Module 23

CONTINUOUS AND ABSOLUTELY CONTINUOUS RANDOM VECTORS

- $\underline{X}$: a p-dimensional random vector (r.v.) defined on a probability space $(\Omega, \mathcal{P}(\Omega), P)$;

- $F_{\underline{X}}$: d.f. of $\underline{X}$.

Definition 1:

(a) The r.v. $\underline{X}$ is said to be continuous if $F_{\underline{X}}(\underline{x})$ is continuous at every point $\underline{x} \in \mathbb{R}^p$;

(b) The r.v. $\underline{X}$ is said to be a absolutely continuous if there exists a non-negative function $f_{\underline{X}} : \mathbb{R}^p \to \mathbb{R}$ such that

$$F_{\underline{X}}(\underline{x}) = \int_{(-\underline{\infty},\ \underline{x}]} f_{\underline{X}}(\underline{y}) d\underline{y}, \quad \underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p,$$

where $(-\underline{\infty}, \underline{x}] = (-\infty, x_1] \times \cdots \times (-\infty, x_p]$, $\underline{y} = (y_1, \ldots, y_p)$ and $d\underline{y} = dy_p \cdots dy_1$.

The function $f_{\underline{X}}(\cdot)$ which is non-negative and is such that

$$\int_{\mathbb{R}^p} f_{\underline{X}}(\underline{y})d\underline{y} = \lim_{\substack{x_i \to \infty \\ i=1,\ldots,p}} F_{\underline{X}}(x_1, \ldots, x_p) = 1$$

is called the joint probability density function (p.d.f.) of $\underline{X}$.

Remark 1 :

(a) It can be shown that if $h : \mathbb{R}^p \to \mathbb{R}$ is any function such that $h(\underline{x}) \geq 0, \ \forall \ \underline{x} \in \mathbb{R}^p$ and

$$\int_{\mathbb{R}^p} h(\underline{x})d\underline{x} = 1,$$

then $h(\cdot)$ is a joint p.d.f. of some A.C. r.v.

(b) Let $\underline{a} = (a_1, \ldots, a_p) \in \mathbb{R}^p$ and $\underline{a}_n = (a_1 - \frac{1}{n}, \ldots, a_p - \frac{1}{n}), n = 1, 2, \ldots$. Then

$$P(\{\underline{X} = \underline{a}\}) \quad = \quad P\left(\bigcap_{n=1}^{\infty} \underline{X}^{-1}\Big((\underline{a}_n, \underline{a}]\Big)\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} P\left(\underline{X}^{-1}\left((\underline{a}_n, \underline{a}]\right)\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{p} (-1)^k \sum_{\underline{z}_n \in \Delta_{k,p}\left((\underline{a}_n, \underline{a}]\right)} F_{\underline{X}}(\underline{z}_n).
\end{aligned}
$$

If $\underline{X}$ is continuous at $\underline{a} \in \mathbb{R}^p$ then, for $\underline{z}_n \in \Delta_{k,p}\left((\underline{a}_n, \underline{a}]\right),\ n = 1, 2, \ldots$ (so that, as $n \to \infty, \underline{z}_n \to \underline{a}$), $F_{\underline{X}}(\underline{z}_n) \to F_{\underline{X}}(\underline{a})$, as $n \to \infty$. Consequently, if $F_{\underline{X}}(\cdot)$ is continuous at $\underline{a} \in \mathbb{R}^p$,

$$
\begin{aligned}
P(\{\underline{X} = \underline{a}\}) &= \sum_{k=0}^{p} (-1)^k \binom{p}{k} F_{\underline{X}}(\underline{a}) \\
&= (1-1)^p F_{\underline{X}}(\underline{a}) \\
&= 0.
\end{aligned}
$$

It follows that if the d.f. $F_{\underline{X}}$ of a p-dimensional r.v. $\underline{X}$ is continuous at $\underline{a} \in \mathbb{R}^p$, then

$$P(\{\underline{X} = \underline{a}\}) = 0.$$

In particular if $\underline{X}$ is a continuous r.v. then

$$P(\{\underline{X} = \underline{a}\}) = 0, \ \forall \ \underline{a} \in \mathbb{R}^p,$$

and

$$P(\{\underline{X} \in S\}) = 0, \ \forall \text{ countable set } S \subseteq \mathbb{R}^p.$$

(c) Let $\underline{X}$ be a p-dimensional A.C. r.v. with d.f. $F_{\underline{X}}(\cdot)$ and p.d.f. $f_{\underline{X}}(\cdot)$. Then

$$F_{\underline{X}}(\underline{x}) = \int_{(-\underline{\infty},\ \underline{x}]} f_{\underline{X}}(\underline{y})d\underline{y},\ \underline{x} \in \mathbb{R}^p$$

is continuous at every point $\underline{x} \in \mathbb{R}^p$. Thus an A.C. r.v. $\underline{X}$ is also continuous and hence

$$P(\{\underline{X} = \underline{a}\}) = 0,\ \forall\ \underline{a} \in \mathbb{R}^p$$

and

$$P(\{\underline{X} \in S\}) = 0,\ \forall \text{ countable set } S \subseteq \mathbb{R}^p.$$

(d) Let $\underline{X}$ be an A.C. r.v. Then it can be shown that, for any $A \subseteq \mathbb{R}^p$,

$$P(\{\underline{X} \in A\}) = \int_A f_{\underline{X}}(\underline{x})d\underline{x} = \int_{\mathbb{R}^p} f_{\underline{X}}(\underline{x})I_A(\underline{x})d\underline{x}.$$

For example if $A = (\underline{a}, \underline{b}] \subseteq \mathbb{R}^p$, then

$$
\begin{aligned}
\int\limits_{(\underline{a},\underline{b}]} f_{\underline{X}}(\underline{x})d\underline{x} &= \int\limits_{a_1}^{b_1}\cdots\int\limits_{a_p}^{b_p} f_{\underline{X}}(\underline{x})d\underline{x} \\
&= \sum_{k=0}^{p}(-1)^k \sum_{\underline{z}_n\in\Delta_{k,p}\left((\underline{a},\ \underline{b}]\right)} \int\limits_{-\infty}^{z_1}\cdots\int\limits_{-\infty}^{z_p} f_{\underline{X}}(\underline{z})d\underline{z} \\
&= \sum_{k=0}^{p}(-1)^k \sum_{\underline{z}\in\Delta_{k,p}\left((\underline{a},\ \underline{b}]\right)} F_{\underline{X}}(\underline{z}) \\
&= P\left(\{\underline{X}\in(\underline{a},\ \underline{b}]\}\right).
\end{aligned}
$$

(e) Let $\underline{X}$ be A.C. and let $A \subseteq \mathbb{R}^p$ comprises of a countable number of curves in $\mathbb{R}^p$. Then

$$
P(\{\underline{X}\in A\}) = \int_A f_{\underline{X}}(\underline{x})d\underline{x} = 0.
$$

In particular

$$P(\{X_i = X_j\}) = 0, \ \forall \ i \neq j.$$

(f) If $\underline{X}$ is an A.C. r.v. then its joint p.d.f. is not unique and there are different various of joint p.d.f. In fact if the values of a joint p.d.f. $f_{\underline{X}}(\cdot)$ of an A.C. r.v. $\underline{X}$ are changed at a finite number of curves with some other non-negative values then the resulting function is again a p.d.f. of $\underline{X}$.

(g) Let $\underline{X}$ be a p-dimensional r.v. with joint d.f. $F_{\underline{X}}(\cdot)$ and joint p.d.f. $f_{\underline{X}}(\cdot)$, so that

$$F_{\underline{X}}(\underline{x}) = \int\limits_{(-\underline{\infty},\ \underline{x}]} f_{\underline{X}}(\underline{y})d\underline{y}.$$

Clearly the joint d.f. of an A.C. r.v. $\underline{X}$ is determined by its joint p.d.f. $f_{\underline{X}}(\cdot)$. Thus to study the induced probability function $P_{\underline{X}}(\cdot)$ it is enough to study the joint p.d.f. $f_{\underline{X}}(\cdot)$.

Definition 2:

For a given $r > 0$ and $\underline{x} \in \mathbb{R}^p$, let

$$N_r(\underline{x}) = \left\{ \underline{t} \in \mathbb{R}^p : \sqrt{\sum_{i=1}^{p} (t_i - x_i)^2} < r \right\}$$

denote the p-dimensional ball of radius r centered at $\underline{x}$. If $\underline{X}$ is continuous then the set

$$S_{\underline{X}} = \{ \underline{x} \in \mathbb{R}^p : P(\{ \underline{X} \in N_r(\underline{x}) \}) > 0, \ \forall \ r > 0 \}$$

is called the support (of distribution) of r.v. $\underline{X}$.

Result 1:

Let $\underline{X} = (X_1, \ldots, X_p)$ be a p-dimensional r.v. with d.f. $F_{\underline{X}}(\cdot)$ such that

$$\frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(x_1, \ldots, x_p)$$

exists everywhere, except (possibly) on a set D comprising of a finite number of curves in $\mathbb{R}^p$, and

$$\int_{\mathbb{R}^p} \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(\underline{x}) I_{D^c}(\underline{x}) d\underline{x} = 1.$$

Then $\underline{X}$ is A.C. with a p.d.f

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(\underline{x}), & \text{if } x \notin D \\ 0, & \text{if } x \in D \end{cases}.$$

Result 2

Let $\underline{X} = (X_1, \ldots, X_{p-1}, X_p)'$ be a p-dimensional r.v. with joint d.f. $F_{\underline{X}}(\cdot)$ and joint p.m.f. $f_{\underline{X}}(\cdot)$. Then the r.v. $\underline{Y} = (X_1, \ldots, X_{p-1})$ is also A.C. with p.d.f.

$$f_{\underline{Y}}(\underline{y}) = \int_{-\infty}^{\infty} f_{\underline{X}}(y_1, \ldots, y_{p-1}, t_p) dt_p, \ \ \underline{y} = (y_1, \ldots, y_{p-1}) \in \mathbb{R}^{p-1}.$$

Proof: The joint d.f. of $\underline{Y} = (X_1, \ldots, X_{p-1})$ is

$$\begin{aligned}
F_{\underline{Y}}(\underline{y}) &= \lim_{y_p \to \infty} F_{\underline{X}}(y_1, \ldots, y_{p-1}, y_p) \\
&= \lim_{y_p \to \infty} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{p-1}} \int_{-\infty}^{y_p} f_{\underline{X}}(t_1, \ldots, t_{p-1}, t_p) dt_p dt_{p-1} \ldots dt_1 \\
&= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{p-1}} \left\{ \int_{-\infty}^{\infty} f_{\underline{X}}(t_1, \ldots, t_{p-1}, t_p) dt_p \right\} dt_{p-1} \ldots dt_1
\end{aligned}$$

$$= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{p-1}} h(t_1, \ldots, t_{p-1}) dt_{p-1} \ldots dt_1,$$

$$\underline{y} = (y_1, \ldots, y_{p-1}) \in \mathbb{R}^{p-1},$$

where

$$h(t_1, \ldots, t_{p-1}) = \int_{-\infty}^{\infty} f_{\underline{X}}(t_1, \ldots, t_{p-1}, t_p) dt_p, \ \underline{t} = (t_1, \ldots, t_{p-1}) \in \mathbb{R}^{p-1}.$$

It follows that $\underline{Y} = (Y_1, \ldots, Y_{p-1})$ is A.C. with p.d.f.

$$h(y_1, \ldots, y_{p-1}) = \int_{-\infty}^{\infty} f_{\underline{X}}(y_1, \ldots, y_{p-1}, t_p) dt_p, \ \underline{y} = (y_1, \ldots, y_{p-1}) \in \mathbb{R}^{p-1}.$$

Remark 2:

The above result suggests that marginal distributions of A.C. r.v. are A.C. Moreover the marginal distribution of A.C. r.v. can be obtained by integrating out the arguments of unwanted variables in joint the p.d.f.

Conditional Distributions

- $\underline{Z} = (\underline{X}, \underline{Y})$: a $(p+q)$-dimensional A.C. r.v. with joint d.f. $F_{\underline{Z}}(\cdot)$, support $S_{\underline{Z}}$ and p.d.f. $f_{\underline{Z}}(\cdot)$,

  where

- $\underline{X} = (X_1, \ldots, X_p)$: a p-dimensional A.C. r.v. with joint d.f. $F_{\underline{X}}(\cdot)$, support $S_{\underline{X}}$ and p.d.f. $f_{\underline{X}}(\cdot)$;

- $\underline{Y} = (Y_1, \ldots, Y_q)$: a q-dimensional A.C. r.v. with joint d.f. $F_{\underline{Y}}(\cdot)$, support $S_{\underline{Y}}$ and p.d.f. $f_{\underline{Y}}(\cdot)$;

- Let $\underline{x} \in S_{\underline{X}}$ be such that $f_{\underline{X}}(\underline{x}) > 0$;
- Then the conditional d.f. of $\underline{Y}$ given $\underline{X} = \underline{x}$ is defined by

$$
\begin{aligned}
F_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) &= P(\{\underline{Y} \leq \underline{y}\}|\{\underline{X} = \underline{x}\}) \qquad \text{(Notation)} \\
&= \lim_{\substack{h_i \downarrow 0 \\ i=1,\ldots,p}} P(\{\underline{Y} \leq \underline{y}\}|\{x_i - h_i < X_i \leq x_i,\ i = 1, \ldots, p\}) \\
&= \lim_{\substack{h_i \downarrow 0 \\ i=1,\ldots,p}} \frac{P(\{x_i - h_i < X_i \leq x_i,\ \underline{Y} \leq \underline{y}, i = 1, \ldots, p\})}{P(\{x_i - h_i < X_i \leq x_i,\ i = 1, \ldots, p\})} \\
&= \lim_{\substack{h_i \downarrow 0 \\ i=1,\ldots,p}} \frac{\int\limits_{x_1-h_1}^{x_1} \cdots \int\limits_{x_p-h_p}^{x_p} \int\limits_{-\infty}^{y_1} \cdots \int\limits_{-\infty}^{y_q} f_{\underline{Z}}(\underline{s}, \underline{t}) d\underline{t} d\underline{s}}{\int\limits_{x_1-h_1}^{x_1} \cdots \int\limits_{x_p-h_p}^{x_p} f_{\underline{X}}(\underline{s}) d\underline{s}} \\
&= \frac{\int\limits_{-\infty}^{y_1} \cdots \int\limits_{-\infty}^{y_q} f_{\underline{Z}}(\underline{x}, \underline{t}) d\underline{t}}{f_{\underline{X}}(\underline{x})}
\end{aligned}
$$

$$= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_q} h_{\underline{x}}(\underline{t}) d\underline{t},$$

where

$$h_{\underline{x}}(\underline{t}) = \frac{f_{\underline{Z}}(\underline{x}, \underline{t})}{f_{\underline{X}}(\underline{x})}, \ \underline{t} = (t_1, \ldots, t_q) \in \mathbb{R}^q.$$

- It follows that the conditional distribution of $\underline{Y}$ given $\underline{X} = \underline{x}$ (where $\underline{x} \in S_{\underline{X}}$ is such that $f_{\underline{X}}(\underline{x}) > 0$) is A.C. with p.d.f.

$$f_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) = \frac{f_{\underline{Z}}(\underline{x}, \underline{y})}{f_{\underline{X}}(\underline{x})}, \ \underline{y} \in \mathbb{R}^q.$$

- Clearly $\int_{\mathbb{R}^q} f_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) d\underline{y} = 1$ and $f_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) \geq 0, \ \forall \ \underline{y} \in \mathbb{R}^q$.
- The definition of conditional d.f. given in (b) above holds for general r.v.s (discrete/continuous/A.C./mixed).

Definition 3:

Let $\underline{Z} = (X_1, \ldots, X_p, Y_1, \ldots, Y_q)'$ be a $(p+q)$-dimensional A.C. r.v. with joint p.d.f. $f_{\underline{Z}}(\cdot)$. Let $\underline{X} = (X_1, \ldots, X_p)$, $\underline{Y} = (Y_1, \ldots, Y_q)$ and let $f_{\underline{X}}(\cdot)$ be the marginal p.d.f. of $\underline{X}$. Let $\underline{x} \in S_{\underline{X}}$ be such that $f_{\underline{X}}(\underline{x}) > 0$. Then the conditional p.d.f. of $\underline{Y}$ given $\underline{X} = \underline{x}$ is defined by

$$f_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) = \frac{f_{\underline{Z}}(\underline{x}, \underline{y})}{f_{\underline{X}}(\underline{x})}, \ \underline{y} \in \mathbb{R}^q.$$

Result 3:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be a p-dimensional A.C. r.v. with joint p.d.f. $f_{\underline{X}}(\cdot)$ and marginal p.d.f.s $f_{X_i}(\cdot)$, $i = 1, \ldots, p$. Then

(a) R.v.s $X_1, \ldots, X_p$ are independent iff

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{p} f_{X_i}(x_i), \ \underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p.$$

(b) R.v.s $X_1, \ldots, X_p$ are independent $\Rightarrow S_{\underline{X}} = S_{X_1} \times \cdots \times S_{X_p}$.

Result 4:

Let $\underline{X}_1, \ldots, \underline{X}_p$ be A.C. r.v.s of (possibly) different dimensions. Then

(a) $\underline{X}_1, \ldots, \underline{X}_p$ are independent iff, for any arbitrary functions $\psi_1(\cdot), \ldots, \psi_p(\cdot)$, $\psi_1(\underline{X}_1), \ldots, \psi_p(\underline{X}_p)$ are independent.

(b) $\underline{X}_1, \ldots, \underline{X}_p$ are independent $\Leftrightarrow$ $P(\{\underline{X}_i \in A_i,\ i = 1, \ldots, p\}) = \prod_{i=1}^{p} P(\{X_i \in A_i\}),\ \forall\ A_1, \ldots, A_p.$

Remark 3:

(a) Result 3 remains valid (with obvious modifications) for independence of A.C. random vectors (of possibly different dimensions);

(b) Random vectors $\underline{X}$ and $\underline{Y}$ are independent

$$\Leftrightarrow F_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) = F_{\underline{Y}}(\underline{y}),\ \forall\ \underline{y} \in \mathbb{R}^q, \underline{x} \in S_{\underline{X}}$$

$$\Leftrightarrow f_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) = f_{\underline{Y}}(\underline{y}),\ \forall\ \underline{y} \in \mathbb{R}^q, \underline{x} \in S_{\underline{X}} \text{ with } f_{\underline{X}}(\underline{x}) > 0.$$

Result 5:

Let $\underline{X} = (X_1, \ldots, X_p)'$ be an A.C. r.v. with joint p.d.f. $f_{\underline{X}}(\cdot)$. Then $X_1, \ldots, X_p$ are independent iff

$$f_{\underline{X}}(\underline{x}) = c \prod_{i=1}^{p} g_i(x_i),\ \underline{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p,$$

for some non-negative functions $g_i(\cdot), i = 1, \ldots, p$, defined on $\mathbb{R}$ and some positive constant c. In that case the marginal p.d.f. of X_i is

$$f_{X_i}(x) = c_i g_i(x),\ x \in \mathbb{R},$$

for some positive constant $c_i,\ i = 1, \ldots, p$.

Note: Result 5 remains valid (with obvious modifications) for independence of A.C. random vectors (of possibly different dimensions);

Example 1:

Let $\underline{X} = (X_1, X_2, X_3)'$ be an A.C. r.v. with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{c}{x_1 x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a fixed real constant.

(a) Determine c and the support of $\underline{X}$.

(b) Find the marginal p.d.f. of (X_2, X_3).

(c) Find the marginal p.d.f.s of X_1, X_2 and X_3.

(d) For fixed $0 < x_3 < x_2 < 1$, find the conditional p.d.f. of X_1 given $(X_2, X_3) = (x_2, x_3)$.

(e) For fixed $x_2 \in (0, 1)$, find the conditional p.d.f. of (X_1, X_3) given that $X_2 = x_2$.

(f) Are X_1, X_2 and X_3 independent r.v.s?

(g) Let $x_2 \in (0, 1)$ be fixed. Are X_1 and X_3 independent given that $X_2 = x_2$?

Solution:

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underline{X}}(x_1, x_2, x_3)\,dx_3\,dx_2\,dx_1 = 1$$

$$\Rightarrow \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{c}{x_1x_2}\,dx_3\,dx_2\,dx_1 = 1$$

$$\Rightarrow c\int_0^1\int_0^{x_1} \frac{1}{x_1}\,dx_2\,dx_1 = 1$$

$$\Rightarrow c\int_0^1 dx_1 = 1$$

$$\Rightarrow c = 1.$$

Also for $c = 1$, $f_{\underline{X}}(\underline{x}) \geq 0, \ \forall \ \underline{x} \in \mathbb{R}^3$.

$$\begin{aligned} S_{\underline{X}} &= \{\underline{x} \in \mathbb{R}^3 : P(\{\underline{X} \in N_r(\underline{x})\}) > 0, \ \forall \ r > 0\} \\ &= \{\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_3 \leq x_2 \leq x_1 \leq 1\}. \end{aligned}$$

(b) For $0 < x_3 < x_2 < 1$,

$$\begin{aligned} f_{X_2,X_3}(x_2, x_3) &= \int_{-\infty}^{\infty} f_{\underline{X}}(x_1, x_2, x_3) dx_1 \\ &= \int_{x_2}^{1} \frac{1}{x_1 x_2} dx_1 \\ &= \frac{-\ln x_2}{x_2}. \end{aligned}$$

Thus

$$f_{X_2,X_3}(x_2, x_3) = \begin{cases} \frac{-\ln x_2}{x_2}, & \text{if } 0 < x_3 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(c) For $0 < x_1 < 1$,

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underline{X}}(x_1, x_2, x_3)\,dx_3\,dx_2 \\ &= \int_0^{x_1}\int_0^{x_2} \frac{1}{x_1 x_2}\,dx_3\,dx_2 \\ &= \int_0^{x_1} \frac{1}{x_1}\,dx_2 \\ &= 1. \end{aligned}$$

Thus

$$f_{X_1}(x_1) = \begin{cases} 1, & \text{if } 0 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

From (b), for $0 < x_2 < 1$,

$$\begin{aligned} f_{X_2}(x_2) &= \int\limits_{-\infty}^{\infty} f_{X_2,X_3}(x_2,x_3)dx_3 \\ &= \int\limits_{0}^{x_2} \frac{-\ln x_2}{x_2} dx_3 \\ &= -\ln x_2. \end{aligned}$$

Thus

$$f_{X_2}(x_2) = \begin{cases} -\ln x_2, & \text{if } 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Similarly, for $0 < x_3 < 1$

$$f_{X_3}(x_3) = \int\limits_{-\infty}^{\infty} f_{X_2,X_3}(x_2,x_3)dx_2$$

$$= \int_{x_3}^{1} -\frac{\ln x_2}{x_2} dx_2$$

$$= \int_{0}^{-\ln x_3} t dt$$

$$= \frac{(\ln x_3)^2}{2}.$$

Thus

$$f_{X_3}(x_3) = \begin{cases} \frac{(\ln x_3)^2}{2}, & \text{if } 0 < x_3 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(d) For fixed $0 < x_3 < x_2 < 1$,

$$f_{X_1|(X_2,X_3)}(x_1|(x_2,x_3)) = \frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_2,X_3}(x_2,x_3)}$$

$$= \begin{cases} -\frac{1}{x_1 \ln x_2}, & \text{if } x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(e) For $x_2 \in (0, 1)$,

$$\begin{aligned} f_{X_1,X_3|X_2}(x_1, x_3|x_2) &= \frac{f_{X_1,X_2,X_3}(x_1, x_2, x_3)}{f_{X_2}(x_2)} \\ &= \begin{cases} -\frac{1}{x_1 x_2 \ln x_2}, & \text{if } x_2 < x_1 < 1, 0 < x_3 < x_2 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

(f) Clearly

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) \neq f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3), \ \forall \ \underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

Thus X_1, X_2 and X_3 not independent.

(g) For fixed $x_2 \in (0,1)$,

$$f_{X_1,X_3|X_2}(x_1,x_3|x_2) = g_1(x_1)g_2(x_3),\ \forall\ (x_1,x_3) \in \mathbb{R}^2,$$

where

$$g_1(x_1) = \begin{cases} \frac{1}{x_1}, & \text{if } x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_2(x_3) = \begin{cases} -\frac{1}{x_2 \ln x_2}, & \text{if } 0 < x_3 < x_2 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, given $X_2 = x_2$, X_1 and X_3 are independent.

Take Home Problem

Let $\underline{X} = (X_1, X_2)'$ be an A.C. r.v. with joint p.d.f.

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{cx_2}{1-x_1}, & \text{if } x_1 > 0, x_2 > 0, x_2 + x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a fixed real constant.

(a) Find the value c and the support of $\underline{X}$.

(b) Find the marginal p.d.f.s of X_1 and X_2.

(c) For fixed $x_2 \in (0, 1)$, find the conditional p.d.f. of X_1 given that $X_2 = x_2$.

(d) For fixed $x_1 \in (0, 1)$, find the conditional p.d.f. of X_2 given that $X_1 = x_1$.

(e) Are X_1 and X_2 independent?

Abstract of Next Module

We will study the expectations and moments of random vectors.

Thank you for your patience