Assignment 4
MSO 203B - Partial Differential Equation

October 22, 2018

Tutorial Problem

1. A general second order linear PDE is given by

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

where the coeffcient functions a, b, c, d, e, f, g are smooth. We characterize the equation based on its discriminant e.g.

- If $\Delta := b^2 - 4ac > 0$ then the equation is Hyperbolic.
- If $\Delta := b^2 - 4ac < 0$ then the equation is Elliptic.
- If $\Delta := b^2 - 4ac = 0$ then the equation is Parabolic.

Use a change of variable to show that the equation

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0,\ x > 0$$

admits only one family of characteristics and remains Parabolic in the new coordinates. This is an example of the fact that Parabolic equations admits only one family of characteristics.

2. Any linear second order PDE with smooth coefficients has a hidden structure in its form. This is generally known as the Canonical Form (aka Standard Form). What this means is that there exists a change of variable which when applied on the equation will yield one of the following:-

- It is of Elliptic type the "prototype" of which is the Laplace equation.
- It is of Parabolic type the "prototype" of which is the Heat equation.
- It is of Hyperbolic type the "prototype" of which is the Wave equation.

Using this information show that the equation given by $u_{xx} + 4u_{xy} + u_x = 0$ is essentially hyperbolic and that the change of variable $(x, y) \to (4x - y, y)$ brings out its canonical form. Try to find the general solution of the equation by solving the canonical form.

3. The main application of the canonical form is that it can be used to find solutions of apparently complicated looking PDE which may otherwise prove very hard to solve. To illustrate this point reduce the PDE $xu_{xx} + 2x^2u_{xy} = u_x - 1$ to its canonical form and find its solution.

4. The nature of a second order linear equation depends on the domain in question. Let us look deeply into it. Consider the equation

$$u_{xx} + x^2 u_{yy} = 0$$

Try to find the region in $\mathbf{R}^2$ where it is elliptic. Describe the family of curves obtained from the characteristic equation and while you are at it try and get the canonical form.

Practise Problem

1. Find the canonical form of the equation $u_{xx} - xu_{yy} = 0$ for both $x > 0$ and $x < 0$.
2. Show that the 1-D wave equation $u_{tt} = u_{xx}$ is hyperbolic and find its general solution.
3. Determine the region where the equation $u_{xx} + yu_{yy} + 2u_y = 0$ is elliptic and find its canonical form.