CS201A/201: Math for CS I/Discrete Mathematics
Quiz-1

Max marks: 40
Time: 40 mins. 30-Aug-2014

1. *Answer all 4 questions. Answer in the separate answer booklet provided and please start each answer on a fresh page.*
2. *Your answers should be* **precise** *and* **brief**.
3. *You can consult* **only your own handwritten notes**. *Other material like photocopies, books, articles, electronic gadgets etc. is* **NOT** *allowed.*

1. A new born pair of rabbits become fertile after two months and produce a new born pair every month after that. If we introduce a new born pair into a large cage on 1^{st} Jan. of a year how many rabbits will there be at the end of the year (that is in the 12th month). Write the recursive formula for $P(n)$ the number of rabbits after n months. Assume no rabbit dies and every new born pair can breed after two months.

Solution:

It is easier to work out the recursive formula first. Also, it is simpler to work with pairs. For the n^{th} month, since no rabbits die, all rabbit pairs present in the $(n-1)^{th}$ month will be present in the n^{th} month, plus the new born pairs will be equal to the number of pairs present in $(n-2)^{th}$ month (it takes 2 months before a pair starts breeding).

So, $P(n) = P(n-1) + P(n-2)$ with $P(1) = 1,\ P(2) = 1$ as the base conditions.

This immediately gives us the following sequence *(month, no. of pairs)*:

$$(1,1)(2,1),(3,2)(4,3),(5,5),(6,8),(7,13),(8,21),(9,34),(10,55),(11,89),(12,144)$$

or 288 rabbits.

Historically, this is exactly the problem that was posed by Fibonacci (pseudonym of Leonardo of Pisa) and is today called the Fibonacci sequence. Though, Fibonacci sequences were known much earlier in Sanskrit tradition.

[7,3=10]

2. For $m, n \geq 1$ which is greater 2^{mn} or m^n. Prove your answer.

Solution:

Claim: $2^{mn} > m^n$.

Here are two proofs. Since lhs and rhs are ≥ 1 taking any positive root (that is n^{th} root for $n \geq 1$) does not change the order relation between lhs and rhs. So, take n^{th} root of both sides to get lhs$=2^m$ and rhs$=m$. $2^m > m$, $m \geq 1$ can be proved by induction very easily. It holds for $m = 1$. Assume true for arbitrary m. We have using induction hypothesis: $2^m + 2^m > m + m \implies 2^{m+1} > m + m \implies 2^{m+1} > m + 1$ since $m \geq 1$.

A second proof is by a double induction. It is clearly true for $m = 1$, $n = 1$.
Assume it is true for arbitrary $m,\ n$.
Now consider three cases a) $(m + 1,\ n)$, b) $(m, n + 1)$ and c) $(m + 1,\ n + 1)$.
Case a):

$$\begin{aligned} 2 &> \frac{m}{2^m} + \frac{1}{2^m} \quad \text{follows from induction hypothesis} \\ \implies 2 \times 2^m &> m + 1 \\ \implies 2^{m+1} &> (m + 1) \\ \implies 2^{(m+1)n} &> (m + 1)^n, \quad n \geq 1 \end{aligned}$$

Case b): By induction hypothesis $2^{mn}2^m > m^n m \implies 2^{m(n+1)} > m^{n+1}$.

Case c): By case a) $2^{(m+1)n} > (m + 1)^n$ and $2^{m+1} > (m + 1)$.
So, $2^{(m+1)n}2^{m+1} > (m + 1)^n(m + 1) \implies 2^{(m+1)(n+1)} > (m + 1)^{(n+1)}$.

[10]

3. Recall that a binary relation R over finite sets A, B is defined as $R \subseteq A \times B$. We can define the inverse of a relation as follows $R^{-1} \subseteq B \times A$ and $b\,R^{-1}\,a$ iff $a\,R\,b$. Complete the following table by filling the blank space:

R is a function	iff R^{-1} is
R is a surjection	iff R^{-1} is
R is an injection	iff R^{-1} is
R is a bijection	iff R^{-1} is
R is total	iff R^{-1} is

Solution:

Since R (and R^{-1}) are relations injection means that if $(a_1, b), (a_2, b) \in R$ then $a_1 = a_2$. That is distinct points are related to distinct elements. Note that it is possible to have

$(a, b_1), (a, b_2) \in R,\ b_1 \neq b_2$. Surjection means for every $b \in B$ there is some $a \in A$ such that $(a, b) \in R$.

R is a function whenever $(a, b_1), (a, b_2) \in R \implies b_1 = b_2$. That is a is related to at most one element of B. This means two distinct points in B cannot be related to the same $a \in A$ which means R^{-1} is injective.

R is a surjection implies every $b \in B$ is related to some $a \in A$ that means R^{-1} is total.

R is an injection means that distinct points in A cannot be related to the same $b \in B$. So, a $b \in B$ can be related to at most one $a \in A$. That is R^{-1} is a function. This also follows from the first property above.

R is a bijection means it is injective and surjective. This is a conjunction of the above two so R^{-1} is a total function.

R is total means every $a \in A$ has some $b \in B$ such that $(a, b) \in R$ which means R^{-1} is a surjection. Also, follows from the second property above.

[2x5=10]

4. Argue that if $a \notin A$ and $f : A \cup \{a\} \rightleftarrows A$ (there is a bijection between $A \cup \{a\}$ and A) then A is an infinite set.

Solution:

Unlike the previous question where we dealt with relations we are dealing with functions now. And by assumption they are implicitly total.

Suppose to the contrary that A is finite, say $A = \{a_1, \ldots, a_n\}$ and let $g : A \cup \{a\} \rightleftarrows A$. Since g is a bijection it is also an injection. That means two distinct points in $A \cup \{a\}$, $a \notin A$ must map to distinct points in A. This implies that $card(A) \geq card(A) + 1$ which is clearly a contradiction. So, A cannot be finite.

We now show that a bijection is possible with an infinite set. Consider the following enumeration for $\{a\} \cup \mathbb{N}$:

$$(a, 1)(1, 2), (2, 3), \ldots, (i, i + 1), \ldots$$

. So, we have a bijection between $\{a\} \cup \mathbb{N}$ and $\mathbb{N}$.

[10]