right)$$
$$= \prod_{i=1}^{p} E\left(e^{tX_i}\right)$$
$$= \prod_{i=1}^{p} M_{X_i}(t_i),\ \ \underline{t} = (t_1, \ldots, t_p) \in A.$$

Result 1 : Suppose that the joint m.g.f. $M_{\underline{X}}(\underline{t})$ is finite in a rectangle $(-\underline{a}, \underline{a}) \subseteq \mathbb{R}^p$. Then, $M_{\underline{X}}(\underline{t})$ possesses partial derivatives of all orders in $(-\underline{a}, \underline{a})$. Furthermore, for any non-negative integers $k_1, \ldots, k_p$, $E(|X_1^{k_1} \cdots X_p^{k_p}|)$ is finite and,

$$E(X_1^{k_1} \cdots X_p^{k_p}) = \left[\frac{\partial^{k_1+\cdots+k_p}}{\partial t_1^{k_1} \cdots \partial t_p^{k_p}} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}.$$

Result 2 :

Under the assumption of Result 1, let $\psi_{\underline{X}}(\underline{t}) = \ln M_{\underline{X}}(\underline{t})$, $\underline{t} \in (-\underline{a}, \underline{a})$. Then

$$E(X_i^m) = \left[\frac{\partial^m}{\partial t_i^m} M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \; i = 1, \ldots, p, \; m = 1, 2, \ldots$$

$$\mathrm{Var}(X_i) = \left[\frac{\partial^2}{\partial t_i^2} \psi_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \; i = 1, \ldots, p$$

$$\text{and } \mathrm{Cov}(X_i, X_j) = \left[\frac{\partial^2}{\partial t_i \partial t_j} \psi_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}, \; i, j = 1, \ldots, p.$$

Proof. The first assertion follows from Result 1.

$$\frac{\partial}{\partial t_i} \psi_{\underline{X}}(\underline{t}) = \frac{1}{M_{\underline{X}}(\underline{t})} \frac{\partial}{\partial t_i} M_{\underline{X}}(\underline{t}),$$

$$\frac{\partial^2}{\partial t_i \partial t_j}\psi_{\underline{X}}(\underline{t}) = \frac{M_{\underline{X}}(\underline{t})\frac{\partial^2}{\partial t_j \partial t_i}M_{\underline{X}}(\underline{t}) - \left[\frac{\partial}{\partial t_i}M_{\underline{X}}(\underline{t})\right]\frac{\partial}{\partial t_j}M_{\underline{X}}(\underline{t})}{[M_{\underline{X}}(\underline{t})]^2}.$$

Thus,

$$\begin{aligned}\left[\frac{\partial^2}{\partial t_i^2}\psi_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} &= \left[\frac{\partial^2}{\partial t_i^2}M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} - \left[\left[\frac{\partial}{\partial t_i}M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}\right]^2 \\ &= E(X_i^2) - (E(X_i))^2 \\ &= \mathrm{Var}(X_i)\end{aligned}$$

and

$$\begin{aligned}\left[\frac{\partial^2}{\partial t_i \partial t_j}\psi_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} &= \left[\frac{\partial^2}{\partial t_j \partial t_i}M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} - \left[\frac{\partial}{\partial t_i}M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}}\left[\frac{\partial}{\partial t_j}M_{\underline{X}}(\underline{t})\right]_{\underline{t}=\underline{0}} \\ &= E(X_iX_j) - E(X_i)E(X_j) = \mathrm{Cov}(X_i, X_j).\end{aligned}$$

Result 3 :

Let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional r.v. with joint m.g.f. $M_{\underline{X}}(\underline{t})$ and marginal m.g.f.s $M_{X_i}(\cdot), i = 1, \ldots, p$. Suppose that, for some $\underline{a} \in \mathbb{R}^p$,

$$M_{\underline{X}}(\underline{t}) = \prod_{i=1}^{p} M_{X_i}(t_i), \; \forall \; \underline{t} \in (-\underline{a}, \underline{a}).$$

Then $X_1, \ldots, X_p$ are independent.

Remark 2 : Remark 1 and Result 3 hold for random vectors as well.

Equality in Distribution :

Definition 2 :

(a) Two p-dimensional r.v.s $\underline{X}$ and $\underline{Y}$ are said to have the same distribution (written as $\underline{X} \stackrel{d}{=} \underline{Y}$) if $F_{\underline{X}}(\underline{x}) = F_{\underline{Y}}(\underline{x}),\ \forall \underline{x} \in \mathbb{R}^p$, where $F_{\underline{X}}(\cdot)$ and $F_{\underline{Y}}(\cdot)$ are d.f.s of $\underline{X}$ and $\underline{Y}$, respectively.

(b) R.V.s $X_1, \ldots, X_p$ are said to be exchangeable iff $(X_1, \ldots, X_p) \stackrel{d}{=} (X_{\beta_1}, \ldots, X_{\beta_p})$, for any permutation $(\beta_1, \ldots, \beta_p)$ of $(X_1, \ldots, X_p)$.

Result 4 :

Let $\underline{X}$ and $\underline{Y}$ be r.v.s having p.m.f./p.d.f. $f_{\underline{X}}(\cdot)$ and $f_{\underline{Y}}(\cdot)$, respectively. Then

(a) $f_{\underline{X}}(\underline{x}) = f_{\underline{Y}}(\underline{x}),\ \forall\ \underline{x} \in \mathbb{R}^p \Rightarrow \underline{X} \stackrel{d}{=} \underline{Y}$;

(b) $\underline{X} \stackrel{d}{=} \underline{Y} \Rightarrow \psi(\underline{X}) \stackrel{d}{=} \psi(\underline{Y})$, for any function $\psi : \mathbb{R}^p \to \mathbb{R}$;

(c) $M_{\underline{X}}(\underline{t}) = M_{\underline{Y}}(\underline{t}),\ \forall\ \underline{t} \in (-\underline{a}, \underline{a})$, for some $\underline{a} \in \mathbb{R}^p$, $\Rightarrow \underline{X} \stackrel{d}{=} \underline{Y}$.

Example 1 :

Let $X_1, \ldots, X_p$ be i.i.d. r.v.s with common p.m.f./p.d.f. $f(\cdot)$. Then

(a) $X_1, \ldots, X_p$ are exchangeable;

(b) $E\left(\frac{X_1}{\sum\limits_{i=1}^{p} X_i}\right) = 1$, provided the expectations exists;

(c) $P(X_1 = \min\{X_1, \ldots, X_p\}) = \frac{1}{p}$, provided X_1 is A.C.

Solution.

(a) Suppose that $X_1, \ldots, X_p$ are i.i.d. r.v.s with common p.m.f./p.d.f. $f(\cdot)$. Let $\underline{X} = (X_1, \ldots, X_p)$, $\underline{Y} = (X_{\beta_1}, \ldots, X_{\beta_p})$ for a permutation $(\beta_1, \ldots, \beta_p)$ of $(1, \ldots, p)$. Then,

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{p} f_{X_i}(x_i) = \prod_{i=1}^{p} f(x_i) = \prod_{i=1}^{p} f_{X_{\beta_i}}(x_i) = f_{\underline{Y}}(\underline{x}), \ \underline{x} \in \mathbb{R}^p,$$

implying that $\underline{X} \stackrel{d}{=} \underline{Y}$.

(b) By (a), we have for any permutation $(\beta_1, \ldots, \beta_p)$ of $(1, \ldots, p)$, $(X_1, \ldots, X_p) \stackrel{d}{=} (X_{\beta_1}, \ldots, X_{\beta_p})$

$$\Rightarrow E\left(\frac{X_1}{\sum\limits_{i=1}^{p} X_i}\right) = E\left(\frac{X_{\beta_1}}{\sum\limits_{i=1}^{p} X_{\beta_i}}\right)$$

$$\Rightarrow E\left(\frac{X_1}{\sum\limits_{i=1}^{p} X_i}\right) = E\left(\frac{X_j}{\sum\limits_{i=1}^{p} X_i}\right),\ j = 1, \ldots, p$$

But
$$\sum_{j=1}^{p} E\left(\frac{X_j}{\sum\limits_{i=1}^{p} X_i}\right) = E\left(\frac{\sum\limits_{j=1}^{p} X_j}{\sum\limits_{i=1}^{p} X_i}\right) = 1$$

$$\Rightarrow E\left(\frac{X_j}{\sum\limits_{i=1}^{p} X_i}\right) = \frac{1}{p},\ j = 1, \ldots, p.$$

(c) Since X_1 is A.C., $\underline{X} = (X_1, \ldots, X_p)'$ is A.C. Let $X_{1:p} \leq \cdots \leq X_{p:p}$ denote the ranked values of $X_1, \ldots, X_p$ ($X_{1:p}, \ldots, X_{p:p}$ are called order statistics of $X_1, \ldots, X_p$). Then

$$P(X_{1:p} < \cdots < X_{p:p}) = 1 \tag{2}$$

$$P(X_1 = \min\{X_1, \ldots, X_p\}) = P(X_i = \min\{X_1, \ldots, X_p\}), i = 1, \ldots, p$$

But due to (2),

$$\sum_{i=1}^{p} P(X_i = \min\{X_1, \ldots, X_p\}) = 1$$

$$\Rightarrow P(X_i = \min\{X_1, \ldots, X_p\}) = \frac{1}{p}, i = 1, \ldots, p.$$

Take Home Problems

1. Let X_1, X_2 and X_3 be i.i.d. random variables with $P(\{X_1 = X_2 = X_3\}) = \frac{1}{6}$ and $P(\{X_1 = X_2 < X_3\}) = \frac{1}{6}$. Find $P(\{X_3 < X_1 < X_2\})$.

2. Let $\underline{X} = (X_1, X_2)$ be an absolutely continuous r.v. with joint m.g.f.

$$M(t_1, t_2) = e^{\frac{t_1^2 + t_2^2 + 2\rho t_1 t_2}{2}}, \quad (t_1, t_2) \in \mathbb{R}^2.$$

Find correlation between X_1 and X_2. Derive conditions under which X_1 and X_2 are independent.

Thank you for your patience