# Recap

## Error Analysis

- True error
  - → Approximate error
  - → Error bound

- Truncation error

  when limiting process truncated

- Data error

$$x \longrightarrow f(x)$$

$$\tilde{x} = x + \Delta x \longrightarrow f(x + \Delta x)$$

$$\Delta f(x) = |\Delta x \, f'(x)|$$

$$\Delta f(x_1, x_2, \ldots x_m) = \left( \sum \left| \Delta x_i \frac{\partial f}{\partial x_i} \right| \right)$$

Quadrature sum

$$\Delta f(x_i, x_2, \ldots x_m) = \sqrt{\sum_{i=1}^{m} \left( \Delta x_i \frac{\partial f}{\partial x_i} \right)^2}$$

# Number representation

- Integer — unsigned, signed
- Fixed-point — XXX . XX
- Floating point

$$x = \pm m b^{p}$$

m — mantissa (significand)

b — base (2, 10)

p — exponent

## How floating point numbers are stored

| Sign | Exponent | Mantissa |
|---|---|---|

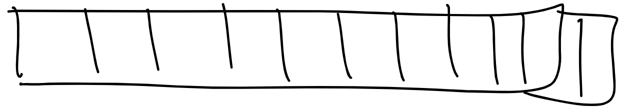
# IEEE 754

64-bit

| 1 bit | 11 bits | 52 bits |
|---|---|---|

~~Decimal 64~~ → −383 to 384 → IEEE 754-r (2008)

11 bits



$2^{11}$ — 0 to 2047

Excess-p $\quad p = 2^{n-1} - 1 \quad$ n-bits

$p = 2^{10} - 1 = 1023$

−1023 to 1024

−1022 to 1024

$$m\,2^{1024} = m\,10^{a}$$

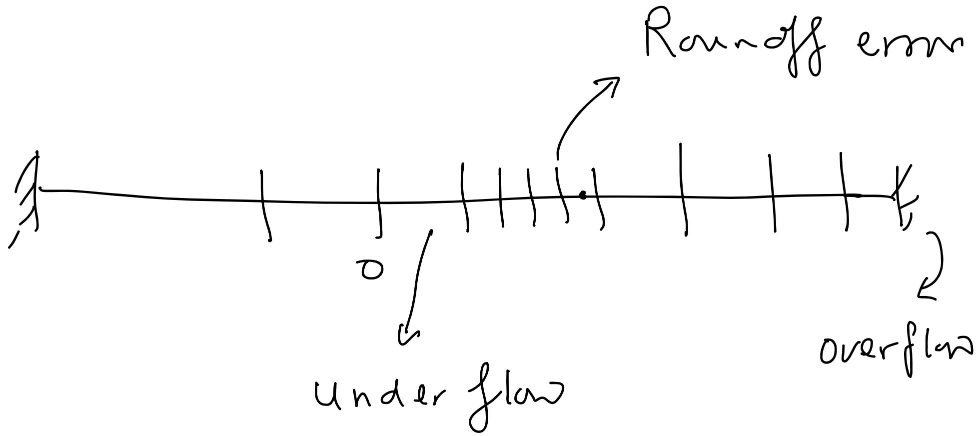
$$a = \frac{\log(2)}{\log(10)}\,1024$$

$$= 308$$

$10^{-308}$ $\quad$ $10^{308}$ $\quad$ floating point

characteristics of computer numbers

1. Finite range
2. Hole near zero
3. Non-uniform gaps




## Round off errors

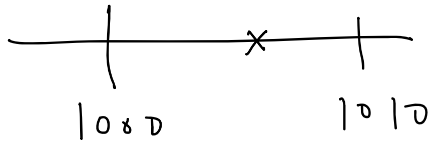
System $\quad$ 3 places for mantissa

1 place for exponent

1000 $\quad$ $0.100 \times 10^4$

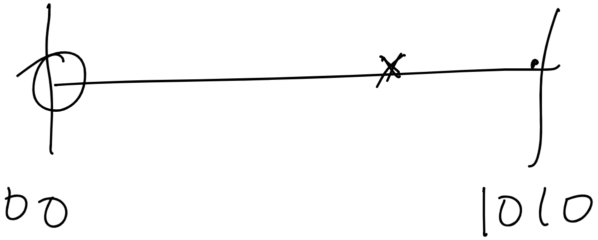
1010 $\quad$ $0.101 \times 10^4$

1007



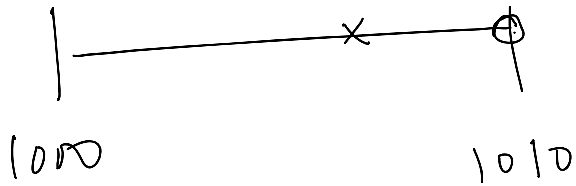
## Chopping

$1007 \rightarrow 0.100\cancel{7} \times 10^4$




$|\Delta x| \le 10$

## Rounding




$|\Delta x| \le 5$

## Relative error

$$\left|\frac{\Delta x}{x}\right| = \frac{7}{1007}$$

Chopping $\left|\frac{\Delta x}{x}\right| \le \frac{10}{1000} = 10^{-2}$

Rounding $\le \frac{1}{2} 10^{-2}$

General Round off $\left|\frac{\Delta x}{x}\right| \le \frac{1}{2} 10^{1-t}$

$t \rightarrow$ number of significant digits in mantissa

$\le \frac{1}{2} 2^{1-t}$

## Relative round off error

$$\left|\frac{\Delta x}{x}\right| \le \frac{1}{2} b^{1-t} = u$$

$t \rightarrow$ number of significant digits in mantissa

machine precision
epsilon
round off units

64-bit binary

$$u = \frac{1}{2} 2^{1-t} \qquad t = 52$$

$$= 0.5 \times 2^{-51}$$

## Addition

$208.00 + 0.25$

$= 208.25$

$$\begin{array}{rl} & 0.208 \times 10^3 \\ + & 0.25 \times 10^0 \end{array}$$

$$\begin{array}{l} 0.208 \times 10^3 \\ 0.00025 \times 10^3 \\ \hline 0.20825 \times 10^3 \\ \hline 0.208 \end{array}$$

$a + 1 - a = 1$

## Substraction

### Two nearly equal numbers

$$x_1 = 0.246 \times 10^3$$
$$x_2 = 0.245 \times 10^3$$

$$= 0.001 \times 10^3$$
$$= 0.100 \times 10^1$$

$\Rightarrow$ Loss of significance

## Forward error analysis

$$x \longrightarrow f(x)$$
$$x + \Delta x \longrightarrow f(x + \Delta x)$$

$$\boxed{\Delta f(x) = |\Delta x \, f'(x)|}$$

### Condition number of the problem

$$C_p = \frac{\text{Relative error in function } f(x)}{\text{Relative error in data } x}$$

$$= \frac{\Delta f(x) / f(x)}{\Delta x / x}$$

$$\boxed{C_p = \left| \frac{x f'(x)}{f(x)} \right|}$$

Well-conditioned $C_p < 1$ (attenuated)

illcond. $C_p > 1$ (as amplified

## Condition number

$$C_p = \left| \frac{x f'(x)}{f(x)} \right|$$

$C_p < 1$ - well

$C_p > 1$ - ill condition

**Example** $f(x) = \sqrt{x}$ $\qquad f'(x) = \frac{1}{2\sqrt{x}}$

$$C_p = \frac{x f'(x)}{f(x)}$$

$$C_p = \frac{1}{2}$$

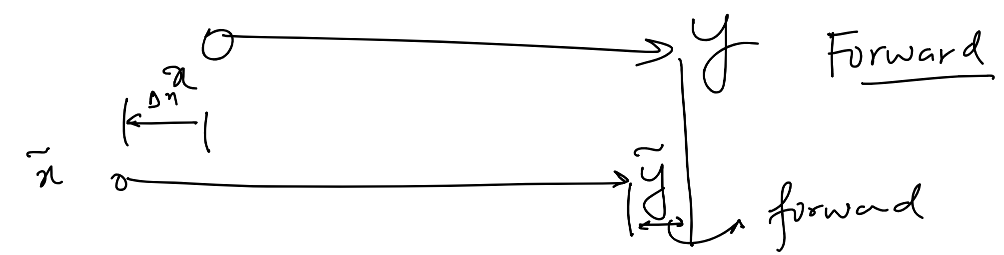
$$f(x) = \frac{10}{1-x^2}$$

$$f'(x) = \frac{20x}{(1-x^2)^2}$$

$$C_p = \left| \frac{2x^2}{(1-x^2)} \right|$$

$x \approx 1$ $\qquad C_p \gg 1$ ill conditioning

# Backward error analysis





$$\left| \frac{x - x_A}{x} \right| \le C_A u$$

$C_A$ – condition number of the algorithm

$u$ – epsilon

## Example

4 digit decimal

$$u = \frac{1}{2} \times 10^{1-4}$$

$$= 0.5 \times 10^{-3}$$

$$f(x) = \sqrt{1 + \sin x} - 1$$

$f(x) = 0.8688 \times 10^{-2}$

$x = 1^\circ$

$fl(\pi/180) = 0.1745 \times 10^{-1}$

$fl(\sin x) = 0.1745 \times 10^{-1}$

$fl(1 + \sin x) = 0.1017 \times 10^{1}$

$fl(\sqrt{1+\sin x}) = 0.1008 \times 10^{1}$

$fl(\sqrt{1+\sin x} - 1) = \underline{\underline{0.8000 \times 10^{-2}}}$

$\sqrt{1+\sin x} - 1 = 0.8000\times10^{-2}$

$x_A = 0.9204\times10^{\cancel{-1}}\ 1$

$\left|\frac{x-x_A}{x}\right| = 0.0796 \le C_A u$

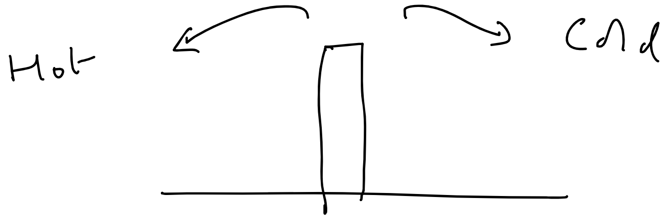
$\boxed{C_A \approx 160}$

$f(x) = (\sqrt{1+\sin x} - 1)\,\frac{(\sqrt{1+\sin x}+1)}{(\sqrt{1+\sin x}+1)}$

$= \frac{\sin x}{\sqrt{1+\sin x}+1}$

$C_A \approx 0.4$

Analogy: Bathroom




$T \qquad \theta$

$\Delta\theta \ll \qquad T \gg$

$\theta \rightsquigarrow T_A$

$\theta_A \leftarrow$

$\left|\frac{\theta_A - \theta}{\theta}\right|$