MSO 201a: Probability and Statistics
2016-2017-II Semester
Assignment-I

A. Illustrative Discussion Problems

1. (i) Let $\Omega = \{0, 1, 2, \ldots\}$. In each of the following cases, verify if $(\Omega, \mathcal{P}(\Omega), P)$ is a probability space:
   (a) $P(A) = \sum_{x \in A} e^{-\lambda}\lambda^x/x!, \quad A \in \mathcal{P}(\Omega),\ \lambda > 0$;
   (b) $P(A) = 0$, if A has a finite number of elements, and $P(A) = 1$, if A has infinite number of elements, $A \in \mathcal{P}(\Omega)$.

   (ii) Let $(\Omega, \mathcal{P}(\Omega), P)$ be a probability space and let $A, B, C, D \in \mathcal{P}(\Omega)$. Suppose that $P(A) = 0.6, P(B) = 0.5, P(C) = 0.4, P(A \cap B) = 0.3, P(A \cap C) = 0.2, P(B \cap C) = 0.2, P(A \cap B \cap C) = 0.1, P(B \cap D) = P(C \cap D) = 0, P(A \cap D) = 0.1$ and P(D)=0.2. Find:
   (a) $P(A \cup B \cup C)$ and $P(A^c \cap B^c \cap C^c)$; (b) $P((A \cup B) \cap C)$ and $P(A \cup (B \cap C))$;
   (c) $P((A^c \cup B^c) \cap C^c)$ and $P((A^c \cap B^c) \cup C^c)$; (d) $P(D \cap B \cap C)$ and $P(A \cap C \cap D)$;
   (e) $P(A \cup B \cup D)$ and $P(A \cup B \cup C \cup D)$; (f) $P((A \cap B) \cup (C \cap D))$.

2. Suppose that $n\ (\geq 3)$ persons $P_1, \ldots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, 2, \ldots, n-2\}$.

3. Suppose that we have $n\ (\geq 2)$ letters and corresponding n addressed envelopes. If these letters are inserted at random in n envelopes, find the probability that no letter is inserted into the correct envelope. Find the approximate value of this probability when there are $n = 50$ letters.

4. Let $\Omega = (0, 1]$ and let probability function P be such that $P((a, b]) = b - a$, where $0 \leq a < b \leq 1$.
   (a) Show that $\{b\} = \cap_{n=1}^{\infty}(b - \frac{1}{n+1}, b], \forall\ b\ \in (0, 1]$;
   (b) Show that $P(\{b\}) = 0, \forall\ b\ \in (0, 1]$;
   (c) Show that, for any countable set $A \in \mathcal{P}(\Omega)$, $P(A) = 0$;
   (d) For $n \in \mathbb{N}$, let $A_n = (0, \frac{1}{n}]$ and $B_n = (\frac{1}{2} + \frac{1}{n+2}, 1]$. Verify that $A_n \downarrow, B_n \uparrow$, $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$ and $P(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} P(B_n)$.

5. Consider four coding machines M_1, M_2, M_3 and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1} $(k = 1, 2, 3)$ which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3 and M_4 change the code with probability $\frac{3}{4}$. Given that the

machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

6. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that probabilities of the student clearing examinations in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. Assuming that the performances of the student in four subjects are independent, find the probability that the student will clear examination(s) of
   (a) all the subjects; (b) no subject; (c) exactly one subject;
   (d) exactly two subjects; (e) at least one subject.

7. For independent events $A_1, A_2, \ldots$ (i.e., any finite sub-collection of $\{A_1, A_2, \ldots\}$ is a collection of independent events), show that:
$$P(\bigcap_{i=1}^{\infty} A_i^c) \leq e^{-\sum_{i=1}^{\infty} P(A_i)}.$$
   Hence show that if $\sum_{i=1}^{\infty} P(A_i) = \infty$, then with certainty atleast one of the events $A_1, A_2, \ldots$ will occur.

8. Let A, B and C be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following:
   (a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$; (b) $P(A \cap B|C) = P(A|C)P(B|C)$ if A and B are independent events.

B. Practice Problems from the Text Book

Problem Nos.: 3.11, 3.13, 3.14, 3.17, 4.6, 4.7, 4.9, 4.14, 4.18, 4.20, 4.21, 4.25, 4.30, 4.31, 4.34.

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Solutions

Problem No. 1 (i) (a) Clearly

$$P(A) \geq 0 \quad \forall A \in \mathcal{P}(\Omega)$$

$$P(\Omega) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{\underline{|x}} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{\underline{|x}} = e^{-\lambda} e^{\lambda} = 1,$$

and for mutually exclusive (disjoint) events $A_1, A_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{x \in \bigcup_{i=1}^{\infty} A_i} \frac{e^{-\lambda}\lambda^x}{\underline{|x}} = \sum_{i=1}^{\infty} \sum_{x \in A_i} \frac{e^{-\lambda}\lambda^x}{\underline{|x}} \quad (\text{since } A_i\text{s are disjoint})$$

$$= \sum_{i=1}^{\infty} P(A_i).$$

(b) Let $A_i = \{i\}$, $i = 0, 1, 2, \ldots$. Then $\bigcup_{i=1}^{\infty} A_i = \Omega$ $= \{0, 1, 2, \ldots\}$ (an infinite set) so

$$1 = P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(\Omega) \neq \sum_{i=1}^{\infty} P(A_i) = 0. \quad \left[\begin{array}{l}\bigcup_{i=1}^{\infty} A_i \text{ is an infinite set} \\ \text{and each } \{i\} \text{ is a finite set}\end{array}\right]$$

Thus $P(\cdot)$ is not a probability function.

(ii) No need to solve in the tutorial. Just provide the answers:
(a) $P(A \cup B \cup C) = 0.9$; $P(A^c \cap B^c \cap C^c) = 0.1$; (b) $P((A \cup B) \cap C) = 0.3$; $P(A \cup (B \cap C)) = 0.7$; (c) $P((A^c \cup B^c) \cap C^c) = 0.4$; $P((A^c \cap B^c) \cup C^c) = 0.7$; (d) $P(D \cap B \cap C) = 0$; $P(A \cap C \cap D) = 0$; (e) $P(A \cup B \cup D) = 0.9$; $P(A \cup B \cup C \cup D) = 1$; (f) $P((A \cap B) \cup (C \cap D)) = 0.3$.

Problem No. 2 Total # of ways in which $P_1, \ldots, P_n$ can stand in a row $= \underline{|n}$
Total # of possible positions for P_1 and P_2 such that there are exactly r positions between P_1 and P_2

$$= \underline{|2} \times (n-r-1)$$

[Corresponds to permuting positions of P_1 and P_2]

[Corresponds to possible positions $\{1, r+2\}, \{2, r+3\}, \ldots, \{n-r-1, n\}$ for P_1 and P_2]

# of ways in which r persons can be chosen to stand between P_1 and P_2 $= \binom{n-2}{r}$

# of ways in which $P_1, \ldots, P_n$ can stand in a row such that there are exactly r persons between P_1 and P_2

$$= \left[\underline{|2} \times (n-r-1)\right] \times \binom{n-2}{r} \times \underline{|r} \times \underline{|n-r-2}$$

Corresponds to permutations of r persons between P_1 and P_2

Corresponds to permutation of $(n-r-2)$ persons excluding P_1, P_2 and r persons between P_1 and P_2

$= 2(n-r-1)\,\underline{|n-2}$

(one can write above figure directly by first fixing the positions of P_1 and P_2 in $\underline{|2} \times (n-r-1)$ ways and then considering $\underline{|n-2}$ permutations of remaining $(n-2)$ persons, excluding P_1 and P_2).

Thus

$$\text{Required probability} = \frac{2(n-r-1)\,\underline{|n-2}}{\underline{|n}} = \frac{2(n-r-1)}{n(n-1)}.$$

Problem 3 Let us label the letters as $L_1, \ldots, L_n$ and respective envelopes as $A_1, \ldots, A_n$. Define events:

E_i: Letter L_i is (correctly) inserted into envelope A_i, $i=1,\ldots,n$.

$$\text{Required probability} = P\left(\bigcap_{i=1}^{n} E_i^c\right) = P\left(\left(\bigcup_{i=1}^{n} E_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} E_i\right)$$

$$= 1 - (p_{1,n} - p_{2,n} + p_{3,n} + \cdots + (-1)^{n-1} p_{n,n}),$$

where, for $r=1,\ldots,n$,

$$p_{r,n} = \sum\sum\cdots\sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}).$$

n letters can be inserted into n envelopes in $\underline{|n}$ ways. For $1 \le i_1 < i_2 < \cdots < i_r \le n$, $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}$ is the event that letters $L_{i_1}, \ldots, L_{i_r}$ are inserted into correct envelopes. Number of cases favorable to this event is $\underline{|n-r}$. Thus

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_r}) = \frac{\underline{|n-r}}{\underline{|n}}, \quad 1 \le i_1 < i_2 < \cdots < i_r \le n$$

$$\Rightarrow p_{r,n} = \sum\sum\cdots\sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \frac{\underline{|n-r}}{\underline{|n}} = \binom{n}{r}\frac{\underline{|n-r}}{\underline{|n}} = \frac{1}{\underline{|r}}, \quad r=1,\ldots,n.$$

Thus

$$\text{Required probability} = 1 - \left(\frac{1}{\underline{|1}} - \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{(-1)^{n-1}}{\underline{|n}}\right)$$

$$= \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \cdots + \frac{(-1)^n}{\underline{|n}}$$

For $n=50$ $(n \to \infty)$,

$$\text{Required probability} = e^{-1} = 0.3678\ldots$$

Problem 4 (a) $\omega \in \bigcap_{n=1}^{\infty}(b-\frac{1}{n+1}, b] \iff b-\frac{1}{n+1} < \omega \leq b, \quad \forall\, n=1,2,\ldots$

$$\iff b \leq \omega \leq b$$
$$\iff \omega = b.$$

Thus $\bigcap_{n=1}^{\infty}(b-\frac{1}{n+1}, b] = \{b\}$.

(b) $(b-\frac{1}{n+1}, b] \downarrow$. On using continuity of probability function we have

$$P\left(\lim_{n\to\infty}(b-\tfrac{1}{n+1}, b]\right) = \lim_{n\to\infty} P\left((b-\tfrac{1}{n+1}, b]\right)$$

i.e., $P\left(\bigcap_{n=1}^{\infty}(b-\frac{1}{n+1}, b]\right) = \lim_{n\to\infty}\left(b-(b-\frac{1}{n+1})\right)$

$\Rightarrow P(\{b\}) = 0.$

(c) A is countable $\Rightarrow A = \{\omega_i : i \in \Delta\} = \bigcup_{i\in\Delta}\{\omega_i\}$ for some countable set Δ. Thus

$$P(A) = P\left(\bigcup_{i\in\Delta}\{\omega_i\}\right) = \sum_{i\in\Delta} P(\{\omega_i\}) = 0.$$

(d) $A_n = (0, \frac{1}{n}] \downarrow$ and $\bigcap_{n=1}^{\infty} A_n = \phi$

$\Rightarrow P(\lim A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(\phi) = 0$ (Continuity of probability function)

Also $\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty}\left(\frac{1}{n}-0\right) = 0.$

Also $B_n = (\frac{1}{2}+\frac{1}{n+2}, 1] \uparrow$ and $\bigcup_{n=1}^{\infty} B_n = (\frac{1}{2}, 1]$

$\Rightarrow P(\lim B_n) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = P\left((\frac{1}{2}, 1]\right) = \frac{1}{2}$

$\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{1}{2}.$

Problem 5 Define events:

E_i: the machine M_i produced code 1, $i=1,2,3,4$

Required probability $= P(E_1^c \mid E_4) = 1 - P(E_1 \mid E_4)$

We are given that $P(E_1) = \frac{3}{4}$. By Bayes' theorem

$$P(E_1 \mid E_4) = \frac{P(E_4\mid E_1)P(E_1)}{P(E_4\mid E_1)P(E_1) + P(E_4\mid E_1^c)P(E_1^c)}$$

$P(E_4 \mid E_1) = P($machines $M_2\ M_3\ M_4$ either make no code changes or make 2 code changes$)$

$$= \left(\frac{1}{4}\right)^3 + \binom{3}{2}\times\left(\frac{3}{4}\right)^2 \times \frac{1}{4} = \frac{7}{16}$$

$P(E_4 | E_1^c) = P($machine M_2, M_3, M_4 either make 1 code change or make 3 code change$)$

$$= \binom{3}{1} \times \frac{3}{4} \times \left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = \frac{9}{16}$$

Required probability $= 1 - \dfrac{\frac{7}{16} \times \frac{3}{4}}{\frac{7}{16} \times \frac{3}{4} + \frac{9}{16} \times \frac{1}{4}} = 1 - \frac{7}{10} = \frac{3}{10}$.

Problem 6 No need to solve in the tutorials. Just provide the answers

(a) $\frac{1}{120}$; (b) $\frac{1}{5}$; (c) $\frac{5}{12}$; (d) $\frac{7}{24}$; (e) $\frac{4}{5}$

Problem 7 $A_1, A_2, \ldots, A_n$ are independent $\Rightarrow A_1^c, A_2^c, \ldots, A_n^c$ are independent

$$\Rightarrow P\left(\bigcap_{i=1}^n A_i^c\right) = \prod_{i=1}^n P(A_i^c) = \prod_{i=1}^n [1 - P(A_i)] \le \prod_{i=1}^n e^{-P(A_i)} \quad (e^{-x} \ge 1-x, \forall x)$$

$$= e^{-\sum_{i=1}^n P(A_i)}$$

$$\Rightarrow \lim_{n\to\infty} P\left(\bigcap_{i=1}^n A_i^c\right) \le \lim_{n\to\infty} e^{-\sum_{i=1}^n P(A_i)} = e^{-\sum_{i=1}^\infty P(A_i)}$$

$$\Rightarrow P\left(\lim_{n\to\infty} \bigcap_{i=1}^n A_i^c\right) \le e^{-\sum_{i=1}^\infty P(A_i)} \quad \left(\bigcap_{i=1}^n A_i^c \downarrow, \text{ using Continuity of probability function}\right)$$

$$\Rightarrow P\left(\bigcap_{n=1}^\infty \bigcap_{i=1}^n A_i^c\right) \le e^{-\sum_{i=1}^\infty P(A_i)} \quad \left(\bigcap_{i=1}^n A_i^c \downarrow\right)$$

$$\Rightarrow P\left(\bigcap_{n=1}^\infty A_n^c\right) \le e^{-\sum_{i=1}^\infty P(A_i)}$$

$$\sum_{i=1}^\infty P(A_i) = \infty \quad \Rightarrow 0 \le P\left(\bigcap_{n=1}^\infty A_n^c\right) \le 0 \Rightarrow P\left(\bigcap_{n=1}^\infty A_n^c\right) = 0$$

$$\Rightarrow P\left(\bigcup_{n=1}^\infty A_n\right) = 1 - P\left(\left(\bigcup_{n=1}^\infty A_n\right)^c\right) = 1 - P\left(\bigcap_{n=1}^\infty A_n^c\right) = 1$$

$\Rightarrow P($at least one of events $A_1, A_2, \ldots$ occurs$)$

$$= P\left(\bigcup_{n=1}^\infty A_n\right) = 1.$$

Problem 8 (a) $P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{P(A | B \cap C)\, P(B \cap C)}{P(C)} = P(A | B \cap C)\, P(B|C)$.

(b) Let $\Omega = \{1, 2, 3, 4\}$ and let $P(\cdot)$ be such that $P(\{i\}) = \frac{1}{4}$, $i = 1, 2, 3, 4$. Clearly $P(\cdot)$ is a proper probability function. Let $A = \{1, 4\}$, $B = \{2, 4\}$ and $C = \{3, 4\}$. Then $P(A \cap B) = P(\{4\}) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)\, P(B)$. Thus A and B are independent.

$P(A \cap B | C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{1/4}{1/2} = \frac{1}{2}$; $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/4}{1/2} = \frac{1}{2}$

Similarly $P(B|C) = \frac{1}{2}$. Clearly $P(A \cap B | C) \ne P(A|C)\, P(B|C)$, although A and B are independent.