Module 22

DISCRETE RANDOM VECTORS

- $\underline{X} = (X_1, \ldots, X_p)$: a p-dimensional random vector (r.v.), defined on a probability space $(\Omega, \mathcal{P}(\Omega), P)$.

- $F_{\underline{X}}(\cdot)$: d.f. of $\underline{X}$.

Definition 1:

(a) The r.v. $\underline{X}$ is said to be a discrete r.v. if there exists a countable set $S_{\underline{X}} \subseteq \mathbb{R}^p$ such that $P(\{\underline{X} = \underline{x}\}) > 0, \forall\ \underline{x} \in S_{\underline{X}}$ and

$$P(\{\underline{X} \in S_{\underline{X}}\}) = \sum_{\underline{x} \in S_{\underline{X}}} P(\{\underline{X} = \underline{x}\}) = 1.$$

(b) Under the notation of (a), the set $S_{\underline{X}}$ is called the support of $\underline{X}$ and the function

$$f_{\underline{X}}(\underline{x}) = P(\{\underline{X} = \underline{x}\}),\ \underline{x} \in \mathbb{R}^p,$$

which is such that $f_{\underline{X}}(\underline{x}) \geq 0,\ \forall\ \underline{x} \in \mathbb{R}^p,\ f_{\underline{X}}(\underline{x}) > 0,\ \forall\ \underline{x} \in S_{\underline{X}}$ and $\sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(\underline{x}) = 1$, is called the joint probability mass function (p.m.f.) of $\underline{X}$.

Remark 1 :

Let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional discrete r.v. with support $S_{\underline{X}}$, p.m.f. $f_{\underline{X}}(\cdot)$ and d.f. $F_{\underline{X}}(\cdot)$.

(a) For given $r > 0$ and $\underline{x} \in \mathbb{R}^p$, let

$$N_r(\underline{x}) = \left\{ \underline{t} \in \mathbb{R}^p : \sqrt{\sum_{i=1}^{p} (t_i - x_i)^2} < r \right\}$$

denote the p-dimensional ball of radius r centered at $\underline{x}$.
It can be shown that

$$S_{\underline{X}} = \{ \underline{x} \in \mathbb{R}^p : P(\{\underline{X} \in N_r(\underline{x})\}) > 0, \ \forall \ r > 0 \}.$$

(b) Since $P(\{\underline{X} \in S_X\}) = 1$, we have $P(\{\underline{X} \in S_X^c\}) = 0$ and therefore $P(\{\underline{X} = \underline{x}\}) = 0, \ \forall \ x \in S_X^c$.

(c) As in the one-dimensional case ($p = 1$) it can be shown that if there is a countable set $S \subseteq \mathbb{R}^p$ and a function $g : \mathbb{R}^p \to \mathbb{R}$ such that, $g(\underline{x}) \geq 0, \ \forall \ \underline{x} \in \mathbb{R}^p, g(\underline{x}) > 0, \ \forall \ \underline{x} \in S$ and $\sum_{\underline{x} \in S} g(\underline{x}) = 1$, then there exists a probability space $(\Omega, \mathcal{P}(\Omega), P)$ and a p-dimensional discrete r.v. $\underline{X}$ defined on $(\Omega, \mathcal{P}(\Omega), P)$ such that $g(\cdot)$ is p.m.f. of $\underline{X}$.

(d) Let $\underline{Y} = (Y_1, \ldots, Y_p)$ be a p-dimensional r.v. (not necessarily discrete or continuous) with d.f. $F_{\underline{Y}}(\cdot)$. For $\underline{a} = (a_1, \ldots, a_p) \in \mathbb{R}^p$, define $\underline{a}_n = (a_1 - \frac{1}{n}, \ldots, a_p - \frac{1}{n}), n = 1, 2, \ldots$. Then

$$\begin{aligned}
\{\underline{X} = \underline{a}\} &= \underline{X}^{-1}(\{\underline{a}\}) \\
&= \underline{X}^{-1}\left(\bigcap_{n=1}^{\infty}(\underline{a}_n, \underline{a}]\right) \\
&= \bigcap_{n=1}^{\infty} \underline{X}^{-1}\Big((\underline{a}_n, \underline{a}]\Big) \\
\Rightarrow \quad P(\{\underline{X} = \underline{a}\}) &= P\left(\bigcap_{n=1}^{\infty} \underline{X}^{-1}\Big((\underline{a}_n, \underline{a}]\Big)\right) \\
&= \lim_{n\to\infty} P\left(\underline{X}^{-1}\Big((\underline{a}_n, \underline{a}]\Big)\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{p} (-1)^k \sum_{\underline{z}_n \in \Delta_{k,p}\left((\underline{a}_n, \underline{a}]\right)} F_{\underline{X}}(\underline{z}_n).
\end{aligned}$$

(e) Using (d) it follows that the joint p.m.f. of a discrete r.v. $\underline{X}$ is determined by its joint d.f. Conversely the joint d.f. of $\underline{X}$

$$\begin{aligned} F_{\underline{X}}(\underline{x}) &= P\big(\{\underline{X} \in (-\underline{\infty}, \underline{x}]\}\big) \\ &= P\big(\{\underline{X} \in (-\underline{\infty}, \underline{x}) \cap S_{\underline{X}}\}\big) \\ &= \sum_{\underline{t} \in (-\underline{\infty}, \underline{x}] \cap S_{\underline{X}}} f_{\underline{X}}(\underline{t}) \end{aligned}$$

is determined by its p.m.f. Thus to study the probability function $P_{\underline{X}}(\cdot)$, induced by a discrete r.v. $\underline{X}$, it is enough to study its p.m.f.

(f) Let $A \subseteq \mathbb{R}^p$. Then

$$\begin{aligned} P(\{\underline{X} \in A\}) &= P\left(\{\underline{X} \in A \cap S_{\underline{X}}\}\right) \\ &= P\left(\bigcup_{\underline{x} \in A \cap S_{\underline{X}}} \{\underline{X} = \underline{x}\}\right) \\ &= \sum_{\underline{x} \in A \cap S_{\underline{X}}} P\left(\{\underline{X} = \underline{x}\}\right) \\ &= \sum_{\underline{x} \in A \cap S_{\underline{X}}} f_{\underline{X}}(\underline{x}) \\ &= \sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(\underline{x}) I_A(\underline{x}). \end{aligned}$$

Result 1:

Let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional ($p \geq 2$) discrete r.v. with support $S_{\underline{X}}$ and p.m.f. $f_{\underline{X}}(\cdot)$. For fixed $k \in \{1, \ldots, p-1\}$, let $\underline{Y} = (X_1, \ldots, X_k)$ and $\underline{Z} = (X_{k+1}, \ldots, X_p)$ so that $\underline{X} = (\underline{Y}, \underline{Z})$. For $y \in \mathbb{R}^k$, define $A_{\underline{y}} = \{\underline{z} \in \mathbb{R}^{p-k} : (\underline{y}, \underline{z}) \in S_{\underline{X}}\}$. Then the r.v. $\underline{Y}$ is discrete with support $S_{\underline{Y}} = \{\underline{y} \in \mathbb{R}^k : (\underline{y}, \underline{z}) \in S_{\underline{X}}, \text{ for some } \underline{z} \in \mathbb{R}^{p-k}\}$ and joint marginal p.m.f.

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} \sum_{\underline{z} \in A_{\underline{y}}} f_{\underline{X}}(\underline{y}, \underline{z}), & \text{if } \underline{y} \in S_{\underline{Y}} \\ 0, & \text{otherwise} \end{cases}.$$

Proof: Note that

$$\{\underline{X} \in S_{\underline{X}}\} = \{(\underline{Y}, \underline{Z}) \in S_{\underline{X}}\} \subseteq \{\underline{Y} \in S_{\underline{Y}}\}.$$

Thus

$$P(\{\underline{Y} \in S_{\underline{Y}}\}) \geq P(\{\underline{X} \in S_{\underline{X}}\}) = 1$$
$$\Rightarrow P(\{\underline{Y} \in S_{\underline{Y}}\}) = 1.$$

Also $S_{\underline{Y}}$ is countable (since $S_{\underline{X}}$ is countable), and for $\underline{y} \in S_{\underline{Y}}$,

$$
\begin{aligned}
P(\{\underline{Y} = \underline{y}\}) &= P\Big(\{\underline{Y} = \underline{y}\} \bigcap \{\underline{X} \in S_{\underline{X}}\}\Big) \\
&= P\Big(\{\underline{Y} = \underline{y}\} \bigcap \{(\underline{y}, \underline{Z}) \in S_{\underline{X}}\}\Big) \\
&= P\Big(\{\underline{Y} = \underline{y}\} \bigcap \{\underline{Z} \in A_{\underline{y}}\}\Big) \\
&= P\Big(\bigcup_{\underline{z} \in A_{\underline{y}}} \{(\underline{Y}, \underline{Z}) = (\underline{y}, \underline{z})\}\Big) \\
&= \sum_{\underline{z} \in A_{\underline{y}}} P\big(\{(\underline{Y}, \underline{Z}) = (\underline{y}, \underline{z})\}\big) \\
&= \sum_{\underline{z} \in A_{\underline{y}}} f_{\underline{X}}(\underline{y}, \underline{z}).
\end{aligned}
$$

Note that, for $\underline{y} \in S_{\underline{Y}}, A_{\underline{y}} \neq \phi$ and, for $\underline{z} \in A_{\underline{y}}, (\underline{y}, \underline{z}) \in S_{\underline{X}}$. Therefore $P(\{\underline{Y} = \underline{y}\}) > 0, \forall\ \underline{y} \in S_{\underline{Y}}$. Hence the assertion follows.

Remark 2:

(a) The marginal distributions of discrete distributions are discrete.

(b) The above results suggests that to get a marginal p.m.f. from joint p.m.f. one needs to sum out the arguments of unwanted variables in the joint p.m.f.

Example 1: Let $\underline{Z} = (X, Y)$ have the joint p.m.f.

$$f_{X,Y}(x, y) = \begin{cases} cy, & \text{if } 1 \leq x \leq y \leq n, \; x, y \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases},$$

where n (≥ 2) is a fixed positive integer and c is a fixed real constant.

(a) Find the value of constant c;

(b) Find the marginal p.m.f.s of X and Y;

(c) Find $P(\{X > Y\}), P(\{X = Y\})$ and $P(\{X < Y\})$.

Solution:

(a) Clearly

$$\begin{aligned} S_{\underline{Z}} &= \text{support of } \underline{Z} \\ &= \{(s,t) \in \mathbb{R}^2 : s,t \in \{1,\ldots,n\},\ s \le t\}. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{\underline{z} \in S_{\underline{Z}}} f_{X,Y}(\underline{z}) &= 1 \\ \Rightarrow \sum_{y=1}^{n} \sum_{x=1}^{y} cy &= 1 \\ \Rightarrow c \sum_{y=1}^{n} y^2 &= 1 \\ \Rightarrow c &= \frac{6}{n(n+1)(2n+1)}. \end{aligned}$$

(b) For $x \in S_X = \{1, 2, \ldots, n\}$

$$\begin{aligned} f_X(x) &= P(\{X = x\}) \\ &= \sum_{y=x}^{n} P(\{X = x, Y = y\}) \\ &= c \sum_{y=x}^{n} y \\ &= c\left[\frac{n(n+1)}{2} - \frac{(x-1)x}{2}\right] \end{aligned}$$

Thus

$$f_X(x) = \begin{cases} \frac{3\left[n(n+1)-(x-1)x\right]}{n(n+1)(2n+1)}, & \text{if } x = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}.$$

For $y \in S_Y = \{1, 2, \ldots, n\}$

$$\begin{aligned} P(\{Y = y\}) &= \sum_{x=1}^{y} cy \\ &= cy^2. \end{aligned}$$

Therefore the marginal p.m.f. of Y is

$$f_Y(y) = \begin{cases} \frac{6y^2}{n(n+1)(2n+1)}, & \text{if } y = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}.$$

(c)

$$\begin{aligned} P(\{X > Y\}) &= \sum_{\substack{(x,y) \in S_{\underline{Z}} \\ x > y}} f_{X,Y}(x, y) \\ &= 0. \end{aligned}$$

$$
\begin{aligned}
P(\{X=Y\}) &= \sum_{\substack{(x,y)\in S_{\underline{Z}} \\ x=y}} f_{X,Y}(x,y) \\
&= c\sum_{y=1}^{n} y \\
&= \frac{3}{2n+1}.
\end{aligned}
$$

$$
\begin{aligned}
P(\{X<Y\}) &= 1-P(\{X=Y\})-P(\{X>Y\}) \\
&= 1-\frac{3}{2n+1} \\
&= \frac{2n-2}{2n+1}.
\end{aligned}
$$

Conditional Distribution of Discrete Random Vectors

- $(\Omega, \mathcal{P}(\Omega), P)$: a given probability space.
- $\underline{X} = (X_1, \ldots, X_p) : \Omega \to \mathbb{R}^p$: a p-dimensional r.v. (not necessarily discrete or A.C.) with d.f. $F_{\underline{X}}(\cdot)$ ($p \geq 2$).

Definition 2:

(a) Let $D \subseteq \mathbb{R}^p$ be such that $P(\{\underline{X} \in D\}) > 0$. Then the conditional d.f. of $\underline{X}$ given that $\underline{X} \in D$ is defined by

$$\begin{aligned} F_{\underline{X}|D}(\underline{x}) &= P\big(\{\underline{X} \leq \underline{x}\} | \{\underline{X} \in D\}\big) \\ &= \frac{P(\{\underline{X} \leq \underline{x}, \underline{X} \in D\})}{P(\{\underline{X} \in D\})}, \ \underline{x} \in \mathbb{R}^p. \end{aligned}$$

(b) Let $\underline{X} = (X_1, \ldots, X_p)$ and $\underline{Y} = (Y_1, \ldots, Y_q)$ be p and q dimensional r.v.s, respectively, and let $\underline{Z} = (\underline{X}, \underline{Y})$ (a $(p+q)$-dimensional r.v.). Let $S_{\underline{Z}}$ and $f_{\underline{Z}}(\cdot)$, respectively, denote the support and joint p.m.f. of $\underline{Z}$. Let $S_{\underline{Y}}$ and $f_{\underline{Y}}(\cdot)$, respectively, denote the support and joint p.m.f. of $\underline{Y}$. For a fixed $\underline{y} \in S_{\underline{Y}}$, define $S_{\underline{X}|\underline{y}} = \{\underline{x} \in \mathbb{R}^p : (\underline{x}, \underline{y}) \in S_{\underline{Z}}\}$. Then the conditional d.f. and conditional p.m.f. of $\underline{X}$ given $\underline{Y} = \underline{y}$ ($\underline{y} \in S_{\underline{Y}}$ is fixed) are defined by

$$\begin{aligned} F_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) &= P(\{\underline{X} \leq \underline{x} | \underline{Y} = \underline{y}\}) \\ &= \frac{P(\{\underline{X} \leq \underline{x}, \underline{Y} = \underline{y}\})}{P(\{\underline{Y} = \underline{y}\})} \\ &= \frac{\sum_{\underline{t} \in S_{\underline{X}|\underline{y}}, \underline{t} \leq \underline{x}} f_{\underline{Z}}(\underline{t}, \underline{y})}{\sum_{\underline{t}:(\underline{t},\underline{y}) \in S_{\underline{Z}}} f_{\underline{Z}}(\underline{t}, \underline{y})}, \ \underline{x} \in \mathbb{R}^p \end{aligned}$$

and

$$\begin{aligned} f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) &= P(\{\underline{X}=\underline{x}|\underline{Y}=\underline{y}\}) \\ &= \frac{P(\{\underline{X}=\underline{x},\underline{Y}=\underline{y}\})}{P(\{\underline{Y}=\underline{y}\})} \\ &= \frac{f_{\underline{Z}}(\underline{x},\underline{y})}{f_{\underline{Y}}(\underline{y})},\ \underline{x}\in\mathbb{R}^p; \end{aligned}$$

respectively.

Remark 3:

(a) It is easy to verify that the function $F_{\underline{X}|D}(\cdot)$, defined in above definition, is a proper d.f. (i.e., it satisfies the four properties of a d.f.).

(b) It is straightforward to establish that, for every fixed $\underline{y}\in S_{\underline{Y}}$, the function $F_{\underline{X}|\underline{Y}}(\cdot|\underline{y})$ is a proper d.f. and $f_{\underline{X}|\underline{Y}}(\cdot|\underline{y})$ is the p.m.f. corresponding to $F_{\underline{X}|\underline{Y}}(\cdot|\underline{y})$ with support $S_{\underline{X}|\underline{y}}$.

Result 3:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be a p-dimensional discrete r.v. with support $S_{\underline{X}}$ and joint p.m.f. $f_{\underline{X}}(\cdot)$. Let S_{X_i} and $f_{X_i}(\cdot)$, respectively, denote the support and marginal p.m.f of $X_i, i = 1, \ldots, p$. Then

(a) $X_1, \ldots, X_p$ are independent iff

$$f_{\underline{X}}(x_1, \ldots, x_p) = \prod_{i=1}^{p} f_{X_i}(x_i), \; \forall \; \underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p. \tag{1}$$

(b) $X_1, \ldots, X_p$ are independent $\Rightarrow S_{\underline{X}} = S_{X_1} \times \cdots \times S_{X_p}$.

Proof:

Take $p = 2$, for simplicity of notation.

(a) Suppose that (1) holds. Then

$$\begin{aligned} S_{\underline{X}} &= \{\underline{x} \in \mathbb{R}^2 : f_{\underline{X}}(\underline{x}) > 0\} \\ &= \{\underline{x} \in \mathbb{R}^2 : f_{X_1}(x_1) f_{X_2}(x_2) > 0\} \\ &= \{x \in \mathbb{R} : f_{X_1}(x) > 0\} \times \{y \in \mathbb{R} : f_{X_2}(y) > 0\} \\ &= S_{X_1} \times S_{X_2}. \end{aligned}$$

Moreover, for $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} F_{X_1, X_2}(x_1, x_2) &= P(\{X_1 \leq x_1, X_2 \leq x_2\}) \\ &= \sum_{\substack{\underline{t} \in S_{\underline{X}} \\ \underline{t} \leq \underline{x}}} f_{\underline{X}}(\underline{t}) \\ &= \sum_{\substack{t_1 \in S_{X_1} \\ t_1 \leq x_1}} \sum_{\substack{t_2 \in S_{X_2} \\ t_2 \leq x_2}} f_{X_1}(t_1) f_{X_2}(t_2) \end{aligned}$$

$$= F_{X_1}(t_1)F_{X_2}(t_2),$$

implying that X_1 and X_2 are independent.
Conversely, suppose that X_1 and X_2 are independent, i.e.,

$$F_{X_1,X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2),\ \forall\ \underline{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Then, for $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned}
f_{\underline{X}}(x_1, x_2) &= P(\{X_1 = x_1, X_2 = x_2\}) \\
&= \lim_{h \downarrow 0} P(\{x_1 - h < X_1 \leq x_1, x_2 - h < X_2 \leq x_2\}) \\
&= \lim_{h \downarrow 0} \big[F_{\underline{X}}(x_1, x_2) - F_{\underline{X}}(x_1 - h, x_2) - F_{\underline{X}}(x_1, x_2 - h) \\
&\qquad + F_{\underline{X}}(x_1 - h, x_2 - h)\big] \\
&= \lim_{h \downarrow 0} \big[F_{X_1}(x_1)F_{X_2}(x_2) - F_{X_1}(x_1 - h)F_{X_2}(x_2) \\
&\qquad - F_{X_1}(x_1)F_{X_2}(x_2 - h) + F_{X_1}(x_1 - h)F_{X_2}(x_2 - h)\big]
\end{aligned}$$

$$
\begin{aligned}
&= F_{X_1}(x_1)F_{X_2}(x_2) - F_{X_1}(x_1-)F_{X_2}(x_2) - F_{X_1}(x_1)F_{X_2}(x_2-) \\
&\qquad + F_{X_1}(x_1-)F_{X_2}(x_2-) \\
&= \left[F_{X_1}(x_1) - F_{X_1}(x_1-)\right]F_{X_2}(x_2) \\
&\qquad - \left[F_{X_1}(x_1) - F_{X_1}(x_1-)\right]F_{X_2}(x_2-) \\
&= \left[F_{X_1}(x_1) - F_{X_1}(x_1-)\right]\left[F_{X_2}(x_2) - F_{X_2}(x_2-)\right] \\
&= f_{X_1}(x_1)f_{X_2}(x_2).
\end{aligned}
$$

(b) The proof of this assertion is contained in the proof of (a).

Result 4:

Let $\underline{X}_1, \ldots, \underline{X}_p$ be discrete random vectors of (possibly) different dimensions. Then

(a) $\underline{X}_1, \ldots, \underline{X}_p$ are independent $\Leftrightarrow$

$$P(\{\underline{X}_i \in A_i, i = 1, \ldots, p\} = \prod_{i=1}^{p} P(\{X_i \in A_i\}), \ \forall \ A_1, \ldots, A_p \subseteq \mathbb{R}^p;$$

(b) $\underline{X}_1, \ldots, \underline{X}_p$ are independent $\Leftrightarrow \psi_1(\underline{X}_1), \ldots, \psi_p(\underline{X}_p)$ are independent, for any functions (not necessarily real-valued) $\psi_1, \ldots, \psi_p$.

Remark 4:

(a) Result 3 above remains valid (with obvious modifications) for independence of discrete random vectors (of possibly different dimensions).

(b) Random vectors $\underline{X}$ and $\underline{Y}$ are independent

$$\Leftrightarrow F_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = F_{\underline{X}}(\underline{x}),\ \forall\ \underline{x} \in \mathbb{R}^p, \underline{y} \in S_{\underline{Y}}$$

$$\Leftrightarrow f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = f_{\underline{X}}(\underline{x}),\ \forall\ \underline{x} \in \mathbb{R}^p, \underline{y} \in S_{\underline{Y}}.$$

Result 5:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be a discrete r.v. with joint p.m.f. $f_{\underline{X}}(\cdot)$. Then $X_1, \ldots, X_p$ are independent iff

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{p} g_i(x_i), \ \underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p, \tag{2}$$

for some non-negative functions $g_i(\cdot), i = 1, \ldots, p$, defined on $\mathbb{R}$. In that case the marginal p.m.f. of X_i is

$$f_{X_i}(x) = c_i g_i(x), \ x \in \mathbb{R},$$

for some positive constant c_i, and support of X_i is

$$S_{X_i} = \{x \in \mathbb{R} : g_i(x) > 0\}, \ i = 1, \ldots, p.$$

Proof: Take $p = 2$, for simplicity of notation.
First suppose that X_1 and X_2 are independent. Then

$$f_{\underline{X}}(\underline{x}) = f_{X_1}(x_1) f_{X_2}(x_2), \ \forall \ \underline{x} = (x_1, x_2) \in \mathbb{R}^2,$$

so that (2) holds.

Conversely suppose that (2) holds. Then

$$\begin{aligned} S_{\underline{X}} &= \{\underline{x} \in \mathbb{R}^2 : f_{\underline{X}}(\underline{x}) > 0\} \\ &= \{x_1 \in \mathbb{R} : g_1(x_1) > 0\} \times \{x_2 \in \mathbb{R} : g_2(x_2) > 0\} \\ &= S_1 \times S_2, \text{ say.} \end{aligned}$$

We have

$$
\begin{aligned}
f_{X_1}(x_1) &= P(\{X_1 = x_1\}) \\
&= \sum_{t:(x_1,t)\in S_{\underline{X}}} P(\{X_1 = x_1, X_2 = t\}) \\
&= \sum_{t:(x_1,t)\in S_1\times S_2} g_1(x_1)g_2(t) \\
&= \begin{cases} g_1(x_1)\sum_{t\in S_2} g_2(t), & \text{if } x_1 \in S_1 \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

Similarly

$$
f_{X_2}(x_2) = \begin{cases} g_2(x_2)\sum_{s\in S_1} g_1(s), & \text{if } x_2 \in S_2 \\ 0, & \text{otherwise} \end{cases}.
$$

Also

$$
\begin{aligned}
\sum_{(s,t)\in S_{\underline{X}}} f_{\underline{X}}(s,t) &= 1 \\
\Rightarrow \sum_{(s,t)\in S_1\times S_2} g_1(s)g_2(t) &= 1 \\
\Rightarrow \left[\sum_{s\in S_1} g_1(s)\right]\left[\sum_{t\in S_2} g_2(t)\right] &= 1 \\
\Rightarrow f_{X_1}(x_1)f_{X_2}(x_2) &= \begin{cases} g_1(x_1)g_2(x_2), & \text{if } x_1\in S_1, x_2\in S_2 \\ 0, & \text{otherwise} \end{cases}. \\
&= f_{\underline{X}}(x_1,x_2),\ \forall\ \underline{x}=(x_1,x_2)\in\mathbb{R}^2,
\end{aligned}
$$

implying that X_1 and X_2 are independent.

Example 2 :

Let $\underline{Z} = (X, Y)$ have the joint p.m.f.

$$f_{X,Y}(x, y) = \begin{cases} \frac{y}{55}, & \text{if } 1 \leq x \leq y \leq 5,\ x, y \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the conditional p.m.f. of X given $Y = y$ ($y \in \{1, \ldots, 5\}$) and of Y given $X = x$ ($x \in \{1, \ldots, 5\}$).

(b) Are X and Y independent?

(c) Find $P(\{Y \geq 3\}|\{X = 2\})$.

Solution : We have $S_{X,Y} = \{(s, t) \in \mathbb{N} \times \mathbb{N} : 1 \leq s \leq t \leq 5\}$.

(a) Fix $y \in \{1, \ldots, 5\}$. Then

$$f_{X|Y}(x|y) = \frac{P(\{X = x, Y = y\})}{P(\{Y = y\})},\ x \in \mathbb{R}.$$

$$\begin{aligned} P(\{Y=y\}) &= \sum_{s:(s,y)\in S_{X,Y}} P(\{X=s, Y=y\}) \\ &= \sum_{s=1}^{y} \frac{y}{55} \\ &= \frac{y^2}{55}. \end{aligned}$$

Thus, for fixed $y \in \{1, \ldots, 5\}$,

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x \in \{1, \ldots, y\} \\ 0, & \text{otherwise} \end{cases}.$$

For fixed $x \in \{1, \ldots, 5\}$,

$$f_X(x) = \sum_{t:(x,t)\in S_{X,Y}} P(\{X=x, Y=t\})$$

$$
\begin{aligned}
&= \sum_{t=x}^{5} \frac{t}{55} \\
&= \frac{(6-x)(5+x)}{110} \\
f_{Y|X}(y|x) &= \frac{P(\{X=x, Y=y\})}{P(\{X=x\})} \\
&= \begin{cases} \frac{2y}{(6-x)(5+x)}, & \text{if } y \in x, x+1, \ldots, 5 \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

(b) Since $f_{X|Y}(x|y)$ is not independent of y, we infer that X and Y are not independent. Also note that

$$S_X = \{1, \ldots, 5\} = S_Y$$

and

$$\begin{aligned} S_{X,Y} &= \{(s,t) \in \mathbb{N} \times \mathbb{N} : 1 \leq s \leq t \leq 5\} \\ &\neq S_X \times S_Y. \end{aligned}$$

(c) We have, from (a),

$$f_{Y|X}(y|2) = \begin{cases} \frac{y}{14}, & \text{if } y = 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}.$$

$$\begin{aligned} P(\{Y \geq 3\}|\{X = 2\}) &= 1 - P(\{Y = 2\}|\{X = 2\}) \\ &= 1 - \frac{1}{7} = \frac{6}{7}. \end{aligned}$$

Take Home Problem

Let $\underline{X} = (X_1, X_2, X_3)'$ be a discrete r.v. with joint p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} c\ x_1 x_2 x_3, & x_1 = 1, 2,\ x_2 = 1, 2, 3,\ x_3 = 1, 3 \\ 0, & \text{otherwise} \end{cases},$$

where c is a fixed real constant.

(a) Find the value of c, support of $\underline{X}$ and support of $X_i, i = 1, 2, 3$;

(b) Find the marginal p.m.f. of $X_i, i = 1, 2, 3$;

(c) Find the marginal p.m.f. of $\underline{Y} = (X_1, X_3)$;

(d) Find $P(\{X_1 = X_2 = X_3\})$;

(e) Find the conditional p.mf. of X_1 given $(X_2, X_3) = (2, 1)$;

(f) Find the conditional p.mf. of (X_1, X_3) given that $X_2 = 3$;

(g) Given $X_2 = 3$, are X_1 and X_3 independent?

(h) Are X_1, X_2 and X_3 independent r.v.s?

(i) Are X_1 and X_3 independent r.v.s?

Abstract of Next Module

- We will introduce continuous and absolutely continuous random vectors and study properties of their probability distributions.

Thank you for your patience