MSO 201a: Probability and Statistics

2016-2017-II Semester
Assignment-IV

A. Illustrative Discussion Problems

1. Let the random variable X have the p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } -2 < x < -1 \\ \frac{x^2}{65}, & \text{if } -1 < x < 4 \\ \frac{2x}{27}, & \text{if } 4 < x < 5 \\ 0, & \text{otherwise} \end{cases}.$$

   Find the distribution functions and hence the p.d.f.s (provided they exist) of $X^+ = \max(X, 0)$ and $Y = |X|$.

2. Let the random variable X have the p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } -2 < x < -1 \\ \frac{1}{6}, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

   Find the p.d.f. and hence the d.f. of $Y = X^2$.

3. Let the random variable X have the p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2x^2}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases},$$

   and let $Y = 1/X$.
   (a) Find the distribution function of Y and hence find its p.d.f.;
   (b) Find the p.d.f. of Y directly (i.e., without finding the distribution function).

4. Let the random variable X have the p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}, & \text{if } -2 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.m.f./p.d.f. of $Y = g(X)$, where

$$g(x) = \begin{cases} -1, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

5. In three independent tosses of a fair coin let X denote the number of tails appearing. Let $Y = X^2$ and $Z = 2X^2 + 1$. Find the mean and variance of random variables Y and Z.

6. (a) Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{c}{x^{2+r}}, & \text{if } x \in \{1, 2, \ldots, \} \\ 0, & \text{otherwise} \end{cases},$$

where $c^{-1} = \sum_{n=1}^{\infty} n^{-2-r}$ and $r \geq 0$ is an integer. For what values of $j \in \{0, 1, 2, \ldots\}$, $E(X^j)$ is finite?
(b) Find a distribution for which no moment exist.

7. Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2}, & \text{if } 1 < x \leq 2 \\ \frac{3-x}{2}, & \text{if } 2 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Find the expected value of $Y = X^2 - 5X + 3$.

8. (a) Let $E(|X|^\beta) < \infty$ for some $\beta > 0$. Then show that $E(|X|^\alpha) < \infty$, $\forall\ \alpha \in (0, \beta]$.
(b) If X is an absolutely continuous random variable with median m, then show that $E(|X - m|) \leq E(|X - c|)$, $\forall\ c \in (-\infty, \infty)$.

9. Consider a target comprising of three concentric circles of radii $1/\sqrt{3}, 1, \sqrt{3}$ feet. Shots within the inner circle earn 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target do not earn any point. Let X denote the distance (in feet) of the hit from the centre and suppose that X has the p.d.f.

$$f_X(x) = \begin{cases} \frac{2}{\pi(1+x^2)}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find the expected score in a single shot.

B. Practice Problems from the Text Book

Chapter 1: Probability and Distributions, Problem Nos.: 7.17, 7.18, 7.20, 7.23, 7.24, 8.2, 8.4, 8.7, 8.8, 8.11, 9.1, 9.18.

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Solutions

Problem 1 $F_{X^+}(x) = P[X^+ \le x] = P(\max(X,0) \le x) = P(X \le x, 0 \le x) = \begin{cases} 0, & \text{if } x<0 \\ F(x) & \text{if } x \ge 0 \end{cases}$

$$F(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{x+2}{3}, & \text{if } -2 \le x < -1 \\ \frac{1}{3} + \frac{x^3+1}{195}, & \text{if } -1 \le x < 4 \\ \frac{2}{3} + \frac{x^3-16}{27}, & \text{if } 4 \le x < 5 \\ 1, & \text{if } x \ge 5 \end{cases}$$

$$F_{X^+}(x) = \begin{cases} 0, & \text{if } x<0 \\ \frac{1}{3} + \frac{x^3+1}{195}, & \text{if } 0 \le x < 4 \\ \frac{2}{3} + \frac{x^3-16}{27}, & \text{if } 4 \le x < 5 \\ 1, & \text{if } x \ge 5 \end{cases} \Rightarrow$$

F_{X^+} is not continuous at $x=0$; X^+ is not AC $\Rightarrow$ p.d.f. of X^+ does not exist

$$F_Y(x) = P(|X| \le x) = P(-x \le X \le x) = \begin{cases} 0, & \text{if } x<0 \\ F_X(x) - F_X(-x) & \text{if } x \ge 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } x<0 \\ \frac{2x^3}{195}, & \text{if } 0 \le x < 1 \\ \frac{x^3+65x+1}{195} - \frac{1}{3}, & \text{if } 1 \le x < 2 \\ \frac{1}{3} + \frac{x^3+1}{195}, & \text{if } 2 \le x < 4 \\ \frac{2}{3} + \frac{x^3-16}{27}, & \text{if } 4 \le x < 5 \\ 1, & \text{if } x \ge 5 \end{cases} \longrightarrow$$

F_{Y_1} is differentiable everywhere except at finite number of points and $\int_{-\infty}^{\infty} F_Y'(x)\,dx = 1$ (check!)
$\Downarrow$
Y is A.C. with a p.d.f.

$$f_Y(x) = \begin{cases} 2x^2/65, & \text{if } 0 < x < 1 \\ \frac{x^2}{65} + \frac{1}{3}, & \text{if } 1 < x < 2 \\ x^2/65, & \text{if } 2 < x < 4 \\ 2x/27, & \text{if } 4 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Problem 2 $S_X = (-2,-1) \cup (0,3)$; $h(x) = x^2$ is $\downarrow$ on $S_{1X} = (-2,-1)$ and strictly $\uparrow$ on $S_{2X} = (0,3)$

$S_{1X} = (-2,-1)$ $\qquad$ $S_{2X} = (0,3)$

$h_1(x) = x^2$ $\qquad$ $h_2(x) = x^2$

$h(S_{1X}) = (1,4)$ $\qquad$ $h(S_{2X}) = (0,9)$

$h_1^{-1}(y) = -\sqrt{y}$ $\qquad$ $h_2^{-1}(y) = \sqrt{y}$

$$f_Y(y) = f_X(h_1^{-1}(y)) \left|\frac{d}{dy} h_1^{-1}(y)\right| I_{h(S_{1X})}(y) + f_X(h_2^{-1}(y)) \left|\frac{d}{dy} h_2^{-1}(y)\right| I_{h(S_{2X})}(y)$$

$$= f_X(-\sqrt{y}) \left|-\frac{1}{2\sqrt{y}}\right| I_{(1,4)}(y) + f_X(\sqrt{y}) \left|\frac{1}{2\sqrt{y}}\right| I_{(0,9)}(y)$$

$$= \begin{cases} \frac{1}{12\sqrt{y}}, & \text{if } y \in (0,1) \cup (4,9) \\ \frac{1}{3\sqrt{y}}, & 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$F_Y(y) = P(Y \le y) = \begin{cases} 0, & \text{if } y < 0 \\ \int_0^y \frac{1}{12\sqrt{t}}\,dt, & \text{if } 0 \le y < 1 \\ \int_0^1 \frac{1}{12\sqrt{t}}\,dt + \int_1^y \frac{1}{3\sqrt{t}}, & \text{if } 1 \le y < 4 \\ \int_0^1 \frac{1}{12\sqrt{t}} + \int_1^4 \frac{1}{3\sqrt{t}} + \int_4^y \frac{1}{12\sqrt{t}}\,dt, & \text{if } 4 \le y < 9 \\ 1, & \text{if } y \ge 1 \end{cases}$$

Problem 3 $S_X = (0, \infty)$, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & \text{if } x < 0 \\ x/2, & \text{if } 0 \le x < 1 \\ 1 - \frac{1}{2x}, & \text{if } x \ge 1 \end{cases}$

(a) Since $P(X > 0) = 1$,

$$F_Y(y) = P\left(\tfrac{1}{X} \le y\right) = P\left(\tfrac{1}{X} \le y,\ X > 0\right) = \begin{cases} 0, & \text{if } y \le 0 \\ P(X \ge \frac{1}{y},\ X > 0), & \text{if } y > 0 \end{cases} = \begin{cases} 0, & \text{if } y \le 0 \\ P(X \ge \frac{1}{y}), & \text{if } y > 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } y < 0 \\ y/2, & \text{if } 0 \le y < 1 \\ 1 - \frac{1}{2y}, & \text{if } y \ge 1 \end{cases} \longrightarrow$$

F_Y is differentiable everywhere except at finite number of points and $\int_{-\infty}^{\infty} F_Y'(y)\,dy = \int_0^1 \frac{1}{2}\,dy + \int_1^{\infty} \frac{1}{2y^2}\,dy = 1$

$\Downarrow$

Y is A.C. with a p.d.f.

$\left(P(X \ge \tfrac{1}{y}) = 1 - F_X(\tfrac{1}{y})\right)$

$$f_Y(y) = \begin{cases} \frac{1}{2}, & \text{if } 0 < y < 1 \\ \frac{1}{2y^2}, & \text{if } 1 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(b) $S_X = (0, \infty)$; $h(x) = \frac{1}{x}$ is strictly $\downarrow$ on S_X,

$h(S_X) = (0, \infty)$, $h^{-1}(y) = \frac{1}{y}$. Thus

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| I_{h(S_X)}(y) = \begin{cases} \frac{1}{2}, & \text{if } 0 < y < 1 \\ 1/2y^2, & \text{if } y > 1 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4 $P(Y = -1) = P(X < 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-2}^{0} \frac{1}{3}\,dx = \frac{2}{3}$

$P(Y = \frac{1}{2}) = P(X = 0) = 0$ (since X is A.C.)

$P(Y = 1) = P(X > 0) = \int_0^{\infty} f(x)\,dx = \int_0^1 \frac{1}{3}\,dx = \frac{1}{3}$.

Thus Y is discrete with p.m.f.

$$f_Y(y) = \begin{cases} \frac{2}{3}, & \text{if } y = -1 \\ 1/3, & \text{if } y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note: Here X is A.C. but $Y = h(X)$ is discrete.

Problem 5 $P(X=0)=\frac{1}{8}$; $P(X=1)=\frac{3}{8}$, $P(X=2)=\frac{3}{8}$, $P(X=3)=\frac{1}{8}$

$$f_X(x)=\begin{cases} 1/8, & \text{if } x=0,3 \\ 3/8, & \text{if } x=1,2 \\ 0, & \text{o.w.}\end{cases} \;;\quad E(X)=\frac{3}{2},\quad E(X^2)=3$$

$E(X^4)=\frac{33}{2}$, $E(Y)=E(X^2)=3$; $Var(Y)=E(Y^2)-(E(Y))^2$

$=E(X^4)-(E(X^2))^2=\frac{33}{2}-9=\frac{15}{2}$

$E(Z)=E(2X^2+1)=2E(X^2)+1=7$

$Var(Z)=Var(2X^2+1)=4\,Var(X^2)=4\,Var(Y)=30.$

(Note: For real constants a and b, $Var(aZ+b)=a^2\,Var(Z)$).

Problem 6 (a) $E(X^j)=c\sum_{n=1}^{\infty}\frac{1}{n^{2+r-j}}<\infty \iff 2+r-j>1 \iff j<1+r$

(b) Let $P(X=n)=\begin{cases}\frac{c_1}{n^2}, & \text{if } n=1,2,\dots \\ 0, & \text{o.w}\end{cases}$; where $c_1=\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1}$.

By (a), $E(X^j)<\infty \iff j<1 \Rightarrow E(X), E(X^2),\dots$ are not finite.

Problem 7 $E(Y)=E(X^2-5X+3)=E(X^2)-5E(X)+3$, provided expectations exist.

$E(X)=\int_0^1\frac{x^2}{2}dx+\int_1^2\frac{x}{2}dx+\int_2^3\frac{x(3-x)}{2}dx=\frac{3}{2}$

$E(X^2)=\int_0^1\frac{x^3}{2}dx+\int_1^2\frac{x^2}{2}dx+\int_2^3 x^2\frac{(3-x)}{2}dx=\frac{8}{3}$

$\Rightarrow E(Y)=\frac{-11}{6}$.

Problem 8 (a) $E(|X|^\alpha)=\int_{-\infty}^{\infty}|x|^\alpha f(x)dx=\int_{|x|\le 1}|x|^\alpha f(x)dx+\int_{|x|>1}|x|^\alpha f(x)dx$

$\le\int_{|x|\le 1}f(x)dx+\int_{|x|>1}|x|^\beta f(x)dx$ $\left(\begin{array}{l}|x|\le 1\Rightarrow |x|^\alpha\le 1\\ |x|>1\Rightarrow |x|^\alpha\le |x|^\beta\end{array}\right)$

$\le\int_{-\infty}^{\infty}f(x)dx+\int_{-\infty}^{\infty}|x|^\beta f(x)dx=1+E(|X|^\beta)<\infty.$

(b) Consider $\Delta=E(|X-c|)-E(|X-m|)=\int_{-\infty}^{\infty}|x-c|f(x)dx-\int_{-\infty}^{\infty}|x-m|f(x)dx$

Case I: $-\infty<c<m$

$\Delta=\int_{-\infty}^{c}(c-x)f(x)dx+\int_{c}^{\infty}(x-c)f(x)dx-\int_{-\infty}^{m}(m-x)f(x)dx-\int_{m}^{\infty}(x-m)f(x)dx$ —(1)

$\int_c^\infty(x-c)f(x)dx=\int_c^m(x-c)f(x)dx+\int_m^\infty(x-c)f(x)dx$

$\int_{-\infty}^{m}(m-x)f(x)dx=\int_{-\infty}^{c}(m-x)f(x)dx+\int_c^m(m-x)f(x)dx$

$\Delta = 2c\,F(c) - c + 2\int_c^m x f(x)\,dx$ (using $F(m) = \frac{1}{2}$)

$\geq 2c\,F(c) - c + 2c\,[F(m) - F(c)] = 0$

<u>Case II</u> $-\infty < m < c < \infty$

$\Delta = 2c\,F(c) - c - 2\int_m^c x f(x)\,dx \geq 2c\,F(c) - c - 2c\,[F(c) - F(m)] = 0$

(using $\int_{-\infty}^{c} = \int_{-\infty}^{m} + \int_m^c$ and $\int_m^\infty = \int_m^c + \int_c^\infty$ in (1))

Problem 9 Let Z = score on a shot. Then $S_Z = \{0, 2, 3, 4\}$

$P(Z=0) = P(X > \sqrt{3}) = \int_{\sqrt{3}}^{\infty} \frac{2}{\pi} \cdot \frac{1}{1+x^2}\,dx = \frac{2}{\pi}\left[\tan^{-1} x\right]_{\sqrt{3}}^{\infty} = \frac{1}{3}$

$P(Z=2) = P(1 < X < \sqrt{3}) = \frac{2}{\pi}\int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{6}$

$P(Z=3) = P\left(\frac{1}{\sqrt{3}} < X < 1\right) = \frac{2}{\pi}\int_{1/\sqrt{3}}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{6}$

$P(Z=4) = P\left(X < \frac{1}{\sqrt{3}}\right) = \frac{2}{\pi}\int_0^{1/\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{3}$

$E(Z) = 0 \times \frac{1}{3} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{3} = \frac{13}{6}.$

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