Module 5

CONTINUITY OF PROBABILITY FUNCTION

AND

EQUALLY LIKELY PROBABILITY MODELS

Review of last Module:

- $$P\left(\bigcup_{i=1}^{n} E_i\right) = p_{1,n} - p_{2,n} + p_{3,n} - \cdots + (-1)^{n-1} p_{n,n},$$

  where

  $$p_{r,n} = \sum_{1\le i_1 < i_2 < \cdots < i_r \le n} \cdots \sum P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}),\ \ r = 1, \ldots, n;$$

- $$P\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} P\left(E_i\right); \qquad \text{(Boole's inequality)}$$

- $$P\left(\bigcap_{i=1}^{n} E_i\right) \ge \sum_{i=1}^{n} P\left(E_i\right) - (n-1). \qquad \text{(Bonferroni's inquality);}$$

- Throughout assume that $(\Omega, \mathcal{P}(\Omega), P)$ is a probability space associated with a random experiment $\mathcal{E}$.

Continuity of Probability Function

Definition 1: Let $\{E_n\}_{n\geq 1}$ be a sequence of events. The sequence $\{E_n\}_{n\geq 1}$ is said to be

(a) increasing (written as $E_n \uparrow$) if $E_n \subseteq E_{n+1},\ n = 1, 2 \ldots.$ In that case the limit of sequence $\{E_n\}_{n\geq 1}$ is defined as $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$;

(b) decreasing (written as $E_n \downarrow$) if $E_{n+1} \subseteq E_n,\ n = 1, 2 \ldots.$ In that case the limit of sequence $\{E_n\}_{n\geq 1}$ is defined as $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$;

(c) monotone if either $E_n \uparrow$ or $E_n \downarrow$.

Example 1:

(a) Let $E_n = \left(0, \frac{1}{n}\right)$, $n = 1, 2 \ldots$. Then $E_n \downarrow$ and $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n = \phi$ (the empty set);

(b) Let $E_n = \left(0, 1 - \frac{1}{n+1}\right]$, $n = 1, 2, \ldots$. Then $E_n \uparrow$ and $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = (0, 1)$;

(c) Let $E_n = \left(1 - \frac{1}{n+1}, 2 + \frac{1}{n+1}\right)$, $n = 1, 2, \ldots$. Then $E_n \downarrow$ and $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n = [1, 2]$.

Result 1: Let $\{E_n\}_{n\geq 1}$ be a monotone sequence of events. Then

$$P\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} P\left(E_n\right), \qquad \text{(Continuity of probability function)}$$

i.e.,

$$\lim_{n\to\infty} P(E_n) = \begin{cases} P\left(\bigcup_{n=1}^{\infty} E_n\right), & \text{if } E_n \uparrow \\ P\left(\bigcap_{n=1}^{\infty} E_n\right), & \text{if } E_n \downarrow \end{cases}.$$

Proof: Suppose $E_n \uparrow$, so that $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$. Define

$$E_0 = \phi,\ F_1 = E_1,\ F_2 = E_2 - E_1,\ F_n = E_n - E_{n-1},\ n = 1, 2, \ldots.$$

Then F_n's are disjoint and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$. Thus

$$\begin{aligned}
P\left(\lim_{n\to\infty} E_n\right) &= P\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= P\left(\bigcup_{n=1}^{\infty} F_n\right) \\
&= \sum_{n=1}^{\infty} P\left(F_n\right) \qquad (F_n^{\prime} s \text{ are disjoint}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} P\left(F_k\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} P\left(E_k - E_{k-1}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[P\left(E_k\right) - P\left(E_{k-1}\right)\right] \qquad (E_{k-1} \subseteq E_k,\ \forall\ k \geq 1) \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n} P\left(E_k\right) - \sum_{k=1}^{n} P\left(E_{k-1}\right)\right] \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n} P\left(E_k\right) - \sum_{k=0}^{n-1} P\left(E_k\right)\right] \\
&= \lim_{n\to\infty} \left[P\left(E_n\right) - P(\,E_0\,)\right] \\
&= \lim_{n\to\infty} P\left(E_n\right). \qquad (E_0 = \phi)
\end{aligned}$$

Now suppose that $E_n \downarrow$, so that $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$. Then $E_n^c \uparrow$ and that

$$\begin{aligned}
P\left(\bigcup_{n=1}^{\infty} E_n^c\right) &= P\left(\lim_{n\to\infty} E_n^c\right) = \lim_{n\to\infty} P\left(E_n^c\right) \\
\Longrightarrow P\left(\lim_{n\to\infty} E_n\right) &= P\left(\bigcap_{n=1}^{\infty} E_n\right) \\
&= 1 - P\left(\left(\bigcap_{n=1}^{\infty} E_n\right)^c\right) \\
&= 1 - P\left(\bigcup_{n=1}^{\infty} E_n^c\right) \\
&= 1 - \lim_{n\to\infty} P\left(E_n^c\right) \\
&= 1 - \lim_{n\to\infty} \left[1 - P\left(E_n\right)\right] \\
&= \lim_{n\to\infty} P\left(E_n\right).
\end{aligned}$$

Result 2: Let $\{E_n\}_{n\geq 1}$ be a sequence of events. Then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} P\left(E_n\right). \quad \text{(generalized Boole's Inequality)}$$

Proof: Let $F_n = \bigcup_{k=1}^{n} E_k$, $n = 1, 2, \ldots$. Then $F_n \uparrow$ and $\lim_{n\to\infty} F_n = \bigcup_{k=1}^{\infty} E_k$. Thus

$$\begin{aligned} P\left(\bigcup_{k=1}^{\infty} E_k\right) &= P\left(\lim_{n\to\infty} F_n\right) \\ &= \lim_{n\to\infty} P\left(F_n\right). \quad \text{(continuity of probability functions)} \end{aligned}$$

Also, for $n = 1, 2, \ldots,$

$$\begin{aligned} P\left(F_n\right) &= P\left(\bigcup_{k=1}^{n} E_k\right) \\ &\leq \sum_{k=1}^{n} P\left(E_k\right) \quad \text{(Boole's inequality)} \\ \Longrightarrow \quad \lim_{n\to\infty} P\left(F_n\right) &\leq \lim_{n\to\infty} \sum_{k=1}^{n} P\left(E_k\right) \\ &= \sum_{k=1}^{\infty} P\left(E_k\right) \\ \Longrightarrow \quad P\left(\bigcup_{k=1}^{\infty} E_k\right) &= \lim_{n\to\infty} P\left(F_n\right) \leq \sum_{k=1}^{\infty} P\left(E_k\right). \end{aligned}$$

Definition 2:

(a) Events $\{E_\alpha : \alpha \in S\}$ (for some index set S) are said to be mutually exclusive if $E_i \cap E_j = \phi$ (the empty set), $\forall\ i \neq j$;

(b) A collection $\{E_\alpha : \alpha \in S\}$ of events is said to be exhaustive if $P\left(\bigcup_{\alpha \in S} E_\alpha\right) = 1$.

Equally Likely Probability Models

- $\Omega = \{\omega_1, \ldots, \omega_k\}$ has k (a finite number) elements;
- For a set A, let $|A|$ denote the number of elements in A;
- Let $P : \mathcal{P}(\Omega) \to \mathbb{R}$ be defined by $P(\{\omega_i\}) = \frac{1}{k},\ i = 1, ..., k$ (each outcome in the sample space Ω is equally likely) and for any event $E \subseteq \Omega$ (note that E is a finite set)

$$P(E) = \sum_{\omega_i \in E} P(\{\omega_i\}) = \frac{|E|}{k} = \frac{\text{number of cases favourable to } E}{\text{total number of cases}}; \quad \begin{pmatrix}\text{part of probability} \\ \text{modelling}\end{pmatrix}$$

- Clearly
  - $P(E) \geq 0, \forall\ E \in \mathcal{P}(\Omega)$;
  - If $E_i,\ i \in S$, is a countable collection of disjoint events then

$$P\left(\bigcup_{i \in S} E_i\right) = \frac{|\bigcup_{i \in S} E_i|}{k} = \frac{\sum_{i \in S} |E_i|}{k} = \sum_{i \in S} \frac{|E_i|}{k} = P(E_i);$$

  - $P(\Omega) = \frac{|\Omega|}{k} = \frac{k}{k} = 1$,

i.e., $P : \mathcal{P}(\Omega) \to \mathbb{R}$ is a probability function.

Remark 1: When we say that a random experiment, with finite sample space, has been performed at random, it means that all outcomes in the sample space Ω are equally likely. In that case, for any event E,

$$P\left(E\right) = \frac{\text{number of cases favourable to } E}{\text{total number of cases}}.$$

Example 2: Five cards are drawn at random and without replacement from a deck of 52 cards. Find the probability of following events.

(a) E_1: each card is spade;

(b) E_2: at least one card is spade;

(c) E_3: three cards are king and two cards are queen;

(d) E_4: two kings, two queens and one jack are drawn.

Solution:

- Total number of favourable cases $= \binom{52}{5}$;
- Number of cases favourable to $E_1 = \binom{13}{5}$;
- $P(E_1) = \frac{\binom{13}{5}}{\binom{52}{5}}$.
- Similarly

$$\begin{aligned} P(E_2) &= 1 - P(E_2^c) \\ &= 1 - P(\text{none of the card is spade}) \\ &= 1 - \frac{\binom{39}{5}}{\binom{52}{5}}; \end{aligned}$$

- $P(E_3) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$;
- $P(E_4) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$.

Convention:

- Unbiased coin/die: all points are equally likely;
- loaded coin/die: all points are not equally likely (not unbiased).

Take Home Problems:

1. Let $\Omega = [0, 1]$ and, for $(a, b] \subseteq [0, 1]$ $(0 \leq a < b \leq 1)$, let $P((a, b]) = b - a$. For $(0 \leq a < b \leq 1)$ and a countable set S, find
   (a) $P([a, b])$, $P([a, b))$ and $P((a, b))$;
   (b) $P(\{a\})$
   (c) $P(S)$.
2. Two slips are drawn together and at random from a box containing 6 slips, numbered 1 to 6. What is the probability that larger of the numbers on drawn slips is 3?

Abstract of Next Module

- Generally chances of occurrence or non-occurrence of one event affect the chances of occurrences or non-occurrences of other events. To understand this dependence we will introduce the concept of conditional probability of an event B given the information that event A has occurred;
- We will also prove two important theorems dealing with conditional probabilities.

Thank you for your patience