MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-VI

A. Illustrative Discussion Problems

1. (a) Let X be a random variable such that $P(X \leq 0) = 0$ and let $\mu = E(X)$ be finite. Show that $P(X \geq 2\mu) \leq 0.5$;

   (b) Let X be a random variable such that $P(X \leq 0) = 0$ and let $E(X^2) = 2$. Show that $P(\sqrt{X} \geq 2) \leq 0.125$;

   (c) If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, determine a lower bound for $P(-2 < X < 8)$.

2. (a) An enquiry office receives, on an average, $25,000$ telephone calls a day. What can you say about the probability that this office will receive at least $30,000$ telephone calls tomorrow?

   (b) An enquiry office receives, on an average, $20,000$ telephone calls per day with a variance of $2,500$ calls. What can be said about the probability that this office will receive between $19,900$ and $20,100$ telephone calls tomorrow? What can you say about the probability that this office will receive more than $20,200$ telephone calls tomorrow?

3. (a) Let X be a random variable with $E(X) = 1$. Show that $E(e^{-X}) \geq \frac{1}{3}$;

   (b) For pairs of positive real numbers (a_i, b_i), $i = 1, \ldots, n$ and $r \geq 1$, show that

$$\left(\sum_{i=1}^{n} a_i^r b_i\right)\left(\sum_{i=1}^{n} b_i\right)^{r-1} \geq \left(\sum_{i=1}^{n} a_i b_i\right)^{r}.$$

   Hence show that, for any positive real number m,

$$\left(\sum_{i=1}^{n} a_i^{2m+1}\right)\left(\sum_{i=1}^{n} a_i\right) \geq \left(\sum_{i=1}^{n} a_i^{m+1}\right)^{2}.$$

4. Let X be a random variable such that $P(X > 0) = 1$. Show that:

   (a) $E(X^{2m+1}) \geq (E(X))^{2m+1}$, $m \in \{1, 2, \ldots\}$;

   (b) $E(Xe^X) + e^{E(X)} \geq E(X)e^{E(X)} + E(e^X)$,

   provided the involved expectations are finite.

5. Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be real constants and let $X_{\mu,\sigma}$ be a random variable having p.d.f. (corresponding distribution is called a normal distribution, denoted by $N(\mu, \sigma^2)$)

$$f_{X_{\mu,\sigma}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < \infty.$$

   (a) Show that $f_{X_{\mu,\sigma}}$ is a p.d.f.;

   (b) Show that the probability distribution function of $X_{\mu,\sigma}$ is symmetric about μ;

   (c) Find the m.g.f. of $X_{\mu,\sigma}$ and hence find the mean, the median, the variance, the coefficient of skewness and kurtosis of $X_{\mu,\sigma}$;

   (d) Plot $f_{X_{\mu,\sigma}}$.

6. For $\mu \in \mathbb{R}$ and $\lambda > 0$, let $X_{\mu,\lambda}$ be a random variable having the p.d.f.

$$f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}, & \text{if } x \geq \mu \\ 0, & \text{otherwise} \end{cases}.$$

   (a) Find $C_r(\mu, \lambda) = E((X-\mu)^r), r \in \{1, 2, \ldots\}$ and $\mu_r'(\mu, \lambda) = E(X_{\mu,\lambda}^r), r \in \{1, 2\}$;

   (b) For $p \in (0, 1)$, find the p-th quantile $\xi_p \equiv \xi_p(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$ ($F_{\mu,\lambda}(\xi_p) = p$, where $F_{\mu,\lambda}$ is the distribution function of $X_{\mu,\lambda}$);

   (c) Find the lower quartile $q_1(\mu, \lambda)$, the median $m(\mu, \lambda)$ and the upper quartile $q_3(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

   (d) Find the mode $m_0(\mu, \lambda)$ of the distribution of $X_{\mu,\sigma}$;

   (e) Find the standard deviation $\sigma(\mu, \lambda)$, the mean deviation about median $\text{MD}(m(\mu, \lambda))$, the inter-quartile range $\text{IQR}(\mu, \lambda)$, the quartile deviation (or semi-inter-quartile range) $\text{QD}(\mu, \lambda)$, the coefficient of quartile deviation $\text{CQD}(\mu, \lambda)$ and the coefficient of variation $\text{CV}(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

   (f) Find the coefficient of skewness $\beta_1(\mu, \lambda)$ and the Yule coefficient of skewness $\beta_2(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

   (g) Find the excess kurtosis $\gamma_2(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

   (h) Based on values of measures of skewness and the kurtosis of the distribution of $X_{\mu,\lambda}$, comment on the shape of $f_{\mu,\sigma}$.

7. Let X be a random variable with p.d.f. $f_X(x)$ that is symmetric about μ ($\in \mathbb{R}$), i.e., $f_X(x+\mu) = f_X(\mu - x)$, $\forall\ x \in (-\infty, \infty)$. Further suppose that the distribution

function of X is strictly increasing on support S_X.

(a) If q_1, m and q_3 are respectively the lower quartile, the median and the upper quartile of the distribution of X then show that $\mu = m = \frac{q_1+q_3}{2}$;

(b) If $E(X)$ is finite then show that $E(X) = \mu = m = \frac{q_1+q_3}{2}$.

B. Practice Problems from the Text Book

Chapter 1: Probability and Distributions, Problem Nos.: 7.9, 7.10, 7.11, 7.12, 7.13, 9.14, 9.15, 10.1, 10.6.

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Assignment-V (Solutions)

$\boxed{\text{Problem 1}}$ (a) $P(X \geq 2\mu) = P(|X| \geq 2\mu)$ (since $P(X>0)=1$)

$$\leq \frac{E(|X|)}{2\mu} = \frac{E(X)}{2\mu} = \frac{1}{2} \quad (P(X>0)=1 \Rightarrow \mu>0)$$

(b) $P(\sqrt{X} \geq 2) = P(|X| \geq 4) \leq \dfrac{E(|X|^2)}{16} = \dfrac{1}{8} = 0.125$ (since $P(X>0)=1$, $P(\sqrt{X} \geq 2) = P(|X| \geq 4)$)

(c) $\mu = E(X) = 3$, $\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = 4$

$$P(-2 < X < 8) = P(-5 < X - \mu < 5) = P(|X-\mu| < 5) \geq 1 - \frac{\sigma^2}{5^2} = \frac{21}{25}.$$

$\boxed{\text{Problem 2}}$

(a) Define r.v. X = # of telephone calls received on a day

$E(X) = 25000$. Thus $P(X \geq 0) = 1$ and

$$P(X \geq 30{,}000) \leq \frac{E(X)}{30{,}000} = \frac{5}{6} = 0.8333\cdots$$

(b) Define r.v. X = # of telephone calls received on a day.

$\mu = E(X) = 20{,}000$, $\sigma^2 = Var(X) = 2{,}500$. So

$$P(19900 \leq X \leq 20100) = P(-100 \leq X-\mu \leq 100) = P(|X-\mu| \leq 100) \geq 1 - \frac{\sigma^2}{100^2} = \frac{3}{4}.$$

Moreover

$$P(X > 20{,}200) \leq \min\left\{\frac{E(X)}{20{,}200}, \frac{E(X^2)}{20{,}200^2}\right\} = \min\left\{\frac{100}{101}, \frac{2500 + 20{,}000^2}{20{,}200^2}\right\}$$

$$= \min\left\{\frac{100}{101}, \frac{4000025}{4080400}\right\} = 0.9803$$

Also

$$P(X > 20{,}200) = P(X-\mu > 200) \leq P(|X-\mu| \geq 200) \leq \frac{\sigma^2}{200^2} = 0.0625$$

Thus knowledge of variance substantially improves the bound.

$\boxed{\text{Problem 3}}$

(a) $g(x) = e^{-x}$, $x \in \mathbb{R}$ is a convex function on $\mathbb{R}$ ($g''(x) = e^{-x} \geq 0 \ \forall x \in \mathbb{R}$).

Using the Jensen inequality we have

$$E(e^{-X}) \geq e^{-E(X)} = \frac{1}{e} \geq \frac{1}{3}.$$

(b) Let $p_i = \dfrac{b_i}{\sum_{j=1}^{n} b_j}$, $i=1,\dots,n$ so that $0<p_i<1$, $i=1(1)n$ and $\sum_{i=1}^{n} p_i = 1$.

Let X be a r.v. having p.m.f. $f(x) = \begin{cases} p_i, & \text{if } x = a_i,\ i=1,\dots,n. \\ 0, & \text{otherwise} \end{cases}$

Then $P(X>0)=1$. Since $g(x) = x^r$, $x \in \mathbb{R}$ is a convex function on $[0, \infty)$ for $r \geq 1$, using Jensen's inequality we have

$$E(X^r) \geq (E(X))^r \Rightarrow \left(\sum_{i=1}^{n} a_i^r p_i\right) \geq \left(\sum_{i=1}^{n} a_i p_i\right)^r \Rightarrow \text{first assertion}$$

Second assertion follows from first by taking $r=2$, $a_i \equiv a_i^m$, $b_i \equiv a_i$, $i=1(1)n$.

Problem 4

(a) For $m \in \{1, 2, \dots\}$, $g(x) = x^{2m+1}$, $x \in [0, \infty)$ is convex on $[0, \infty)$. By Jensen's inequality $E(g(X)) \geq g(E(X)) \Rightarrow$ assertion.

(b) Let $h(x) = (x-1)e^{x}$, $x \geq 0$. Then $h'(x) = xe^{x} \uparrow$ on $[0, \infty)$, implying that h is convex on $[0, \infty)$. By the Jensen inequality $E(h(X)) \geq h(E(X)) \Rightarrow$ assertion.

Problem 6 Notation: $X_{\mu,\lambda} \equiv X$; $F_{\mu,\lambda} \equiv F$, $f_{\mu,\lambda} \equiv f$

(a) For $r \in \{1, 2, \dots\}$, $c_r(\mu,\lambda) = E((X-\mu)^r) = \int_\mu^\infty (x-\mu)^r \frac{1}{\lambda} e^{-\frac{(x-\mu)}{\lambda}} dx = \lambda^r \int_0^\infty t^r e^{-t} dt = \lfloor r \, \lambda^r$.

$E(X-\mu) = \lambda \Rightarrow \mu_1'(\mu,\lambda) = E(X) = \mu + \lambda$; $\mu_2'(\mu,\lambda) = E(X^2) = E((X-\mu)^2) + 2\mu E(X) - \mu^2 = 2\lambda^2 + 2\mu\lambda + \mu^2$.

(b) $F(\xi_p) = p \Rightarrow \int_\mu^{\xi_p} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}} dx = p \Rightarrow 1 - e^{-\frac{\xi_p - \mu}{\lambda}} = p \Rightarrow \xi_p = \mu - \lambda \ln(1-p)$.

(c) Using (b): $q_1(\mu,\lambda) = \xi_{1/4} = \mu - \lambda \ln \frac{3}{4}$, $m(\mu,\lambda) = \xi_{1/2} = \mu - \lambda \ln \frac{1}{2}$

$q_3(\mu,\lambda) = \xi_{3/4} = \mu - \lambda \ln \frac{1}{4}$

(d) Clearly $f \downarrow$ on $[\mu, \infty) \Rightarrow \sup\{f(x) : x \in \mathbb{R}\} = f(\mu) = \frac{1}{\lambda}$

$\Rightarrow mo(\mu,\lambda) = \mu$.

(e) $\sigma(\mu,\lambda) = \sqrt{c_2(\mu,\lambda)} = \sqrt{2}\lambda$ (from (a))

$MD(m(\mu,\lambda)) = E(|X - \mu + \lambda \ln \frac{1}{2}|) = E(|X - \mu - \lambda \ln 2|)$.

For $d > 0$, $E(|X - \mu - d\lambda|) = \int_\mu^\infty |x - \mu - d\lambda| \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}} dx = \lambda \int_0^\infty |z - d| e^{-z} dz$

$= \lambda(d - 1 + 2e^{-d})$

$\Rightarrow MD(m(\mu,\lambda)) = \lambda \ln 2$

$IQR(\mu,\lambda) = q_3(\mu,\lambda) - q_1(\mu,\lambda) = \lambda \ln 3$

$CQD(\mu,\lambda) = \frac{(q_3 - q_1)}{q_3 + q_1} = \frac{\lambda \ln 3}{2\mu - \lambda \ln \frac{3}{16}}$

$CV(\mu,\lambda) = \frac{\sigma(\mu,\lambda)}{\mu_1'(\mu,\lambda)} = \frac{\sqrt{2}\lambda}{\mu + \lambda}$

(f) $\mu_3 = E((X-\mu)^3) = 6\lambda^3$ (see (a)) $\Rightarrow \beta_1(\mu,\lambda) = \frac{\mu_3^2}{\mu_2^3} = \frac{36\lambda^6}{8\lambda^6} = \frac{9}{2}$

$\beta_2(\mu,\lambda) = \frac{q_3 - 2m + q_1}{q_3 - q_1} = \frac{\ln(4/3)}{\ln 3}$

(g) $\mu_4 = E((X-\mu)^4) = 24\lambda^4$; $\gamma_1(\mu,\lambda) = \frac{\mu_4}{\mu_2^2} = \frac{24\lambda^4}{4\lambda^4} = 6$; $\gamma_2(\mu,\lambda) = \gamma_1(\mu,\lambda) - 3 = 3$

(h) $\beta_1(\mu,\lambda) > 0$, and $\beta_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of X is positively skewed $\Rightarrow f$ has longer tails on the right side

$\gamma_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of X is leptokurtic $\Rightarrow f$ is more peaked around mode than normal distribution

Problem 5 (a) Clearly $f_{X_{\mu,\sigma}}(x) \geq 0 \quad \forall x \in \mathbb{R}$. Also

$$\int_{-\infty}^{\infty} f_{X_{\mu,\sigma}}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz = I, \text{ say.}$$

Clearly $I \geq 0$ and

$$I^2 = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}}\,dy\,dz$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} r e^{-\frac{r^2}{2}}\,d\theta\,dr$$ [On making the transformation $y = r\cos\theta$, $z = r\sin\theta$, $r \geq 0$, $0 \leq \theta < 2\pi$ so that the Jacobian of transformation is r]

$$= \int_0^{\infty} r e^{-\frac{r^2}{2}}\,dr = \int_0^{\infty} e^{-z}\,dz = 1.$$

(b) Clearly

$$f_{X_{\mu,\sigma}}(\mu - x) = f_{X_{\mu,\sigma}}(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

$f_{X_{\mu,\sigma}}$ is symmetric about μ

$\Rightarrow$ distribution of $X_{\mu,\sigma}$ is symmetric about μ (it will be shown in (c) that $E(X_{\mu,\sigma})$ is finite)

$\Rightarrow E(X_{\mu,\sigma}) = \mu$

(c) $$M_{X_{\mu,\sigma}}(t) = E(e^{tX_{\mu,\sigma}}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\sigma t)^2}{2}}\,dz = e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R} \quad \left(\text{since by (a) } \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx = \sigma\sqrt{2\pi}\right)$$

$\Rightarrow$ moments of all orders are finite.

$X-\mu \stackrel{d}{=} \mu - X \Rightarrow P(X-\mu \leq 0) = P(\mu - X \leq 0) \Rightarrow F_{\mu,\sigma}(\mu) = \frac{1}{2}$.

$\Rightarrow$ Mean $=$ Median $= E(X_{\mu,\sigma}) = \mu$ (by (b)) [d.f. $F_{\mu,\sigma}(x) = \int_{-\infty}^{x} f_{\mu,\sigma}(t)\,dt$ is strictly $\uparrow$ in S_X and $F_{\mu,\sigma}(\mu) = \frac{1}{2}$]

Let $Z = \frac{X_{\mu,\sigma} - \mu}{\sigma}$. Then

$$M_Z(t) = E(e^{tZ}) = e^{-\frac{\mu t}{\sigma}} M_{X_{\mu,\sigma}}\left(\frac{t}{\sigma}\right) = e^{\frac{t^2}{2}} = \sum_{r=0}^{\infty} \frac{t^{2r}}{2^r \lfloor r}$$

$\Rightarrow E(Z^{2r+1}) = $ Coefficient of $\frac{t^{2r+1}}{\lfloor 2r+1}$ in $M_Z(t) = 0$, $r = 0, 1, 2, \ldots$

$E(Z^{2r}) = $ Coefficient of $\frac{t^{2r}}{\lfloor 2r}$ in $M_Z(t) = \frac{\lfloor 2r}{2^r \lfloor r}$, $r = 1, 2, \ldots$

$\Rightarrow E((X-\mu)^{2r+1}) = 0$, $r = 0, 1, 2, \ldots$, $E((X-\mu)^{2r}) = \frac{\lfloor 2r}{2^r \lfloor r}\sigma^{2r}$, $r = 1, 2, \ldots$

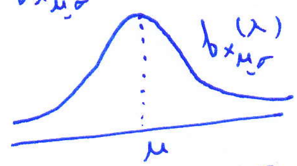
$\Rightarrow E(X) = \mu$, $\mu_2 = E((X-\mu)^2) = \sigma^2$, $\mu_3 = E((X-\mu)^3) = 0$,

$\mu_4 = E((X-\mu)^4) = 3\sigma^4$

$\Rightarrow$ Coefficient of skewness $\beta_1(\mu, \sigma) = \frac{\mu_3}{\mu_2^{3/2}} = 0$

Kurtosis $= \frac{\mu_4}{\mu_2^2} = 3$

(d) It is easy to verify that $f_{X_{\mu,\sigma}}(x) \uparrow$ in $(-\infty, \mu)$, $\downarrow$ in (μ, ∞) and $\lim_{x \to \pm\infty} f_{X_{\mu,\sigma}}(x) = 0$. Moreover $f_{X_{\mu,\sigma}}(x)$ is convex in $(-\infty, \mu-\sigma) \cup (\mu+\sigma, \infty)$ and concave otherwise.



Problem 7 We have $X - \mu \overset{d}{=} \mu - X$.

(a) $X-\mu \overset{d}{=} \mu - X \Rightarrow P(X-\mu \le 0) = P(\mu - X \le 0) \Rightarrow F_X(\mu) = \frac{1}{2}$ ($F_X(\cdot)$ is continuous)

Also F_X is strictly $\uparrow$ on S_X and is continuous $\Rightarrow m = \inf\{x \in \mathbb{R} : F_X(x) \ge \frac{1}{2}\}$

$\Rightarrow m = \mu$.

F_X is strictly $\uparrow$ on S_X and is continuous $\Rightarrow F_X(q_1) = \frac{1}{4}$ and $F_X(q_3) = \frac{3}{4}$.

We have

$P(X - \mu \le q_3 - \mu) = P(\mu - X \le q_3 - \mu) = P(X \le q_3) = F_X(q_3) = \frac{3}{4}$

$\Rightarrow P(\mu - X \le q_3 - \mu) = \frac{3}{4} \Rightarrow F_X(2\mu - q_3) = \frac{1}{4} \Rightarrow 2\mu - q_3 = q_1$

$\Rightarrow \mu = m = \frac{q_3 + q_1}{2}$

(b) $X - \mu \overset{d}{=} \mu - X$ and $E(X)$ is finite $\Rightarrow E(X) = \mu = m = \frac{q_3 + q_1}{2}$.

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