MSO 201a: Probability and Statistics

2015-2016-II Semester
Assignment-XII

A. Illustrative Discussion Problems

1. (a) Let $X_1, X_2, \ldots$ be a sequence of r.v.s, such that $X_n \sim \mathrm{U}(-n, n)$, $n = 1, 2, \ldots$. Does the $F_n(\cdot)$ of X_n converge to a d.f., as $n \to \infty$
   (b) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $\mathrm{U}(0, \theta)$, $\theta > 0$ r.v.s and let $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $Y_n = nX_{1:n}$, $n = 1, 2, \ldots$. Find the limiting distributions of $X_{1:n}$ and Y_n (as $n \to \infty$).

2. (a) Show that $\lim_{n\to\infty} 2^{-n} \sum_{k=0}^{r_n} \binom{n}{k} = \frac{1}{2}$, where r_n is the largest integer $\leq \frac{n}{2}$.
   (b) Let $X_n \sim \text{Poisson}(4n)$, $n = 1, 2, \ldots$, and let $Y_n = \frac{X_n}{n}$, $n = 1, 2, \ldots$.
   (i) Show that $Y_n \xrightarrow{p} 4$;
   (ii) Show that $Y_n^2 + \sqrt{Y_n} \xrightarrow{p} 18$;
   (iii) Show that $\frac{n^2Y_n^2+nY_n}{nY_n+n^2} \xrightarrow{p} 16$.

3. (a) Let $X_n \sim N(\frac{1}{n}, 1 - \frac{1}{n})$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{d} Z \sim N(0, 1)$.
   (b) Let $f(x) = \frac{1}{x^2}$, if $1 \leq x < \infty$, and $= 0$, elsewhere, be the p.d.f. of a r.v. X. Consider the random sample of size 72 from the distribution having p.d.f. $f(\cdot)$. Compute, approximately, the possibility that more than 50 of the items of the random sample are less than 3.
   (c) Let $X_1, X_2, \ldots$ be a random sample from Poisson(3) distribution and let $Y = \sum_{i=1}^{100} X_i$. Find, approximately, $P(100 \leq Y \leq 200)$.
   (d) Let $X \sim \text{Bin}(25, 0.6)$. Find, approximately, $P(10 \leq X \leq 16)$. What is the exact value of this probability?

4. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $P(X_n = x) = \frac{1}{2}$, if $x = -n^{\frac{1}{4}}$, $n^{\frac{1}{4}}$, and $= 0$, otherwise. Show that $\overline{X}_n \xrightarrow{p} 0$, as $n \to \infty$.

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s having the common Cauchy p.d.f. $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$, $-\infty < x < \infty$.
   (a) For any $\alpha \in (0, 1)$, show that $Y = \alpha X_1 + (1 - \alpha)X_2$ again has a Cauchy p.d.f. $f(\cdot)$.
   (b) Note that $\overline{X}_{n+1} = \frac{n}{n+1}\overline{X}_n + \frac{1}{n+1}X_{n+1}$ and hence, using induction, conclude that $\overline{X}_n$ has the same distribution as X_1.
   (c) Show that $\overline{X}_n$ does not converge in probability to any constant. (Note that $E(X_1)$ does not exist and hence the WLLN is not guaranteed).

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. In each of the following situations, find the M.M.E. and the M.L.E. of $g(\underline{\theta})$.

   (a) $f(x|\theta) = \theta(1-\theta)^{x-1}$, if $x = 1, 2, \ldots$, and $= 0$, otherwise; $\Theta = (0,1)$; $g(\theta) = \theta$.

   (b) $X_1 \sim \text{Poisson}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_\theta(X_1 + X_2 + X_3 = 0)$.

   (c) $X_1 \sim U(-\frac{\theta}{2}, \frac{\theta}{2})$; $\Theta = (0, \infty)$; $g(\theta) = (1+\theta)^{-1}$.

   (d) $X_1 \sim N(\mu, \sigma^2)$; $\underline{\theta} = (\mu, \sigma^2)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \frac{\mu^2}{\sigma^2}$.

   (e) $f(x|\underline{\theta}) = \sigma^{-1}\exp(-\frac{x-\mu}{\sigma})$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\theta)$, where $\theta \in \Theta$ is an unknown parameter. In each of the following situations, find the M.L.E. of θ.

   (a) $X_1 \sim N(\theta, 1)$, $\Theta = [0, \infty)$. (b) $X_1 \sim \text{Bin}(1, \theta)$, $\Theta = [\frac{1}{4}, \frac{3}{4}]$.

8. The lifetimes of a brand of a component are assumed to be exponentially distributed with mean (in hours) θ, where $\theta \in \Theta = (0, \infty)$ is unknown. Ten of these components were independently put in test. The only data recorded were the number of components that had failed in less than 100 hours versus the number that had not failed. It was found that three had failed before 100 hours. What is the M.L.E. of θ?

9. Let $X_1, \ldots, X_n$ be a random sample from a distribution having mean μ and finite variance σ^2. Show that $\overline{X}$ and S^2 are unbiased estimators of μ and σ^2, respectively.

10. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$.

    (a) $n \geq 2$, $f(x|\underline{\theta}) = \frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}}$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \mu$.

    (b) $X_1 \sim U(0, \theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

    (c) $X_1 \sim N(\theta, 1)$; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta^2$.

11. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from $\text{Exp}(\theta)$ distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Let the estimand be $g(\theta) = \theta^r$, where r is

a fixed positive integer. Find the M.L.E. $\delta_M(\underline{X})$ of $g(\theta)$ and also find the unbiased estimator based on M.L.E.

B. Practice Problems

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $N(\mu, \sigma^2)$ r.v.s, where $-\infty < \mu < \infty$ and $0 < \sigma < \infty$. Show that $Z_n = \sum_{i=1}^{n} X_i$, does not have a limiting distribution, as $n \to \infty$,.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $\text{Exp}(\theta)$, $\theta > 0$, r.v.s and let $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $Y_n = nX_{1:n}$, $n = 1, 2, \ldots$. Find the limiting distributions of $X_{1:n}$ and Y_n (as $n \to \infty$).

3. (a) Show that $X_n \xrightarrow{p} a \;\Leftrightarrow\; X_n - a \xrightarrow{p} 0 \;\Leftrightarrow\; |X_n - a| \xrightarrow{p} 0$.
(b) If $X_n \xrightarrow{p} a$ and $X_n \xrightarrow{p} b$, then show that $a = b$.
(c) Let a and $r > 0$ be real numbers. If $E(|X_n - a|^r) \to 0$, as $n \to \infty$, then show that $X_n \xrightarrow{p} a$.

4. (a) For $r > 0$ and $t > 0$, show that $E(\frac{|X|^r}{1+|X|^r}) - \frac{t^r}{1+t^r} \le P(|X| \ge t) \le \frac{1+t^r}{t^r} E(\frac{|X|^r}{1+|X|^r})$.
(b) Show that $X_n \xrightarrow{p} 0 \;\Leftrightarrow\; E(\frac{|X_n|^r}{1+|X_n|^r}) \to 0$, for some $r > 0$.

5. (a) If $\{X_n\}_{n\ge1}$ are identically distributed and $a_n \to 0$, then show that $a_n X_n \xrightarrow{p} 0$.
(b) If $Y_n \le X_n \le Z_n$, $n = 1, 2, \ldots$, $Y_n \xrightarrow{p} a$ and $Z_n \xrightarrow{p} a$, then show that $X_n \xrightarrow{p} a$.
(c) If $X_n \xrightarrow{p} c$ and $a_n \to a$, then show that $a_n X_n \xrightarrow{p} ac$.
(d) Let $X_n = \min(|Y_n|, a)$, $n = 1, 2, \ldots$, where a is a positive constant. Show that $X_n \xrightarrow{p} 0 \Leftrightarrow Y_n \xrightarrow{p} 0$.

6. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s with mean μ and finite variance. Show that:
(a) $\frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i \xrightarrow{p} \mu$;
(b) $\frac{6}{n(n+1)(2n+1)} \sum_{i=1}^{n} i^2 X_i \xrightarrow{p} \mu$.

7. Let $X_n \sim \text{Gamma}(\frac{1}{n}, n)$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{p} 1$.

8. (a) If $T_n = \max(|X_1|, \ldots, |X_n|) \xrightarrow{p} 0$, as $n \to \infty$, then show that $\overline{X}_n \xrightarrow{p} 0$. Is the conclusion true if only $S_n = \max(X_1, \ldots, X_n) \xrightarrow{P} 0$.
(b) If $\{X_n\}_{n\ge1}$ are i.i.d. $U(0,1)$ r.v.s. and $Z_n = (\prod_{i=1}^{n} X_i)^{\frac{1}{n}}$, $n = 1, 2, \ldots$. Find a real α such that $Z_n \xrightarrow{p} \alpha$.

9. Let $\{E_n\}_{n\ge1}$ be a sequence of i.i.d. Exp(1) r.v.s.
(a) Show that $T_n \equiv \sum_{i=1}^{n} E_i \sim \text{Gamma}(n, 1)$, $n = 1, 2, \ldots$.

(b) For any real number x, show that $\lim_{n\to\infty} \int_0^{n+x\sqrt{n}} \frac{e^{-t}t^{n-1}}{\Gamma(n)}dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$.
(c) For large values of n, show that an approximation (called the Stirling approximation) to the gamma function is: $\Gamma n \approx \sqrt{2\pi}e^{-n}n^{n-\frac{1}{2}}$.

10. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. In each of the following situations, find the M.M.E. and the M.L.E..

    (a) $f(x|\underline{\theta}) = \theta_1$, if $x = 1$, $= \frac{1-\theta_1}{\theta_2 - 1}$, if $x = 2, 3, \ldots, \theta_2$, and $= 0$, otherwise; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : 0 < z_1 < 1, z_2 \in \{2, 3, \ldots\}\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

    (b) $f(x|\theta) = K(\theta)x^\theta(1-x)$, if $0 \le x \le 1$, and $= 0$, otherwise; $\Theta = (-1, \infty)$; $g(\theta) = \theta$; here $K(\theta)$ is the normalizing factor.

    (c) $X_1 \sim \text{Gamma}(\alpha, \mu)$; $\underline{\theta} = (\alpha, \mu)$; $\Theta = (0, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\alpha, \mu)$.

    (d) $f(x|\underline{\theta}) = (\sigma\sqrt{2\pi})^{-1}x^{-1}\exp(-\frac{1}{2\sigma^2}(\ln x - \mu)^2)$, if $x > 0$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

    (e) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_\theta(X_1 \le 1)$.

    (f) $X_1 \sim U(\theta_1, \theta_2)$; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : -\infty < z_1 < z_2 < \infty\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

11. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$.

    (a) $f(x|\theta) = e^{-(x-\theta)}$, if $x > \theta$, and $= 0$, otherwise; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta$.

    (b) $n \ge 2$, $f(x|\underline{\theta}) = \frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}}$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \sigma$.

    (c) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta$.

12. Consider a single observation X from a distribution having p.m.f. $f(x|\theta) = \theta$, if $x = -1$, $= (1-\theta)^2\theta^x$, if $x = 0, 1, 2, \ldots$, and $= 0$, otherwise, where $\theta \in \Theta = (0, 1)$ is an unknown parameter. Determine all unbiased estimators of θ.

13. Let X_1, X_2 be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. Show that given any unbiased estimator, say $\delta(\underline{X})$, which is not permutation symmetric (i.e., $P_{\underline{\theta}}(\delta(X_1, X_2) = \delta(X_2, X_1)) < 1$, for some $\underline{\theta} \in \Theta$), there exists a permutation symmetric and unbiased estimator $\delta_U(\underline{X})$ which is better than $\delta(\cdot)$. Can you extend this result to the case when we have a random sample consisting of n (≥ 2) observations?

14. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$ distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Of the two estimators, the M.M.E. and the M.L.E, of θ, which one would you prefer with respect to the criterion of the bias?

15. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution having p.d.f.

$$f(x|\underline{\theta}) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, & \text{if } x > \mu \\ 0, & \text{otherwise,} \end{cases}$$

where $\underline{\theta} = (\mu, \sigma) \in \Theta = (-\infty, \infty) \times (0, \infty)$ is unknown. Let the estimand be $g(\underline{\theta}) = \mu$. Find an unbiased estimator of $g(\underline{\theta})$ which is based on the M.L.E..

MSO 201a: Probability and Statistics
2016-2017 - II Semester
Assignment No. XII
Solutions

Problem No. 1 (a) $X_n \sim U(-n, n) \Rightarrow f_n(x) = \begin{cases} \frac{1}{2n}, & \text{if } -n \le x \le n \\ 0, & \text{otherwise} \end{cases}$;

$$F_n(x) = \int_{-\infty}^{x} f_n(t)\,dt = \begin{cases} 0, & \text{if } x < -n \\ \frac{x+n}{2n}, & \text{if } -n \le x < n; \\ 1, & \text{if } x \ge n \end{cases}$$

$\lim_{n\to\infty} F_n(x) = \frac{1}{2} = F(x)$ $\forall x \in \mathbb{R}$. Clearly F is not a d.f.
$(F(-\infty) \ne 0 \quad F(\infty) \ne 1)$.

(b) The common p.d.f. and d.f. of X_i are $f(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < 0 \\ \frac{x}{\theta}, & 0 \le x < \theta \\ 1, & x \ge \theta \end{cases}$$

$$F_{X_{1:n}}(x) = P(X_{1:n} \le x) = 1 - P(X_{1:n} > x) = 1 - P(X_i > x,\ i = 1, \dots, n)$$
$$= 1 - \prod_{i=1}^{n} P(X_i > x) = 1 - (1 - F(x))^n = \begin{cases} 0, & x < 0 \\ 1 - (1 - \frac{x}{\theta})^n, & 0 \le x < \theta \\ 1, & x \ge \theta \end{cases}$$

Clearly $\lim_{n\to\infty} F_{X_{1:n}}(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$.

Let X be the r.v. degenerate at 0. Then d.f. of X is
$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$ and $\lim_{n\to\infty} F_{X_{1:n}}(x) = F(x)$ $\forall x \in C(F)$

$\Rightarrow X_{1:n} \xrightarrow{P} 0$.

Also
$$F_{Y_n}(x) = P(X_{1:n} \le \tfrac{x}{n}) = \begin{cases} 0, & x < 0 \\ 1 - (1 - \frac{x}{n\theta})^n, & 0 \le x < n\theta \\ 0, & \text{otherwise} \end{cases} \xrightarrow{n\to\infty} \begin{cases} 0, & x \le 0 \\ 1 - e^{-\frac{x}{\theta}}, & x > 0 \end{cases}$$

$= \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{\theta}}, & x \ge 0 \end{cases} = F(x)$ (say), which is d.f. $\text{Exp}(\theta)$. Therefore

$$Y_n \xrightarrow{d} Y \sim \text{Exp}(\theta).$$

$\boxed{\text{Problem No. 2}}$ (a) Let $X_1, X_2, \dots$ be i.i.d. $Bin(1, \frac{1}{2})$ r.v.s. and let $Z_n = \sum_{i=1}^{n} X_i = n\bar{X}$ (say). Then $Z_n \sim Bin(n, \frac{1}{2})$. By CLT (Central Limit Theorem) $\frac{\sqrt{n}(\bar{X} - \frac{1}{2})}{\sqrt{\frac{1}{4}}} \longrightarrow N(0, 1)$, (as $n \to \infty$)

($E(X_1) = \frac{1}{2}$, $Var(X_1) = \frac{1}{4}$). Thus $\frac{\sqrt{n}\left(\frac{Z_n}{n} - \frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} N(0, 1)$

as $n \to \infty$. and $\lim_{n\to\infty} P\left(\frac{\sqrt{n}\left(\frac{Z_n}{n} - \frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \le 0\right) = \Phi(0) = \frac{1}{2}$, i.e.

$\lim_{n\to\infty} P(Z_n \le \frac{n}{2}) = \frac{1}{2} \Rightarrow \lim_{n\to\infty} P(Z_n \le r_n) = \frac{1}{2} \Rightarrow \lim_{n\to\infty} \sum_{k=0}^{r_n} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{n-k}$

$= \frac{1}{2}$.

(b) $E(X_n) = 4n = Var(X_n) \Rightarrow E(Y_n) = 4$, $Var(Y_n) = \frac{4}{n}$.

(i) $E(Y_n) = 4$, $Var(Y_n) \to 0 \Rightarrow Y_n \xrightarrow{p} 4$

(ii) $Y_n \xrightarrow{p} 4$ and $g(x) = x^2 + \sqrt{x}$, $x \ge 0$ is continuous function

$\Rightarrow Y_n^2 + \sqrt{Y_n} \xrightarrow{p} 4^2 + \sqrt{4} = 18$

(iii) $Y_n \xrightarrow{p} 4 \Rightarrow \frac{n^2 Y_n^2 + n Y_n}{n Y_n + n^2} = \frac{Y_n^2 + \frac{Y_n}{n}}{\frac{Y_n}{n} + 1} \xrightarrow{p} \frac{4^2 + 0}{0 + 1} = 16$

($Y_n \xrightarrow{p} 4 \Rightarrow \frac{Y_n}{n} \xrightarrow{p} 0$, as $\frac{1}{n} \to 0$).

$\boxed{\text{Problem No. 3}}$ (a) For $x \in \mathbb{R}$, $F_n(x) = P(X_n \le x) = \Phi\left(\frac{x - \frac{1}{n}}{\sqrt{1 - \frac{1}{n}}}\right) \longrightarrow \Phi(x)$ as $n \to \infty$.

$\Rightarrow X_n \xrightarrow{d} Z \sim N(0, 1)$.

(b) Let $X_1, X_2, \dots, X_{72}$ be i.i.d. r.v.s with p.d.f. $f(x) = \begin{cases} \frac{1}{x^2}, & 1 < x < \infty \\ 0, & \text{o.w.} \end{cases}$

Define

$$Y_i = \begin{cases} 1, & \text{if } X_i < 3 \\ 0, & \text{if } X_i \ge 3 \end{cases}, \quad i = 1, 2, \dots$$

Then $S_{72} = \sum_{i=1}^{72} Y_i$ = # of observations in random sample that are less than 3

Clearly $Y_1, Y_2, \dots$ are i.i.d. $Bin(1, \theta)$ r.v.s where $\theta = P(X_i < 3)$

$= \int_1^3 \frac{1}{x^2}\, dx = \frac{2}{3}$. By CLT $\frac{\sqrt{n}(\bar{Y} - \theta)}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0, 1)$, i.e.

$\frac{S_{72} - 48}{4} \approx N(0, 1)$ (for $n = 72$ (large) and $\theta = \frac{2}{3}$)

$P(S_{72} > 50) = 1 - P(S_{72} \le 50) \overset{*}{=} 1 - \Phi\left(\frac{50 - 48}{4}\right) = 1 - \Phi(0.5)$

[* Note: One could use continuity correction (as dist. of discrete rv S_{72} is approximated by AC rv $Z \sim N(0,1)$) to write $P(S_{72} > 50)$ $= 1 - P(S_{72} \le 50) = 1 - P(S_{72} \le 50.5) = 1 - \Phi\left(\frac{50.5 - 48}{4}\right) = 1 - \Phi(.625)$ $= .26595$

(c) $\bar{X} = \frac{Y}{100}$, $E(X_i) = 3 = Var(X_i)$. Thus $\frac{\sqrt{n}(\bar{X}-\theta)}{\sigma} \xrightarrow{d} N(0,1)$, $\frac{10(\bar{X}-3)}{\sqrt{3}} \xrightarrow{d} N(0,1)$.

$P(100 \le Y \le 200) = P(Y \le 200) - P(Y < 100) = P(Y \le 200.5) - P(Y \le 99.5)$ (using continuity correction) $= P(\bar{X} \le 2.005) - P(\bar{X} \le .995) \approx \Phi\left(\frac{10(2.005-3)}{\sqrt{3}}\right) - \Phi\left(\frac{10(.995-3)}{\sqrt{3}}\right) =$

$\Phi(-5.7445) - \Phi(-11.5755) \approx 0.0 = 0$.

(d) By CLT (since $X \stackrel{d}{=} \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are i.i.d. $Bin(1, 0.6)$ rvs; here $n = 25$ is reasonably large),

$$\frac{5\left(\frac{X}{25} - 0.6\right)}{\sqrt{0.6 \times 0.4}} \approx N(0,1) \quad \text{i.e.} \quad \frac{X-15}{\sqrt{6}} \approx N(0,1)$$

$P(10 \le X \le 16) = P(X \le 16) - P(X < 10) = P(X \le 16.5) - P(X \le 9.5)$

$\approx \Phi\left(\frac{16.5-15}{\sqrt{6}}\right) - \Phi\left(\frac{9.5-15}{\sqrt{6}}\right) = \Phi(.6124) - \Phi(-2.2454) = .7291 - (1 - .9871) = .7162$.

Problem No. 4 $E(X_i) = 0$, $Var(X_i) = E(X_i^2) = \frac{\sqrt{i}}{2} + \frac{\sqrt{i}}{2} = \sqrt{i}$

$E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 0$, $Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sqrt{i}$

Clearly $0 \le Var(\bar{X}_n) \le \frac{n\sqrt{n}}{n^2} \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} E(\bar{X}_n) = 0$ and $\lim_{n\to\infty} Var(\bar{X}_n) = 0 \Rightarrow \bar{X}_n \xrightarrow{p} 0$.

Problem No. 5 (a) Joint p.d.f. of X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{\pi^2} \cdot \frac{1}{1+x_1^2} \times \frac{1}{1+x_2^2}, \quad -\infty < x_i < \infty, \ i = 1, 2$$

Consider the 1-1 transformation $(X_1, X_2) \to (Y, Z)$ defined by $Y = \alpha X_1 + (1-\alpha) X_2$, $Z = X_2 \Rightarrow x_1 = \frac{y - (1-\alpha) z}{\alpha}$, $x_2 = z$, $S_{X_1, X_2} = \mathbb{R}^2 \to S_{Y,Z} = \mathbb{R}^2$.

$$J = \begin{vmatrix} \frac{1}{\alpha} & -\frac{1-\alpha}{\alpha} \\ 0 & 1 \end{vmatrix} = \frac{1}{\alpha} > 0.$$

Thus joint p.d.f. of (Y, Z) is

$$f_{Y,Z}(y, z) = \frac{1}{\alpha} \cdot \frac{1}{\pi^2} \cdot \frac{1}{1 + \left(\frac{y}{\alpha} - \frac{(1-\alpha)}{\alpha} z\right)^2} \cdot \frac{1}{1+z^2}, \quad -\infty < y < \infty, \ -\infty < z < \infty$$

For $-\infty < y < \infty$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y, z)\, dz = \frac{1}{\alpha \pi^2} \int_{-\infty}^{\infty} \frac{1}{1+z^2} \cdot \frac{1}{1 + \left(\frac{y}{\alpha} - \frac{1-\alpha}{\alpha} z\right)^2}\, dz$$

$$= \frac{\beta}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{1+z^2} \cdot \frac{1}{1 + ((\beta-1)z - \beta y)^2}\, dz, \quad \text{where } \beta = \frac{1}{\alpha}.$$

We can write

$$\frac{1}{1+z^2}\cdot\frac{1}{1+\{(\beta-1)z-\beta y\}^2}=\frac{2Az}{1+z^2}+\frac{B}{1+z^2}+\frac{2(\beta-1)\{(\beta-1)z-\beta y\}C}{1+\{(\beta-1)z-\beta y\}^2}$$

$$+\frac{(\beta-1)D}{1+\{(\beta-1)z-\beta y\}^2},\quad \text{where } A=-C,\quad C=\frac{-\beta(\beta-1)y}{(1+y^2)[\beta^2y^2+\beta^2(\beta-2)^2]},$$

$$B=\frac{1+\beta^2y^2-(\beta-1)^2}{\beta^2[\beta^2y^2+(\beta-2)^2][1+y^2]}$$

$$D=-\frac{(\beta-1)[1+\beta^2y^2-(\beta-1)^2]}{\beta^2\{\beta^2y^2+(\beta-2)^2\}(1+y^2)}+\frac{2\beta^2(\beta-1)y^2}{\beta^2\{\beta^2y^2+(\beta-2)^2\}(1+y^2)}$$

Thus, for $-\infty<y<\infty$,

$$f_Y(y)=\frac{\beta}{\pi^2}\left[A\ln\left\{\frac{1+z^2}{1+\{(\beta-1)z-\beta y\}^2}\right\}\Bigg|_{-\infty}^{\infty}+(B+D)\pi\right]$$

$$=\frac{\beta}{\pi}\cdot(B+D)=\frac{1}{\pi}\cdot\frac{1}{1+y^2},\ -\infty<y<\infty.$$

(b) Result is clearly true for $n=2$ (using (a) with $\alpha=\frac{1}{2}$). Assuming that the result is true for $n=m$.

$$\bar{X}_{m+1}=\frac{m}{m+1}\bar{X}_m+\frac{1}{m+1}X_{m+1}$$

But $\bar{X}_m$ and X_{m+1} are i.i.d. with p.d.f. $f(\cdot)$. Now on using (a) with $\alpha=\frac{m}{m+1}$ we conclude that $\bar{X}_{m+1}$ also has p.d.f. $f(\cdot)$. Hence the result follows by induction.

(c) $\bar{X}_n \overset{d}{=} X_1$, $n=1,2,\ldots \Rightarrow \bar{X}_n \xrightarrow{d} X_1$. Here $\bar{X}_n$ does not converge in probability to any constant.

$\boxed{\text{Problem No. 6}}$ (a) M.L.E. = M.M.E = $\frac{1}{\bar{X}}$ $\begin{cases} E_\theta(X_1)=\frac{1}{\theta} \\ \ln L_{\underline{x}}(\theta)=n\ln\theta+(\sum x_i-n)\ln(1-\theta) \end{cases}$

(b) $g(\theta)=P(X_1+X_2+X_3=0)=e^{-3\theta}$

M.L.E. of θ is $\bar{X}$ $\Rightarrow$ M.L.E. of $g(\theta)$ is $\delta_{ML}(\underline{X})=e^{-3\bar{X}}$

$E_\theta(X_1)=\theta,\ \forall\ \theta>0 \Rightarrow$ M.M.E. of θ is $\bar{X}$ $\Rightarrow$ M.M.E. of $g(\theta)$ is $\delta_{MME}(\underline{X})=e^{-3\bar{X}}$

(c) $L_{\underset{\sim}{x}}(\theta)=\prod_{i=1}^{n}f_{X_i}(x_i|\theta)=\begin{cases}\frac{1}{\theta^n}, & \text{if } \theta\geq 2t \\ 0, & \text{o.w.}\end{cases}$, where

$t=\max\{|x_1|,\ldots,|x_n|\}$.

$\Rightarrow$ M.L.E. of θ is $2T=2\max\{|X_1|,\ldots,|X_n|\}$

$\Rightarrow$ M.L.E of $g(\theta)=\frac{1}{1+\theta}$ is $\delta_{ML}(\underline{X})=(1+2T)^{-1}$.

$E_\theta(X_1) = 0,\ \forall\ \theta \in \mathbb{R}$. So method of moments for estimation fails.

But $E_\theta(X_1^2) = \frac{\theta^2}{12},\ \forall\ \theta$. So modified M.M.E. of θ can be obtained from $A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\hat{\theta}^2}{12} \Rightarrow \hat{\theta} = \sqrt{12 A_2}$. So modified M.M.E is $\delta_{MME}(\underline{X}) = (1 + \sqrt{12 A_2})^{-1}$.

(d) M.L.E. of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2)$
$= (A_1, A_2 - A_1^2)$. So M.L.E. of $g(\underline{\theta})$ is $\delta_{ML}(\underline{X}) = \dfrac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}$

$E(X_1) = \mu,\ E(X_1^2) = \mu^2 + \sigma^2 \Rightarrow$ M.M.E. of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (A_1, A_2 - A_1^2). \Rightarrow$ M.M.E. of $g(\underline{\theta})$ is $\delta_{MME}(\underline{X})$

$$= \frac{\bar{X}}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \delta_{ML}(\underline{X}).$$

(e) $$L_{\underline{X}}^*(\mu,\sigma) = \ln L_{\underline{X}}(\mu,\sigma) = \begin{cases} -n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(X_i - \mu), & \text{if } \mu \le X_{(1)} \\ 0, & \text{o.w.} \end{cases},\ \sigma > 0$$

where $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

Clearly $L_{\underline{X}}^*(\mu,\sigma) \le L_{\underline{X}}^*(X_{(1)},\sigma),\ \forall\ \mu \le X_{(1)},\ \sigma > 0$

$L_X^*(X_{(1)},\sigma) = -n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(X_i - X_{(1)}),\ \sigma > 0$

$\frac{\partial}{\partial\sigma} L_{\underline{X}}^*(X_{(1)},\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - X_{(1)})$

$L_{\underline{X}}^*(X_{(1)},\sigma)\ \uparrow\ (\downarrow)$ if $\sigma < (>)\ \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}) = \hat{\sigma}$ (say).

Thus
$$L_{\underline{X}}^*(\mu,\sigma) \le L_{\underline{X}}^*(X_{(1)},\sigma) \le L_{\underline{X}}^*\left(X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})\right),\ \forall\ \mu \le X_{(1)},\ \sigma > 0$$

$\Rightarrow$ M.L.E of (μ,σ) is $\delta_{MLE}(\underline{X}) = (\delta_{1MLE}(\underline{X}), \delta_{2MLE}(\underline{X})) = (X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}))$.

One can see that $E_{\underline{\theta}}(X_1) = \mu + \sigma,\ \forall\ \underline{\theta} \in \Theta,\ E_{\underline{\theta}}(X_1^2) = (\mu+\sigma)^2 + \sigma^2\ \forall\ \underline{\theta} \in \Theta$. So M.M.E. $\delta_{MME}(\underline{X}) = (\delta_{1MME}(\underline{X}), \delta_{2MME}(\underline{X}))$ is given by

$A_1 = \delta_{1MME} + \delta_{2MME}$ $\quad A_2 = (\delta_{1MME} + \delta_{2MME})^2 + \delta_{2MME}^2$

$\Rightarrow \delta_{2MME} = \sqrt{\frac{n-1}{n}}\, S$ $\quad$ and $\delta_{1MME} = \bar{X} - \sqrt{\frac{n-1}{n}}\, S$,

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})$.

Problem No. 7

(a) For a fixed realization $\underline{x}$

$$L^*_{\underline{x}}(\theta) = \ln L_{\underline{x}}(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2, \quad \theta \geq 0$$

$$\frac{d}{d\theta} L^*_{\underline{x}}(\theta) = \sum_{i=1}^{n}(x_i-\theta) = n\bar{x} - n\theta$$

Thus $L^*_{\underline{x}}(\theta) \uparrow (\downarrow)$ if $\theta < \bar{x}$ $(\theta > \bar{x})$

Case I $\bar{x} \geq 0$

$L^*_{\underline{x}}(\theta)$, $\theta \in [0,\infty)$ is maximized at $\theta = \bar{x}$

Case II $\bar{x} < 0$

$L^*_{\underline{x}}(\theta)$, $\theta \in [0,\infty)$, is maximized at $\theta = 0$.

Thus the M.L.E. of θ $(\theta \in \Theta = [0,\infty))$ is $\delta_{MLE}(\underline{X}) = \max(\bar{X}, 0)$.

(b) For a fixed realization $\underline{x}$

$$L^*_{\underline{x}}(\theta) = \ln L_{\underline{x}}(\theta) = \ln\left\{\prod_{i=1}^{n}\binom{1}{x_i}\theta^{x_i}(1-\theta)^{1-x_i}\right\} = \ln\left\{c_{\underline{x}}\,\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\right\},$$

where $c_{\underline{x}} = \prod_{i=1}^{n}\binom{1}{x_i}$.

$$L^*_{\underline{x}}(\theta) = \ln c_{\underline{x}} + \left(\sum_{i=1}^{n} x_i\right)\ln\theta + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-\theta)$$

$$\frac{d}{d\theta} L^*_{\underline{x}}(\theta) = \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{1-\theta} \geq (\leq)\, 0 \iff \theta \leq (\geq)\, \bar{x}$$

Thus $L^*_{\underline{x}}(\theta) \uparrow (\downarrow)$ if $\theta \leq \bar{x}$ $(\theta \geq \bar{x})$.

Case I $0 \leq \bar{x} < \frac{1}{4}$

$L^*_{\underline{x}}(\theta)$ is maximized at $\theta = \frac{1}{4}$

Case II: $\frac{1}{4} \leq \bar{x} \leq \frac{3}{4}$

$L^*_{\underline{x}}(\theta)$ is maximized at $\theta = \bar{x}$

Case III $\bar{x} \geq 3/4$

$L^*_{\underline{x}}(\theta)$ is maximized at $\theta = \frac{3}{4}$.

Thus the M.L.E. of θ $(\theta \in \Theta = [\frac{1}{4}, \frac{3}{4}])$ is

$$\delta_{MLE}(\underline{X}) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq \bar{X} < \frac{1}{4} \\ \bar{X}, & \text{if } \frac{1}{4} \leq \bar{X} \leq \frac{3}{4} \\ \frac{3}{4}, & \text{if } \bar{X} > \frac{3}{4}. \end{cases}$$

Problem No. 8 Let X = # of items that have failed in less than 100 hrs.

$X \sim Bin(10, \mu)$, where $\mu = \frac{1}{\theta}\int_0^{100} e^{-x/\theta}\, dx = 1 - e^{-\frac{100}{\theta}}$

$\Rightarrow \theta = \frac{-100}{\ln(1-\mu)}$. Given $X=3$, $\hat{\mu} = \frac{3}{10} = 0.3$ is the M.L.E of μ. Thus the M.L.E of θ is $\hat{\theta} = \frac{-100}{\ln(0.7)}$.

Problem 9 Let $\underline{\theta} = (\mu, \sigma^2)$. Then

$E_{\underline{\theta}}(\bar{X}) = E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu \quad \forall \underline{\theta} \in \Theta$

$E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i^2) = E_{\underline{\theta}}(X_1^2) = Var(X_1) + (E_{\underline{\theta}}(X_1))^2$

$= \sigma^2 + \mu^2$

$\Rightarrow E_{\underline{\theta}}(S^2) = E_{\underline{\theta}}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = \frac{n}{n-1} E_{\underline{\theta}}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right]$

$E_{\underline{\theta}}(\bar{X}^2) = Var(\bar{X}) + (E_{\underline{\theta}}(\bar{X}))^2 = \frac{\sigma^2}{n} + \mu^2 \quad \left(Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)\right)$

$E_{\underline{\theta}}(S^2) = \frac{n}{n-1}\left[\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2\right] = \sigma^2.$

Problem No. 10 (a) By Problem 6(e) M.L.E. of $\underline{\theta} = (\mu, \sigma)$ is $\left(X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})\right)$. So M.L.E. of $g(\underline{\theta})$ is $\delta_{MLE}(\underline{X}) = X_{(1)}$.

Clearly $E(X_i) = \frac{1}{\sigma}\int_{\mu}^{\infty} x e^{-\frac{x-\mu}{\sigma}} dx = \mu + \sigma$ and p.d.f. of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = \begin{cases} \frac{n}{\sigma} e^{-\frac{n}{\sigma}(x-\mu)}, & x > \mu \\ 0, & \text{o.w} \end{cases}$$

(easy to derive using d.f. $F_{X_{(1)}}(x) = 1 - P(X_i > x, i = 1, \ldots, n)$)

Clearly $E(X_{(1)}) = \mu + \frac{\sigma}{n}$. Thus

$E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\mu + \sigma - \mu - \frac{\sigma}{n}\right) = \frac{n-1}{n}\sigma, \quad \forall \underline{\theta}$

$\Rightarrow E_{\underline{\theta}}\left(X_{(1)} - \frac{1}{n(n-1)}\sum_{i=1}^{n}(X_i - X_{(1)})\right) = \mu, \quad \forall \underline{\theta}$

$\Rightarrow$ M.L.E based unbiased estimator of $g(\underline{\theta})$ is

$$\delta^{*}_{MLE}(\underline{X}) = X_{(1)} - \frac{1}{n(n-1)}\sum_{i=1}^{n}(X_i - X_{(1)}).$$

(b) As we have seen in lectures M.L.E. of θ is $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Thus the M.L.E. of $g(\theta)$ is $\delta_{MLE}(\underline{X}) = X_{(n)}^r$. The d.f. of $X_{(n)}$ is

$$F_{X_{(n)}}(x) = [F(x)]^n = \begin{cases} 0, & x<0 \\ \left(\frac{x}{\theta}\right)^n, & 0 \le x < \theta \\ 1, & x \ge \theta \end{cases}$$ and the p.d.f. of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$$

$$E_{\theta}(\delta_{MLE}(X)) = E(X_{(n)}^r) = \int_0^\theta x^r \frac{n x^{n-1}}{\theta^n} dx = \frac{n}{n+r}\theta^r, \ \forall \theta>0$$

$\Rightarrow E_{\theta}\left(\frac{n+r}{n} X_{(n)}^r\right) = \theta^r, \ \forall \theta > 0$

$\Rightarrow$ M.L.E based unbiased estimator of $g(\theta) = \theta^r$ is

$\delta^*_{MLE}(X) = \frac{n+r}{n} X_{(n)}^r$.

(c) It is straightforward to see that M.L.E. of θ is $\bar{X}$. Thus M.L.E. of $g(\theta)=\theta^2$ is $\bar{X}^2 = \delta_{MLE}(X)$, say.

$E_\theta(\delta_{MLE}(X)) = E_{\theta}(\bar{X}^2) = Var(\bar{X}) + (E_\theta(\bar{X}))^2 = \frac{1}{n} + \theta^2, \ \forall \theta$

$\Rightarrow E_{\theta}\left(\bar{X}^2 - \frac{1}{n}\right) = \theta, \ \forall \theta \in \mathbb{R}$

$\Rightarrow$ M.L.E based unbiased estimator of $g(\theta)$ is

$\delta^*_{MLE}(X) = \bar{X}^2 - \frac{1}{n}$.

Problem 12.11 For a fixed realization

$$L_x(\theta) = \begin{cases} \frac{1}{\theta^n} e^{-\sum_{i=1}^n x_i/\theta}, & \text{if } \theta > 0 \\ 0, & \text{o.w} \end{cases}$$

is maximized at $\theta = \bar{x}$. Thus the M.L.E. of $g(\theta)=\theta^r$ is $\delta_{MLE}(X) = \bar{X}^r = \frac{1}{n^r} T^r$, where $T = \sum_{i=1}^n X_i$ $\sim$ Gamma(n, θ) (reproductive property of gamma distn)

$$\Rightarrow E_\theta(T^r) = \int_0^\infty t^r \frac{e^{-t/\theta} t^{n-1}}{\theta^n \Gamma n} dt = \frac{\theta^{n+r}\Gamma(n+r)}{\theta^n \Gamma n}$$

$$\Rightarrow E_\theta\left(\frac{n^r \bar{X}^r (n-1)!}{(n+r-1)!}\right) = \theta^r, \ \forall \theta > 0$$

$\Rightarrow$ M.L.E. based unbiased estimator of $g(\theta) = \theta^r$ is

$$\delta^*_{MLE}(X) = \frac{n^r (n-1)!}{(n+r-1)!} \bar{X}^r.$$